Properties of Languages with Catenation and Shuffle

Nils Erik Flick and Manfred Kudlek

Fachbereich Informatik, MIN-Fakultät, Universität Hamburg, DE
email: {flick,kudlek}@informatik.uni-hamburg.de

Abstract. We present finite automata for shuffle languages, iteration lemmata for languages with catenation and shuffle, closure properties of such language classes, as well as decidability results.

Keywords: shuffle catenation languages, structural and closure properties

1 Introduction

In [4] language classes based on the two operators catenation and shuffle were investigated, and a complete hierarchy was established. Such classes are important in the area of concurrency, in particular for modelling client/server systems [2, 3]. This paper actually is a continuation of [4]. For definitions and results we refer to it. Here we deal in particular with the *upper hierarchy* of that article.

In section 2 we introduce a finite automaton with shuffle as bascic operation as an analogon to well known finite automata with catenation as basic operation and show the equivalence to shuffle-rational sets. Section 3 deals with structural properties as iteration lemmata and semilinearity. Finally, in section 4 closure properties under language operations are investigated, as union, intersection, catenation, shuffle, their iterations, intersection with regular and shuffle-rational sets, complement, set difference, reversal, prefix, and quotient. In section 5 some decidability results are presented. Some complexity results are given in [1].

In this article, $\odot$ always denotes concatenation of words or languages and $\sqcup\!\sqcup$ denotes the shuffle of words, $w \sqcup\!\sqcup\, v = \{u_1v_1 \odot \cdots \odot u_nv_n | w = \bigodot u_i, v = \bigodot v_i\}$, also extended to languages by $L_1 \sqcup\!\sqcup\, L_2 = \bigcup_{v\in L_1, w\in L_2} v \sqcup\!\sqcup\, w$.

2 Shuffle Finite Automata

In case of shuffle as basic operator one can also define a finite automaton, the *shuffle finite automaton*. Remember that $\mathcal{SHUF} = \mathcal{RAT}(\sqcup\!\sqcup) = (\cup, \sqcup\!\sqcup, ^{\sqcup\!\sqcup})(\mathcal{FIN})$ is the closure of the finite languages under union, shuffle and iterated shuffle.

Definition 1. *Shuffle Finite Automaton*

A non-deterministic shuffle finite automaton $NSFA$ is defined similar to finite automata as $A = (Q, \Sigma, \delta, Q_0, Q_f)$ with Q a finite set of states, $Q_0 \subseteq Q$

the initial states, $Q_f \subseteq Q$ *the final states,* Σ *a finite set of symbols, and* $\delta \subseteq Q \times \Sigma^* \times Q$ *a finite set of transitions.*

A configuration of A *is a pair* $(q, w) \in Q \times \Sigma^*$*. A direct step of* A *is defined by* $(q, w) \vdash (q', w')$ *iff* $(q, \alpha, q') \in \delta \wedge \exists w' \in \Sigma^* : w \in \{\alpha\} \sqcup\!\sqcup \{w'\}$*. Let* $\vdash^*$ *denote the reflexive and transitive closure of* $\vdash$*. The language accepted by* A *is*

$L(A) = \{w \in \Sigma^* \mid \exists q_0 \in Q_0 \exists q_f \in Q_f : (q_0, w) \vdash^* (q_f, \lambda)\}$.

Let $\mathcal{NSFA}$ *denote the class of all languages accepted by* $NSFA$*.*

Lemma 1. $\mathcal{RAT}(\sqcup\!\sqcup) \subseteq \mathcal{NSFA}$*.*

Proof. $L = \{w\}$ with $w \in \Sigma^*$. Then L is accepted by the shuffle finite automaton $A = (\{q_0, q_f\}, \Sigma, \{(q_0, w, q_f)\}, \{q_0\}, \{q_f\})$.
$L = L_1 \cup L_2$. Let $A_i = (Q_i, \Sigma_i, \delta_i, Q_{0i}, Q_{fi})$ accept L_i. Wlog assume $Q_1 \cap Q_2 = \emptyset$. Then L is accepted by $A = (Q, \Sigma, \delta, Q_0, Q_f)$ where $Q = Q_1 \cup Q_2$, $Q_0 = \{q_0\}$, $Q_f = Q_{f1} \cup Q_{f2}$, $\Sigma = \Sigma_1 \cup \Sigma_2$, and

$\delta = \{(q_0, \lambda, q_{0i}) \mid q_{0i} \in Q_{0i}, = i \in \{1, 2\}\} \cup \delta_1 \cup \delta_2$.

$L = L_1 \sqcup\!\sqcup L_2$. Let $A_i = (Q_i, \Sigma_i, \delta_i, Q_{0i}, Q_{fi})$ accept L_i. Wlog assume $Q_1 \cap Q_2 = \emptyset$. Then L is accepted by $A = (Q, \Sigma, \delta, Q_0, Q_f)$ where

$Q = Q_1 \cup Q_2$, $Q_0 = Q_{01}$, $Q_f = Q_{f2}$, $\Sigma = \Sigma_1 \cup \Sigma_2$, and

$\delta = \{(q_{f1}, \lambda, q_{02}) \mid q_{f1} \in Q_{f1}, q_{02} \in Q_{02}\} \cup \delta_1 \cup \delta_2$

$L = L_1^{\sqcup\!\sqcup}$. Let $A' = (Q, \Sigma, \delta', Q_0, Q_f)$ accept M. Then L is accepted by the NSFA $A = (Q, \Sigma, \delta, Q_0, Q'_f)$ where $Q'_f = Q_0 \cup Q_f$, and

$\delta = \{(q_f, \lambda, q_0) \mid q_0 \in Q_0, q_f \in Q_f\} \cup \delta'$ □

Lemma 2. $\mathcal{NSFA} \subseteq \mathcal{RAT}(\sqcup\!\sqcup)$*.*

Proof. Let $A = (Q, \Sigma, \delta, Q_0, Q_f)$ be a NSFA. Construct a system of equations, using the set of states Q as variables: $q = q' \sqcup\!\sqcup \alpha$ iff $(q, \alpha, q') \in \delta$, $q = \alpha$ iff $(q, \alpha, q_f) \in \delta^* \wedge q_f \in Q_f$. Then $L(A) = \bigcup_{q_0 \in Q_0} q_0$. □

From this we get

Theorem 1. $\mathcal{NSFA} = \mathcal{SHUF}$*.*

As an example, consider $(\{q_0, q_1\}, \{a, b\}, \{(q_0, ab, q_1), (q_1, c, q_1), (q_1, cd, q_0)\}, \{q_0\}, \{q_1\})$, which accepts the language $\{ab\} \sqcup\!\sqcup (\{cd\} \sqcup\!\sqcup \{ab\} \cup \{c\})^{\sqcup\!\sqcup}$.

3 Structural Properties

In this section we exhibit iteration lemmata for the rational, linear and algebraic languages on catenation and shuffle, which generalize the well known lemmata for languages based on catenation (see also [7]), and semilinearity of all classes considered. Structural properties are used to show part of the relationships of $\mathcal{RAT}(\odot, \sqcup\!\sqcup), \mathcal{LIN}(\odot, \sqcup\!\sqcup), \mathcal{ALG}(\odot, \sqcup\!\sqcup)$ to the Chomsky hierarchy $\mathcal{REG} \subset \mathcal{CF} \subset \mathcal{CS}$.

3.1 Iteration Lemmata

Let the classes defined by systems of equations be given by grammars in normal form, i.e. all productions are of the form $X \to Y \circ \alpha$, $X \to \alpha$ for $\mathcal{RAT}(\odot,\sqcup)$, $X \to Y \circ \alpha$, $X \to \beta \circ Y$, $X \to \alpha$ for $\mathcal{LIN}(\odot,\sqcup)$, and $X \to Y \circ Z$, $X \to \alpha$ for $\mathcal{ALG}(\odot,\sqcup)$, where $X, Y, Z \in \mathcal{V}$ are variables, $\alpha, \beta \in \mathcal{C}$ singletons, $\circ \in \{\odot,\sqcup\}$.

Let $||w||$ denote the length of the word $||w||$ and for $\alpha \subseteq \Sigma^*$, $||\alpha|| := \max\{||w|| \mid w \in \alpha\}$. Let $X_1 \in \mathcal{V}$ be a distinguished (*initial*) variable,

$k = min\{||\alpha|| \mid \alpha \in \mathcal{C}, \alpha \neq \{\lambda\}\}$, and $K = max\{||\alpha|| \mid \alpha \in \mathcal{C}\}$.

Furthermore, wlog it can be assumed that $X \to \{\lambda\}$ implies $X = X_1$ and in that case X_1 doesn't occur on any righthand side of another production.

Consider now a derivation tree with depth d. Then for the set A defined by the yield of the tree, i.e. the set it evaluates to when seen as a term tree using $\odot$ and $\sqcup$ on the leaves, the following facts hold for $d > 1$:

$\mathcal{RAT}(\odot,\sqcup)$: $(d-1) \cdot k \leq ||A|| \leq d \cdot K$,

$\mathcal{LIN}(\odot,\sqcup)$: $(d-1) \cdot k \leq ||A|| \leq d \cdot K$,

$\mathcal{ALG}(\odot,\sqcup)$: $(d-1) \cdot k \leq ||A|| \leq 2^{d-1} \cdot K$.

Note that, by the property of $\odot$ and $\sqcup$, all $x \in A$ have the same length: $\forall x \in A \ : \ ||x|| = ||A||$.

In the derivation tree nodes are labelled by pairs $(X, \circ) \in \mathcal{V} \times \{\odot,\sqcup\}$, in case $X \to Y \circ Z$ with labels of children $(Y, \circ'), (Z, \circ'')$, in case $X \to Y \circ \alpha$ or $X \to \beta \circ Y$ with labels of children $(Y, \circ'), \alpha$ or $\beta, (Y, \circ')$, respectively, and in case $X \to \alpha$ with label of child α.

Let $\kappa = 2 \cdot |\mathcal{V}|$ for $\mathcal{RAT}$ or $\mathcal{LIN}$, and $\kappa = 2^{2 \cdot |\mathcal{V}|-1}$ for $\mathcal{ALG}$.

If now $\exists x \in L \ : \ ||x|| > \kappa \cdot K$ then there exists a derivation tree for which a longest path from the root to the leaf has $> 2 \cdot |\mathcal{V}|$ nodes labelled with some $(X, \circ)$. Therefore there exist 2 nodes with the same label on such a path. Numbering the labels from root to leaf with $(X_1, \circ_1)$ to $(X_d, \circ_d)$ one has $(X_i, \circ_i) = (X_j, \circ_j)$ for some $1 \leq i < j \leq d$ where X_ℓ are variables and $\circ_\ell \in \{\odot,\sqcup\}$. Note that for $\mathcal{RAT}$ and $\mathcal{LIN}$ there is just one longest path.

In the sequel we shall use prefix notation.

The first case to be considered is $\mathcal{RAT}$. Consider a derivation for a language L:

$(X_1, \circ_1) \to \circ_1(X_2, \circ_2)\alpha_1 \to^* \circ_1 \cdots \circ_{i-1}(X_i, \circ_i)\alpha_{i-1} \cdots \alpha_1$

$\to^* \circ_1 \cdots \circ_{i-1}\circ_i \cdots \circ_{j-1}(X_j, \circ_j)\alpha_{j-1} \cdots \alpha_i\alpha_{i-1} \cdots \alpha_1$

$\to \circ_1 \cdots \circ_{i-1}\circ_i \cdots \circ_{j-1}\circ_j(X_{j+1}, \circ_{j+1})\alpha_{j-1} \cdots \alpha_i\alpha_{i-1} \cdots \alpha_1$

$\to^* \circ_1 \cdots \circ_{i-1}\circ_i \cdots \circ_{j-1}\circ_j\circ_{j+1} \cdots \circ_{d-1}X_d\alpha_{d-1} \cdots \alpha_j\alpha_{j-1} \cdots \alpha_i\alpha_{i-1} \cdots \alpha_1$

$\to \circ_1 \cdots \circ_{i-1}\circ_i \cdots \circ_{j-1}\circ_j\circ_{j+1} \cdots \circ_{d-1}\alpha_d\alpha_{d-1} \cdots \alpha_j\alpha_{j-1} \cdots \alpha_i\alpha_{i-1} \cdots \alpha_1$

which can be written

$(X_1, \circ_1) \to^* \varrho(X_i, \circ_i)\xi \to^* \varrho\sigma(X_j, \circ_j)\eta\xi \to^* \varrho\sigma\tau\eta\xi$

where $\varrho = \circ_1 \cdots \circ_{i-1}$, $\sigma = \circ_i \cdots \circ_{j-1}$, $\tau = \circ_j \cdots \circ_{d-1}\alpha_d\alpha_{d-1} \cdots \alpha_j$,

$\xi = \alpha_{i-1} \cdots \alpha_1$, $\eta = \alpha_{j-1} \cdots \alpha_i$,

and τ, $\sigma\tau\eta$, $\varrho\sigma\tau\eta\xi$ are terms.

Since $(X_i, \circ_i) = (X_j, \circ_j)$ all $\sigma^k\tau\eta^k$ and $\varrho\sigma^k\tau\eta^k\xi$ are also terms and possible yields of derivation trees, and $\varrho\sigma^k\tau\eta^k\xi \subseteq L$ for $k \geq 0$.

Choosing now (i,j) with $(X_i, \circ_i) = (X_j, \circ_j)$ as the lowermost pair on the path, then $0 < ||\sigma\tau\eta|| \leq \kappa$. Furthermore,

$$0 < ||\eta|| =: \sum_{k=j-1}^{i} ||\alpha_k|| \leq \kappa \cdot K \ .$$

Therefore

Theorem 2. *For any $L \in \mathcal{RAT}(\odot, \sqcup\!\sqcup)$ there exists some $\kappa' \in \mathbb{N}$ such that for any $x \in L$ with $||x|| > \kappa'$ holds:*
$x \in \varrho\sigma\tau\eta\xi$ where $\varrho = \circ_1 \cdots \circ_{i-1}$, $\sigma = \circ_i \cdots \circ_{j-1}$,
$\tau = \circ_j \cdots \circ_{d-1}\alpha_d\alpha_{d-1} \cdots \alpha_j$, $\eta = \alpha_{j-1} \cdots \alpha_i$, $\xi = \alpha_{i-1} \cdots \alpha_1$
$\circ \in \{\odot, \sqcup\!\sqcup\}$, $\alpha \in \mathcal{C}$ constants
$0 < ||\sigma\tau\eta|| \leq \kappa'$
$\forall k \in \mathbb{N} \ : \ \varrho\sigma^k\tau\eta^k\xi \subseteq L$.

Lemma 3. *$L = \{a^n b^n \mid n \geq 0\} \notin \mathcal{RAT}(\odot, \sqcup\!\sqcup)$.*

Proof. If the iterated part in Theorem 2 consists either only of a's or b's then iteration would only increase the number of a's or b's, producing a word $\notin L$. If the iterated part contains both a's and b's then, since $A \odot B \subseteq A \sqcup\!\sqcup B$ would produce some word $uavbwaxby \notin L$ or $ubvawbxay \notin L$. □

This implies

Theorem 3. *$\mathcal{LIN} \not\subseteq \mathcal{RAT}(\odot, \sqcup\!\sqcup)$.*

The case $\mathcal{LIN}(\odot, \sqcup\!\sqcup)$ is similar. The difference is that the expressions ϱ and σ consists not only of operators $\circ$ but also constants α, η and ξ not only of constants α but also operators $\circ$. τ, $\sigma\tau\eta$, and $\varrho\sigma\tau\eta\xi$, as well as $\sigma^k\tau\eta^k$ and $\varrho\sigma^k\tau\eta^k\xi$ $(k \geq 0)$, are terms as in case $\mathcal{RAT}(\odot, \sqcup\!\sqcup)$, and $\varrho\sigma^k\tau\eta^k\xi \subseteq L$.

Taking now the uppermost pair $(X_i, \circ_i) = (X_j, \circ_j)$ $(i < j)$ one gets

$$||\varrho\sigma\eta\xi|| =: \sum_{k=1}^{j-1} ||\alpha_k|| \leq \kappa \cdot K \ .$$

From this one gets

Theorem 4. *For any $L \in \mathcal{LIN}(\odot, \sqcup\!\sqcup)$ there exists some $\kappa' \in \mathbb{N}$ such that for any $x \in L$ with $||x|| > \kappa'$ holds:*
$x \in \varrho\sigma\tau\eta\xi$ where ϱ, σ, η, ξ are expressions with operators and constants
$0 < ||\varrho\sigma\eta\xi|| \leq \kappa'$
$\forall k \in \mathbb{N} \ : \ \varrho\sigma^k\tau\eta^k\xi \subseteq L$.

Lemma 4. *$L_1 = \{a^m b^m c^n d^n \mid m, n \geq 0\} \notin \mathcal{LIN}(\odot, \sqcup\!\sqcup)$,*
$L_2 = \{a^m b^m a^n b^n \mid m, n \geq 0\} \notin \mathcal{LIN}(\odot, \sqcup\!\sqcup)$,
$L_3 = \{a^m b^m \mid m \geq 0\}^{\odot} \notin \mathcal{LIN}(\odot, \sqcup\!\sqcup)$.

Proof. L_1: If an iterated part contains 2 different symbols, e.g. a, b, then because of $A \odot B \subseteq A \sqcup\!\sqcup B$, some word *uavbwaxby* or *ubvawbxay* would be obtained. Therefore each iterated part can contain only 1 symbol. Now τ can contain only words $a^j b^k c^\ell d^m$. Hence the first iterated part can contain only a's or b's. But then words with different numbers of a's and b's would be obtained. Similarly for the second iterated part. Similar for L_2 and L_3. □

From this follows

Theorem 5. $\mathcal{CF} \not\subseteq \mathcal{LIN}(\odot, \sqcup\!\sqcup)$.

The case $\mathcal{ALG}(\odot, \sqcup\!\sqcup)$ is even more general. Here the expressions ϱ and σ consists not only of operators $\circ$ but also constants α and entire *terms*, η and ξ not only of constants α but also operators $\circ$ and entire terms. τ, $\sigma\tau\eta$, and $\varrho\sigma\tau\eta\xi$, as well as $\sigma^k\tau\eta^k$ and $\varrho\sigma^k\tau\eta^k\xi$ $(k \geq 0)$, are terms as in case $\mathcal{RAT}(\odot, \sqcup\!\sqcup)$, and $\varrho\sigma^k\tau\eta^k\xi \subseteq L$.

Choosing the lowermost pair $(X_i, \circ_i) = (X_j, \circ_j)$ $(i < j)$ yields $||\sigma\tau\eta|| \leq \kappa{\cdot}K$.

From this follows

Theorem 6. *For any* $L \in \mathcal{ALG}(\odot, \sqcup\!\sqcup)$ *there exists some* $\kappa' \in \mathbb{N}$ *such that for any* $x \in L$ *with* $||x|| > \kappa'$ *holds:*
$x \in \varrho\sigma\tau\eta\xi$ *where* ϱ, σ, η, ξ *are expressions with operators, constants, and terms*
$0 < ||\sigma\tau\eta|| \leq \kappa'$
$\forall k \in \mathbb{N}\ :\ \varrho\sigma^k\tau\eta^k\xi \subseteq L.$

Lemma 5. $L_1 = \{a^n b^n c^n \mid n \geq 0\} \notin \mathcal{ALG}(\odot, \sqcup\!\sqcup)$,
$L_2 = \{a^m c^n b^m d^n \mid m, n \geq 0\} \notin \mathcal{ALG}(\odot, \sqcup\!\sqcup)$.

Proof. Similar to the case $\mathcal{LIN}(\odot, \sqcup\!\sqcup)$. □

3.2 Semilinearity

Theorem 7. *Any* $L \in \mathcal{SHUF}$ *can be represented in the form*

$$L = \bigcup_{i=1}^{k} (\alpha_i \sqcup\!\sqcup A_i^{\sqcup\!\sqcup})$$

where $k < \infty$, α_i *are singletons, and* A_i *finite sets.*

Proof. Since $\sqcup\!\sqcup$ is commutative this is a special case of a general theorem in [8].

From this follows that $\mathcal{SHUF}$ is the same family that one gets when the operators iterated shuffle, shuffle and finally union of languages are applied any number of times, but in that order.

Lemma 6. $\mathcal{SHUF} = \mathcal{RAT}(\sqcup\!\sqcup) = (\cup)(\sqcup\!\sqcup)(^{\sqcup\!\sqcup})(\mathcal{FIN})$.

Any of the classes considered contain only semilinear sets.

Theorem 8. *Any* $L \in \mathcal{ALG}(\odot,\sqcup\!\sqcup)$ *is a semilinear set.*

Proof. Any $L \in \mathcal{ALG}(\odot,\sqcup\!\sqcup)$ can be generated by a grammar in normal form. From that construct a multiset grammar (also in normal form) as follows: for $X \to Y{\circ}Z$ take $X \to X \oplus Y$, for $X \to \alpha$ take $X \to \psi(\alpha)$, where $\psi(\alpha)$ is the Parikh image of α and $\oplus$ multiset addition. By the property of $\odot$ and $\sqcup\!\sqcup$ with respect to addition of multiplicites of symbols and [9] one gets a semilinear set, more precisely

$$\psi(L) = \bigcup_{i}^{k} (\psi(\alpha_i) \oplus \psi(A_i)^{\oplus}) \ .$$

□

Since $\{a^{2^n} \mid n \geq 0\} \in \mathcal{CS}$ is not semilinear one gets

Lemma 7. $\{a^{2^n} \mid n \geq 0\} \notin \mathcal{ALG}(\odot,\sqcup\!\sqcup)$.

Theorem 9. $\mathcal{ALG}(\odot,\sqcup\!\sqcup) \subset \mathcal{CS}$.

4 Closure Properties

In this section we consider closure properties of language classes under certain operators, primarily for the more important classes from [4]. These are, apart from $\mathcal{REG}$, $\mathcal{LIN}$ and $\mathcal{CF}$, the classes $\mathcal{ER}$ (*extended regular* expressions of [6]), $\mathcal{SHUF}$, $\mathcal{ES} = (\cup,\sqcup\!\sqcup,{}^{\odot},{}^{\sqcup\!\sqcup})(\mathcal{FIN})$ (*extended shuffle* expressions, in analogy to $\mathcal{ER}$), $\mathcal{SE} = (\cup,\odot,\sqcup\!\sqcup,{}^{\odot},{}^{\sqcup\!\sqcup})(\mathcal{FIN})$ (the languages defined by the so-called *shuffle expressions* of [5]), $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$, $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$, and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$. In the sequel $\wedge$ applied on language classes has the meaning $\mathcal{L}_1 \wedge \mathcal{L}_2 = \{L_1 \cap L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$. Similarly for operators $\vee$ (for $\cup$), $\odot$, and $\sqcup\!\sqcup$.

All closure results achieved so far are summarized in Table 1 on the facing page, where $\cup$, $\cap$, $\odot$, $\sqcup\!\sqcup$, ${}^{\odot}$, ${}^{\sqcup\!\sqcup}$, $\cap R$, $\cap S$, ${}^{-}$, $\setminus$, ^{R}, π, and $/_1$ denote union, intersection, catenation, shuffle, catenation and shuffle iteration, intersection with regular and shuffle-rational sets, complement, set difference, reversal, prefix, and left quotient with single words, respectively.

1. Union

Trivially, the classes $\mathcal{REG}, \mathcal{LIN}, \mathcal{CF}, \mathcal{SHUF} = \mathcal{RAT}(\sqcup\!\sqcup) = (\cup,\sqcup\!\sqcup,{}^{\sqcup\!\sqcup})(\mathcal{FIN})$, $\mathcal{ER} = (\cup,\odot,{}^{\odot},{}^{\sqcup\!\sqcup})(\mathcal{FIN})$, $(\cup,\sqcup\!\sqcup,{}^{\odot},{}^{\sqcup\!\sqcup})(\mathcal{FIN})$, $\mathcal{SE}$ are closed under $\cup$. For the classes $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$, $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$, and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ this follows by construction of a grammar as follows:

Let L_1, L_2 be generated by grammars in normal form with initial disjoint sets of variables and initial variables X_{11}, X_{21}. Then $L = L_1 \cup L_2$ is generated by a grammar with the union of variables for L_1, L_2, with additional initial variable X_1 and additional productions $X_1 \to X_{11}, X_1 \to X_{21}$.

2. Intersection

It is well known that $\mathcal{REG}$ is closed under $\cap$, whereas $\mathcal{LIN}$ and $\mathcal{CF}$ are not.

Lemma 8. $\mathcal{REG} \wedge \mathcal{SHUF} \not\subseteq \mathcal{ALG}(\odot,\sqcup\!\sqcup)$.

Table 1

	$\cup$	$\cap$	$\odot$	$\sqcup\!\sqcup$	${}^{\odot}$	${}^{\sqcup\!\sqcup}$	$\cap R$	$\cap \mathcal{S}$	${}^{-}$	$\setminus$	^{R}	π	$/_1$
$\mathcal{REG}$	Y	Y	Y	Y	Y	N	Y	N	Y	Y	Y	Y	Y
$\mathcal{LIN}$	Y	N	N	N	N	N	Y	N	N	N	Y	Y	Y
$\mathcal{CF}$	Y	N	Y	N	Y	N	Y	N	N	N	Y	Y	Y
$\mathcal{ER}$	Y	N	Y	N	Y	Y	N	N	N	N	Y	Y	N
$\mathcal{SHUF}$	Y	N	N	Y	N	Y	N	N	N	N	Y	Y	Y
$\mathcal{ES}$	Y	N	N	Y	Y	Y	N		N	N	Y	N	N
$\mathcal{SE}$	Y	N	Y	Y	Y	Y	N	N	N	N	Y	Y	Y
$\mathcal{RAT}(\odot,\sqcup\!\sqcup)$	Y	N	Y	Y	Y	Y	N	N	N	N	Y	Y	
$\mathcal{LIN}(\odot,\sqcup\!\sqcup)$	Y	N	N	N	N	N	N	N	N	N	Y	Y	
$\mathcal{ALG}(\odot,\sqcup\!\sqcup)$	Y	N	Y	Y	Y	Y	N	N	N	N	Y	Y	

Proof. Consider $L_1 = \{a\}^{\odot} \odot \{b\}^{\odot} \odot \{c\}^{\odot}$ and $L_2 = \{abc\}^{\sqcup\!\sqcup}$.
Then $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\} \notin \mathcal{ALG}(\odot,\sqcup\!\sqcup)$ by Lemma 5.
Note that $L_1 \in (\odot, {}^{\odot})(\mathcal{FIN})$ and $L_2 \in ({}^{\sqcup\!\sqcup})(\mathcal{FIN})$. □

The hierarchy results in [4] imply that $\mathcal{ER}$, $\mathcal{ES}$, $\mathcal{SE}$, $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$, $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$, and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ are not closed under $\cap$. It remains to investigate $\mathcal{SHUF}$.

Lemma 9. *$\mathcal{SHUF} \wedge \mathcal{SHUF} \not\subseteq \mathcal{SHUF}$.*

Proof. Consider $L_1 = \{da, bd, ba, c\}^{\sqcup\!\sqcup} \in \mathcal{SHUF}$ and $\{dd, acb\}^{\sqcup\!\sqcup} \in \mathcal{SHUF}$, and assume $L_1 \cap L_2 \in \mathcal{SHUF}$. It is easy to see that $x \in L_1 \cap L_2 \Rightarrow x = \lambda \vee x = dx'd$.

Furthermore $X = \{x_{mn} = d^n a^n c^n b^n d^n d(acb)^m d \mid m, n \geq 1\} \subseteq L_1 \cap L_2$ since $x_{mn} \in \{da\}^{(\sqcup\!\sqcup)n} \sqcup\!\sqcup \{bd\}^{(\sqcup\!\sqcup)n} \sqcup\!\sqcup \{c\}^{(\sqcup\!\sqcup)n} \sqcup\!\sqcup \{da\} \sqcup\!\sqcup \{bd\} \sqcup\!\sqcup \{ba\}^{(\sqcup\!\sqcup)(m-1)} \sqcup\!\sqcup \{c\}^{(\sqcup\!\sqcup)m}$ and $x_{mn} \in \{dd\}^{(\sqcup\!\sqcup)n} \sqcup\!\sqcup \{acb\}^{(\sqcup\!\sqcup)n} \sqcup\!\sqcup \{dd\} \sqcup\!\sqcup \{acb\}^{(\sqcup\!\sqcup)m}$.

By the normal form theorem for $\mathcal{SHUF}$

$$L_1 \cap L_2 = \bigcup_{i=1}^{k} (\{x_i\} \sqcup\!\sqcup A_i^{\sqcup\!\sqcup}) \ .$$

The structure of $L_1 \cap L_2$ implies $x_i = \lambda \vee x_i = dy_i d$ and $\forall z \in A_i \ : \ z = dz'd$. Since $|X| = \infty$, $\exists \ell \exists n \ : \ |\{x_{nm} \mid x_{nm} \in \{x_\ell \sqcup\!\sqcup A_\ell^{\sqcup\!\sqcup}\}| = \infty$ (otherwise $|L_1 \cap L_2 \cap X| < \infty$). There is a maximal m with $u(acb)^m d \in \{x_\ell\} \cup A_\ell$ since $\{x_\ell\} \cup A_\ell$ is finite. To get some $v(acb)^{m'} d \in X$ with $m' > m$ the only possibility is to use some $w(acb)^j d \in A_\ell$. But that cannot produce $u(acb)^{m'} d$ with $m' > m$. □

3. Catenation

From the well known closure of $\mathcal{REG}$ and $\mathcal{CF}$ under $\odot$, and the definition of the language classes it follows that $\mathcal{ER}$ and $\mathcal{SE}$ are closed under $\odot$. For $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$, $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$, and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ this is shown in a similar way as for union, adding a new production $X_1 \to X_{11} \odot X_{21}$. From the hierarchy results in [4] follows that $\mathcal{SHUF}$ and $\mathcal{ES}$ are not closed under catenation.

4. Shuffle

The hierarchy result in [4] and the definition of the language classes imply that $\mathcal{REG}$, $\mathcal{SHUF}$, $\mathcal{ES}$, $\mathcal{SE}$ are closed under $\sqcup\!\sqcup$, and that $\mathcal{ER}$ is not. For

$\mathcal{RAT}(\odot,\sqcup\!\sqcup)$, $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$, and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ closure is shown as for catenation, replacing $\odot$ by $\sqcup\!\sqcup$.

$\mathcal{LIN}$ and $\mathcal{CF}$ are not closed under $\sqcup\!\sqcup$ since $L_1 = \{a^m b^m \mid m \geq 0\} \in \mathcal{LIN}$ and $L_1 = \{c^n d^n \mid n \geq 0\} \in \mathcal{LIN}$ but

$(L_1 \sqcup\!\sqcup L_2) \cap (\{a\}^{\odot} \odot \{c\}^{\odot} \odot \{b\}^{\odot} \odot \{d\}^{\odot}) = \{a^m b^n c^m d^n \mid m, n \geq 0\} \notin \mathcal{CF}$.

5. Catenation Iteration

Closure under catenation iteration is well known for $\mathcal{REG}$ and $\mathcal{CF}$. For $\mathcal{ER}$, $\mathcal{ES}$ and $\mathcal{SE}$ this follows by definition of the classes. For $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$ and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ this is shown as follows:

Let L be generated by a grammar in normal form with initial variable X_1. For $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$ replace all productions $Y \to \alpha$ (α a constant) by productions $Y \to X_1 \odot \alpha$. For $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ add a production $X_1 \to X_1 \odot X_1$.

It is also well known that $\mathcal{LIN}$ is not closed under catenation closure. $\mathcal{SHUF}$ is not closed under $^{\odot}$ by the hierarchy result in [4]. $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$ is not closed under $^{\odot}$ by Lemma 4.

6. Shuffle Iteration

$\mathcal{ER}$, $\mathcal{ES}$, $\mathcal{SE}$ are closed under shuffle iteration by definition of the classes. For $\mathcal{RAT}(\odot,\sqcup\!\sqcup)$ and $\mathcal{ALG}(\odot,\sqcup\!\sqcup)$ this is shown as for catenation iteration, replacing $\odot$ by $\sqcup\!\sqcup$. $\mathcal{REG}$ is not closed under $^{\sqcup\!\sqcup}$ by the hierarchy results in [4] and definition.

$\mathcal{LIN}$, $\mathcal{CF}$ and $\mathcal{LIN}(\odot,\sqcup\!\sqcup)$ are not closed under shuffle iteration since

$L = \{a^m b^m, c^n d^n \mid m, n \geq 0\} \in \mathcal{LIN}$ but

$L^{\sqcup\!\sqcup} \cap (\{a\}^{\odot} \odot \{c\}^{\odot} \odot \{b\}^{\odot} \odot \{d\}^{\odot}) = \{a^m c^n b^m d^n \mid m, n \geq 0\} \notin \mathcal{ALG}(\odot,\sqcup\!\sqcup)$.

7. Intersection with Regular Sets

Closure of $\mathcal{REG}$, $\mathcal{LIN}$ and $\mathcal{CF}$ under intersection with regular sets is well known. All other classes, except for $\mathcal{ER}$, are not closed under $\cap R$ by Lemma 8.

8. Intersection with Shuffle-rational Sets

An analogous closure property is intersection with shuffle-rational sets ($\cap S$). Lemma 8 implies that apart from $\mathcal{RAT}$ and $\mathcal{ES}$ all other classes are not closed under $\cap S$.

9. Complement

Since except for $\mathcal{REG}$ all classes are not closed under intersection these classes are not closed under complement either. Thus the existence of an equivalent deterministic shuffle finite automaton is rather improbable.

10. Set Difference

Since all classes contain the language Σ^* they are not closed under set difference either, except for $\mathcal{REG}$.

11. Reversal

All language classes considered are closed under reversal (or mirror image). For the classes defined by algebraic closure this is a consequence from reversing all constants, and for those defined by grammars in normal form a consequence from reversing all constants and replacing productions $X \to Y \circ Z$ by $X \to Z \circ Y$.

12. Prefix

The set of prefixes of a language is defined as $\pi(L) = \{x \mid \exists y : xy \in L\}$. Similarly suufixes are given by $\sigma(L) = \{x \mid \exists y : yx \in L\}$. Then one has:

Lemma 10. *1.* $\pi(L_1 \cup L_2) = \pi(L_1) \cup \pi(L_2)$,

2. $\pi(L_1 \odot L_2) = \pi(L_1) \cup L_1 \odot \pi(L_2)$,
3. $\pi(L^{\odot}) = L^{\odot} \odot \pi(L)$,
4. $\pi(L_1 \sqcup\!\sqcup L_2) = \pi(L_1) \sqcup\!\sqcup \pi(L_2)$,
5. $\pi(L^{\sqcup\!\sqcup}) = (\pi(L))^{\sqcup\!\sqcup}$.

Proof. 1., 2.: trivial.

3.: $\pi(L^{\odot}) = \pi(L^{\odot} \odot L \cup \{\lambda\}) = \pi(L^{\odot} \odot L) \cup \{\lambda\}$
$= L^{\odot} \odot \pi(L) \cup \pi(L) \cup \{\lambda\} = L^{\odot} \odot \pi(L)$.

4.: Wlog the shuffle of two languages can be expressed as

$$L_1 \sqcup\!\sqcup L_2 = \{x \in \Sigma^* \mid x = \bigodot_{i=1}^{n} y_i z_i \wedge \bigodot_{i=1}^{n} y_i \in L_1 \wedge \bigodot_{i=1}^{n} z_i \wedge ||y_i|| \le 1 \wedge ||z_i|| \le 1\}$$

where $\bigodot$ denotes finite application of $\odot$. Then

$$\pi(L_1 \sqcup\!\sqcup L_2) = \{u \in \Sigma^* \mid ((u = (\bigodot_{i=1}^{m-1} y_i z_i) \odot y_m \wedge v = z_m \odot \bigodot_{i=m+1}^{n} y_i z_i)$$

$$\vee\ (u = \bigodot_{i=1}^{m} y_i z_i \wedge\ v = \bigodot_{i=m+1}^{n} y_i z_i)) \wedge \bigodot_{i=1}^{n} x_i \in L_1 \wedge \bigodot_{i=1}^{n} y_i \in L_2\}$$

$$= \{u \in \Sigma^* \mid (u = \bigodot_{i=1}^{m-1} y_i z_i) \odot y_m \wedge \bigodot_{i=1}^{m} y_i \in \pi(L_1) \wedge \bigodot_{i=1}^{m-1} z_i \in \pi(L_2))$$

$$\vee\ (u = \bigodot_{i=1}^{m} y_i z_i \wedge \bigodot_{i=1}^{m} y_i \in \pi(L_1) \wedge \bigodot_{i=1}^{m} z_i \in \pi(L_2))\} = \pi(L_1) \sqcup\!\sqcup \pi(L_2)\ .$$

5.: By induction from 4. □

From Lemma 10, the algebraic definition of classes and by induction on the structure of the expression follows that $\mathcal{REG}$, $\mathcal{ER}$, $\mathcal{SHUF}$ and $\mathcal{SE}$ are closed under prefix operation.

For the closure of $\mathcal{LIN}$ and $\mathcal{CF}$ we recall the following proofs:

$\mathcal{LIN}$: Consider a normal form grammar. Define a copy $\hat{\mathcal{V}}$ of the set of variables $\mathcal{V}$, with $\hat{X}_1$ as new initial variable, add a production $\hat{X} \to \hat{Y}$ for any production $X \to Y \odot \alpha$, $\hat{X} \to \alpha \odot \hat{Y}$ for any production $Y \to \alpha \odot Y$, and $\hat{X} \to X$ for any variable.

$\mathcal{CF}$: Similar to $\mathcal{LIN}$, to a grammar in normal form add new productions $\hat{X} \to \hat{Y} \cup Y \odot \hat{Z}$ for any production $X \to Y \odot Z$, and $\hat{X} \to X$ for any variable.

$\mathcal{RAT}(\odot, \sqcup\!\sqcup)$, $\mathcal{LIN}(\odot, \sqcup\!\sqcup)$ and $\mathcal{ALG}(\odot, \sqcup\!\sqcup)$ are also closed under the prefix operation. Transform the equations of the equation system in normal form as follows: For $X \to Y \sqcup\!\sqcup Z$, add a new production $X' \to Y' \sqcup\!\sqcup Z'$. Otherwise proceed as above. The correctness follows from lemma 10.

$\mathcal{ES}$ is not closed under prefix since $\pi(\{ab\}^{\odot}) = \{ab\}^{\odot} \odot \{a, \lambda\} \notin \mathcal{ES}$. This holds because for $x \in \{ab\}^{\odot} \odot \{a\}$ one has $||x||_a = ||x||_b + 1$, and any iteration by $^{\odot}$ or $^{\sqcup\!\sqcup}$ yields some y with $||y||_a > ||y||_b + 1$. Therefore a can only be added after an iteration. But for that only $\sqcup\!\sqcup$ is available, giving e.g. $uaav \notin \pi(\{ab\}^{\odot})$.

13. Quotient

The left (catenation) quotient of a language A by B is defined by $A/B = \{y \in \Sigma^* \mid \exists x \in B : xy \in A\}$. A special case is $A = \{x\}$ with $x \in \Sigma^*$. Trivially one has

Lemma 11. $A/(B \cup C) = A/B \cup A/C$, $\quad (A \cup B)/C = A/C \cup B/C$,
$A \subseteq B \odot (A/B)$, $\quad A/\{\lambda\} = A$,
$\lambda \in B \Rightarrow A \subseteq A/B$, $\quad A/B \subseteq \sigma(A)$,
$||\{x\}/B|| < \infty$.

Furthermore

Lemma 12. $A/(B \odot C) = (A/B)/C$, $\quad (A \odot B)/C = (A/C) \odot B \,\cup\, B/(C/A)$.

Proof.
$$\begin{aligned} z \in A/(B \odot C) &\Leftrightarrow \exists x \in B\ \exists y \in C\ :\ (xy \in B \odot C \wedge xyz \in A) \\ &\Leftrightarrow \exists x \in B\ \exists y \in C\ :\ (xy \in B \odot C \wedge yz \in A/B \wedge xyz \in A) \\ &\Leftrightarrow z \in (A/B)/C \end{aligned}$$
$$\begin{aligned} z \in (A \odot B)/C &\Leftrightarrow \exists x \in C\ \exists y' \in \Sigma^*\ \exists z' \in B\ :\ (xy' \in A \wedge z = y'z') \\ &\qquad \vee\ \exists x \in C\ \exists y \in A\ \exists z' \in \Sigma^*\ :\ (x = yz' \wedge z'z \in B) \\ &\Leftrightarrow \exists x \in C\ \exists y' \in \Sigma^*\ \exists z' \in B\ :\ (y' \in A/C \wedge z = y'z') \\ &\qquad \vee\ \exists x \in C\ \exists y \in A\ \exists z' \in \Sigma^*\ :\ (z' \in C/A \wedge z'z \in B) \\ &\Leftrightarrow z \in (A/C) \odot B \vee z \in B/(C/A). \end{aligned}$$
□

This implies

Corollary 1. $(A \odot B)/\{x\} = (A/\{x\}) \odot B \,\cup\, B/(\{x\}/A)$.

Note that $||\{x\}/A|| < \infty$. For $\circ \in \{\odot, \sqcup\!\sqcup\}$ define

$$A^{(\circ)0} = \{\lambda\}\ ,\ A^{(\circ)(j+1)} = A \circ A^{(\circ)j}\ .$$

Lemma 13.
$$A/B^{\odot} = A/\bigcup_{i=0}^{\infty} B^{(\odot)i} = \bigcup_{i=0}^{\infty}(A/B^{(\odot)i})\ .$$

Proof. Applying Lemma 11. □

Note that $A/B^{\odot}$ is the least fix point of the monotone operator τ defined by $\tau(A) = A \cup A/B$.

Lemma 14.
$$A^{\odot}/\{x\} = (\bigcup_{i=0}^{m}(A^{(\odot)i}/\{x\})) \odot A^{\odot}\ \ ,\ m \leq ||x||\ .$$

Proof. Wlog assume $x \neq \lambda$. x as a prefix can only be cut off from a a finite product $A^{(\odot)m}$ with $m \leq ||x||$. □

Lemma 15.
$$(A \sqcup\!\sqcup B)/\{x\} = \bigcup_{y \in \pi(A), z \in \pi(B), x \in \{y\} \sqcup\!\sqcup \{z\}} (A/\{y\} \sqcup\!\sqcup B/\{z\})\ .$$

Proof. $w \in (A ш B)/\{x\} \Leftrightarrow xw \in A ш B$
$\Leftrightarrow \exists y \in A\ \exists z \in B\ :\ xw \in \{y\} ш \{z\}$
$\Leftrightarrow \exists y_1 \in A\ \exists y_2 \in A\ \exists z_1 \in B\ \exists z_2 \in B\ :$
$(y = y_1 y_2 \wedge z = z_1 z_2 \wedge x \in \{y_1\} ш \{z_1\} \wedge w \in \{y_2\} ш \{z_2\})$
$\Leftrightarrow y_1 \in \pi(A) \wedge z_1 \in \pi(B) \wedge x \in \{y_1\} ш \{z_1\} \wedge w \in (A/\{y_1\}) ш (B/\{z_1\})$.
Note that for given x the union is finite. □

Lemma 16. $$A^{ш}/\{x\} = \bigcup_{y_j \in \pi(A), x \in W} \coprod_{i=1}^{\infty} (A/\{y_i\}) \ ,\ \ W = \coprod_{i=1}^{\infty} \{y_i\}\ .$$

Proof. By induction. Wlog let $x \neq \lambda$. $A^{(ш)0}/\{x\} = \{\lambda\}/\{x\} = \emptyset$.

$$A^{(ш)1}/\{x\} = A/\{x\} = \bigcup_{y_1 \in \pi(A), x \in \{y_1\}} A/\{x\}$$

$$A^{(ш)k}/\{x\} = \bigcup_{y_j \in \pi(A), x \in W_k} \coprod_{i=1}^{k} (A/\{y_i\})\ ,\ W_k = \coprod_{i=1}^{k} \{y_i\}\ .$$

$$A^{(ш)(k+1)}/\{x\} = (A^{(ш)k} ш A)/\{x\}$$

$$= \bigcup_{y \in A^{(\coprod)k}, y_{(k+1)} \in \pi(A), x \in \{y\} ш \{y_{(k+1)}\}} (A^{(ш)k}/\{y\}) ш (A/\{y_{(k+1)}\})$$

$$= \bigcup_{y \in \pi(A^{(\coprod)k}), y_{(k+1)} \in \pi(A), x \in \{y\} ш \{y_{(k+1)}\}} \ \bigcup_{y_j \in \pi(A), y \in W_k} \coprod_{i=1}^{k} (A/\{y_i\}) ш (A/\{y_{(k+1)}\})$$

$$= \bigcup_{y_j \in \pi(A), x \in W_{(k+1)}} \coprod_{i=1}^{k+1} (A/\{y_i\})\ ,\ W_{(k+1)} = \coprod_{i=1}^{k} \{y_i\}\ .$$

Here $\coprod$ denotes the finite application of ш. Again, for given x union and $\coprod$ are finite since there are only finitely many possible $y_j \in \pi(A)$. Therefore, for some m

$$A^{ш}/\{x\} = \bigcup_{y_j \in \pi(A), x \in W} \coprod_{i=1}^{m} (A/\{y_i\}) \ ,\ \ W = \coprod_{i=1}^{m} \{y_i\}\ .$$

□

From the considerations above follows

Lemma 17. *$\mathcal{SHUF}$ is closed under left quotient with a single word.*

Proof. By structural induction using lemmata 11, 15, 16. □

Lemma 18. *$\mathcal{SE}$ is closed under left quotient with a single word.*

Proof. By structural induction using lemmata 11, 1, 15, 14, 16. □

Symmetrically,both are also closed under right quotient with a single word.

For the classes $\mathcal{REG}$, $\mathcal{LIN}$, and $\mathcal{CF}$ it is well known that they are closed under left and right quotient with regular sets, hence also under left and right quotient with a single word.

Since $\{ab\}^{\odot}/_1\{a\} = \{b\} \odot \{ab\}^{\odot} \notin \mathcal{ES}$ which can be shown as for the nonclosure of $\mathcal{ES}$ under prefix, $\mathcal{ES}$ is not closed under left quotient with a single word. $\mathcal{ER}$ is not because $\{abc\}^{\sqcup\!\sqcup}/\{a\} = \{bc\} \sqcup\!\sqcup \{abc\}^{\sqcup\!\sqcup} \notin \mathcal{ER}$ by [4] Lemma 7.

5 Decidability Properties

In this section we present some decidability results.

Since $\mathcal{ALG}(\odot, \sqcup\!\sqcup) \subset \mathcal{CS}$ it follows that

Theorem 10. *The membership problem for* $\mathcal{ALG}(\odot, \sqcup\!\sqcup)$ *is decidable.*

Theorem 11. *Emptiness and finiteness for* $\mathcal{ALG}(\odot, \sqcup\!\sqcup)$ *are decidable.*

Proof. For the emptiness problem check whether there exists some $w \in L$ with $||w|| \leq 2 \cdot \kappa'$, and for the finiteness problem whether there exists some $w \in L$ with $\kappa' < ||w|| \leq 2 \cdot \kappa'$, using the iteration lemma for $\mathcal{ALG}(\odot, \sqcup\!\sqcup)$. □

6 Outlook

Further research should be done in particular for more decidability problems, as language equivalence, whether the intersection of two languages is empty, or given a language of some class whether it belongs to a lower one. Such results are important in the area of concurrency.

References

1. Berglund, Martin; Björklund, Henrik; Högberg, Johanna: *Recognizing Shuffle Languages.* **LNCS 6638**, pp. 142-154, 2011.
2. Czaja, Ludwik: *On Deadlock and Fairness Decision Problems for Computations on Client-server Systems.* **FI 109 (3)** pp. 255-264, 2011.
3. Czaja, Ludwik; Kudlek, Manfred: *Language Theoretic Properties of Client/Server Systems.* Proceedings of CS&P 2011, pp. 79-84, 2011.
4. Flick, Nils Erik; Kudlek, Manfred: *On A Hierarchy of Languages with Catenation and Shuffle.* DLT, **LNCS 7410/2012**, pp. 452–458, Springer 2012.
5. Gischer, Jay: *Shuffle Languages, Petri Nets, and Context-sensitive Grammars.* **CACM 24 (9)**, pp. 597–605, 1981.
6. Jantzen, Matthias: *Extending Regular Expressions with Iterated Shuffle.* **TCS 38**, pp. 223–247, 1985.
7. Kudlek, Manfred: *On General Iteration Lemmata for Certain Classes of Word, Trace and Graph Languages.* **FI 37 (4)**, pp. 413–422, 1999.
8. Kudlek, Manfred: *On Semilinear Sets over Commutative Semirings.* **FI 79 (3-4)**, pp. 447–452, 2007.
9. Kudlek, Manfred; Martín Vide, Carlos; Păun, Gheorghe: *Toward FMT (Formal Macroset Theory).* In: Multiset Processing, eds. Calude, Cristian; Păun, Gheorghe; Rozenberg, Grzegorz; Salomaa, Arto, **LNCS 2235**, pp. 123-133, 2000.