Elements of Representation Theory for Pawlak Information Systems

Marcin Wolski[1] and Anna Gomolińska[2]

[1] Maria Curie-Skłodowska University, Department of Logic and Philosophy of Science, Pl. Marii Curie-Skłodowskiej 4, 20-031 Lublin, Poland,
marcin.wolski@umcs.lublin.pl

[2] Białystok University, Institute of Computer Science, Sosnowa 64, 15-887 Białystok, Poland,
anna.gom@math.uwb.edu.pl

Abstract. Representation theory is a branch of mathematics whose original purpose was to represent information about abstract algebraic structures by means of methods of linear algebra (usually, by linear transformations and matrices). Rota in his famous "Foundations" defined a representation of a locally finite partially ordered set (poset) P in terms of a module over a ring $\mathbb{A}$, which can be further extended to an associative $\mathbb{A}$-algebra called incidence algebra of P. He applied this construction to solve a number of important problems in combinatorics. Our goal in this paper is to apply Rota's construction of incidence algebras to (arbitrary) Pawlak information systems. To be more precise, we analyse both incidence algebras and information systems in the context of granular computing. Therefore, starting from objects and an indiscernibility relation, we focus our attention upon information granules (i.e. equivalence classes) and a corresponding incidence algebra; finally, we discuss a lattice of closed ideals of this algebra (whose maximal elements serve as a representation of information granules). In this way we obtain a (partially ordered) set of maximal (closed) ideals which is isomorphic to the set of information granules of a Pawlak information system (also equipped with a natural information order).

1 Introduction

Representation theory is a branch of mathematics which studies abstract algebraic structures by means of methods of linear algebra, with special emphasis put upon vector spaces and linear transformations. Roughly speaking, since the 1930s when Noether gave to the theory its modern settings, studying representations of an algebra has been actually studying modules over this algebra. In the paper we focus our attention on partially ordered sets (posets) regarded as abstract algebraic structures and incidence algebras regarded as their linear algebraic counterparts. To be more precise, in representation theory representations of posets are studied by means of modules of corresponding incidence algebras. The concept of an incidence algebra understood as an associative algebra over a ring, which corresponds to a poset, has been introduced by Rota in *Foundations I*. However, Rota actually established an incidence algebra a fundamental concept in combinatorics rather than in representation theory. The primary object of his study was a Möbius function of a given poset P, that is a special element of the incidence algebra $INC(P)$ of P; it turned out that the Möbius function was a fundamental invariant of posets. In contrast to his approach, in the present paper we are not interested in combinatorial properties of posets and their incidence algebras, but in their mutual relationships and possible applications to data analysis. For this reason we simplify a bit also representation theory and we focus our attention on special features of incidence algebras proved by Rota in [3]. Thus, in what follows, we discuss objects and the corresponding incidence ring, then we focus on equivalence classes of indiscernible objects (information granules) and the corresponding incidence algebra, and finally we shortly discuss a lattice of ideals of these rings whose elements (closed ideals) may serve as a representation of information granules.

2 Rough Sets and Granular Computing

In this section we recall basic concepts from rough sets, however, we frame the presentation so as to fit the granular computing approach to data. Generally speaking, granular computing is a paradigm of (or better still, theoretical perspective on) information processing. It focuses on the idea that the knowledge which is present in data can be processed at various levels of resolution or scales; in other words, at various levels of granulation. Therefore in this section, while considering a partition of a set of objects (induced by an indiscernibility relation), we make further steps and equip this partition (regarded as a set of information granules) with some additional structure (e.g. order), which would reflect the information about objects on the level of granules (i.e. at a lower resolution).

Definition 1 (Information System). *A quadruple $\mathcal{I} = \langle U, Att, Val, f\rangle$ is called an* information system*, where*

- *U is a non–empty finite set of objects;*
- *Att is a non–empty finite set of attributes;*
- *$Val = \bigcup_{A \in Att} Val_A$, where Val_A is a (non-empty) value–domain of the attribute A;*
- *$f : U \times Att \to \mathcal{P}(Val)$ is an information function, such that for all $A \in Att$ and $a \in U$, $f(a, A) \subseteq Val_A$.*

If $f(a, A) \neq \emptyset$ for all $a \in U$ and $A \in Att$, then the information system $\mathcal{I}$ is called complete*; otherwise, it is called* incomplete*. If $f(a, A)$ is a singleton (i.e. $\{val\}$, where $val \in Val_A$), for all $a \in U$ and $A \in Att$, then it is called deterministic.*

Usually, an information system is supposed to be complete and deterministic. However, in what follows by an information system it is meant a system without any additional requirements.

The language for object description in an information system $\langle U, Att, Val, f\rangle$ is the descriptor language $\mathcal{L}_{Desc}$ [8]; its primitive formulas (atoms) are descriptors of the form $[A = val]$, where $A \in Att$ and $val \in Val_A$, which is read as *an attribute A has a value val*:

$$\alpha ::= [A = val] \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta$$

The set of atoms is denoted by Φ, and $\mathcal{F}_{Desc}$ denotes the set of all well-formed formulas.

Definition 2 **($\mathcal{L}_{Desc}$ Model).** *Let $\mathcal{L}_{Desc}$ be a descriptor language over an information system $\langle U, Att, Val, f\rangle$. Then $\langle U, v\rangle$ is a model, where $v : U \times \Phi \to \{0,1\}$ is a function assigning to each pair (a, p), where $a \in U$ and $p \in \Phi$, a truth value. We usually write $v(a, p) = 1$ (or $v_a(p) = 1$), which is read as* for the object a, p is true. *Let us define:*

– $v_a([A = val]) = 1$ *if $val \in f(a, A)$, and 0 otherwise.*

As usual, the function v is extended to every formula $\alpha \in \mathcal{F}_{Desc}$ in the standard way.

It is worth noticing that in the case of an incomplete information system, if $f(a, A) = \emptyset$, then $v_a([A = val]) = 0$ for all values $val \in Val_A$.

With each object $a \in U$ we associate the set of atoms of $\mathcal{L}_{Desc}$ which are true for the object a:

$$|a|_{\mathcal{L}_{Desc}} = \{p \in \Phi \mid v_a(p) = 1\}$$

Whenever the context of an information system and the descriptor language over this system is clear, we shall omit the subscript and simply write $|a|$. The sets of this form induce a preorder on the set of objects U:

$$a \leq b \quad \text{iff} \quad |a| \subseteq |b|,$$

for all $a, b \in U$. Intuitively, $a \leq b$ means that we have more pieces of information about the object b than about a. As usual, when we have the same knowledge about two objects, then these objects are indiscernible, denoted by $a\ IND\ b$:

$$a\ IND\ b \quad \text{iff} \quad a \leq b \quad \text{and} \quad b \leq a.$$

Clearly, IND is an equivalence relation.

Proposition 1. *Let $\langle U, Att, Val, f\rangle$ be a complete and deterministic information system. Then $\leq$ and IND coincide.*

As usual, the partition of U induced by IND is denoted by U/IND. The proposition above tells that in the case of a complete information system, the quotient set U/IND inherits no order (different than the identity) from the preordered set $(U, \leq)$. However, in the case of incomplete and nondeterministic systems there is non-trivial inheritance of information. Let $[a]_{IND} \in U/IND$ denote the equivalence class induced by a, then let us define $\leq$ on U/IND (by abuse of notation we use the same symbol as for U):

$$[a]_{IND} \leq [b]_{IND} \quad \text{iff} \quad a \leq b.$$

Proposition 2. *Let $\langle U, Att, Val, f\rangle$ be an information system. Then $(U/IND, \leq)$ is a partially ordered set.*

Actually, any preordered set $P = (U, \leq)$ can be viewed as a partially ordered set U/IND with the order inherited from P. Furthermore, for finite sets the preordered set P can be viewed as a pair $(U/IND, \gamma)$, where $\gamma : U/IND \to \mathbb{N}$ returns the number of elements of a given equivalence class. The following proposition comes from [1].

Proposition 3. *Let $P = (U, \leq)$ and $P' = (U', \leq')$ be two finite preordered sets. Sets P and P' will be isomorphic if there exists an isomorphism f of partially ordered sets $(U/IND, \gamma)$ and $(U'/IND', \gamma')$ such that $\gamma = f \circ \gamma'$.*

In consequence, in the case of a finite preordered set $P = (U, \leq)$, we can pass between P and its representation $(U/IND, \gamma)$ forth and back by means of a partially ordered set $(U/IND, \leq)$ without any loss of information.

Example 1. Let us consider a simple example of a data table describing patients and their mammography test results and the medical diagnosis concerning cancer (as depicted by Fig. 1). This information system is nondeterministic and for the object 5 we have

$$v_5([Test\ \ result = Positive]) = v_5([Test\ \ result = Negative]) = 1.$$

In consequence, the relations $\leq$ and IND are given by

$$\leq = \{(1,1),(2,2),(1,2),(1,5),(2,1),(2,5),(3,3),(4,4),(3,4),$$

$$(3,5),(4,3),(4,5),(5,5)\},$$

$$IND = \{(1,1),(2,2),(1,2),(2,1),(3,3),(4,4),(3,4),(4,3),(5,5)\},$$

respectively. The quotient set U/IND is equal to $\{[1]_{IND}, [3]_{IND}, [5]_{IND}\}$, where $[1]_{IND} = [2]_{IND} = \{1,2\}$, $[3]_{IND} = [4]_{IND} = \{3,4\}$ and $[5]_{IND} = \{5\}$. Because these two relations are different we obtain a non-trivial poset $(U/IND, \leq)$, depicted in Fig. 1.

Patient	Test result	Cancer
1	Positive	No
2	Positive	No
3	Negative	No
4	Negative	No
5	Positive, Negative	No

Fig. 1. A data table (i.e. information system) corresponding to the mammography test.

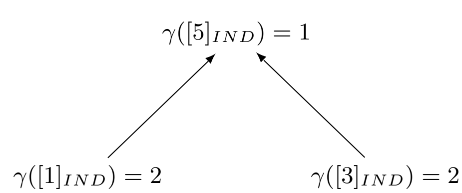

Fig. 2. Hasse diagram of $(U/IND, \leq)$.

Intuitively, the set U/IND consists of basic information granules induced by an information system $\langle U, Att, Val, f\rangle$. Thus the poset $(U/IND, \gamma)$ actually represents information encoded by $\langle U, Att, Val, f\rangle$ from granular perspective. As said earlier, we can regard $(U/IND, \leq)$ as $(U, \leq)$ presented at a lower resolution.

3 Incidence Algebras

Incidence algebras were introduced by Rota in [10] and elaborated in [3]. In order to make the presentation self-contained we recall some basic concepts from algebra (and only them) useful for a better understanding of the paper.

Definition 3 (Module). *A left module M over a ring $\mathbb{A}$ consists of an Abelian group $(M, +)$ and an operation $* : \mathbb{A} \times M \to M$ such that for all $r, s \in \mathbb{A}$ and $a, b \in M$ we have:*

1. $r * (a + b) = r * a + r * b$,
2. $(r + s) * a = r * a + s * a$,
3. $(r * s) * a = r * (s * a)$,
4. $1 * a = a$ *if $\mathbb{A}$ has a multiplicative identity* 1.

A right $\mathbb{A}$-module M is defined similarly, only the ring acts on the right; if $\mathbb{A}$ is commutative, then left $\mathbb{A}$-modules are the same as right $\mathbb{A}$-modules and are simply called $\mathbb{A}$-modules.

Definition 4 (Associative Algebra). *Let $\mathbb{A}$ be a fixed commutative ring. An associative $\mathbb{A}$-algebra is an additive Abelian group M which has the structure of both a ring and an $\mathbb{A}$-module in such a way that ring multiplication is $\mathbb{A}$-bilinear:*

$$r * (a * b) = (r * a) * b = a * (r * b),$$

for all $r \in \mathbb{A}$ and $a, b \in M$.

In other words, an $\mathbb{A}$-algebra is an $\mathbb{A}$-module equipped with the operation of multiplication (convolution) $* : M \times M \to M$ of elements of M such that the above equations hold. By the standard abuse of notation, we have used the same symbol $*$ for multiplication by scalars (i.e. $r * a$, $r \in \mathbb{A}$ and $a \in M$) and multiplication (convolution) of elements of M (i.e. $a * b$, $a, b \in M$). Obviously, the context always makes clear the meaning of the symbol $*$.

Definition 5 (Ideal). *A subset $I \subseteq M$ will be called a left ideal of $\mathbb{A}$-algebra if the following conditions hold:*

- $a + b \in I$, *for all* $a, b \in I$,
- $r * a \in I$, *for all* $r \in \mathbb{A}$ *and* $a \in I$,
- $a * b \in I$, *for all* $a \in M$ *and* $b \in I$.

In other words, a left ideal $I \subseteq M$ is closed under addition, multiplication by scalars, and left multiplication by arbitrary elements of M. Replacing $a * b \in I$ by $b * a \in I$ we define a right ideal. A two-sided ideal is a subset that is both a left and a right ideal. Usually a two-sided ideal is referred to as an ideal.

Definition 6 (Product of Ideals). *Let I and J be ideals in a ring $\mathbb{A}$. The ideal product is defined by*

$$IJ = \{\sum a_i b_i \mid a_i \in I,\ \ b_i \in J\}.$$

Let $P = (U, \leq)$ be a locally finite partially ordered set (poset), that is the relation $\leq$ is a reflexive, transitive and antisymmetric, such that every interval $[a, b] = \{c \in U \mid a \leq c \leq b\}$ is finite.

Definition 7 (Incidence Algebra). *An incidence algebra $INC_{\mathbb{A}}(P)$ of a locally finite partially ordered set $P = (U, \leq)$ is a set of all functions $f : U \times U \to \mathbb{A}$, where $\mathbb{A}$ is a commutative ring with multiplicative identity such that*

$$f(a, b) = 0 \quad \textit{if} \quad a \not\leq b.$$

Of course, the sum and multiplication by scalars of these functions are defined in the standard way:

$$(f + g)(a, b) = f(a, b) + g(a, b) \quad \text{and} \quad (r * f)(a, b) = r * f(a, b)$$

for all $f, g \in Inc_{\mathbb{A}}(P)$, $a, b \in U$ and $r \in \mathbb{A}$. Formally, it means that $Inc_{\mathbb{A}}(P)$ is an $\mathbb{A}$-module, which can be made an $\mathbb{A}$-algebra by the addition of convolution, justifying the term *incidence algebra*:

$$(f * g)(a, b) = \sum_{c:a \leq c \leq b} f(a, c)g(c, b)$$

If $a \not\leq b$, then there will be no c such that $a \leq c \leq b$ and $(f * g)(a, b) = 0$; such a sum we shall call degenerated. It is also worth noting that if P is finite and we present $f \in Inc_{\mathbb{A}}(P)$ as $n \times n$ matrix $\mathfrak{m}_f$, where n is the number of elements of U, with (a, b) entry $f(a, b)$, then the operation of convolution will be the standard multiplication of matrices.

Definition 8 (Standard Topology). *The standard topology on an incidence algebra $INC(P)$ of a locally finite partially ordered set $P = (U, \leq)$ is defined as follows: A sequence $f_1, f_2, f_3, \ldots$ converges to a function f if and only if $f_n(a, b)$ converges to $f(a, b)$ in the field $\mathbb{A}$ for every $a, b \in U$, when n goes to the infinity.*

When studying of incidence algebras, a special role is played by ideals. When P is finite, then any two-sided ideal I is closed, i.e. I is a closed set in the standard topology. The following propositions proved by Rota in [3] are of special importance.

Proposition 4. *In a locally finite poset P, let $S(P)$ be the set of all segments of P ordered by inclusion. Then there is a natural anti-isomorphism between the lattice of closed ideals of $INC(P)$ and the lattice of order ideals of $S(P)$.*

Since the proposition actually reveals the structure of closed ideals, let us sketch a proof (whose full form can be found in [3]). Firstly, let us recall that an order ideal I of a poset $P = (U, \leq)$ is a set $I \subseteq U$ such that if $a \leq b$ and $b \in I$, then $a \in I$. Let J be an ideal of $INC(P)$ and $Z(J)$ be the family of all segments $[a, b]$ such that $f(a, b) = 0$, for all $f \in J$. Then $Z(J)$ is an order ideal of $S(P)$. Conversely, let Z be an order ideal of $S(P)$, and let J be the set of all $f \in INC(P)$ such that $f(a, b) = 0$ if $[a, b] \in Z$. Then J is a closed ideal of $INC(P)$.

Proposition 5. *The closed maximal ideals of incidence algebra $INC_{\mathbb{A}}(P)$ are those of the form*

$$J_a = \{f \in INC(P) \mid f(a,a) = 0\}.$$

Actually, what is important from our perspective is a one-to-one correspondence between elements of P and maximal closed ideals of $INC(P)$ which have a very simple form. Therefore, we do not actually need to go into details of standard topologies of posets. It suffices to keep in mind that J_a is a set of special functions f such that $f(a,a) = 0$. The proposition above suggests how to represent the elements of P while building the representation of P in terms of an incidence algebra; we shall come back to this idea soon.

The following proposition comes from Stanley (see e.g. [11]) – the proof is also presented in [3] – and shows that an ordered set P is uniquely determined by its incidence algebra.

Proposition 6. *Let P and P' be (locally finite) partially ordered sets, and let $\mathbb{A}$ be a field. If $INC_{\mathbb{A}}(P)$ and $INC_{\mathbb{A}}(P')$ are isomorphic as $\mathbb{A}$-algebras, then P and P' will be isomorphic.*

The concept of an incidence algebra originally introduced by Rota for partially ordered sets can naturally be extended for preordered sets (just by replacing a poset by a preordered set in the above definitions). Belding in [2] proved that:

Proposition 7. *Let P and P' be finite preordered sets and let $\mathbb{A}$ be a field. Then if $INC_{\mathbb{A}}(P)$ and $INC_{\mathbb{A}}(P')$ are isomorphic as $\mathbb{A}$-algebras, then P and P' will be isomorphic as preordered sets.*

In the case of preordered sets and rings, $INC_{\mathbb{A}}(P)$ is often referred to as an *incidence ring*. To complete the presentation, let us recall the connection between incidence rings of a preordered set and its corresponding partial order (see e.g. [5]).

Proposition 8. *Let $P = (U, \leq)$ be a finite preordered set and $\mathbb{A}$ a commutative ring with multiplicative identity, and $\hat{P} = (U/IND, \leq)$. Then $INC_{\mathbb{A}}(P)$ and $INC_{\mathbb{A}}(\hat{P})$ are Morita equivalent.*

Recall that two unital rings $\mathbb{A}$ and $\mathbb{A}'$ will be Morita equivalent if the categories of their left modules are equivalent. In the category of commutative rings the Morita equivalence is exactly an isomorphism. Of course, the operation of convolution need not be commutative. However, Morita equivalent rings have isomorphic lattices of ideals. Thus, since maximal (closed) ideals are supposed to serve as representation for points according to Proposition 5, we will not loose much information when we confine our interest to the case of partial orders and their incidence algebras.

4 A Linear Algebra of Pawlak Systems

In Sect. 2 we presented an information system $\mathcal{I} = \langle U, Att, Val, f\rangle$ as a partially ordered set $P^{\mathcal{I}} = (U/IND, \leq)$, or better still, as $(U/IND, \gamma)$. In consequence, we can associate with every information system its incidence algebra $INC(\mathcal{I}) = INC(P^{\mathcal{I}})$. As already mentioned, the points of $P^{\mathcal{I}}$ correspond to maximal closed ideals of the algebra $INC(\mathcal{I})$. Therefore, before we will proceed in this study we introduce some auxiliary concepts allowing us to focus attention upon these ideals. For elements a, b of a poset P define

$$S(a,b) = \{[c,d] \mid [c,d] \in S(P) \quad \text{and } a \leq c \leq d \leq b\},$$
$$J_{S(a,b)} = \{f \in INC(P) \mid f(c,d) = 0 \quad \text{for all} \quad [c,d] \in S(a,b)\}.$$

It is obvious that

$$J_a = J_{S(a,a)}, \quad \text{and} \;\; a \not\leq b \;\; \text{implies} \quad J_{S(a,b)} = INC(P).$$

Of course, $J_{S(a,b)}$ is a closed ideal of $INC(P)$ for all $a, b \in P$, as required.

Proposition 9. *Let $\mathcal{I} = \langle U, Att, Val, f\rangle$ be an information system. Then $\mathcal{I}$ is complete and deterministic if and only if for all $a, b \in U$, such that $a \neq b$, $J_{S(a,b)} = INC(\mathcal{I})$.*

The property of being deterministic and complete system can also be expressed by means of the operation of convolution. Let us recall that convolution $f * g$ was defined by

$$(f * g)(a,b) = \sum_{c: a \leq c \leq b} f(a,c)g(c,b). \tag{1}$$

Hence, given that $\mathcal{I}$ is complete and deterministic, if $a \neq b$ then $(f * g)(a,b) = 0$ for all f, g. In the case $a = b$ we obtain $(f * g)(a,a) = f(a,a)g(a,a)$. Since the underlying ring $\mathbb{A}$ is commutative, we get that for such an information system $\mathcal{I}$ the convolution $*$ is also commutative.

Proposition 10. *Let $\mathcal{I} = \langle U, Att, Val, f\rangle$ be an information system. Then $\mathcal{I}$ is complete and deterministic if and only if its incidence algebra $INC(\mathcal{I})$ is commutative.*

Thus, incompleteness or nondeterminism of an information system means that the corresponding algebra is not commutative. Although this case is not necessarily desirable in data analysis, it is mathematically much more interesting than the case of complete and deterministic systems. In what follows we analyse the non-commutative cases.

In order to make the presentation less abstract we shall take advantage of the correspondence of incidence algebras with rings of square matrices (e.g. [11]), which will allow us to visualise some features of these algebras. In the case of a finite set $P = (U, \leq)$, we can present P by its incidence matrix $\mathfrak{c}_P$ as usual, that is, a square matrix $n \times n$, where n is the number of elements of U:

$$\mathfrak{c}_P = [c_{ab}]_{a,b \in U} \quad \text{with} \quad c_{ab} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{if } a \not\leq b \end{cases}$$

For example, the incidence matrix of $(U/IND, \leq)$ from Sect. 2 (see Fig. 1), where $U = \{[1]_{IND}, [3]_{IND}, [5]_{IND}\}$, is given by (to simplify the matrix we omit the subscript IND):

$$\mathfrak{c}_P = \begin{pmatrix} c_{[1][1]} = 1 & c_{[1][3]} = 0 & c_{[1][5]} = 1 \\ c_{[3][1]} = 0 & c_{[3][3]} = 1 & c_{[3][5]} = 1 \\ c_{[5][1]} = 0 & c_{[5][3]} = 0 & c_{[5][5]} = 1 \end{pmatrix}$$

As usual, given a commutative ring $\mathbb{A}$ and a finite non-empty set U with n elements, we can consider the associative algebra $\mathfrak{M}(\mathbb{A})$ of all square $n\times n$ matrices $\mathfrak{m} = [m_{ab}]_{a,b\in U}$ with entries from $\mathbb{A}$. As was mentioned in Sect. 3, elements of an incidence algebra $INC(P)$ of a finite partially ordered set $P = (U, \leq)$, can be viewed as $n \times n$ matrices:

$$INC(P) = \{\mathfrak{m} = [m_{ab}]_{a,b\in U} \mid m_{ab} = 0 \quad \text{if} \quad a \not\leq b\}$$

In other words, $INC(P)$ is a subalgebra of $\mathfrak{M}(\mathbb{A})$. It consists of matrices $\mathfrak{m}^f$ (labelled by elements of $INC(P)$), which are consistent with the incidence matrix $\mathfrak{c}_P$:

$$\mathfrak{m}^f = [m_{ab}]_{a,b\in U} \quad \text{with} \quad m_{ab} = \begin{cases} f(a,b) & \text{if } a \leq b \\ 0 & \text{if } a \not\leq b \end{cases}$$

That is, for a finite P we have

$$INC(P) = \{f : U \times U \to \mathbb{A} \mid f(a,b) = 0 \quad \text{if} \quad x \not\leq y\}$$

which is isomorphic to

$$\{\mathfrak{m}^f \mid m_{ab} = f(a,b) \quad \text{where} \quad f \in INC(P)\}.$$

For instance, the incidence algebra of our example consists of matrices of the following form:

$$\mathfrak{m}^f = \begin{pmatrix} f([1],[1]) \in \mathbb{A} & f([1],[3]) = 0 & f([1],[5]) \in \mathbb{A} \\ f([3],[1]) = 0 & f([3],[3]) \in \mathbb{A} & f([3],[5]) \in \mathbb{A} \\ f([5],[1]) = 0 & f([5],[3]) = 0 & f([5],[5]) \in \mathbb{A} \end{pmatrix}$$

As was proved by Rota [3], the elements of P are in one-to-one correspondence with maximal elements of the lattice of the closed ideals J_a of $INC(P)$. Fortunately, they have a very simple characteristic and we do not need to pay much attention to the standard topology τ associated with the poset P. In terms of matrices, elements of J_a have the form $\mathfrak{m}^f$ with an additional requirement that $m_{aa} = f(a,a) = 0$.

Given representations for points, we now find out how the partial order $\leq$ is related to the maximal closed ideals. As usual, $a < b$ means $a \leq b$ and $a \neq b$. We say that b covers a, symbolically $a \ll b$ if $a < b$ and no element $c \in U$ satisfies $a < c < b$. Thus, what a Hasse diagram of P actually illustrates is the induced covering relation $\ll$. Of course, $a \not\ll a$ for all $a \in U$.

Since we deal with ideals of $INC(P)$, the only operation we have at hand is the product of ideals J_aJ_b:

$$J_aJ_b = \{\sum f_i * g_i \mid f_i \in J_a, \ \ g_i \in J_b\}$$

Previously we introduced closed ideals of the form $J_{S(a,b)}$; so, let us ask when it is true that $J_aJ_b \subseteq J_{S(a,b)}$. It turns out that this inclusion holds in three simple cases:

1. If $a \not\leq b$, then for any element $f \in INC(P)$ it holds that $f(a,b) = 0$.
2. If $a = b$, then $h(a,a) = f(a,a)g(a,a) = 0$ and $h(b,b) = f(b,b)g(b,b) = 0$.
3. The last case is when $a \ll b$:

$$(f * g)(a,b) = f(a,a)g(a,b) + f(a,b)g(b,b) = 0$$

The first case may be ruled out by the restriction that $J_{S(a,b)} \neq INC(P)$. On the other hand, the other two cases are consistent with the order of P.

In this way we obtained the following order on ideals.

Definition 9 (Order on Ideals). *Let $INC(P)$ be an incidence algebra of finite partially ordered set P over a field $\mathbb{A}$. We shall say that $J_a \sim J_b$ if*

$$J_{S(a,b)} \neq INC(P) \quad \textit{and} \quad J_a J_b = J_{S(a,b)}.$$

Now define $\leq_J$ to be a transitive closure of $\sim$.

Generally, when $a \neq b$ then for every $f \in J_a J_b$ it will hold that $f(a) = 0 = f(b)$, and $f(a,b)$ will be some value from $\mathbb{A}$. However, in the case when $a \ll b$, we shall also get that $f(a,b) = 0$. Thus, if $a \ll b$, then $Z(J_a J_b) = \{[a,a],[b,b],[a,b]\} = S(a,b)$ and hence $J_a J_b = J_{S(a,b)}$. On the other hand, if $a \leq b$ and $a \neq b$, then $\{[a,a],[b,b],[a,b]\} \subseteq S(a,b)$. But $J_a J_b = J_{S(a,b)}$ and hence $S(a,b) = Z(J_a J_b) \subseteq \{[a,a],[b,b],[a,b]\}$. It is possible only when $a \ll b$.

Proposition 11. *Let $INC(P)$ be an incidence algebra of a finite partially ordered set P over a field $\mathbb{A}$. Then $\leq_J$ is a partial order on $\mathcal{J}$. If we define $J_a \ll J_b$ when $a \neq b$ and $J_a \sim J_b$, then $\ll$ will be a covering relation induced by $\leq_J$.*

Proposition 12. *Let $INC(P)$ be an incidence algebra of finite partially ordered set P over a field $\mathbb{A}$, and let $\mathcal{J}$ be a set of closed maximal ideals equipped with $\leq_J$. Then P and J are isomorphic as partial orders.*

The easy computation brings us the following Hasse diagram of $(\mathcal{J}, \leq_J)$ induced by our example (Fig. 3). Summing up, when we start with an information system $\mathcal{I}$ we

$$\begin{tikzcd} & J_{[5]_{IND}} = J_c & \\ J_{[1]_{IND}} = J_a \arrow[ur] & & J_{[3]_{IND}} = J_b \arrow[ul] \end{tikzcd}$$

Fig. 3. Hasse diagram of $(\mathcal{J}, \leq_J)$.

can associate with $\mathcal{I}$ two incidence algebras: (1) an incidence ring $INC(P)$ where $P = (U, \leq)$, and (2) an incidence algebra $INC(P')$ where $P' = (U/IND, \leq)$. As we know, $INC(P)$ and $INC(P')$ are Morita equivalent. Therefore, their granular representations given in terms of maximal closed ideals $\mathcal{J}$ are isomorphic. It can also be derived from Proposition 4 in [3]: For a preordered set $P = (U, \leq)$ and the corresponding poset $P' = (U/INC, \leq)$, their lattices of order ideals $S(P)$ and $S(P')$ are isomorphic, which amounts to the isomorphism of the lattices of closed ideals of incidence algebras. Hence, without any loss of important information we can proceed with

$INC(P')$ and use the results proved by Rota to build the corresponding partial order defined on maximal closed ideals.

It is worth emphasising that the above representation works virtually for all structures used in rough sets. Consider e.g. a tolerance space (U, T) where U is a set of objects and T is a tolerance relation, that is T is symmetric and reflexive. Define

$$T(a) = \{b \in U \mid aTb\} \quad \text{and} \quad a \leq b \quad \text{iff} \quad T(a) \subseteq T(b).$$

Then $\leq$ is a preorder and hence, as presented in the paper, we can produce two incidence rings whose lattices of ideals are isomorphic. Recently, Peters [9] suggested to use perceptual systems instead of information systems when dealing with visual information. The special role is played there by a perceptual tolerance relation which, as in the case of tolerance spaces, is an easy object to representation.

In the last paragraph let us sketch some possible applications of the above algebras. As the reader may know, *interaction* is an emerging paradigm of models of computation. It is supposed to reflect some recent changes in technology such as multiagent systems and object-based distributed systems [4]. However, rough set framework seems to lack theoretical tools to deal with interactions: what we have at hand are standard set-theoretical operations (which have already got simple interpretations). In our approach, we not only have an intersection of ideals at hand, but also their product, which may provide a means for modelling some kind of interaction between granules. Of course, the product *has not provided* such a means, but it *may provide* such a tool. Nonetheless, it seems that to deal with the problem of interactions we must at least seek some (conservative) enrichments of rough set framework, what we actually did in the article.

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