

# Block relations in fuzzy setting<sup>\*</sup>

Jan Konecny, Michal Krupka

Department of Computer Science  
Palacký University in Olomouc  
17. listopadu 12, CZ-77146 Olomouc  
Czech Republic  
jan.konecny@upol.cz, michal.krupka@upol.cz

**Abstract.** One of the main problems in FCA (especially in fuzzy setting) is to reduce a concept lattice to appropriate size to make it graspable and understandable. We generalize known results on the correspondence of block relations of formal contexts and complete tolerances on concept lattices to fuzzy setting. Using an illustrative example we show that the results can be used for reduction of fuzzy concept lattices.

Keywords: Fuzzy concept lattice, Tolerance, Block relation, Factorization

## 1 Introduction

The problem of the size of concept lattices is recognized as one of the most important problems of Formal Concept Analysis (FCA). The main aim of this paper is to describe approximation of formal concepts by fuzzy tolerances in fuzzy setting.

In [3, 2], a method of factorization of fuzzy concept lattices is presented. A similarity threshold  $a$  is supplied and the method outputs a factor lattice instead of the whole concept lattice which might be large. The elements of the factor lattice are maximal blocks of concepts from the whole concept lattice which are pairwise similar to degree at least  $a$ . We generalize the results to a more general family of similarities (complete fuzzy tolerances) than those induced by a single threshold. We also show that the factor lattice can be computed from a special kind of superrelations of the incidence relation, called fuzzy block relations; and the relationship between complete fuzzy tolerances and fuzzy block relations. That way we generalize a Wille's results on (crisp) tolerances and block relations in [17] (see also [7]).

In [13], Meschke proposed a method of approximation of (crisp) concepts by tolerances and block relations. We use some of his ideas (namely the selection of important objects and attributes) in an illustrative example at the end of this paper.

The organization of the paper is as follows. In Section 2 we recall the fundamental notions and results on the basic fuzzy structures used in the paper. Section 3 is devoted to study of fuzzy tolerance relations: we define complete tolerance relations on completely lattice ordered fuzzy sets and show fundamental properties of the corresponding factor sets. In Section 4 we describe fuzzy block relations and prove the main result

---

<sup>\*</sup> Supported by grant no. P103/10/1056 of the Czech Science Foundation.

of the paper: the correspondence between fuzzy block relations and factor structures of concept lattices. Section 5 contains an illustrative example.

Due to space limitations, we abbreviate, resp. omit, some proofs.

## 2 Preliminaries

We recall basic notions and facts of residuated lattices, fuzzy sets and fuzzy relations, fuzzy Galois connections and fuzzy concept lattices.

### 2.1 Residuated lattices and fuzzy sets

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice [3, 9, 16] is a structure  $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$  such that

- (i)  $\langle L, \wedge, \vee, 0, 1 \rangle$  is a complete lattice, i.e., a partially ordered set in which arbitrary infima and suprema exist;
- (ii)  $\langle L, \otimes, 1 \rangle$  is a commutative monoid, i.e.,  $\otimes$  is a binary operation which is commutative, associative, and  $a \otimes 1 = a$  for each  $a \in L$ ;
- (iii)  $\otimes$  and  $\rightarrow$  satisfy adjointness, i.e.,  $a \otimes b \leq c$  iff  $a \leq b \rightarrow c$ .

0 and 1 denote the least and greatest elements. The partial order of  $\mathbf{L}$  is denoted by  $\leq$ . Throughout the paper,  $\mathbf{L}$  denotes an arbitrary complete residuated lattice.

Elements  $a$  of  $L$  are called truth degrees.  $\otimes$  and  $\rightarrow$  (truth functions of) “fuzzy conjunction” and “fuzzy implication”.

Common examples of complete residuated lattices include those defined on a unit interval, (i.e.,  $L = [0, 1]$ ) or on a finite chain in the unit interval, e.g.  $L = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ ,  $\wedge$  and  $\vee$  being minimum and maximum,  $\otimes$  being a left-continuous t-norm with the corresponding  $\rightarrow$ .

The three most important pairs of adjoint operations on the unit interval are

$$\begin{array}{ll} \text{Łukasiewicz:} & \begin{array}{l} a \otimes b = \max(a + b - 1, 0) \\ a \rightarrow b = \min(1 - a + b, 1) \end{array} \\ & a \otimes b = \min(a, b) \\ \text{Gödel:} & a \rightarrow b = \begin{cases} 1 & a \leq b \\ b & \text{otherwise} \end{cases} \\ & a \otimes b = a \cdot b \\ \text{Goguen (product):} & a \rightarrow b = \begin{cases} 1 & a \leq b \\ \frac{b}{a} & \text{otherwise} \end{cases} \end{array}$$

An  $\mathbf{L}$ -set (or fuzzy set)  $A$  in a universe set  $X$  is a mapping assigning to each  $x \in X$  some truth degree  $A(x) \in L$  where  $L$  is the support of a complete residuated lattice. The set of all  $\mathbf{L}$ -sets in a universe  $X$  is denoted  $L^X$ .

The operations with  $\mathbf{L}$ -sets are defined componentwise. For instance, the intersection of  $\mathbf{L}$ -sets  $A, B \in L^X$  is an  $\mathbf{L}$ -set  $A \cap B$  in  $X$  such that  $(A \cap B)(x) = A(x) \wedge B(x)$  for each  $x \in X$ , etc. An  $\mathbf{L}$ -set  $A \in L^X$  is also denoted  $\{A^{(x)}/x \mid x \in X\}$ . If for all  $y \in X$  distinct from  $x_1, x_2, \dots, x_n$  we have  $A(y) = 0$ , we also write  $\{A^{(x_1)}/x_1, A^{(x_2)}/x_2, \dots, A^{(x_n)}/x_n\}$ .

Binary **L**-relations (binary fuzzy relations) between  $X$  and  $Y$  can be thought of as **L**-sets in the universe  $X \times Y$ . That is, a binary **L**-relation  $I \in L^{X \times Y}$  between a set  $X$  and a set  $Y$  is a mapping assigning to each  $x \in X$  and each  $y \in Y$  a truth degree  $I(x, y) \in L$  (a degree to which  $x$  and  $y$  are related by  $I$ ). An **L**-set  $A \in L^X$  is called crisp if  $A(x) \in \{0, 1\}$  for each  $x \in X$ . Crisp **L**-sets can be identified with ordinary sets. For a crisp  $A$ , we also write  $x \in A$  for  $A(x) = 1$  and  $x \notin A$  for  $A(x) = 0$ . An **L**-set  $A \in L^X$  is called empty (denoted by  $\emptyset$ ) if  $A(x) = 0$  for each  $x \in X$ . For  $a \in L$  and  $A \in L^X$ ,  $a \otimes A \in L^X$  and  $a \rightarrow A \in L^X$  are defined by

$$(a \otimes A)(x) = a \otimes A(x) \text{ and } (a \rightarrow A)(x) = a \rightarrow A(x).$$

For an **L**-set  $A \in L^X$  and  $a \in L$ , the  $a$ -cut of  $A$  is a crisp subset  ${}^a A \subseteq X$  such that  $x \in {}^a A$  iff  $a \leq A(x)$ . This definition applies also to binary **L**-relations, whose  $a$ -cuts are interpreted as classical (crisp) binary relations.

For universe  $X$  we define **L**-relation *graded subsethood*  $L^X \times L^X \rightarrow L$  by:

$$S(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x) \quad (1)$$

Graded subsethood generalizes the classical subsethood relation  $\subseteq$ ; indeed, in the crisp case (i.e.  $L = \{0, 1\}$ ) (1) becomes  $S(A, B) = 1$  iff  $\forall x \in X : x \in A$  implies  $x \in B$ . Note that unlike  $\subseteq$ ,  $S$  is a binary **L**-relation on  $L^X$ . Described verbally,  $S(A, B)$  represents a degree to which  $A$  is a subset of  $B$ . In particular, we write  $A \subseteq B$  iff  $S(A, B) = 1$ . As a consequence, we have  $A \subseteq B$  iff  $A(x) \leq B(x)$  for each  $x \in X$ .

Further we set

$$A \approx^X B = S(A, B) \wedge S(B, A). \quad (2)$$

The value  $A \approx^X B$  is interpreted as the degree to which the sets  $A$  and  $B$  are similar.

A binary **L**-relation  $R$  on a set  $X$  is called *reflexive* if  $R(x, x) = 1$  for any  $x \in X$ , *symmetric* if  $R(x, y) = R(y, x)$  for any  $x, y \in X$ , and *transitive* if  $R(x, y) \otimes R(y, z) \leq R(x, z)$  for any  $x, y, z \in X$ .  $R$  is called an **L**-tolerance, if it is reflexive and symmetric, **L**-equivalence if it is reflexive, symmetric and transitive. If  $R$  is an **L**-equivalence such that for any  $x, y \in X$  from  $R(x, y) = 1$  it follows  $x = y$ , then  $R$  is called an **L**-equality on  $X$ . **L**-equalities are often denoted by  $\approx$ . The similarity  $\approx^X$  of **L**-sets (2) is an **L**-equality on  $L^X$ .

Let  $\sim$  be an **L**-equivalence on  $X$ ,  $A \in L^X$  an **L**-set. We say that  $A$  is *compatible with*  $\sim$  (or *extensional with respect to*  $\sim$ ) if the following condition holds for any  $x, x' \in X$ :

$$A(x) \otimes (x \sim x') \leq A(x').$$

For each  $A \in L^X$  there exists a minimal (with respect to inclusion of **L**-sets) **L**-set  $C_{\sim}(A) \in L^X$ , compatible with  $\sim$ , and such that  $A \subseteq C_{\sim}(A)$ . It holds

$$C_{\sim}(A)(x) = \bigvee_{x' \in X} A(x') \otimes (x' \sim x). \quad (3)$$

We say that a binary **L**-relation  $R$  on  $X$  is *compatible with*  $\sim$  if for each  $x, x', y, y' \in X$ ,

$$R(x, y) \otimes (x \sim x') \otimes (y \sim y') \leq R(x', y').$$

For a mapping  $f : X \rightarrow Y$ , where  $Y$  is endowed with an  $\mathbf{L}$ -equality  $\approx$ , and  $\mathbf{L}$ -set  $A \in L^X$  we define the *image of  $A$  with respect to  $f$*  as an  $\mathbf{L}$ -set  $f(A) \in L^Y$  satisfying

$$f(A)(y) = \bigvee_{x \in X} A(x) \otimes (f(x) \approx y).$$

$f(A)$  is compatible with  $\approx$  and  $f(A) = C_{\approx}(B)$ , where  $B$  is an  $\mathbf{L}$ -set in  $Y$  given by  $B(y) = \bigvee_{x \in f^{-1}(\{y\})} A(x)$ .

In the following we use well-known properties of residuated lattices and fuzzy structures which can be found e.g. in [3, 9].

## 2.2 $\mathbf{L}$ -ordered sets

In this section, we recall basic definitions and results of the theory of  $\mathbf{L}$ -ordered sets. Basic references are [1, 3] and the references therein.

An  $\mathbf{L}$ -order on a set  $U$  with an  $\mathbf{L}$ -equality relation  $\approx$  is a binary  $\mathbf{L}$ -relation  $\preceq$  on  $U$  which is compatible with  $\approx$ , reflexive, transitive and satisfies  $(u \preceq v) \wedge (v \preceq u) \leq u \approx v$  for any  $u, v \in U$  (*antisymmetry*). The tuple  $\mathbf{U} = \langle \langle U, \approx \rangle, \preceq \rangle$  is called an  $\mathbf{L}$ -ordered set. An immediate consequence of the definition is that for any  $u, v \in U$  it holds

$$u \approx v = (u \preceq v) \wedge (v \preceq u). \quad (4)$$

If  $\mathbf{U} = \langle \langle U, \approx \rangle, \preceq \rangle$  is an  $\mathbf{L}$ -ordered set, then the tuple  $\langle U, {}^1\preceq \rangle$ , where  ${}^1\preceq$  is the 1-cut of  $\preceq$ , is a (partially) ordered set. We sometimes write  $\leq$  instead of  ${}^1\preceq$  and use the symbols  $\wedge, \bigwedge$  resp.  $\vee, \bigvee$  for denoting infima resp. suprema in  $\langle U, {}^1\preceq \rangle$ .

For two  $\mathbf{L}$ -ordered sets  $\mathbf{U} = \langle \langle U, \approx_U \rangle, \preceq_U \rangle$  and  $\mathbf{V} = \langle \langle V, \approx_V \rangle, \preceq_V \rangle$ , a mapping  $f : U \rightarrow V$  is called an *isomorphism of  $\mathbf{U}$  and  $\mathbf{V}$* , if it is a bijection and  $(u_1 \preceq_U u_2) = (f(u_1) \preceq_V f(u_2))$  for any  $u_1, u_2 \in U$ .  $\mathbf{U}$  and  $\mathbf{V}$  are then called *isomorphic*.

For an  $\mathbf{L}$ -ordered set  $\mathbf{U}$  and an  $\mathbf{L}$ -set  $V \in L^U$  we define  $\mathbf{L}$ -sets  $\mathcal{L}V, \mathcal{U}V \in L^U$  by

$$\mathcal{L}V(u) = \bigwedge_{v \in U} V(v) \rightarrow (u \preceq v), \quad \mathcal{U}V(u) = \bigwedge_{v \in U} V(v) \rightarrow (v \preceq u).$$

Using basic rules of fuzzy logic,  $\mathcal{L}V$  (resp.  $\mathcal{U}V$ ) can be interpreted as the set of elements which are smaller (resp. greater) than or equal to elements of  $V$ .  $\mathcal{L}V$  (resp.  $\mathcal{U}V$ ) is called *the lower cone* (resp. *the upper cone*) of  $U$ .

For any  $\mathbf{L}$ -set  $V \in L^U$  there exists at most one element  $u \in U$  such that  $\mathcal{L}V(u) \wedge \mathcal{U}(\mathcal{L}V)(u) = 1$  (resp.  $\mathcal{U}V(u) \wedge \mathcal{L}(\mathcal{U}V)(u) = 1$ ) [1, 3]. If there is such an element, we call it *the infimum of  $V$*  (resp. *the supremum of  $V$* ) and denote  $\inf V$  (resp.  $\sup V$ ); otherwise we say, that the infimum (resp. supremum) does not exist.

If  $u, v \in U$ ,  $v \preceq u$ , then the  $\mathbf{L}$ -set  $[v, u] = \mathcal{U}\{v\} \cap \mathcal{L}\{u\}$  is called an *interval* in  $\mathbf{U}$ . It holds  $\inf[v, u] = v$ ,  $\sup[v, u] = u$ .

An  $\mathbf{L}$ -ordered set  $\mathbf{U}$  is called *completely lattice  $\mathbf{L}$ -ordered*, if for each  $V \in L^U$ , both  $\inf V$  and  $\sup V$  exist.

Let  $\mathbf{U}$  be an  $\mathbf{L}$ -ordered set. For  $a \in L$  and  $u \in U$  we set  $a \rightarrow u = \inf\{^a/u\}$  and  $a \otimes u = \sup\{^a/u\}$ .  $a \rightarrow u$  is called *the  $a$ -shift of  $u$* ,  $a \otimes u$  is called *the  $a$ -multiple of  $u$* .  $\mathbf{U}$  is called *shift complete* (resp. *multiple complete*) if  $a \rightarrow u$  (resp.  $a \otimes u$ ) exists for any  $a \in L, u \in U$ .

The notions of shift and multiple coincide with the notions of *cotensor* and *tensor* [10, 15, 18] and shift-complete resp. multiple-complete  $\mathbf{L}$ -ordered sets are cotensored resp. tensored  $\Omega$ -categories from [15, 18]. In this paper, we use basic properties of shifts and multiples, which can be found in these papers and also in [12]. We summarize some of them below.

For any  $u, v \in U$  and  $a \in L$  it holds  $v = a \rightarrow u$  iff for each  $w \in U$ ,

$$(w \preceq v) = a \rightarrow (w \preceq u). \quad (5)$$

Similarly,  $v = a \otimes u$  iff for each  $w \in U$ ,

$$(v \preceq w) = a \rightarrow (u \preceq w). \quad (6)$$

Shifts and multiples satisfy the following adjointness condition: If  $a \rightarrow v$  and  $a \otimes u$  both exist then  $(u \preceq (a \rightarrow v)) = ((a \otimes u) \preceq v)$ .

The following theorem has been proved in [18] (Propositions 3.12, 3.13) and shows how shifts and multiples can be efficiently used for proving that an  $\mathbf{L}$ -ordered set is completely lattice  $\mathbf{L}$ -ordered.

**Theorem 1.**  $\mathbf{V}$  is a completely lattice  $\mathbf{L}$ -ordered set iff  $\mathbf{V}$  is shift complete and multiple complete and  $\langle V, \overset{1}{\preceq} \rangle$  is a complete lattice.

### 2.3 Isotone $\mathbf{L}$ -Galois connections

We introduce the notion of isotone  $\mathbf{L}$ -Galois connections. For details, see [8].

An *isotone  $\mathbf{L}$ -Galois connection* between  $\mathbf{L}$ -ordered sets  $\mathbf{U}$  and  $\mathbf{V}$  is a pair  $\langle f, g \rangle$ , where  $f: U \rightarrow V$ ,  $g: V \rightarrow U$  are mappings such that for each  $u \in U$ ,  $v \in V$  it holds

$$(f(u) \preceq v) = (u \preceq g(v)). \quad (7)$$

A pair  $\langle u, v \rangle$ , where  $u \in U$  and  $v \in V$ , is called a *fixpoint of  $\langle f, g \rangle$*  if  $f(u) = v$  and  $g(v) = u$ . More generally, *the degree to which  $\langle u, v \rangle$  is a fixpoint of  $\langle f, g \rangle$*  is defined by

$$\text{Fix}_{\langle f, g \rangle}(\langle u, v \rangle) = (f(u) \approx v) \wedge (g(v) \approx u). \quad (8)$$

$\text{Fix}_{\langle f, g \rangle}$  is an  $\mathbf{L}$ -set in  $U \times V$  and is called *the  $\mathbf{L}$ -set of fixpoints of  $\langle f, g \rangle$* .

**Theorem 2 (basic properties of isotone  $\mathbf{L}$ -Galois connections).** *Let  $\langle f, g \rangle$  be an isotone  $\mathbf{L}$ -Galois connection between  $\mathbf{L}$ -ordered sets  $\mathbf{U}$  and  $\mathbf{V}$ . Then*

- (a)  $u \preceq g(f(u))$  for each  $u \in U$ ,  $f(g(v)) \preceq v$  for each  $v \in V$ .
- (b)  $f$  and  $g$  are increasing:  $(u_1 \preceq u_2) \leq (f(u_1) \preceq f(u_2))$  and  $(v_1 \preceq v_2) \leq (g(v_1) \preceq g(v_2))$ .
- (c)  $f(g(f(u))) = f(u)$ ,  $g(f(g(v))) = g(v)$ .
- (d) If  $U' \in L^U$  then  $f(\inf U') \leq \inf f(U')$  and  $f(\sup U') \geq \sup f(U')$ . If  $V' \in L^V$  then  $g(\inf V') \leq \inf g(V')$  and  $g(\sup V') \geq \sup g(V')$ .
- (e) If  $\mathbf{U}$  and  $\mathbf{V}$  are completely lattice  $\mathbf{L}$ -ordered sets and  $K \in L^{U \times V}$ , then

$$S(K, \text{Fix}_{\langle f, g \rangle}) \leq \text{Fix}_{\langle f, g \rangle}(\langle \inf pr_U(K), f(g(\inf pr_V(K))) \rangle), \quad (9)$$

$$S(K, \text{Fix}_{\langle f, g \rangle}) \leq \text{Fix}_{\langle f, g \rangle}(\langle g(f(\sup pr_U(K))), \sup pr_V(K) \rangle), \quad (10)$$

where  $pr_U: U \times V \rightarrow U$  and  $pr_V: U \times V \rightarrow V$  are Cartesian projections.

*Proof.* Omitted. Some of the properties are proved in [8].

For an  $\mathbf{L}$ -ordered set  $\mathbf{U}$ , an isotone  $\mathbf{L}$ -Galois connection  $\langle f, g \rangle$  between  $\mathbf{U}$  and  $\mathbf{U}$  is called an *inflationary Galois connection on  $\mathbf{U}$*  if  $f$  is deflationary and  $g$  inflationary:

$$f(u) \leq u \quad \text{and} \quad g(u) \geq u \quad (11)$$

for each  $u \in U$ . Notice that if one of the conditions (11) holds true, then the second one follows from (7).

## 2.4 Composition Operators and Concept Lattices Associated to $I$

We use three relational composition operators,  $\circ$ ,  $\triangleleft$ , and  $\triangleright$ . The composition operators are defined by

$$(A \circ B)(x, y) = \bigvee_{z \in Z} A(x, z) \otimes B(z, y), \quad (12)$$

$$(A \triangleleft B)(x, y) = \bigwedge_{z \in Z} A(x, z) \rightarrow B(z, y), \quad (13)$$

$$(A \triangleright B)(x, y) = \bigwedge_{z \in Z} B(z, y) \rightarrow A(x, z). \quad (14)$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [11] to which we refer for an overview of knowledge processing applications. One may easily see that  $\triangleright$  can be defined in terms of  $\triangleleft$  and vice versa. They have natural verbal descriptions. For instance,  $(A \circ B)(x, y)$  is the truth degree of the proposition “there is a factor  $z$  such that  $z$  applies to the object  $x$  and the attribute  $y$  is a manifestation of  $z$ ”;  $(A \triangleleft B)(x, y)$  is the truth degree of “for every factor  $z$ , if  $z$  applies to the object  $x$  then the attribute  $y$  is a manifestation of  $z$ ”. Note also that for  $L = \{0, 1\}$ ,  $A \circ B$  coincides with the well-known composition of binary relations.

A *formal  $\mathbf{L}$ -context* (or just context) is a triple  $\langle X, Y, I \rangle$  where  $X$  and  $Y$  are sets whose elements are called objects and attributes, respectively and  $I$  is an  $L$ -relation between  $X$  and  $Y$ . Consider the following pairs of operators between  $L^X$  and  $L^Y$  induced by an  $L$ -relation  $I \in L^{X \times Y}$ :

$$A^\uparrow(y) = \bigwedge_{x \in X} A(x) \rightarrow I(x, y), \quad B^\downarrow(x) = \bigwedge_{y \in Y} B(y) \rightarrow I(x, y), \quad (15)$$

$$A^\cap(y) = \bigvee_{x \in X} A(x) \otimes I(x, y), \quad B^\cup(x) = \bigwedge_{y \in Y} I(x, y) \rightarrow B(y), \quad (16)$$

$$A^\wedge(y) = \bigwedge_{x \in X} I(x, y) \rightarrow A(x), \quad B^\vee(x) = \bigvee_{y \in Y} B(y) \otimes I(x, y), \quad (17)$$

for  $A \in L^X$ ,  $B \in L^Y$ . We call (15) antitone concept-forming operators and (16), (17) isotone concept-forming operators (these are less common if FCA). Furthermore, denote the set of fixpoints of  $\langle \uparrow, \downarrow \rangle$  by  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$ . We have

$$\mathcal{B}(X^\uparrow, Y^\downarrow, I) = \{ \langle A, B \rangle \mid A^\uparrow = B, B^\downarrow = A \},$$

$\mathcal{B}(X^\uparrow, Y^\downarrow, I)$  is a completely lattice  $\mathbf{L}$ -ordered set with the  $\mathbf{L}$ -equality  $\approx$  and  $\mathbf{L}$ -order  $\preceq$  defined by

$$\langle A_1, B_1 \rangle \approx \langle A_2, B_2 \rangle = (A_1 \approx^X A_2) \quad (= B_1 \approx^Y B_2), \quad (18)$$

$$\langle A_1, B_1 \rangle \preceq \langle A_2, B_2 \rangle = S(A_1, A_2) \quad (= S(B_2, B_1)), \quad (19)$$

and is called the **L**-concept lattice associated to  $I$ . Its elements are called *formal L-concepts*. **L**-concept lattices are the fundamental structures of formal concept analysis in fuzzy setting [7, 3]. For a formal **L**-concept  $\langle A, B \rangle$ ,  $A$  and  $B$  are called the *extent* and the *intent* and they represent the collection of objects and attributes to which the formal concept applies. The sets of all extents and intents of the respective concept lattices are denoted by  $\text{Ext}(X^\uparrow, Y^\downarrow, I)$  and  $\text{Int}(X^\uparrow, Y^\downarrow, I)$ , respectively. It may be shown that

$$\begin{aligned}\text{Ext}(X^\uparrow, Y^\downarrow, I) &= \{A \in L^X \mid A = A^{\uparrow\downarrow}\}, \\ \text{Int}(X^\uparrow, Y^\downarrow, I) &= \{B \in L^Y \mid B = B^{\downarrow\uparrow}\}.\end{aligned}$$

The above-defined operators and their sets of fixpoints have been extensively studied, see e.g. [1, 3, 8, 14].

*Example 1.* Let **L** be the 6-element Łukasiewicz chain (i.e.,  $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ ) and  $\langle X, Y, I \rangle$  be a formal **L**-context, where  $X = \{\text{BV}, \text{LH}, \text{MD}, \text{TSS}\}$ ,  $Y = \{\text{c1}, \text{c2}, \text{c3}, \text{c4}, \text{c5}\}$ , and  $I$  is depicted in Fig. 1. Elements of  $X$  are four selected movies by director David Lynch, elements of  $Y$  are five film critics and values of  $I$  are ratings the critics assigned to the movies, taken from [www.metacritic.com](http://www.metacritic.com) and rescaled to the six-element scale. The context is our central example in this paper and we describe it deeper and work with it in Section 5. For now, we use it just to give an example of an **L**-concept lattice.

Note that our data are suitable for interpreting by means of fuzzy logic. A movie rating (usually given by number of “stars” or by a percentage) can be interpreted as an answer to the question: “Do you like this movie?”, given in degrees. In most cases, it is not possible to answer the above question with simple “Yes” or “No”, and, in the same time, there are no doubts about the answer (especially, if given by a film critic).

	c1	c2	c3	c4	c5
BV	0.4	0.4	0.2	0.4	0.6
LH	0.8	0.8	0.8	1	1
MD	0.8	0.4	1	0.8	1
TSS	0.4	0.4	0.8	0.6	0.4

**Fig. 1.** Formal context of movies (“Blue Velvet” (BV), “Lost Highway” (LH), “Mulholland Drive” (MD), and “The Straight Story” (TSS)), reviewers (David Sterritt (c1), Desson Thomson (c2), Jonathan Rosenbaum (c3), Owen Gleiberman (c4) and Roger Ebert (c5)) and their ratings on the 6-element Łukasiewicz chain  $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ . Data taken from [www.metacritic.com](http://www.metacritic.com) on March 20, 2011.

The **L**-concept lattice  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$  is depicted in Fig. 2. When displaying **L**-concept lattices, we use labeled Hasse diagrams to include all the information on extents and intents. For any  $x \in X$ ,  $y \in Y$  and formal **L**-concept  $\langle A, B \rangle$  we have  $A(x) \geq a$  and  $B(y) \geq b$  if and only if there is a formal concept  $\langle A_1, B_1 \rangle \leq \langle A, B \rangle$ , labeled by  $a/x$  and a formal concept  $\langle A_2, B_2 \rangle \geq \langle A, B \rangle$ , labeled by  $b/y$ . We use labels  $x$  resp.  $y$  instead of  $1/x$  resp.  $1/y$  and omit redundant labels (i.e., if a concept has both the labels  $a/x$  and  $b/x$  then we keep only that with the greater degree; dually for attributes). The whole structure of **L**-ordered set on  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$  can be determined from the labeled diagram using the results from [1] (see also [3]).

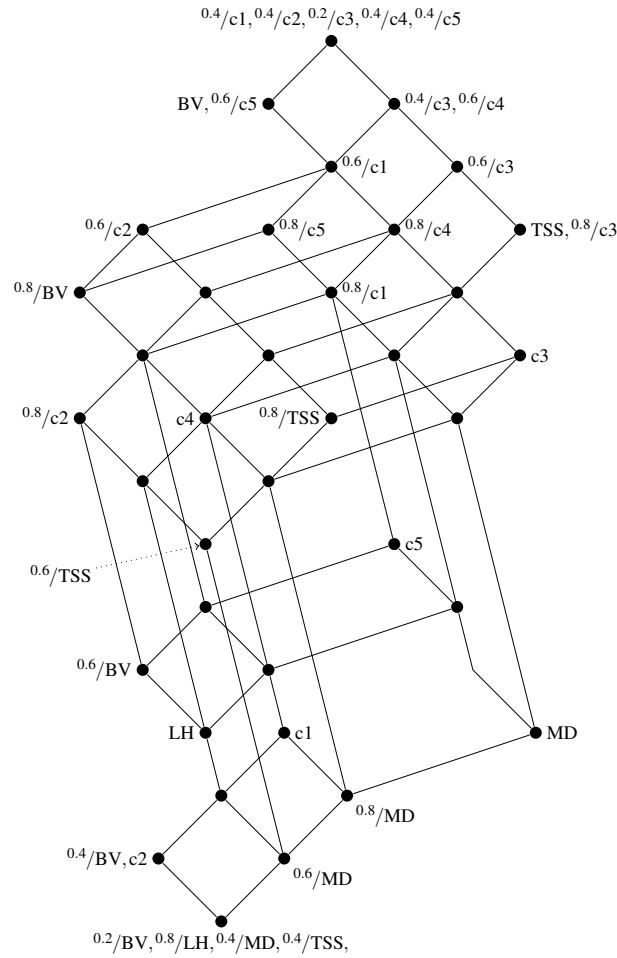


Fig. 2. L-concept lattice of movies, critics and ratings (from Fig. 1).

### 3 Complete L-tolerances

In this section we develop a generalization of the known results on factorization of complete lattices by complete tolerances [6, 17, 7] to fuzzy setting. We give a definition of a complete L-tolerance on a completely lattice L-ordered set, introduce a structure of an L-ordered set on the corresponding factor set and show that together with this structure, the factor set is a completely lattice L-ordered set (Theorem 7). In addition, we show that complete L-tolerances on a completely lattice L-ordered set are in one-to-one correspondence with inflationary L-Galois connections (Theorem 8).

For an L-tolerance  $\sim$  on a set  $X$ , an L-set  $B \in L^X$  is called a *block of the L-tolerance*  $\sim$  if for each  $x_1, x_2 \in X$  it holds  $B(x_1) \otimes B(x_2) \leq (x_1 \sim x_2)$ . A block  $B$  is called *maximal*



if for each block  $B'$  from  $B \subseteq B'$  it follows  $B = B'$ . The set of all maximal blocks of  $\sim$  always exists by Zorn's lemma, is called *the factor set of  $X$  by  $\sim$*  and denoted by  $X/\sim$ .

Further we set for each  $x \in X$ ,  $[[x]]_{\sim}(y) = x \sim y$ , obtaining an  $\mathbf{L}$ -set  $[[x]]_{\sim}$  called *the class of  $\sim$  determined by  $x$* .

An  $\mathbf{L}$ -tolerance  $\sim$  on a completely lattice  $\mathbf{L}$ -ordered set  $\mathbf{U} = \langle \langle U, \approx \rangle, \preceq \rangle$  is called *complete* if it is compatible with  $\approx$  and for each  $R \subseteq \sim$  it holds

$$\sup(pr_1(R)) \sim \sup(pr_2(R)) = \inf(pr_1(R)) \sim \inf(pr_2(R)) = 1, \quad (20)$$

where  $pr_1$  (resp.  $pr_2$ ) is the projection  $U \times U \rightarrow U$  to the first (resp. second) component.

Note that in the classical case the condition (20) becomes the well-known condition of completeness of  $\sim$  [17, 7]: Any subset  $R \subseteq \sim$  is a set of pairs  $\{\langle u_j, v_j \rangle \mid j \in J\}$  such that for each  $j \in J$  it holds  $u_j \sim v_j$  and  $\sup(pr_1(R)) = \bigvee_{j \in J} u_j$ ,  $\sup(pr_2(R)) = \bigvee_{j \in J} v_j$  and similarly for infima.

For the rest of this section we suppose  $\sim$  is a complete  $\mathbf{L}$ -tolerance on a completely lattice  $\mathbf{L}$ -ordered set  $\mathbf{U} = \langle \langle U, \approx \rangle, \preceq \rangle$ .

**Lemma 1.** *If  $u \sim v = 1$  then  $(a \otimes u) \sim (a \otimes v) = 1$  and  $(a \rightarrow u) \sim (a \rightarrow v) = 1$ .*

*Proof.* Follows directly from the definition.

**Lemma 2.** *If  $\sim$  is a complete  $\mathbf{L}$ -tolerance on a complete lattice  $\mathbf{L}$ -ordered set  $\mathbf{U} = \langle \langle U, \approx \rangle, \preceq \rangle$  then so is  $a \rightarrow \sim$ , for each  $a \in L$ .*

*Proof.* Since  $\sim$  is compatible with  $\approx$  then for  $u, u', v, v' \in U$  we have  $(u \approx u') \otimes (a \rightarrow (u' \sim v')) \otimes (v' \approx v) \leq a \rightarrow ((u \approx u') \otimes (u' \sim v') \otimes (v' \approx v)) \leq a \rightarrow (u \sim v)$ , proving  $a \rightarrow \sim$  is compatible with  $\approx$ .

Let  $R \subseteq a \rightarrow \sim$ . We have for each  $u, v \in U$ ,  $(uRv) \leq a \rightarrow (u \sim v)$ , which is equivalent to  $a \otimes (uRv) \leq (u \sim v)$ . Thus,  $a \otimes R \subseteq \sim$  and we can use (20). We have  $\sup(pr_1(a \otimes R)) = \sup(a \otimes pr_1(R)) = a \otimes \sup(pr_1(R))$ ,  $\inf(pr_1(a \otimes R)) = \inf(a \otimes pr_1(R)) = a \rightarrow \inf(pr_1(R))$  and similarly for  $pr_2$ . Now the result follows from Lemma 1.

**Theorem 3.** *If  $\sim$  is a complete  $\mathbf{L}$ -tolerance on  $\mathbf{U}$ , then for each  $a \in L$  the  $a$ -cut  ${}^a\sim$  is a complete tolerance on the complete lattice  $\langle U, {}^1\preceq \rangle$ .*

*Proof.* For  $a = 1$  the assertion follows easily from the definition of complete  $\mathbf{L}$ -tolerances. The rest follows from Lemma 2.

We set for any  $u \in U$ ,

$$u^{\sim} = \sup[[u]]_{\sim}, \quad u_{\sim} = \inf[[u]]_{\sim}. \quad (21)$$

**Theorem 4.** *The pair  $\langle \sim, \sim \rangle$  is an inflationary isotone  $\mathbf{L}$ -Galois connection on  $\mathbf{U}$ .*

*Proof.* We show that for  $u, v \in U$  it holds  $(v_{\sim} \preceq u) \leq (v \preceq u^{\sim})$ , the proof of the opposite inequality is similar. Denote  $(v_{\sim} \preceq u) = a$ . Since  $\mathcal{U}\{^a/v_{\sim}\}(u) = 1$  (by the definition of upper cone), then we have  $a \otimes v_{\sim} \leq u$ . Now,  $(a \otimes v) \sim (a \otimes v_{\sim}) = 1$  (Lemma 1) and  $u \sim (u \vee (a \otimes v)) = 1$ . Thus,  $u \vee (a \otimes v) \leq u^{\sim}$  which means that  $(a \otimes v) \leq u^{\sim}$  or  $v \preceq u^{\sim} \geq a$ . The fact that  $\langle \sim, \sim \rangle$  is inflationary is trivial.

**Lemma 3.** *Let  $u, v \in U$  be such that  $v \leq u$ ,  $u \sim v = 1$ . If  $w_1 \leq u$ , then  $(v \preceq w_1) \leq (u \sim w_1)$  and if  $v \leq w_2$ , then  $(w_2 \preceq u) \leq (w_2 \sim v)$ .*

*Proof.* By completeness of  $\sim$ ,  $(u \vee w_1) \sim (v \vee w_1) = 1$ . Thus,

$$(v \preceq w_1) = (v \vee w_1) \approx w_1 = (u \sim (v \vee w_1)) \otimes ((v \vee w_1) \approx w_1) \leq (u \sim w_1).$$

Similarly for  $w_2$ .

**Theorem 5.** *For each  $u \in U$  the class  $\llbracket u \rrbracket_{\sim}$  is equal to the interval  $[u_{\sim}, u^{\sim}]$  in  $\mathbf{U}$ .*

*Proof.* The inclusion  $\llbracket u \rrbracket_{\sim} \subseteq [u_{\sim}, u^{\sim}]$  follows from basic properties of cones (for example,  $\llbracket u \rrbracket_{\sim} \subseteq \mathcal{L}\mathcal{U}\llbracket u \rrbracket_{\sim} = \mathcal{L}\{u^{\sim}\}$ ).

We prove the opposite inclusion. For  $w \in U$  denote  $u_{\sim} \preceq w = a$  and  $w \preceq u^{\sim} = b$ . We have  $[u_{\sim}, u^{\sim}](w) = a \wedge b$ . By Lemma 3,  $(u_{\sim} \preceq (w \wedge u)) \leq ((w \wedge u) \sim u)$  and  $((w \vee u) \preceq u^{\sim}) \leq ((w \vee u) \sim u)$ . But  $u_{\sim} \preceq (w \wedge u) = a$  and  $(w \vee u) \preceq u^{\sim} = b$  which means that  $[u_{\sim}, u^{\sim}](w) = a \wedge b \leq ((w \wedge u) \sim u) \wedge ((w \vee u) \sim u)$ . Using Theorem 3 we easily obtain that the right-hand side is less than or equal to  $w \sim u$ .

**Theorem 6.** *Maximal blocks of  $\sim$  are exactly  $\mathbf{L}$ -intervals  $[v, u]$  where  $\langle u, v \rangle$  are fix-points of  $\langle \sim, \sim \rangle$ .*

*Proof.* Let  $w_1, w_2 \in U$ . We have

$$\begin{aligned} [v, u](w_1) \otimes [v, u](w_2) &= ((v \preceq w_1) \wedge (w_1 \preceq u)) \otimes ((v \preceq w_2) \wedge (w_2 \preceq u)) \\ &\leq ((v \preceq w_1) \otimes (w_2 \preceq u)) \wedge ((w_1 \preceq u) \otimes (v \preceq w_2)) \\ &= ((v \preceq w_1) \otimes (w_2 \sim v)) \wedge ((w_1 \preceq u) \otimes (u \preceq w_2^{\sim})) \\ &\leq (w_2 \sim w_1) \wedge (w_1 \preceq w_2^{\sim}) = \llbracket w_2 \rrbracket_{\sim}(w_1) = (w_1 \sim w_2), \end{aligned}$$

showing  $[v, u]$  is a block.

Conversely, if  $W \in L^U$  is a block then so is the interval  $[\inf W, \sup W]$  which contains  $W$  as a subset. Among all intervals  $[v, u]$ , the maximal blocks are those satisfying  $v^{\sim} = u$  and  $u_{\sim} = v$ .

We define two binary  $\mathbf{L}$ -relations  $\approx$  and  $\preceq$  on  $U/\sim$  by

$$[v_1, u_1] \approx [v_2, u_2] = v_1 \approx v_2 \quad (= u_1 \approx u_2), \quad (22)$$

$$[v_1, u_1] \preceq [v_2, u_2] = v_1 \preceq v_2 \quad (= u_1 \preceq u_2) \quad (23)$$

(the equalities in the parentheses follow directly from (4), Theorem 4, and Theorem 2 (b))

**Theorem 7.** *The tuple  $\langle \langle U/\sim, \approx \rangle, \preceq \rangle$  is a completely lattice  $\mathbf{L}$ -ordered set.*

*Proof.* Sketch: we use Theorem 1, (5), (6).

Let  $\langle f, g \rangle$  be an inflationary Galois connection on  $\mathbf{U}$ . Set for each  $u, v \in U$

$$u \sim_{\langle f, g \rangle} v = (f(u) \preceq v) \wedge (v \preceq g(u)). \quad (24)$$

**Theorem 8.**  $\sim_{\langle f,g \rangle}$  is an  $\mathbf{L}$ -complete tolerance, satisfying  $u \sim_{\langle f,g \rangle} = f(u)$ ,  $u \sim^{\langle f,g \rangle} = g(u)$ , for each  $u \in U$ . The assignment  $\langle f,g \rangle \mapsto \sim_{\langle f,g \rangle}$  is a bijection between the set of all inflationary isotone  $\mathbf{L}$ -Galois connections  $\mathbf{U}$  and the set of all  $\mathbf{L}$ -complete tolerances on  $\mathbf{U}$ .

*Proof.* Sketch: We use basic properties of isotone  $\mathbf{L}$ -Galois connections (Theorem 2) and the definition of an  $\mathbf{L}$ -complete tolerance on  $\mathbf{U}$ .

## 4 Block $\mathbf{L}$ -relations

This section introduces block  $\mathbf{L}$ -relations on  $\mathbf{L}$ -formal contexts, their properties and the relationship to complete  $\mathbf{L}$ -tolerances on  $\mathbf{L}$ -concept lattices. The main results are contained in Theorems 11 and 12, where we show that block  $\mathbf{L}$ -relations of any formal  $\mathbf{L}$ -context are in one-to-one correspondence with complete  $\mathbf{L}$ -tolerances on the associated  $\mathbf{L}$ -concept lattice and that the  $\mathbf{L}$ -concept lattice of each of the block relations is isomorphic to the original  $\mathbf{L}$ -concept lattice, factorized by the corresponding complete  $\mathbf{L}$ -tolerance.

We define block relations as follows: *Block  $\mathbf{L}$ -relation* of  $I \in L^{X \times Y}$  is an  $\mathbf{L}$ -relation  $J \supseteq I$  such that  $\text{Ext}(X^\uparrow, Y^\downarrow, J) \subseteq \text{Ext}(X^\uparrow, Y^\downarrow, I)$  and  $\text{Int}(X^\uparrow, Y^\downarrow, J) \subseteq \text{Int}(X^\uparrow, Y^\downarrow, I)$ .

In crisp setting, [17] defines block relation as a relation  $J \supseteq I$  where each row is an intent of  $I$  and each column is an extent of  $I$ . Lemma 4 says that block  $\mathbf{L}$ -relation is a proper generalization of crisp block relation. The reason we define the notion that way is to allow an analogous definition in the isotone case.

**Lemma 4.**  $J \in L^{X \times Y}$ ,  $J \supseteq I$  is a block  $\mathbf{L}$ -relation of  $I$  iff  $\{x\}^{\uparrow I} \in \text{Int}(X^\uparrow, Y^\downarrow, I)$  for each  $x \in X$  and  $\{y\}^{\downarrow I} \in \text{Ext}(X^\uparrow, Y^\downarrow, I)$  for each  $y \in Y$ .

The following theorem provide characterization of block  $\mathbf{L}$ -relations.

**Theorem 9.** Let  $I \in L^{X \times Y}$  be an  $\mathbf{L}$ -relation. The following statements are equivalent:

- (a)  $J$  is a block relation of  $I$ .
- (b)  $J = I \triangleright S_i$  with  $S_i \in L^{Y \times Y}$  and for the induced mapping  $^{\wedge S_i}$  we have  $B^{\wedge S_i} \in \text{Int}(X^\uparrow, Y^\downarrow, I)$  and  $B \subseteq B^{\wedge S_i}$  for each  $B \in \text{Int}(X^\uparrow, Y^\downarrow, I)$ .
- (c)  $J = S_e \triangleleft I$  with  $S_e \in L^{X \times X}$  and for the induced mapping  $^{\cup S_e}$  we have  $A^{\cup S_e} \in \text{Ext}(X^\uparrow, Y^\downarrow, I)$  and  $A \subseteq A^{\cup S_e}$  for each  $A \in \text{Ext}(X^\uparrow, Y^\downarrow, I)$ .

*Proof.* (sketch, equivalence of (a) and (b)): We have  $\text{Ext}(X^\uparrow, Y^\downarrow, J) \subseteq \text{Ext}(X^\uparrow, Y^\downarrow, I)$  iff there exists a matrix  $S_i$  such that  $J = I \triangleright S_i$  by [4, Theorem 7]. We have  $A^{\uparrow J} = A^{\uparrow I \triangleright S_i} = (A^{\uparrow I})^{\wedge S_i}$  for  $A \in L^X$ . Since  $A^{\uparrow I}$  is any intent  $B$  in  $\text{Int}(X^\uparrow, Y^\downarrow, I)$  and  $A^{\uparrow J} = B^{\wedge S_i} \in \text{Int}(X^\uparrow, Y^\downarrow, I)$  we obtain  $B^{\wedge S_i} \in \text{Int}(X^\uparrow, Y^\downarrow, I)$  for each  $B \in \text{Int}(X^\uparrow, Y^\downarrow, I)$ . By (16) and (14) we have that  $J(x, y)$  is equal to  $\{x\}^{\uparrow I \wedge S_i}(y)$  and with the property  $B \subseteq B^{\wedge S_i}$  we get  $I \subseteq J$  proving that  $J$  is a block  $\mathbf{L}$ -relation.

**Theorem 10.** Let  $I \in L^{X \times Y}$  be an  $\mathbf{L}$ -relation between  $X$  and  $Y$ .

- (a) The set of all block  $\mathbf{L}$ -relations  $J$  of  $I$  is an  $\mathbf{L}$ -closure system (i.e. it is closed under  $\bigwedge$  and  $\rightarrow$ ).
- (b) The set of all  $S_e$  (from Theorem 9) is an  $\mathbf{L}$ -interior system (i.e. it is closed under  $\bigvee$  and  $\otimes$ ).
- (c) The set of all  $S_i$  (from Theorem 9) is an  $\mathbf{L}$ -interior system.

*Proof.* (a) Let  $J_i$  be block  $\mathbf{L}$ -relations of  $I$ . Let  $J = \bigwedge_i J_i$  and let  $B \in \text{Int}(X^\uparrow, Y^\downarrow, J)$ , hence  $B = A^{\uparrow J}$  for some  $A \in L^X$ . By definition of  $\uparrow J$  and properties of residuated lattices we have

$$A^{\uparrow J}(y) = \bigwedge_{x \in X} A(x) \rightarrow J(x, y) = \bigwedge_{x \in X} A(x) \rightarrow \bigwedge_i J_i(x, y) = \bigwedge_i \bigwedge_{x \in X} A(x) \rightarrow J_i(x, y) = \bigwedge_i A^{\uparrow J_i}.$$

Thus we have  $A^{\uparrow J} = \bigcap_i A^{\uparrow J_i} \in \text{Int}(X^\uparrow, Y^\downarrow, I)$ . Similarly,  $B^{\downarrow J} = \bigcap_i B^{\downarrow J_i} \in \text{Ext}(X^\uparrow, Y^\downarrow, I)$ ,  $A^{\uparrow a \rightarrow J_i} = a \rightarrow A^{\uparrow J_i} \in \text{Int}(X^\uparrow, Y^\downarrow, I)$ , and  $B^{\downarrow a \rightarrow J_i} = \bigcap_i a \rightarrow B^{\downarrow J_i} \in \text{Ext}(X^\uparrow, Y^\downarrow, I)$ .

Finally, from  $I \subseteq J_i$  we have  $I \subseteq \bigwedge_i J_i$  and  $I \subseteq a \rightarrow J_i$ . Whence  $\bigwedge_i J_i$  and  $I \subseteq a \rightarrow J_i$  are block  $\mathbf{L}$ -relations proving that set of all block  $\mathbf{L}$ -relations  $J$  of  $I$  is an  $\mathbf{L}$ -closure system.

(b) and (c) can be proved similarly.

The induced operators  $\cap, \cup$  of the matrices  $S_e$  and  $S_i$  have following properties:

**Lemma 5.** *Let  $I \in L^{X \times Y}$  be an  $\mathbf{L}$ -relation between  $X$  and  $Y$  and let  $J$  be its block  $\mathbf{L}$ -relation:*

- (a)  $A^{\cup S_e} = A^{\uparrow \downarrow J}$  for any  $A \in \text{Ext}(X^\uparrow, Y^\downarrow, I)$ ,  $B^{\wedge S_i} = B^{\downarrow \uparrow J}$  for any  $B \in \text{Int}(X^\uparrow, Y^\downarrow, I)$ .
- (b)  $\langle A^{\cup S_e}, A^{\cup S_e \cap S_e \uparrow I} \rangle \in \mathcal{B}(X^\uparrow, Y^\downarrow, J)$  and  $\langle B^{\wedge S_i \vee S_i \downarrow I}, B^{\wedge S_i} \rangle \in \mathcal{B}(X^\uparrow, Y^\downarrow, J)$  for each  $\langle A, B \rangle \in \mathcal{B}(X^\uparrow, Y^\downarrow, I)$ .
- (c)  $A^{\cap S_e \uparrow I} = A^{\uparrow \wedge S_i} = A^{\uparrow J}$  for any  $A \in L^X$  and  $B^{\vee S_i \downarrow I} = B^{\downarrow \cup S_e} = B^{\downarrow J}$  for any  $B \in L^Y$ .

Let  $I \in L^{X \times Y}$  be an  $\mathbf{L}$ -relation between  $X$  and  $Y$  and let  $J$  be its block relation. Denote by  $\theta_J$  a mapping  $\theta_J : \mathcal{B}(X^\uparrow, Y^\downarrow, I) \rightarrow \mathcal{B}(X^\uparrow, Y^\downarrow, I)$  defined by

$$\langle A, B \rangle^{\theta_J} = \langle A^{\cup S_e}, A^{\cup S_e \uparrow I} \rangle \quad (25)$$

and denote by  $\theta_J$  a mapping  $\theta_J : \mathcal{B}(X^\uparrow, Y^\downarrow, I) \rightarrow \mathcal{B}(X^\uparrow, Y^\downarrow, I)$  defined by

$$\langle A, B \rangle_{\theta_J} = \langle B^{\wedge S_i \downarrow I}, B^{\wedge S_i} \rangle \quad (26)$$

Notice, that different block  $\mathbf{L}$ -relations  $J$  of an  $\mathbf{L}$ -relation  $I$  induce different mappings  $\theta_J, \theta_J$ . In what follows we omit subscript in the notation  $\theta_J, \theta_J$  and write just  $\theta, \theta$ . Obviously, for each  $\langle A, B \rangle \in \mathcal{B}(X^\uparrow, Y^\downarrow, I)$  we have  $\langle A, B \rangle_{\theta} \leq \langle A, B \rangle \leq \langle A, B \rangle^{\theta}$ .

Now, we explain the relationship between block  $\mathbf{L}$ -relations and compatible  $\mathbf{L}$ -tolerances.

**Theorem 11.** (a) *Let  $J \supseteq I$  be a block  $\mathbf{L}$ -relation. Then the mappings  $\theta, \theta$  form an inflationary  $\mathbf{L}$ -Galois connection on  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$ .*

(b) *For any inflationary  $\mathbf{L}$ -Galois connection  $\langle f, g \rangle$  on  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$  the  $\mathbf{L}$ -relation  $J \in L^{X \times Y}$  given by*

$$\begin{aligned} J(x, y) &= f(\langle \{x\}^{\uparrow \downarrow I}, \{x\}^{\uparrow I} \rangle) \preceq \langle \{y\}^{\downarrow I}, \{y\}^{\downarrow \uparrow I} \rangle \\ & (= \langle \{x\}^{\uparrow \downarrow I}, \{x\}^{\uparrow I} \rangle \preceq g(\langle \{y\}^{\downarrow I}, \{y\}^{\downarrow \uparrow I} \rangle)) \end{aligned} \quad (27)$$

*is a block  $\mathbf{L}$ -relation of  $\langle X, Y, I \rangle$  such that its mappings  $\theta$  and  $\theta$  are equal to the mappings  $f$  and  $g$ , respectively.*

*Proof.* (a) Since  $J$  is a block  $\mathbf{L}$ -relation then  $A^{\uparrow J}$  is an intent of  $\langle X, Y, I \rangle$  and  $f(\langle A, B \rangle) \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ . Similarly for  $g$ .

The pair  $\langle \uparrow^J, \downarrow^J \rangle$  is an antitone  $\mathbf{L}$ -Galois connection between  $L^X$  and  $L^Y$ . Thus, for each  $A \in L^X$  and  $B \in L^Y$  we have  $S(A, B^{\downarrow J}) = S(B, A^{\uparrow J})$ . Now,

$$f(\langle A_1, B_1 \rangle) \preceq \langle A_2, B_2 \rangle = S(B_2, A_1^{\uparrow J}) = S(A_1, B_2^{\downarrow J}) = \langle A_1, B_1 \rangle \preceq g(\langle A_2, B_2 \rangle)$$

proving  $\langle f, g \rangle$  is an isotone  $\mathbf{L}$ -Galois connection. Since  $I \subseteq J$  then  $A^{\uparrow I} \subseteq A^{\uparrow J}$  and  $f$  is deflationary.

(b) First we need to show that each object intent of  $\langle X, Y, J \rangle$  is an intent of  $\langle X, Y, I \rangle$  and each attribute extent of  $\langle X, Y, J \rangle$  is an extent of  $\langle X, Y, I \rangle$ . We shall prove the part for extents. For each  $x \in X$  we have

$$\begin{aligned} \{x\}^{\uparrow J}(y) &= J(x, y) = S(f_X(\{x\}^{\uparrow I \downarrow J}), \{y\}^{\downarrow J}) = \bigwedge_{x' \in X} f_X(\{x\}^{\uparrow I \downarrow J})(x') \rightarrow \{y\}^{\downarrow J}(x') \\ &= \bigwedge_{x' \in X} f_X(\{x\}^{\uparrow I \downarrow J})(x') \rightarrow I(x', y) = f_X(\{x\}^{\uparrow I \downarrow J})^{\uparrow I}(y) \end{aligned}$$

and  $\{x\}^{\uparrow J}$  is an intent of  $\langle X, Y, I \rangle$ . That proves that  $\text{Ext}(X^{\uparrow}, Y^{\downarrow}, J) \subseteq \text{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ ; similarly can be shown that  $\text{Int}(X^{\uparrow}, Y^{\downarrow}, J) \subseteq \text{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ . From properties of inflationary  $\mathbf{L}$ -Galois connections and antitone Galois connections, we have  $f_X(\{x\}^{\uparrow I \downarrow J})^{\uparrow I}(y) \geq \{x\}^{\uparrow I \downarrow J \uparrow I}(y) = \{x\}^{\uparrow I}(y)$ ; hence  $J \supseteq I$ . The equality of  $\theta^{\uparrow, \downarrow}$  and  $f, g$  is immediate.

Theorem 11 and Theorem 8 generalize [7, Theorem 15]. The following theorem represents the main result of this paper.

**Theorem 12.** (a) Complete  $\mathbf{L}$ -tolerances are in one-to-one correspondence with block  $\mathbf{L}$ -relations.

(b) If complete  $\mathbf{L}$ -tolerance  $\sim$  and block  $\mathbf{L}$ -relation  $J$  are in that correspondence, then  $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, J)$  is isomorphic to the lattice of blocks of  $\mathbf{L}$ -tolerance  $\sim$ .

*Proof.* The one-to-one correspondence follows from Theorem 11 and the fact that different block relation produce different mappings  $\theta^{\uparrow, \downarrow}$ . The isomorphism of  $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, J)$  and the lattice of  $\sim$  then follows from definition of  $\theta^{\uparrow, \downarrow}$  and their equality to mappings  $f, g$  of  $\sim$ .

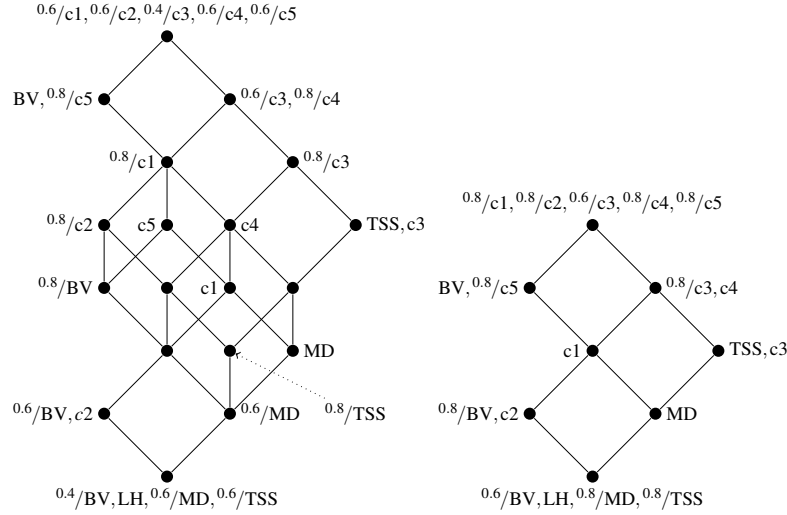
## 5 Illustrative Example

In this section, we use results developed in the previous sections to factorize the  $\mathbf{L}$ -concept lattice  $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$  from Example 1.

Our aim is to reduce the size of the  $\mathbf{L}$ -concept lattice  $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$  by factorization by complete  $\mathbf{L}$ -tolerances. First we take the approach from [2]. This approach is based on a choice of a threshold  $a \in L$  and using the  $a$ -cut  $a \approx$  of the  $\mathbf{L}$ -equality  $\approx$  on  $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$  for factorization. The  $a$ -cut  $a \approx$  is a complete tolerance on the complete lattice  $\langle \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I), \overset{1}{\preceq} \rangle$  and, as noted in [5], the factor lattice is isomorphic to the crisp part of the concept lattice  $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, a \rightarrow I)$ . As it can be easily seen, these results are a special case of the results of this paper. Namely, one can take the complete

$\mathbf{L}$ -tolerance  $a \rightarrow \approx$  on  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$  (the fact that it is indeed a complete  $\mathbf{L}$ -tolerance easily follows from Lemma 2) and construct the associated block  $\mathbf{L}$ -relation  $J \supseteq I$ , which is equal to  $a \rightarrow I$ . The factorized  $\mathbf{L}$ -concept lattice  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)/^a \approx$  is isomorphic to  $\mathcal{B}(X^\uparrow, Y^\downarrow, a \rightarrow I)$ .

As an example, we present in Fig. 3 results we obtained for our formal context of movies, critics and ratings with values of the threshold  $a$  equal to 0.4 and 0.3.



**Fig. 3.**  $\mathbf{L}$ -concept lattice of movies, critics and ratings (from Fig. 1), factorized with respect to the threshold  $a = 0.8$  (left) and  $a = 0.6$  (right).

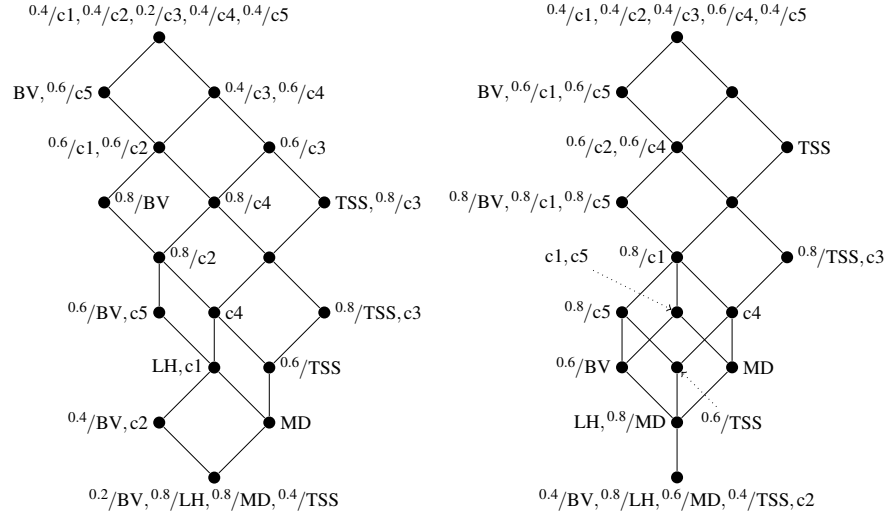
We also tried a more sophisticated approach, based on [13]. The author’s method is based on using a complete tolerance on a complete lattice induced by an interior and closure operator. As an example, the author uses a complete tolerance on a (crisp) concept lattice, obtained by selecting subsets  $X' \subseteq X$  and  $Y' \subseteq Y$  of important objects and important attributes, respectively.

Although a proper fuzzification of the results from [13] remains to be developed, it is possible to try some experiments. We provide a short outline of our method here and leave the details to a forthcoming paper.

For a formal  $\mathbf{L}$ -context  $\langle X, Y, I \rangle$  we select two  $\mathbf{L}$ -sets  $X' \in L^X$  and  $Y' \in L^Y$  and interpret them as  $\mathbf{L}$ -sets of “important objects” and “important attributes”, respectively. Thus, for an object  $x \in X$ , the value  $X'(x)$  is the degree to which  $x$  is important, and similarly for attributes. Further we define two  $\mathbf{L}$ -relations  $I_{X'}, I_{Y'} \in L^{X \times Y}$  by

$$I_{X'}(x, y) = X'(x) \rightarrow I(x, y), \quad I_{Y'}(x, y) = Y'(y) \rightarrow I(x, y).$$

As it can be easily seen from (15),  $\text{Int}(X^\uparrow, Y^\downarrow, I_{X'}) \subseteq \text{Int}(X^\uparrow, Y^\downarrow, I)$  and  $\text{Ext}(X^\uparrow, Y^\downarrow, I_{Y'}) \subseteq \text{Ext}(X^\uparrow, Y^\downarrow, I)$ . Thus, the  $\mathbf{L}$ -relations  $I_{X'}$  and  $I_{Y'}$  select some intents and extents of formal



**Fig. 4.** L-concept lattice of movies, critics and ratings (form Fig. 1), factorized with respect to an L-set of important objects  $X' = \{0.8/BV, LH, MD, TSS\}$  and L-set of important attributes  $Y' = \{c1, 0.8/c2, c3, c4, c5\}$  (left) and with respect to an L-set of important objects  $X' = \{BV, LH, MD, TSS\}$  and L-set of important attributes  $Y' = \{c1, 0.6/c2, c3, c4, c5\}$  (right).

concepts from  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$ . These intents and extents are interpreted as “important”. Now, let  $J_{X', Y'} \in L^{X' \times Y'}$  be the minimal (with respect to L-set inclusion) block L-relation of  $I$  such that both intent and extent of each formal L-concept from  $\mathcal{B}(X^\uparrow, Y^\downarrow, J_{X', Y'})$  are important ( $J_{X', Y'}$  always exists because of Theorem 10(a); details are omitted). Using the results of this paper (Theorem 12) we obtain that  $J_{X', Y'}$  induces a complete L-tolerance on the L-concept lattice  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)$  and the corresponding factor completely lattice L-ordered set is isomorphic to the L-concept lattice  $\mathcal{B}(X^\uparrow, Y^\downarrow, J_{X', Y'})$ .

Note that this approach contains the approach from [2] as a special case. The factor completely lattice L-ordered set  $\mathcal{B}(X^\uparrow, Y^\downarrow, I)/^{a \approx}$  is isomorphic to  $\mathcal{B}(X^\uparrow, Y^\downarrow, J_{X', Y'})$  for  $X' = \{a/x \mid x \in X\}$  and  $Y' = \{a/y \mid y \in Y\}$ .

We apply the above considerations to our example. Suppose we consider the film BV less important than the other films (perhaps because we have not seen BV) and the critic c2 less important than the other critics (because we do not like his opinion on MD). More precisely, set  $X' = \{a/BV, LH, MD, TSS\}$  and  $Y' = \{c1, b/c2, c3, c4, c5\}$ , where  $a, b \in L$ . In Fig. 4 we can see the resulting concept lattices in two cases: first  $a = b = 0.8$  and second  $a = 1$  and  $b = 0.6$ .

## 6 Conclusions

We presented a generalization of the theory of complete tolerances on complete lattices and the relationship of block relations on formal contexts and complete tolerances on the corresponding concept lattices [6, 17, 7] to fuzzy setting. Our results can be used

for reducing the size of fuzzy concept lattices by means of factorization and offer a greater degree of variability than the known approach from [2]. In the future we will focus namely on the investigation of block relations in the isotone case and developing the theory of approximations in fuzzy concept lattices which would give a proper theoretical background to our experiments from Sec. 5.

## References

1. Belohlavek, R.: Concept lattices and order in fuzzy logic. *Ann. Pure Appl. Log.* 128(1-3), 277–298 (2004)
2. Belohlavek, R.: Similarity relations in concept lattices. *J. Log. Comput.* 10(6), 823–845 (2000)
3. Belohlavek, R.: *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic Publishers, Norwell, USA (2002)
4. Belohlavek, R., Konecny, J.: Operators and spaces associated to matrices with grades and their decompositions, to appear, submitted to *Fundamenta Informaticae*
5. Belohlavek, R., Outrata, J., Vychodil, V.: Direct factorization by similarity of fuzzy concept lattices by factorization of input data. In: Yahia, S.B., Nguifo, E.M., Belohlavek, R. (eds.) *CLA. Lecture Notes in Computer Science*, vol. 4923, pp. 68–79. Springer (2006)
6. Czédli, G.: Factor lattices by tolerances. *Acta Sci. Math.* 44, 35–42 (1982)
7. Ganter, B., Wille, R.: *Formal Concept Analysis – Mathematical Foundations*. Springer (1999)
8. Georgescu, G., Popescu, A.: Non-dual fuzzy connections. *Arch. Math. Log.* 43(8), 1009–1039 (2004)
9. Hájek, P.: *Metamathematics of Fuzzy Logic (Trends in Logic)*. Springer (November 2001)
10. Kelly, G.M.: *Basic Concepts of Enriched Category Theory*. Cambridge University Press (1982)
11. Kohout, L. J.; Bandler, W.: Relational-product architectures for information processing. *Information Sciences* 37(1-3), 25–37 (1985)
12. Krupka, M.: An alternative version of the main theorem of fuzzy concept lattices. In: Trappl, R. (ed.) *Cybernetics and Systems 2010*. pp. 9–14. Austrian Society for Cybernetic Studies, Vienna (2010), extended version submitted
13. Meschke, C.: Approximations in concept lattices. In: Kwuida, L., Sertkaya, B. (eds.) *Formal Concept Analysis, Lecture Notes in Computer Science*, vol. 5986, pp. 104–123. Springer Berlin / Heidelberg (2010)
14. Pollandt, S.: *Fuzzy Begriffe: Formale Begriffsanalyse von unscharfen Daten*. Springer-Verlag, Berlin–Heidelberg (1997)
15. Stubbe, I.: Categorical structures enriched in a quantaloid: tensored and cotensored categories. *Theory Appl. Categ.* 16, No. 14, 283–306 (electronic) (2006), <http://www.ams.org/mathscinet-getitem?mr=2223039>
16. Ward, M., Dilworth, R.P.: Residuated lattices. *Transactions of the American Mathematical Society* 45, 335–354 (1939)
17. Wille, R.: Complete tolerance relations of concept lattices. In: Eigenthaler, G., et al. (eds.) *Contributions to General Algebra*, vol. 3, pp. 397–415. Hölder-Pichler-Tempsky, Wien (1985)
18. Zhao, H., Zhang, D.: Many valued lattices and their representations. *Fuzzy Sets and Systems* 159(1), 81–94 (Jan 2008)