The bipartite rationing problem

Herve Moulin* and Jay Sethuraman**

May 2011

Abstract

In the *bipartite rationing* problem, a set of agents share a single resource available in different "types", each agent has a claim over only a subset of the resource-types, and these claims overlap in arbitrary fashion. The goal is to divide fairly the various types of resource between the claimants, when resources are in short supply.

With a single type of resources, this is the classic rationing problem (O'Neill [20]), of which the three benchmark solutions are the proportional, uniform gains, and uniform losses methods. We extend these and other methods to the bipartite context, imposing the familiar *consistency* requirement: the division is unchanged if we remove an agent (resp. a node), and take away at the same time his share of the various resources (resp. reduce the claims of the relevant agents).

We find that most parametric rationing methods (Young [25]) cannot be consistently extended, and come close to characterize the subset of those that can. The latter reduce essentially to the *loss calibrated* rationing methods, a new family of methods containing the proportional method, and the uniform gains and uniform losses as limit points. They admit a single consistent extension, although uniform gains and uniform losses admit infinitely many.

* Rice University, moulin@rice.edu

** Columbia University, jay@ieor.columbia.edu

1 The problem and the punchline

Consider the problem of dividing a max-flow in an arbitrary bipartite graph between source and sink nodes. Each source holds a finite amount of the commodity (say homogenous freight; more examples below), each sink has a finite capacity to store freight, and all edges have infinite capacity. If each node wishes to send or receive as much freight as possible, it is optimal to implement a max-flow, but there are typically many of those: our goal is to propose a fair way to select one max-flow in any such problem.

A familiar consequence of the max-flow min-cut theorem ([1]) is that we can decompose any max-flow problem in (at most) two simpler subproblems that can be treated separately. In one subproblem the sink nodes are *overdemanded*, in the sense that they can be filled to capacity while the *underdemanded* source nodes must be rationed (ship less freight than they could); the situation is reversed in the other subproblem, where the overdemanded source nodes can unload all their freight to the underdemanded storage nodes. The key fact is that there is no edge between two undermanded nodes. This decomposition cuts our fair division problem in half: we need only to propose a rule for problems where the sinks are overdemanded, then exchange the role of sources and sinks to apply the same rule to problems with overdemanded sources (see Remark 1 in Section 2).

To fix ideas we discuss throughout the paper the case of overdemanded sinks. In the simple case of a single sink, this is the simplest and oldest fair division problem, going back to Aristotle and the Talmud [2], and known as the *rationing* model since its formalization by O'Neill [20]. A storage facility with a given capacity must be shared between users who each have a claim on a certain quantity of storage space, but the available space falls short of the sum of all claims. Our generalization consists of introducing multiple storage facilities and arbitrary bipartite constraints between agents wishing to ship their freight and storage sinks, while maintaining the assumption that storage is overdemanded. We speak of the *bipartite* rationing problem, to contrast with the *standard* problem with a single sink.

It is convenient to think of each sink as a different "type" of resource, so that each agent can only consume a subset of the resource-types, and these claims overlap in arbitrary fashion. Inasmuch as not every agent can consume every type of the resource, different types are heterogenous commodities. However from the point of view of a given agent, all types he can consume are perfect substitutes, so his claim applies to all these types and he only cares about the total amount of resource he receives, irrespective of type. Our goal is to propose a fair division of all types of resource between the claimants, when resources are in short supply.

The normative literature initiated by O'Neill [20] emphasizes the relevance of standard rationing to taxation schedules ([25], [26]) and bankruptcy rules ([14]). It develops a rich axiomatic analysis confirming the central role of the three benchmark methods emerging from the empirical social-psychology literature ([21], [12], [11]), where the model is applied to the division of any resource according to individual characteristics. Individual shares can be proportional to claims (*proportional* method), as equal as possible provided that they do not exceed one's claim (*uniform gains* method), or the individual losses can be as equal as possible under the same provision (*uniform losses* method). Unlike its predecessors, the axiomatic analysis identifies rich families of new methods, providing flexible compromises between the three benchmarks: a good example is the family of equal sacrifice methods due to Young [25]. (For an overview of the axiomatic approach, see Moulin [18] and Thomson [23].)

Examples where the rationed commodity comes in several types, and bilateral constraints restrict which agents can claim/consume which resource, include:

1. *Load balancing:* The resources are different types of work, each one with a given size (processing time); the agents are workers, each one able to execute

only certain types of work, and with a capacity constraint on total individual workload. We want to divide the workload between the workers who all care about their total load (they all prefer more, or they all prefer less).

2. *Earmarked funds:* The resources are sponsors with a given total budget to fund the research of some of the agents; each agent submits a project with a total price tag, and each sponsor attaches some strings to the projects it will consider (e.g., must have an environmental dimension, must involve minorities, etc.); each project is submitted to all the sponsors of which it meets the constraints. Agents care about their total funding, irrespective of origin.

3. *Cleaning polluted sites:* The resource-types are different sites, each with a known clean-up cost for the pollution generated by several firms (who are the agents); experts determine which firm pollutes which sites, and firms are jointly liable for cleaning "their" sites. In the absence of data breaking up costs at each site between the liable firms, the judge can only use a one-dimensional proxy (e.g., total output) of each firm's *total* liability.

4. *Blood transfusion:* Each agent needs a certain amount of blood, and blood is available in limited quantity, and in four types O, A, B, AB; an agent with blood type O can only receive O-blood, one of type AB can receive from all types, and so on.

5. *Distribution of utilities (water, power,..) after a natural disaster:* Some of the normal lines and sources of supply are unavailable creating new bilateral constraints; individual claims are the normal consumption levels, that must now be rationed.

We ask if and how it is possible, in the contexts just described, to extend a standard rationing method to a bipartite one.

One way to approach this question is to regard the compatibility constraints as devoid of any normative content: agents are not held responsible for their inability to consume certain resources, so no one should derive an advantage from being compatible with more types than others. This interpretation is compelling in the blood transfusion and emergency distribution examples. It leads to extend the uniform gains method, for instance, by simply picking the most egalitarian profile of (total) shares, under the bilateral feasibility constraints[1]. One can similarly define an extension of the proportional or uniform losses methods by equalizing individual shares divided by individual claims, or equalizing individual losses, respectively.

Yet if it is natural in some problems to regard the compatibility constraints as normatively neutral, in other cases it is ethically compelling to hold claimants responsible for their bilateral constraints and reward accordingly those who can consume more types of resources: This is very clear in the clean-up example; in the funding story, a project relevant to more agencies deserves better funding, and the same remark applies to load balancing.

To make this key point formally, consider two resource-types a, b, both with 2 units to share, Ann and Bob each with a claim of 4 units, and suppose Ann

[1]This selction is unambiguous because there is a profile of individual total shares that *Lorenz dominates* any other feasible profile: see Bochet et al. [6].

can consume both types, while Bob can only consume the b type. If Bob should not be penalized for his limited options, both would get 2 units by the symmetry of the relevant parameters. But if Bob is held responsible for being unable to consume the a type, then Ann should get all of type a, and she will still have a claim of 2 units over the type b resource.

The celebrated *consistency* axiom captures in our model the idea that agents are responsible for their own compatibilities with the different types of resources. This axiom has emerged in a variety of contexts (including TU games, matching, assignment, etc.) as a compelling rationality property for fair division (see e.g., [16] and [24]). A standard rationing method is consistent if, when we take away one agent from the set of participants, and subtract his share from the available resources, the division among the remaining set of claimants does not change. It is satisfied by the three benchmark methods above, and many more.

It is easy to define a version of consistency appropriate for bilateral rationing methods. We can now take away either an agent or a type of resource: if the former, we subtract from each resource-type the share previously assigned to the departing agent; if the latter, we subtract from the claim of each agent the share of the departing resources he was previously receiving; in each case we insist that the division in the reduced problem remain as before. A stronger form of consistency can be applied to each edge of the graph: when we remove a certain edge, we subtract its flow from the capacity of both end nodes, and require as above that the solution choose the same flow in the reduced problem.

To see why a consistent method rewards agents compatible with more resource-types, assume agents have identical claims, very large with repect to the available resources. After dropping all but one resource-type, the remaining claims of the agents will still be nearly identical, relative to the remaining resource, so it will be shared nearly equally. We end up sharing equally each resource-type among all "its" agents.[2]

In addition to the consistency requirement, we insist on a symmetric treatment of agents and types, in the usual sense of equal treatment of equals. Moreover we do not want the artificial creation of new resource-types to matter, in the following sense: if two types are compatible with exactly the same set of agents, merging them into a single type while adding their resources is of no consequence to any agent. We call this property *Invariance to the Merging of Types.*

1.1 Overview of our results

We are looking for symmetric and continuous *standard* methods that can be extended to a *consistent bipartite* method, also invariant to the merging of types. We simply call such a method *extendable.*

[2] We stress that there are also non consistent methods capturing the responsibility idea in a different way. One instance is the Shapley value of the *stand alone* cooperative game, where the value of a coalition is the total amount it can consume, given the constraints. See a numerical example below.

We observe first (Lemma 1) that an extendable standard method must satisfy a property in the spirit of, though not logically related to, consistency and the lower composition axiom (see Moulin [18] and Thomson [23]): If we distribute 15% of the final shares, reduce claims and resources accordingly, then in the smaller problem everyone gets the remaining 85% of his original share[3]. We use this technical property to deduce that many familiar rationing methods are *not* extendable as desired. Examples include the Talmudic and most equal sacrifice methods.

Our first main result (Theorem 1) is that the standard proportional method is uniquely extendable. Its extension can be described in two equivalent ways: it minimizes the sum of two entropies, that of a max-flow plus that of the corresponding profile of losses (claim minus actual share); alternatively it assigns a positive weight w_i to each agent i and divides each resource-type between agents who can consume it, in proportion to the w_i; moreover individual *losses* are proportional to the w_i-s as well. The weights are *not* the individual claims, instead they solve a system of nonlinear equations. We give a numerical example in the next subsection.

We define next a family of standard methods, new to the literature, that we call *loss calibrated.* Every such method is defined by a weakly increasing real-valued function β of a single variable such that $\beta(0) = 0$ and such that $\beta(z)$ is positive if z is[4]. Given a profile of claims x, the shares y are chosen so that the ratio $\frac{y_i}{\beta(x_i - y_i)}$ is the same for all agents. Thus the loss calibrated methods equalize individual shares weighted by a measure of individual losses. For $\beta(z) = z$ we find the proportional method.

Our second main result (Theorem 2) goes a long way toward characterizing *all* extendable standard methods. Each loss calibrated method is uniquely extendable. Conversely if a standard method is *strictly resource monotonic*[5] and extendable, it must be a loss calibrated method.

We also show that the standard proportional method is the only loss calibrated one satisfying any one of three familiar axioms: Lower Composition, Upper Composition, and Self-Duality. This in turn yields several characterizations of the bipartite proportional method.

We discuss finally the extendability of the two other benchmark methods, uniform gains and uniform losses. They are both limit points of the loss calibrated family, yet not strictly resource monotonic. Each is extendable, but in infinitely many ways (Propositions 1 and 2).

---

[3]Formally, if the profile of shares is y when claims are x and we divide r units of resource, then for any $\lambda \in [0, 1]$, in the problem with claims $x - \lambda y$ and resources $(1 - \lambda)r$, the shares must be $(1 - \lambda)y$.

[4]If our method is scale invariant, the function β is a power function; see Section 4.

[5]That is, if we have more resources to share, every individual share increases strictly. Clearly every loss calibrated method meets this property.

1.2 A numerical example

We have three agents $1, 2, 3$ and two resource-types a, b. Agent 1 can consume type a only; agent 3 can consume type b only; and agent 2 can consume both types. Claims are identical $x_1 = x_2 = x_3 = 2$, but there is 1 unit of type a and 2 units of type b. The resources are strictly overdemanded (as defined in section 2). See Figure 1.

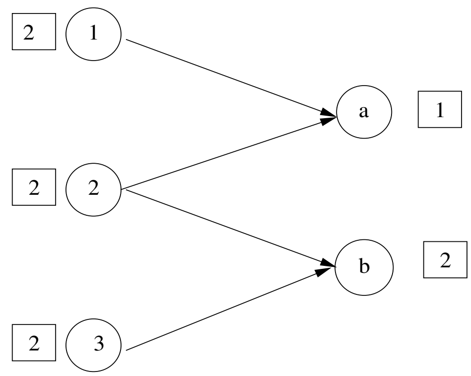

Figure 1: An example with 3 sources and 2 sinks

When agents are not held responsible for their ability or inability to consume certain resources, the egalitarian viewpoint equalizes the total share of each participant as much as permitted by the feasibility constraints. In our example this gives 1 unit of resource to each one of the three agents by means of the flow

$$\varphi_{1a} = 1; \varphi_{2a} = 0; \varphi_{2b} = \varphi_{3b} = 1, \tag{1}$$

where φ_{1a} is agent 1's consumption of type a, etc.

But this is not *consistent* with the egalitarian viewpoint on smaller problems: dropping resource b and subtracting 1 from agent 2's claim, we are left with a standard rationing problem where 1 unit is shared between agents 1 and 2 with claims 2 and 1, so the egalitarian outcome $\widetilde{\varphi}_{1a} = \widetilde{\varphi}_{2a} = \frac{1}{2}$ contradicts the above shares.

One consistent extension of the standard uniform gains method selects the flow φ minimizing $(\varphi_{1a}^2 + \varphi_{1b}^2 + \varphi_{2a}^2 + \varphi_{2b}^2)$ over all max-flows (Proposition 1). This gives the shares

$$\widetilde{\varphi}_{1a} = \frac{1}{2}; \widetilde{\varphi}_{2a} = \frac{1}{2}; \widetilde{\varphi}_{2b} = \widetilde{\varphi}_{3b} = 1 \tag{2}$$

where agent 2 is rewarded for being able to claim both resource-types. Dropping resource b and subtracting 1 from agent 2's claim prompts no revision in the division of resource a.[6]

[6]The Shapley value of the stand alone game (Footnote 2) gives, coincidentally, the same allocation.

Turning to the proportionality viewpoint, we can similarly hold that agents are not responsible for their edges, or lack thereof, and assign *total* shares in proportion to individual claims. As claims are equal in our example, this would lead to the flow (1), and to a similar contradiction of consistency: after dropping b the reduced claims of agents $1,2$ over a are 2 and 1, so proportional division gives $\frac{2}{3}$ to agent 1.

The consistent proportional method described in Theorem 1 relies on three strictly positive weights w_1, w_2, w_3 summing to 1; it divides each resource-type in proportion to the weights of agents claiming this resource. In our example:

$$\overline{\varphi}_{ia} = \frac{w_i}{w_1 + w_2} \cdot 1 \text{ for } i = 1,2;\ \overline{\varphi}_{ib} = \frac{w_i}{w_2 + w_3} \cdot 2 \text{ for } i = 2,3$$

In addition we require individual losses to be proportional to these same weights. In the example total deficit is 3, therefore we need

$$\frac{2 - \overline{\varphi}_{1a}}{w_1} = \frac{2 - (\overline{\varphi}_{2a} + \overline{\varphi}_{2b})}{w_2} = \frac{2 - \overline{\varphi}_{3b}}{w_3} = 3$$

It is easy to check that the above system has the unique solution

$$w_1 = \frac{1}{3}(4 - \sqrt{7}); w_2 = \frac{1}{9}(5\sqrt{7} - 11); w_3 = \frac{2}{9}(4 - \sqrt{7})$$

leading to the flow

$$\overline{\varphi}_{1a} = \frac{3}{\sqrt{7}+2} = 0.646; \overline{\varphi}_{2a} = \frac{2}{\sqrt{7}+3} = 0.354$$

$$\overline{\varphi}_{2b} = \frac{6}{\sqrt{7}+4} = 0.903; \overline{\varphi}_{3b} = \frac{4}{\sqrt{7}+1} = 1.097$$

When we keep the same compatibilities but vary individual claims and resources, the above system can be solved explicitely because it remains quadratic. This is no longer true with three or more resource-types.

1.3 Related literature

As mentioned earlier, Bochet et al. [6] take a neutral view of the compatibility constraints. Furthermore, they work with a model in which each agent has single-peaked preferences (instead of a "claim"), all the resources must be divided between the agents, so an agent may end up with more than her preferred share. The division of the resources achieving the most egalitarian individual shares (in the sense of Lorenz dominance) is strategyproof: truthful report of one's preferred share is a dominant strategy. This is a key ingredient in the characterization of this method in [6]. It is easy to check that the consistent extensions of uniform gains (Subsection 8.1) are also strategyproof in the model of [6].

Strategyproofness is not our concern in this paper (where it would select the consistent extensions of the uniform gains method).

Random assignment with dichotomous preferences, studied by Bogomolnaia and Moulin [5], is the special case of our model where all claims are for one unit and there is one unit of each resource-type: each agent (e.g., worker) can be matched to only some of the resources (jobs). Efficiency requires to implement a convex combination of maximal matchings (a max-flow), and we must choose one such combination on equity grounds. Bogomolnaia and Moulin [5] and Roth et al. [22], take the same viewpoint as Bochet et al. [6]: they show that the same Lorenz dominant division method induces the truthful revelation of the compatibility constraints. In that model as here, the assumption of unit claims and unit types does not significantly simplify the computations.

Inspired by the network exchange theory from sociology, Kleinberg & Tardos [15] and Chakraborty et al. [8, 9] develop models of bargaining on networks where each agent/node engages in bilateral negotiations with other agents/nodes to which he is connected on a fixed graph. The division problem is quite different in [15] than in ours because each agent can strike only one deal. But in [8], [9], each pair of connected agents strike a bargain to share their pair-specific surplus. This is like in the special case of our model where each resource-type is connected to exactly two agents, and represents the amount of surplus over which these two agents bargain. Then agent i's disagreement point in his negotiation with j is determined by the sum of his shares in all other bilateral negotiations. Given an exogenous bargaining rule for two-person problems, an equilibrium profile of bilateral surplus divisions is defined by a consistency property formally similar to ours. However the qualitative effect is exactly opposite: in [8, 9], the bigger my disagreement outcome, the larger my share of the surplus, whereas in our model a bigger share of resource-types other than a decreases my claim on, and my share of a. The intersection of the two models is the uninteresting case with linear utility and very large equal claims, so that each pairwise surplus is divided equally, irrespective of the graph.

2 Model and Notation

We have a set $\mathcal{N}$ of potential agents and a set $\mathcal{Q}$ of potential resource-types (or simply types). An *instance* of the rationing problem is obtained by first picking a set N of n agents, a set Q of q types, and a bipartite graph $G \subseteq N \times Q$; an edge $(i, a) \in G$ indicates that agent i can consume the type a. We do not assume that G is connected. We define $f(i)$ to be the set of types that i is connected to, and $g(a)$ to be the set of agents that connect to type a. That is, $f(i) = \{a \in Q | ia \in G\}$ and $g(a) = \{i \in N | ia \in G\}$.

Next, each agent i has a *claim* x_i and each type a has a *capacity* (amount it can supply) r_a; these are arbitrary non negative numbers.

Notation: for a subset B and a vector y, we let $y_B := \sum_{i \in B} y_i$; and $y_{[B]}$ is the projection of y on $\mathbb{R}^B$.

Definition 1 *A bipartite rationing problem is* $P = (N, Q, G, x, r)$ *such that* *the resources are overdemanded, namely:*

$$\text{for all } B \subseteq Q\text{: } r_B \leq x_{g(B)}. \tag{3}$$

Let $\mathcal{P}$ denote the set of bipartite rationing problems[7] $P = (G, x, r)$.

Given a problem $P \in \mathcal{P}$, a *flow* φ specifies a non-negative real number φ_{ia} for each edge (i, a) in G. It is well known that the system of inequalities (3) characterizes the existence of a flow φ exhausting all resources and transferring at most his claim to agent i:

$$\varphi_{g(a)a} = r_a \text{ for all } a \in Q;\ \ \varphi_{if(i)} \leq x_i \text{ for all } i \in N,$$

where $\sum_{i \in g(a)} \varphi_{ia} \stackrel{def}{=} \varphi_{g(a),a}$, and $\sum_{a \in f(i)} \varphi_{ia} \stackrel{def}{=} \varphi_{i,f(i)}$. We call such a flow *feasible* and define $\mathcal{F}(P)$, or $\mathcal{F}(G, x, r)$, to be the set of feasible flows for problem $P = (G, x, r)$. We also speak of $\varphi \in \mathcal{F}(P)$ as a *solution* to the problem P.

Agent i's total transfer $y_i = \varphi_{i,f(i)}$ is called his *allocation*, or *share*. Although agents care only about their allocation, not its flow decomposition, we must nevertheless work with flows, on which our key axioms bear.

Three subsets of $\mathcal{P}$ play an important role below. A problem $P \in \mathcal{P}$ is *strictly overdemanded* if

$$\text{for all } B \subseteq Q\text{: } r_B < x_{g(B)}.$$

Let $\mathcal{P}^{str}$ be the set of strictly overdemanded problems. A problem $P \in \mathcal{P}$ is *irreducible* if every subproblem is strictly overdemanded:

$$r_Q \leq x_N;\ \ \text{for all } B \subsetneq Q\text{: } r_B < x_{g(B)}.$$

Let $\mathcal{P}^{ir}$ be the set of irreducible problems. Finally, a $P \in \mathcal{P}$ is *balanced* if $r_Q = x_{g(Q)}$. Note that a problem $P \in \mathcal{P} \diagdown \mathcal{P}^{ir}$ should contain a balanced subproblem, and so can be further decomposed. This is the key to the canonical decomposition of an arbitrary problem in $\mathcal{P}$ into a union of irreducible problems, all but at most one of them balanced: see Lemma 4 in section 10.

Note further that an irreducible and balanced problem must have a connected graph, however a strictly overdemanded problem need not be connected.

Definition 2 *A bipartite rationing method (or simply method) H associates to each overdemanded problem $P \in \mathcal{P}$, where $N \subset \mathcal{N}, Q \subset \mathcal{Q}$, a feasible flow $\varphi = H(P) \in \mathcal{F}(P)$.*

Note that any agent with zero claim, and any type with zero resource gets no flow in any method.

Definition 3 *A rationing problem is standard if it involves a single resource type to which all agents are connected. It is a triple $P^0 = (N, x, t)$, where $x \in \mathbb{R}_+^N$ is the profile of claims, and t units of resource are overdemanded: $t \leq x_N$. A standard rationing method h is a method applying only to standard*

[7] It will cause no confusion to omit most of the time the sets N, Q in the notation of a problem, even though these sets will vary in the key consistency axiom.

problems. Thus $h(N,x,t) \in \mathbb{R}_+^N$ is a division of t among the agents in N such that $h_i(N,x,t) \leq x_i$ for all $i \in N$. We write $\mathcal{P}^0$ for the set of standard problems.

We recall the definition of the three benchmark standard rationing methods, proportional h^{pro}, uniform gains h^{ug}, uniform losses h^{ul}:

$$
\begin{aligned}
h^{pro}(x,t) &= \frac{x_i}{x_N} \cdot t; \\
h_i^{ug}(x,t) &= \min\{x_i, \lambda\} \text{ where } \lambda \text{ solves } \sum_{i \in N} \min\{x_i, \lambda\} = t; \\
h_i^{ul}(x,t) &= \max\{x_i - \mu, 0\} \text{ where } \mu \text{ solves } \sum_{i \in N} \max\{x_i - \mu, 0\} = t.
\end{aligned}
$$

For each resource a, a method $H \in \mathcal{H}$ defines a standard rationing method ${}^a h$ by the way it deals with the complete graph $G = N \times \{a\}$ with this resource:

$$
{}^a h(N,x,r_a) = H(N \times \{a\}, x, r_a)
$$

Remark 1: Suppose there are no constraints linking the claims x of the source nodes and the resources r of the sink nodes. We must choose a fair max-flow between sources and sinks. As mentioned in the introdction, we can then decompose the problem (G,x,r) into an overdemanded part and an over-supplied part. Formally we can partition N as N_+, N_-, and Q as Q_+, Q_-, in such a way that $Q_+ = f(N_-)$, $N_+ = g(Q_-)$, and $G(N_-, Q_-) = \varnothing$. Moreover in $(N_-, Q_+, G(N_-, Q_+), x, r)$ the resources are overdemanded, while in $(N_+, Q_-, G(N_+, Q_-), x, r)$ they are underdemanded. See, e.g., Lemma 2 in [7]*. Then we apply first one of our bipartite rationing methods to the overdemanded problem in $N_- \times Q_+$; next, exchanging the roles of agents and types, we use another (or the same) method for the underdemanded problem in $N_+ \times Q_-$.*

3 Basic axioms

As discussed in the introduction, our goal is to understand which standard methods can be extended to bipartite methods, while respecting a consistency property. As in most of the literature on standard methods (see e.g., [18], [23]), we restrict attention to symmetric and continuous rationing methods.

Symmetry (SYM). A method $H \in \mathcal{H}$ is symmetric if the labels of the agents and types do not matter. Formally, given a permutation π of the agents and a permutation σ of the types, define $G^{\pi,\sigma}$ to be the graph such that $(\pi(i), \sigma(a)) \in G^{\pi,\sigma}$ if and only if $(i,a) \in G$. The claims x^π of the agents and resources r^π of the types are similarly defined. Suppose $H(G,x,r) = \varphi$ and $H(G^{\pi,\sigma}, x^\pi, r^\sigma) = \varphi'$. Then the method H is symmetric if and only if $\varphi_{ia} = \varphi'_{\pi(i),\sigma(a)}$ for all $(i,a) \in G$.

The standard method associated with a symmetric H is symmetric as well, thus independent of the choice of type a. In keeping with the rest of our notation, we write it simply as $h(x,t)$ instead of $h(|N|,x,t)$.

Continuity (CONT). A method $H \in \mathcal{H}$ is *continuous* if for all N, Q, and G, the mapping $(x, r) \to H(G, x, r)$ is continuous in the relevant subset of $\mathbb{R}_+^N \times \mathbb{R}_+^Q$.

Definition 4 *We write $\mathcal{H}$ (resp. $\mathcal{H}^0$) for the set of symmetric and continuous bipartite (resp. standard) rationing methods. We use the notation $\mathcal{H}(A, B, \cdots), \mathcal{H}^0(A, B, \cdots)$ for the subset of methods in $\mathcal{H}$ or $\mathcal{H}^0$ satisfying properties $A, B, \cdots$.*

We give two versions of the crucial consistency property, both generalizing consistency for standard methods.

We use the following notation. For a given graph $G \subseteq N \times Q$, and subsets $N' \subseteq N$, $Q' \subseteq Q$, the *restricted graph* of G is $G(N', Q') := G \cap \{N' \times Q'\}$, again not necessarily connected, and the *restricted problem* obtains by also restricting x to N' and r to Q'.

Node Consistency (Node-CSY). Fix an agent $i \in N$ and a problem $P \in \mathcal{P}$, and define the reduced claims and resources under method $H \in \mathcal{H}$ after this agent (and all the edges involving this agent) is removed:

$$x_j^H(-i) = x_j, \text{ for all } j \neq i$$

and

$$r_a^H(-i) = r_a - \varphi_{ia} \text{ for all } a \in f(i); \quad r_b^H(-i) = r_b, \text{ for } b \notin f(i).$$

Let $N^* = N \setminus \{i\}$, and $Q^* = f(N^*) \cap \{b \mid r_b^H(-i) > 0\}$. The reduced problem is $(G(N^*, Q^*), x^H(-i), r^H(-i))$. Similarly, fix a type $a \in Q$ and define the reduced claims and resources under method H after this type (and all the edges involving this type) is removed:

$$x_j^H(-a) = x_j - \varphi_{ja} \text{ for all } j \in g(a); \quad x_j^H(-a) = x_j, \text{ for } j \notin g(a).$$

and

$$r_b^H(-a) = r_b, \text{ for all } b \neq a$$

Let $Q^{**} = Q \setminus \{a\}$, $N^{**} = g(Q^*) \cap \{j \mid x_j^H(-a) > 0\}$. The reduced problem is $(G(N^{**}, Q^{**}), x^H(-a), r^H(-a))$. Clearly, the properties of a problem being overdemanded, strictly overdemanded, or balanced, are preserved under either of these reductions. However the reduced problem may not be connected or irreducible, even if the original problem is.

Suppose $H(G(N, Q), x, r) = \varphi$, $H(G(N^*, Q^*), x^H(-i), r^H(-i)) = \varphi'$, and $H(G(N^{**}, Q^{**}), x^H(-a), r^H(-a)) = \varphi''$. The method $H \in \mathcal{H}$ is *node-consistent* if for all $N \subset \mathcal{N}, Q \subset \mathcal{Q}$, all $(G, x, r) \in \mathcal{P}$, all $i \in N$, $a \in Q$: $\varphi_{jb} = \varphi'_{jb}$ for all $jb \in G(N^*, Q^*)$ and $\varphi_{jb} = \varphi''_{jb}$ for all $jb \in G(N^{**}, Q^{**})$.

Edge Consistency (Edge-CSY). *Edge-consistency* is stronger than node-consistency. Fix an edge $ia \in G$ and define the *reduced claims and resources* under method H after this edge is removed:

$$x_i^H(-ia) = x_i - \varphi_{ia};\ x_j^H(-ia) = x_j \text{ for } j \neq i$$

$$r_a^H(-ia) = r_a - \varphi_{ia};\ r_b^H(-ia) = r_b \text{ for } b \neq a$$

The corresponding reduced problem is $(G \diagdown \{ia\}, x^H(-ia), r^H(-ia))$, where the set of agents is $N^* = N$ unless $f(i) = \{a\}$ in which case $N^* = N \diagdown \{i\}$; similarly the set of types is $Q^* = Q$ unless $g(a) = \{i\}$ in which case $Q^* = Q \diagdown \{a\}$. Clearly the reduced problem is overdemanded if the initial problem is, but not necessarily strictly overdemanded if the initial problem is. Note also that $G \diagdown \{ia\}$ may not be connected even if G is connected.

Suppose $H(G, x, r) = \varphi$ and $H(G \setminus \{ia\}, x^H(-ia), r^H(-ia)) = \varphi'$. The method $H \in \mathcal{H}$ is *edge-consistent* if for all $N \subset \mathcal{N}, Q \subset \mathcal{Q}$, all $(G, x, r) \in \mathcal{P}$, and $ia \in G$: $\varphi_{jb} = \varphi'_{jb}$ for all $jb \in G \diagdown \{ia\}$.

Our next invariance property states that two types compatible with precisely the same set of agents need not be treated as separate types: merging them into a single type is of no consequence, and neither is relabeling a subset of a certain type of resource as a separate type. Note that the next two axioms are only defined for symmetric methods.

Invariance to the Merging of Types (IMT). Fix a problem $P \in \mathcal{P}$ and suppose that in the graph $G \subseteq N \times Q$, two types a_1, a_2 are such that $g(a_1) = g(a_2)$. Let $G^* \subseteq N \times (Q \diagdown \{a_1, a_2\} \cup \{a^*\})$ be the graph obtained by *merging* those two types into a new node labeled a^* with the same connections. The corresponding *merged* problem (G^*, x, r^*) has $r^*_{a^*} = r_{a_1} + r_{a_2}$, $r^*_a = r_a$ for all $a \in Q \diagdown \{a_1, a_2\}$.

Suppose $H(G, x, r) = \varphi$ and $H(G^*, x, r^*) = \varphi^*$. The method $H \in \mathcal{H}$ is *invariant to the merging of types* if for all $N \subset \mathcal{N}, Q \subset \mathcal{Q}$, all $(G, x, r) \in \mathcal{P}$, and a_1, a_2 s.t. $g(a_1) = g(a_2)$: $\varphi^*_{ia^*} = \varphi_{ia_1} + \varphi_{ia_2}$ for all $i \in g(a^*)$, $\varphi^*_{ja} = \varphi_{ja}$ for all $a \in Q \diagdown \{a_1, a_2\}$, $ja \in G$. In particular individual shares y_i are unchanged.

Invariance to Full Merging of Types (IFM). Fix a problem $P = (N \times Q, x, r) \in \mathcal{P}$ where the graph G is complete. If the method $H \in \mathcal{H}$, with associated standard method h, is invariant to the merging of types, repeated applications of this property imply

$$\varphi_{iQ} = h(x, r_Q) \tag{4}$$

i.e., the shares $y(P)$ obtain by merging all resources into a single type. We say that the method $H \in \mathcal{H}$ is *invariant to full merging* if equation (4) holds for all $N \subset \mathcal{N}, Q \subset \mathcal{Q}$, and all $(N \times Q, x, r) \in \mathcal{P}$. Thus IFM is weaker than IMT.

We stress that the proportional method (Theorem 1), and all the loss-calibrated methods (Theorem 2), are edge-consistent and invariant to the merging of types, but their characterization uses only the weaker properties of node-consistency and invariance to full merging (the latter is not even needed in Theorem 1).

Remark 2: When two agents have identical connections, we can similarly merge them into a single agent, and define the corresponding invariance property. This property forces the proportional method for standard problems ([3]), therefore it reduces our general problem to that of extending this method to the bipartite context, the object of Section 5: see Remark 4 at the end of that section.

Remark 3: When the compatibility constraints are nested, Node-CSY and IFM are enough to determine the individual shares $y_i = \varphi_{iQ}$*, if not necessarily the entire flow. Suppose there is a partition* $N = N^1 \cup \cdots \cup N^T$ *such that for any* $i \in N^t$*,* $j \in N^{t'}$*,* $\{t = t' \Rightarrow f(i) = f(j)\}$*, and* $\{t < t' \Rightarrow f(i) \subsetneq f(j)\}$*. If we drop all resource-types but those in* $Q^T = f(N^T) \diagdown f(N^1 \cup \cdots \cup N^{T-1})$*, we are left with the complete graph* $N^T \times Q^T$*, so by IFM the standard method* h^{n^T} *determines* φ_{iQ^T} *for each* $i \in N^T$*. Dropping now all types in* Q^T*, the reduced graph still has nested constraints for the partition* $N^1 \cup \cdots \cup N^{T-2} \cup \{N^{T-1} \cup N^T\}$*, and reduced claims for agents in* N^T*, so we can repeat the argument.*

4 A necessary condition for extendability

Recall from Definition 4 that in $\mathcal{H}$ all methods are continuous and symmetric.

Lemma 1 *Assume the set* $\mathcal{Q}$ *of potential resource types is infinite. If the bipartite rationing method* H *is invariant to full merging and node consistent (*$H \in \mathcal{H}(Node-CSY, IFM)$*), then the corresponding standard method* $h \in \mathcal{H}^0$ *satisfies the following: for all* $(N, x, t) \in \mathcal{P}^0$ *and all* $\lambda \in [0, 1]$

$$h(x - \lambda \cdot h(x, t), (1 - \lambda)t) = (1 - \lambda) \cdot h(x, t) \tag{5}$$

Proof Fix $(N, x, t) \in \mathcal{P}^0$, two integers $p, q, 1 \leq p < q$, and a set Q of types with cardinality q. Consider the problem $P = (N \times Q, x, r)$ where $r_a = \frac{t}{q}$ for all $a \in Q$, with associated profile of shares y at $\varphi = H(P)$. By IFM $y = h(x, t)$ and by SYM $\varphi_{ia} = \frac{y_i}{q}$ for all $i \in N$. Drop now p of the nodes and let Q' be the remaining set of types. Applying Node-CSY p successive times gives

$$H(N \times Q', x', r') = \varphi'$$

where $x' = x - \frac{p}{q}y$, $r'_a = \frac{t}{q}$ for all $a \in Q'$, and φ' is the restriction of φ to $N \times Q'$. Therefore $y' = \frac{q-p}{q}y$. Now IFM applied to the reduced problem gives $y' = h(x', \frac{q-p}{q}t)$. We just showed $\frac{q-p}{q}y = h(x - \frac{p}{q}y, \frac{q-p}{q}t)$, precisely (5) for $\lambda = \frac{p}{q}$. Finally CONT implies (5) for other real values of λ.■

We leave it to the reader to check that the three benchmark standard methods satisfy property (5). We show below that they are all extendable to bipartite methods satisfying Edge-CSY and IMT.

Despite its similarity with Consistency and the Lower Composition (see Lemma 3 below) axiom, property (5) is not normatively compelling: the reduced problem where agents receive a given fraction of their shares in the original problem has no clear interpretation. Yet this technical property is key to our characterization result (Theorem 2) below. That result shows that most consistent standard rationing methods are not extendable in our terms to the bipartite context.

For now, we simply mention a few prominent consistent standard methods ruled out by property (5). To define an *equal sacrifice* method ([26], [18]), we fix a function $u : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{-\infty\}$, differentiable and strictly increasing. Given $(N, x, t) \in \mathcal{P}^0$, the shares y are defined by budget balance and

$$\text{for all } i\text{: } y_i > 0 \Rightarrow u(x_i) - u(y_i) = \max_N \{u(x_j) - u(y_j)\}$$

The *dual* of an equal sacrifice method is similarly defined by the system

$$\text{for all } i\text{: } y_i < x_i \Rightarrow u(x_i) - u(x_i - y_i) = \max_N \{u(x_j) - u(x_j - y_j)\}$$

The proportional method is the only method in both families; uniform losses is an equal sacrifice method; uniform gains is a dual equal sacrifice method.[8]

Corollary to Lemma 1 *The following cannot be extended to a node consistent bipartite method invariant to full merging: the Talmudic method ([2]), any equal sacrifice methods and or any dual equal sacrifice method, except for the three benchmark methods.*

Proof Fix u. Then property (5) for the corresponding equal sacrifice method implies, for any $\varepsilon_i, i = 1, 2$ positive and small enough, any $y_i, i = 1, 2$, and any $\lambda \in [0, 1]$:

$$\begin{aligned} \{u(y_1 + \varepsilon_1) - u(y_1) &= u(y_2 + \varepsilon_2) - u(y_2)\} \Rightarrow \\ u(\lambda y_1 + \varepsilon_1) - u(\lambda y_1) &= u(\lambda y_2 + \varepsilon_2) - u(\lambda y_2) \end{aligned} \tag{6}$$

Fixing y_i λ, and letting ε_i go to zero, we get

$$\frac{u'(y_2)}{u'(y_1)} = \frac{u'(\lambda y_2)}{u'(\lambda y_1)}$$

Note that an affine transformation of u gives the same equal sacrifice method. In the argument below, we use this repeatedly to deduce a simple form for u. Fixing y_2 the equality above gives $u'(\lambda y_1) = u'(y_1) \cdot f(\lambda)$, as well as $u'(\frac{y_1}{\lambda}) = \frac{u'(y_1)}{f(\lambda)}$, therefore after rescaling u so that $u'(1) = 1$, we get $u'(a \cdot b) = u'(a) \cdot u'(b)$ for all positive a, b. Thus after one more rescaling of u, u' is a positive power function $u'(z) = z^p$ for some $p \in \mathbb{R}$ (recall u is increasing). Now u is (after affine rescaling) a power function $u(z) = z^p$ for $p > 0$, or $u(z) = -z^p$ for $p < 0$, or $u(z) = \log(z)$. The latter is the proportional method, for which (5) is true.

[8] Equal sacrifice methods are essentially characterized by Consistency and Lower Composition; their dual by CSY and Upper Composition. See [26], [18].

Ditto for the uniform losses method, corresponding to $u(z) = z$. But for any other method, (5) fails to be true. A simple way to check this is to choose $y_1 = 2$, $y_2 = 4$, $\lambda = \frac{1}{2}$ in (6), so that for all $a, b > 0$ we must have

$$\{(a+2)^p - 2^p = (b+4)^p - 4^p\} \Rightarrow (a+1)^p - 1 = (b+2)^p - 2^p$$

Then one checks that the two curves defined respectively by the left equation and the right equation are distinct if $p \neq 0, 1$.■

5 The bipartite proportional method

The bipartite proportional method H^{pro} plays a key role in the construction of bipartite loss-calibrated methods and our characterization result, both in Section 7. Theorem 1 gives two equivalent definitions of this method, one for any overdemanded problem as the solution of a maximization problem, the other for irreducible problems only. The latter definition is then extended to any overdemanded problem by means of its canonical decomposition in irreducible subproblems (Definition 6 in Subsection 10.1). The latter definition gives much more insight into the structure of our method.

We use two new pieces of notation. The unit simplex of $\mathbb{R}^N$ is written below as $\mathcal{S}(N)$, and its interior as $\overset{\circ}{\mathcal{S}}(N) = \{w | w_N = 1 \text{ and } w_i > 0 \text{ for all } i\}$. The function $En(z) = z\ln(z)$ is strictly convex and $En(0) = 0$. The sum $\sum_k En(z_k)$ is the familiar *entropy* of the vector z.

Theorem 1 *For any problem $P = (G, x, r) \in \mathcal{P}$, the proportional flow is defined as the unique solution of*

$$\widehat{\varphi} = \arg \min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} En(\varphi_{ia}) + \sum_{i \in N} En(x_i - y_i) \tag{7}$$

(i) For any irreducible problem $P = (G, x, r) \in \mathcal{P}^{ir}$, the following system with unknown $w \in \mathcal{S}(N)$

$$x_i = w_i \cdot (x_N - r_Q) + \sum_{a \in f(i)} \frac{w_i}{w_{g(a)}} r_a \text{ for all } i \in N \tag{8}$$

has a unique solution $\widehat{w}$ in $\overset{\circ}{\mathcal{S}}(N)$, and the proportional flow is

$$\widehat{\varphi}_{ia} = \frac{\widehat{w}_i}{\widehat{w}_{g(a)}} r_a \tag{9}$$

(ii) The proportional method H^{pro} is edge-consistent, and invariant to the merging of types ($H^{pro} \in \mathcal{H}(Edge - CSY, IMT)$).

(iii) It is the only continuous and node-consistent method that is proportional for standard problems.

Proof

Problem (7) has a unique solution $\widehat{\varphi}$ for any $P \in \mathcal{P}$ because the objective function is strictly convex and finite, so our method is well defined. Symmetry is clear, and continuity follows from Berge's Maximum Theorem ([4]). The objective function is continuous, and the correspondence $(x, r) \to \mathcal{F}(G, x, r)$ is compact-valued, and continuous as well (upper and lower hemicontinuous); therefore the argmin correspondence is continuous as well.

Step 1: Statement ii) For Edge-CSY, we fix $P = (G, x, r)$ and an edge $ia \in G$. For any $\varphi' \in \mathcal{F}(G \setminus \{ia\}, x^H(-ia), r^H(-ia))$, adding ia to G and $\widehat{\varphi}_{ia}$ to φ' yields a flow $(\varphi', \widehat{\varphi}_{ia})$ in $\mathcal{F}(G, x, r)$. The objective function at $(\varphi', \widehat{\varphi}_{ia})$ is the same as at φ' plus the single term $En(\widehat{\varphi}_{ia})$, because $x_i - y_i = x_i^H(-ia) - y_i^H(-ia)$. Thus if the restriction of c to $P^H(-ia)$ is not optimal in that problem, we can construct a flow $(\varphi', \widehat{\varphi}_{ia})$ beating $\widehat{\varphi}$ in P. We postpone the proof of IMT until step 3.

Step 2: Statement i) We fix an irreducible problem $P = (G, x, r)$. It will be convenient to replace problem (7) by the equivalent problem

$$\min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} Ln(\varphi_{ia}) + \sum_{i \in N} Ln(x_i - y_i) \tag{10}$$

where $Ln(z) = z(\ln(z) - 1)$ is still strictly convex and has derivative $\ln(z)$. The equivalence follows from the fact that we are substracting two constant terms to the objective function: $\sum_{ia \in G} \varphi_{ia} = r_Q$ and $\sum_{i \in N}(x_i - y_i) = x_N - r_Q$.

Step 2.1 We assume in this substep $x_N = r_Q$: P is balanced. By irreducibility, for every $(i, a) \in G$, there is a solution $\varphi \in \mathcal{F}(G, x, r)$ with $\varphi_{ia} > 0$. Also, because the problem is balanced, $y_i = x_i$ for every $\varphi \in \mathcal{F}(G, x, r)$. Thus Problem (10) becomes

$$\min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} Ln(\varphi_{ia}),$$

whose Lagrangean is given by

$$L(\varphi, \lambda, \mu) = \sum_{(i,a) \in G} \varphi_{ia}[\ln(\varphi_{ia}) - 1] + \sum_{i \in N} \lambda_i (x_i - \sum_{a \in Q} \varphi_{ia}) + \sum_{a \in Q} \mu_a (r_a - \sum_{i \in N} \varphi_{ia}),$$

where $\lambda = (\lambda_i)_{i \in N} \in \mathbb{R}_+^N$ and $\mu = (\mu_a)_{a \in Q} \in \mathbb{R}^Q$. Define

$$q(\lambda, \mu) = \min_{\varphi \geq 0} L(\varphi, \lambda, \mu). \tag{11}$$

It is easy to check that for any fixed λ and μ, the minimum is attained in (11) uniquely by the solution $\varphi^*_{ia} = e^{\lambda_i + \mu_a}$, using which we get

$$q(\lambda, \mu) = L(\varphi^*, \lambda, \mu) = -\sum_{(i,a) \in G} \varphi^*_{ia} + \sum_{i \in N} \lambda_i x_i + \sum_{a \in Q} \mu_a r_a.$$

The associated dual problem is thus given by

$$\max_{\lambda, \mu} \Big\{ -\sum_{(i,a) \in G} e^{\lambda_i + \mu_a} + \sum_{i \in N} \lambda_i x_i + \sum_{a \in Q} \mu_a r_a \Big\}. \tag{12}$$

It is clear that (12) has a *unique* optimal solution that is given by the solution to the following system of equations:

$$-\sum_{a\in f(i)} e^{\lambda_i+\mu_a} + x_i = 0, \;\; \forall i \in N,$$

and

$$-\sum_{i\in g(a)} e^{\lambda_i+\mu_a} + r_a = 0, \;\; \forall a \in Q.$$

Letting λ^* and μ^* be the optimal solutions, we have

$$e^{\lambda_i^*} = \frac{x_i}{\sum_{a\in f(i)} e^{\mu_a^*}}; \;\; e^{\mu_a^*} = \frac{r_a}{\sum_{i\in g(a)} e^{\lambda_i^*}}.$$

Finally,

$$\varphi_{ia}^* = e^{\lambda_i^*} e^{\mu_a^*}.$$

In particular, taking $w_i = \frac{e^{\lambda_i^*}}{\sum_N e^{\lambda_j^*}}$ verifies (8) and (9).

Step 2.2 We assume now that $P = (G, x, r)$ is not only irreducible, but also strictly overdemanded, i.e. $x_N > r_Q$. We proceed as before by writing the Lagrangean of Problem (10), which is now

$$\begin{aligned} L(\varphi, \lambda, \mu) \quad = \quad & \sum_{(i,a)\in G} Ln(\varphi_{ia}) + \sum_{i\in N} Ln\Big(x_i - \sum_{a\in f(i)} \varphi_{ia}\Big) \\ & + \sum_{i\in N} \lambda_i (x_i - \sum_{a\in Q} \varphi_{ia}) + \sum_{a\in Q} \mu_a (r_a - \sum_{i\in N} \varphi_{ia}). \end{aligned}$$

As before, for any fixed λ and μ, the minimum in the problem

$$q(\lambda, \mu) = \min_{\varphi\geq 0} L(\varphi, \lambda, \mu)$$

is attained uniquely by the solution of

$$\frac{\varphi_{ia}^*}{x_i - \sum_{a\in f(i)} \varphi_{ia}^*} = e^{\lambda_i+\mu_a}.$$

An implication of this is that in the minimizer of $q(\lambda, \mu)$, each agent's allocation y_i is such that $y_i < x_i$. This implies that the optimal choice of λ in the associated dual problem $\max_{\lambda\geq 0,\mu} q(\lambda, \mu)$ is $\lambda^* = 0$. Also, it is straightforward to check that the dual is a maximization problem with a strictly concave objective function, and so has a unique optimal solution μ^*. Using this, the optimal φ_{ia}^* satisfy $(x_i - \sum_{b:b\in f(i)} \varphi_{ib}^*) e^{\mu_a^*} = \varphi_{ia}^*$. Letting $y_i^* = \sum_{b:b\in f(i)} \varphi_{ib}^*$, we see, in particular, that

$$\frac{\varphi_{ia}^*}{x_i - y_i^*} = \frac{\varphi_{ja}^*}{x_j - y_j^*} = \frac{r_a}{x_{g(a)} - y_{g(a)}^*}, \quad \text{for all } a \text{ and } i, j \in g(a) \tag{13}$$

Setting $\widehat{w}_i = \frac{x_i - y_i^*}{x_N - r_Q}$, so that $\widehat{w} \in \overset{\circ}{\mathcal{S}}(N)$, we see that $\widehat{w}$ is a solution of system (8). Moreover (13) implies (9) as well.

Step 3. Our solution H^{pro} is invariant to the merging of types.

Fix an irreducible problem (G, x, r) with weights $w_i = \frac{x_i}{x_N}$ solving (8), and assume $g(a_1) = g(a_2)$. After merging a_1 and a_2 into a, the weights $\widetilde{w}_a = w_{a_1} + w_{a_2}$, $\widetilde{w}_b = w_b$ for $b \neq a_1, a_2$, satisfy the corresponding system (8) in the merged problem, so statement $i)$ implies that IMT holds in $\mathcal{P}^{ir}$.

Next we fix $(G, x, r) \in \mathcal{P}$ and use its canonical decomposition in irreducible problems (Lemma 4 in the appendix): clearly two nodes such that $g(a_1) = g(a_2)$ must be in the same component Q^k of the decomposition, where IMT applies, and the merging of thses two nodes reduce Q^k by one type and preserves the rest of the decomposition.

Step 4: Statement iii)

Let H be a continuous and node consistent method, proportional for standard problems. Pick first a strictly overdemanded $P = (G, x, r) \in \mathcal{P}^{str}$. Fix a type a and reduce P by dropping successively all nodes but a. Then Node-CSY and the fact that H is proportional for one-type problems imply:

$$\text{for all } i \in g(a) : \varphi_{ia} = h^{pro}(x - y + \varphi_{\cdot a}, r_a) = \frac{x_i - y_i + \varphi_{ia}}{x_{g(a)} - y_{g(a)} + r_a} r_a; \text{ or } \varphi_{ia} = 0 \tag{14}$$

If $y_i = x_i$ this implies either $\varphi_{ia} = 0$ or $\{\varphi_{ia} = r_a$ and $\{y_j = x_j$ and $\varphi_{ja} = 0$ for all $j \in g(a)\}\}$. Restricting attention to a connected component of G, this implies that every resource goes to a single agent and they all have $y_j = x_j$, contradiction.

So $y \ll x$. Then (14) implies $\varphi_{ia} > 0$ for all $ia \in G$. It also reduces to

$$\varphi_{ia} = \frac{x_i - y_i}{x_{g(a)} - y_{g(a)}} r_a \Rightarrow \frac{\varphi_{ia}}{x_i - y_i} = \frac{\varphi_{ja}}{x_j - y_j} = \frac{r_a}{x_{g(a)} - y_{g(a)}} \text{ for all } i, j \in g(a)$$

These are precisely the KKT optimality conditions, so $\varphi = \varphi^*$.

Pick next $P = (G, x, r)$ irreducible and balanced. Both H and the proportional bipartite method H^{pro} are continuous, and P is the limit of strictly overdemanded problems. Thus $H = H^{pro}$ on $\mathcal{P}^{ir}$.

Finally, both methods H and H^{pro} are node-consistent on $\mathcal{P}$, so as explained after Definition 6 in Section 10, they are the canonical extension of their projection on $\mathcal{P}^{ir}$, where they coincide.■

Remark 4. The classic characterization of the standard proportional method relies on its invariance when we merge two agents and endow this superagent with the sum of their claims ([3], [19]): the shares of each non merged agent are unaffected. These properties generalize easily to the bipartite proportional method: if we merge two agents with identical connections ($f(i) = f(j)$) and add their claims, the flow in all edges not involving i or j is unchanged, the flow in a merged edge is the sum of the two earlier flows. Our method h^{pro} meets this property, as one checks easily from the system (8). Thus the proportional *method is characterized by continuity, node consistency, and invariance to the merging of agents.*

6 A new family of standard methods

We recall first some classic properties of standard rationing methods.

Definition 5 *A standard rationing method $h \in \mathcal{H}^0$ is Claim Monotonic (CM) if $x_i \to h_i^n(x,t)$ is weakly increasing for all i, x_{-i}, t. It is Resource Monotonic (RM) —resp. Strictly Resource Monotonic (SRM)— if $t \to h^n(x,t)$is weakly —resp. strictly— increasing for all x.*

Claim and Resource Monotonicity are satisfied by all familiar standard methods (see [18, 23]), including the three benchmarks, the Talmudic and the equal sacrifice methods. Strict Resource Monotonicity excludes uniform gains, uniform losses, and the Talmudic methods, although these methods are the limit of methods meeting SRM (for instance equal sacrifice methods approach uniform gains and losses).

Lemma 2 *Fix a function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, weakly increasing, continuous, and such that $\beta(0) = 0$, $\beta(z) > 0$ for $z > 0$. Define h^β, the standard method **loss-calibrated** by β, as follows:*

$$y = h^\beta(x,t) \overset{def}{\Leftrightarrow} \left\{ \frac{y_i}{\beta(x_i - y_i)} = \frac{y_j}{\beta(x_j - y_j)} \text{ for all } i,j \text{ s.t. } x_i, x_j > 0 \right\} \qquad (15)$$

and, as required in $\mathcal{H}$, $y_N = t$ and $x_i = 0 \Rightarrow y_i = 0$. Then $h^\beta \in \mathcal{H}^0(CSY, CM, SRM)$. We write $\mathcal{H}^0_{[lc]}$ for the set of loss calibrated methods.

Note that if the problem (x,t) is strictly overdemanded, $t < x_N$, system (15) implies $y_i < x_i$ whenever $x_i > 0$. If (x,t) is balanced, this system gives $y = x$ as all ratios must be ∞.

Proof For any $x_i > 0$, the function $z \to \frac{z}{\beta(x_i - z)}$ is continuous and strictly increasing from $[0, x_i]$ to $\mathbb{R}_+ \cup \{\infty\}$, therefore it is invertible as $z = \theta(x_i, \lambda) \overset{def}{\Leftrightarrow} \lambda = \frac{z}{\beta(x_i - z)}$, where θ is continous, strictly increasing in $\lambda \in \mathbb{R}_+ \cup \{\infty\}$, weakly increasing in $x_i > 0$, and $\theta(x_i, 0) = 0$, $\theta(x_i, \infty) = \infty$. Thus an equivalent definition of h^β is

$$y_i = \theta(x_i, \lambda) \text{ where } \lambda \text{ solves } \sum_N \theta(x_i, \lambda) = t$$

This is the classic format of a parametric method ([25]), implying CSY at once. Next CM, and SRM follow easily from the monotonicity properties of θ.■

For our characterization result below, the key property is equation (5) in Lemma 1. It is satisfied by all loss calibrated methods. This follows at once from the representation of $y = h(x,t)$ as the unique solution of system (15). Three more properties common all methods in $\mathcal{H}^0_{[lc]}$ are:

1. *Ranking (RK)*: $x_i \leq x_j \Rightarrow y_i \leq y_j$,

2. *Ranking* (RK*)*: $x_i \leq x_j \Rightarrow x_i - y_i \leq x_j - y_j$,
3. *Cross Monotonicity (CRM): $x_i \to h^n_j(x,t)$ is weakly decreasing for $j \neq i$.*
(we omit the easy proof; see [18, 23] for a normative discussion of RK and CRM)

Loss-calibrated standard methods are a new subfamily of parametric rationing methods, resembling formally the dual equal sacrifice methods, particularly when the latter solve the system

$$u(x_i) - u(x_i - y_i) = u(x_j) - u(x_j - y_j) \text{ for all } i,j \text{ s.t. } x_i, x_j > 0$$

Yet the Corollary to Lemma 1 implies that the proportional method, corresponding to $\beta(z) = z$, is the only loss calibrated method in the equal sacrifice family or its dual.

The familiar Scale Invariance (SI) property, $h(\mu \cdot x, \mu t) = \mu \cdot h(x,r)$ for all x,t, and $\mu > 0$, expresses that small problems are resolved like large ones. It cuts a one-dimensional subset of loss calibrated methods. Indeed SI and (15) imply for all strictly positive numbers y_1, y_2, z_1, z_2, μ:

$$\frac{y_1}{\beta(z_1)} = \frac{y_2}{\beta(z_2)} \Rightarrow \frac{y_1}{\beta(\mu z_1)} = \frac{y_2}{\beta(\mu z_2)}$$

In turn this shows that a constant multiple of β, denoted β for simplicity because it generates the same rationing method, satisfies $\beta(ab) = \beta(a)\beta(b)$ for all $a, b > 0$, hence $\beta(z) = z^p$ for some $p, 0 < p < \infty$. We write h^p for the corresponding method, defined by the system $\frac{y_i}{(x_i - y_i)^p} = \frac{y_j}{(x_j - y_j)^p}$ for all i, j.

The method h^p approaches uniform gains when $p \to 0$, and uniform losses when $p \to \infty$. For $p = 2$ and $p = \frac{1}{2}$, and for those values only, h^p has an explicit parametric representation:

$$y_i = \theta^2(x_i, \lambda) = x_i(1 - \frac{2}{1 + \sqrt{\lambda x_i + 1}})$$

$$y_i = \theta^{\frac{1}{2}}(x_i, \lambda) = x_i(1 - \frac{4}{(\lambda + \sqrt{\lambda^2 + 4x_i})^2})$$

We observe finally that the proportional method stands out within the loss calibrated family for several axiomatic reasons. the following result is proven in Subsection 10.2.

Lemma 3 *The proportional method h^{pro} is the only method in $\mathcal{H}^0_{[lc]}$ satisfying any one of*

Lower Composition: for all $(N, x, t) \in \mathcal{P}^0$ and $t' < t$: $h(x,t) = h(x,t') + h(x - h(x,t'), t - t')$;

Upper Composition: for all $(N, x, t) \in \mathcal{P}^0$ and $t' < t$: $h(x,t') = h(h(x,t),t')$;

Self-duality: for all $(N, x, t) \in \mathcal{P}^0$: $h(x,t) + h(x, x_N - t) = x$.

7 Characterization result

Ideally we would like to understand which methods in $\mathcal{H}^0(CSY, CM, RM)$ are extendable to $\mathcal{H}(Edge - CSY, IMT)$. In fact we give a complete answer for the smaller family $\mathcal{H}^0(CSY, CM, SRM)$, where individual shares increase strictly with the resource, a property that is not especially compelling. We find that the loss calibrated methods, and only those, are extendable as desired.

In statement *i)* of Theorem 2, we fix a function β as in Lemma 2, and define $B(z) = \int_1^z \ln(\beta(t))dt$, where $B(z) \in \mathbb{R}$ for $z > 0$, and $B(0) \in \mathbb{R} \cup \{-\infty\}$. Recall from Theorem 1 the notation $En(z) = \int_1^z (\ln(t) + 1)dt$. We define a bipartite method first for irreducible problems, by distinguishing in $\mathcal{P}^{ir}$ strictly overdemanded and balanced problems; then we extend as usual the method to $\mathcal{P}$ by Definition 6 in Section 10.

Theorem 2 *Assume the set $\mathcal{Q}$ of potential resource types is infinite.*
i) Fix a function β as in Lemma 2, and set $B(z) = \int_1^z \ln(\beta(t))dt$. Define the method H^β on $\mathcal{P}^{ir}$ by $H^\beta(P) = \varphi^\beta$ such that

$$\varphi^\beta = \arg \min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} En(\varphi_{ia}) + \sum_{i \in N} B(x_i - y_i) \text{ if } P \in \mathcal{P}^{str} \quad (16)$$

$$\varphi^\beta = \widehat{\varphi} \text{ (the proportional flow) if } P \text{ is balanced}$$

Then the (canonical extension of) H^β extends the loss calibrated method h^β to a bipartite method in $\mathcal{H}(Edge-CSY, IMT)$. We write $\mathcal{H}_{[lc]}$ for the set of such bipartite loss calibrated methods.
ii) For any function β as in Lemma 2, H^β is the only extension of h^β that is continuous and node-consistent.
iii) Fix a bipartite method $H \in \mathcal{H}(Node - CSY, IFM)$, such that the corresponding standard method h is claim monotonic and strictly resource monotonic, $h \in \mathcal{H}^0(CM, SRM)$. Then $h \in \mathcal{H}^0_{[lc]}$ and $H \in \mathcal{H}_{[lc]}$ is constructed from h as in statement i).

Note that if $B(0) > -\infty$, we can define H^β directly on $\mathcal{P}$ as the solution of the problem (16), because that problem is well defined even when P is balanced (and irreducible). The proportional method is an example.

Combining Lemma 3 and statement *iii)* in Theorem 2, yields three new characterizations of the proportional bipartite method H^{pro}: it is the only method in $\mathcal{H}(Node - CSY, IFM)$ of which the associated standard method is claim monotonic, strictly resource monotonic, and meets one of Lower or Upper Composition, or Self-duality.

Proof of Theorem 2
Step 1: Statement i) We fix β and give first an alternatve definition of H^β in terms of the KT system for irreducible problems. If $P \in \mathcal{P}^{str}$ the function maximized in (16) is strictly concave and $\mathcal{F}(G, x, r)$ contains strictly positive flows φ such that $y \ll x$, therefore φ^β is well defined. We proceed as in the proof of Theorem 1 by replacing for convenience $En(z)$ by $Ln(z) = z(\ln z - 1)$,

and writing the Lagrangean of the (modified) problem (16) as

$$L(\varphi,\lambda,\mu) = \sum_{(i,a)\in G} Ln(\varphi_{ia}) + \sum_{i\in N} B\left(x_i - \sum_{a\in f(i)} \varphi_{ia}\right) + \sum_{i\in N} \lambda_i(x_i - \sum_{a\in Q}\varphi_{ia}) + \sum_{a\in Q}\mu_a(r_a - \sum_{i\in N}\varphi_{ia}).$$

We mimic the argument in Step 2.2 of the proof of Theorem 1. For fixed λ and μ, the minimum of $L(\varphi,\lambda,\mu)$ is attained uniquely by a flow φ^* solving

$$\frac{\varphi^*_{ia}}{\beta(x_i - \sum_{a\in f(i)}\varphi^*_{ia})} = e^{\lambda_i+\mu_a}.$$

This implies $y_i^* < x_i$, so for any $P \in \mathcal{P}^{str}$, the optimal choice of λ is zero, and the solution φ^β of problem (16) is characterized by the KT conditions

$$\frac{\varphi^\beta_{ia}}{\beta(x_i - y_i^\beta)} = \frac{\varphi^\beta_{ja}}{\beta(x_j - y_j^\beta)} = \frac{r_a}{\sum_{k\in g(a)}\beta(x_k - y_k^\beta)} \text{ for all } a \text{ and } i,j\in g(a). \tag{17}$$

Next if $P \in \mathcal{P}^{ir}$ is balanced, problem (16) is the same as problem (7), so φ^β must be the proportional flow.

If Q contains a single resource, we only need to consider problems in $\mathcal{P}^{str}$, for which system (17) picks precisely the standard h^β.

We check now that H^β meets SYM, CONT, IMT, and Edge-CSY. SYM requires no proof. For IMT, suppose two types a, b can be merged in problem P, $g(a) = g(b)$. If $P \in \mathcal{P}^{ir}$ is balanced, merging a and b yields a balanced problem in $\mathcal{P}^{ir}$, and the proportional method is IMT; if $P \in \mathcal{P}^{str}$, merging gives another problem in $\mathcal{P}^{str}$, and preserves system (17). Finally the canonical extension preserves IMT.

For Edge-CSY in the case of a balanced problem $P \in \mathcal{P}^{ir}$, note that a reduction $P \to P(-ia)$ preserves balancedness, and that the proportional method is Edge-CSY. If P is strictly overdemanded this is again preserved in a reduction, except that the reduced problem may have more connected components. It is then easy to adapt the argument in Step 1 of the proof of Theorem 1.

Continuity is clear at $P \in \mathcal{P}^{str}$ by Berge's Maximum Theorem. Fix now an irreducible and balanced problem P, for which $H^\beta(P)$ is the proportional flow, and a sequence of problems P^q with the same graph G and $(x^q, r^q) \to (x, r)$. For q large enough P^q is irreducible as well. Setting $\varphi^q = H^\beta(P^q)$, we check that there exists $w^q \in \mathring{\mathcal{S}}(N)$ such that $\varphi^q_{ia} = \frac{w^q_i}{w^q_{g(a)}} r_a$ for all $ia \in G$. If $P^q \in \mathcal{P}^{str}$, by (17) we can take $w^q_i = \frac{\beta(x_i - y_i^\beta)}{\sum_N \beta(x_k - y_k^\beta)}$; if P^q is balanced we use $\widehat{w}$ in statement $i)$ of Theorem 1. For any subsequence of w^q converging to $w \in \mathcal{S}(N)$, we claim $w_{g(a)} > 0$ for all a. Suppose not, then $Q_0 = \{a \in Q | w_{g(a)} = 0\}$ is non empty;

also $N_1 = N \diagdown g(Q_0)$ is non empty (as $w \in \mathcal{S}(N)$) and a strict subset of N. For any $a \in f(N_1)$ we have $w_{g(a)} > 0$, hence for any $i \in N_1$ we can take limits:

$$\varphi_{ia}^{q\beta} = \frac{w_i^q}{w_{g(a)}^q} r_a \to \varphi_{ia}^{\beta} = \frac{w_i}{w_{g(a)}} r_a \Rightarrow x_i = y_i = \sum_{f(i)} \frac{w_i}{w_{g(a)}} r_a$$

Summing the last equation over $i \in N_1$, and recalling that $w_i = 0$ for any $i \notin N_1$, we get finally $x_{N_1} = r_{f(N_1)} = r_{Q \diagdown Q_0}$; by balancedness this implies $r_{Q_0} = x_{N \diagdown N_1} = x_{g(Q_0)}$, a contradiction of our assumption that P is irreducible.

Using the canonical decomposition, it is now easy to check continuity for any overdemanded problem P.

Step 2: Statement ii)

We fix H, equal to h^β for one-resource problems, continuous, and node-consistent (in our notation, $H \in \mathcal{H}(Node - CSY)$, except that we do not need to assume Symmetry).

Fix $P \in \mathcal{P}^{str}$, a type $a \in Q$, and set $\varphi = H(P)$. Apply Node-CSY repeatedly by dropping all other types. Agent i is left with the claim $x_i - y_i + \varphi_{ia}$, so with the notation $\varphi_a = (\varphi_{ja})_{j \in g(a)} \in \mathbb{R}_+^{g(a)}$ we have

$$\text{for all } a \in Q\text{: } \varphi_a = h^\beta(x - y + \varphi_a, r_a) \tag{18}$$

Fix a and set $D(a) = \{i \in g(a) | \varphi_{ia} > 0\}$. Assume $y_i = x_i$ for some $i \in D(a)$ and derive a contradiction. The systems (18) and (15) imply $\frac{\varphi_{ja}}{\beta(x_j - y_j)} = \frac{\varphi_{ia}}{\beta(x_i - y_i)} = \infty$ for any $j \in g(a)$ such that $x_j - y_j + \varphi_{ja} > 0$, therefore $x_j - y_j > 0$ is impossible, i.e., $x_j = y_j$ for all $j \in g(a)$. As $r_a < x_{g(a)} = y_{g(a)}$, there is a positive flow from some $i' \in g(a)$ to another type b, for which the same argument gives $x_j = y_j$ for all $j \in g(b)$. And so on until we reach $x = y$, contradiction.

We have shown $y \ll x$, so φ solves $\frac{\varphi_{ia}}{\beta(x_i - y_i)} = \frac{\varphi_{ja}}{\beta(x_j - y_j)}$ for all a, i, j s.t. $i, j \in g(a)$, precisely the KT conditions (17) for φ^β. This proves $H = H^\beta$ for strictly overdemanded problems. For irreducible and balanced problems, this equality follows from the continuity of both methods. Finally we explain in Section 11 that two node consistent methods that coincide over $\mathcal{P}^{ir}$ must coincide everywhere, completing the proof.

Step 3: Statement iii) Fix $H \in \mathcal{H}(Node - CSY, IFM)$, with corresponding standard method $h \in \mathcal{H}^0(CM, SRM)$. Young's theorem in [**?**] says that h has a parametric representation θ

$$y = h(x, t) \Leftrightarrow \{y_i = \theta(x_i, \lambda) \text{ for all } i \text{ and } \sum_N \theta(x_i, \lambda) = t\}$$

where $\theta : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous, and $\theta(x_i, 0) = 0$; $\theta(x_i, \infty) = x_i$. CM implies that $\theta(x_i, \lambda)$ is weakly increasing in x_i, and SRM that it is strictly increasing in λ for fixed $x_i > 0$.

Thus for $x_i > 0$ the function $\lambda \to \theta(x_i, \lambda)$ from $[0, \infty]$ to $[0, x_i]$ has the inverse ρ defined by:

$$\theta(x_i, \lambda) = y_i \Leftrightarrow \lambda = \rho(x_i, y_i)$$

Clearly $\rho(z,v)$ is continuous on its domain $\mathcal{D} = \{(z,v)|0 \le v \le z,\ 0 < z < \infty\}$, strictly increasing in v, weakly decreasing in z, and $\rho(z,0) = 0$, $\rho(z,z) = \infty$. Finally h is defined as

$$y = h(x,t) \Leftrightarrow \{\rho(x_i,y_i) = \rho(x_j,y_j) \text{ for all } x_i,x_j > 0; \text{ and } y_N = t\} \tag{19}$$

Choose for $i = 1,2$, $(a_i,y_i) \in \mathbb{R}^2_+ \diagdown \{0\}$ such that $\rho(a_1+y_1,y_1) = \rho(a_2+y_2,y_2)$. In the problem $N = \{1,2\}$, $x_i = a_i+y_i$, $t = y_1+y_2$, we have $h(x,t) = y$. By Lemma 1 equation (5) applies to this problem and to any $\lambda \in [0,1]$. With the notation $\mu = 1-\lambda$ this equation is $h(a+\mu\cdot y,\mu t) = \mu\cdot y$, so in view of (19) we have $\rho(a_1+\mu y_1,\mu y_1) = \rho(a_2+\mu y_2,\mu y_2)$. If we now define $\widetilde{\rho}(a,v) = \rho(a+v,v)$ for all $(a,v) \in \mathbb{R}^2_+ \diagdown \{0\}$, we just showed that for any $(a_i,y_i) \in \mathbb{R}^2_+ \diagdown \{0\}$, $i = 1,2$, and any $\mu \in [0,1]$:

$$\widetilde{\rho}(a_1,y_1) = \widetilde{\rho}(a_2,y_2) \Rightarrow \widetilde{\rho}(a_1,\mu y_1) = \widetilde{\rho}(a_2,\mu y_2) \tag{20}$$

where $\widetilde{\rho}$ is continuous on $\mathbb{R}^2_+ \diagdown \{0\}$, weakly decreasing in a_i, and $\widetilde{\rho}(a,0) = 0, \widetilde{\rho}(0,v) = \infty$ for $a,v > 0$.

We fix now $a > 0$, $v > 0$ and consider $\overline{v} = \inf\{v'|\widetilde{\rho}(a,v') = \widetilde{\rho}(a,v)\}$. By continuity of $\widetilde{\rho}$ we have $\widetilde{\rho}(a,\overline{v}) = \widetilde{\rho}(a,v)$. Note that $\widetilde{\rho}(a,v) = \rho(a+v,v) > 0$ because $\rho(z,v)$ is strictly increasing in v for $z > 0$. Now (20) implies $\widetilde{\rho}(a,\frac{\overline{v}}{v}\overline{v}) = \widetilde{\rho}(a,\frac{\overline{v}}{v}v) = \widetilde{\rho}(a,v)$, so $\overline{v} < v$ would contradict the definition of $\overline{v}$. This proves that for all $a > 0$ the function $v \to \widetilde{\rho}(a,v)$ is one-to-one. It is also continuous and $\widetilde{\rho}(a,0) = 0$, therefore it is strictly increasing. We claim that it is also onto $\mathbb{R}_+$.

Suppose the claim fails at some $a > 0$, namely $\sup_{v\in\mathbb{R}_+}\widetilde{\rho}(a,v) = M < \infty$. As $b \to \widetilde{\rho}(b,1)$ is continuous from $\widetilde{\rho}(a,1) < M$ to $\widetilde{\rho}(0,1) = \infty$, there is a $b, 0 < b < a$, s.t. $\widetilde{\rho}(b,1) = M$. Because $\widetilde{\rho}(b,v)$ increases strictly in v, for every $\varepsilon > 0$ small enough, there is $v'_\varepsilon, v'_\varepsilon < 1$, s.t. $\widetilde{\rho}(b,v'_\varepsilon) = M-\varepsilon$. There is also v''_ε s.t. $\widetilde{\rho}(a,v''_\varepsilon) = M-\varepsilon$. By construction $v'_\varepsilon \to 1$ and $v''_\varepsilon \to \infty$ as $\varepsilon \to 0$. Applying (20) again we get $\widetilde{\rho}(b,\frac{1}{v''_\varepsilon}v'_\varepsilon) = \widetilde{\rho}(a,\frac{1}{v''_\varepsilon}v''_\varepsilon) = \widetilde{\rho}(a,1)$, a contradiction because the left-hand term goes to 0.

We have shown that for all $a > 0$, $v \to \widetilde{\rho}(a,v)$ is an increasing homeomorphism from $\mathbb{R}_+$ into itself. So the equation $\widetilde{\rho}(a,v) = 1$ defines a strictly positive function $v = \beta(a)$ on $\mathbb{R}_+ \diagdown \{0\}$. It is continuous because its graph is closed. It is weakly increasing because $\widetilde{\rho}(a,v)$ is weakly decreasing in a (by CM) and strictly increasing in v. Check finally $\lim_{a\to 0}\beta(a) = 0$ by assuming, to the contrary, $\beta(a) \ge \varepsilon > 0$ for all $a > 0$. When z goes from 0 to $\frac{\varepsilon}{2}$, $\widetilde{\rho}(\frac{\varepsilon}{2}-z,z)$ goes continuously from 0 to ∞, therefore there is a z s.t. $\widetilde{\rho}(\frac{\varepsilon}{2}-z,z) = 1 \Leftrightarrow z = \beta(\frac{\varepsilon}{2}-z)$, contradicting our assumption that β is bounded below by ε. Thus we extend the definition of β by $\beta(0) = 0$, and this function satisfies all the properties listed in Lemma 2.

It remains to check that $h = h^\beta$: by statement *ii)* this implies $H = H^\beta$. Set $\Delta_1 = \{(a,v)|a > 0 \text{ and } v = \beta(a)\} \subset \mathbb{R}^2_+ \diagdown 0$. We constructed Δ_1 as a level curve of $\widetilde{\rho}(a,v)$ in $\mathbb{R}^2_+ \diagdown 0$. Observe that for all $\lambda \ge 1$, the set $\Delta_\lambda = \{(a,v)|\lambda v = \beta(a)\}$ is another level curve of $\widetilde{\rho}$: this follows from applying (20) to $(a_1,\lambda v_1)$, $(a_2,\lambda v_2)$, and $\mu = \frac{1}{\lambda}$. The same is true if $\lambda < 1$: assuming $\widetilde{\rho}(a_1,v_1) < \widetilde{\rho}(a_2,v_2)$, together (20) and the fact that $\widetilde{\rho}(a,v)$ strictly increases in v, imply $\widetilde{\rho}(a_1,\lambda v_1) < \widetilde{\rho}(a_2,\lambda v_2)$.

We now define $\Delta_\infty = \{(a,0)|a>0\}, \Delta_0 = \{(0,v)|v>0\}$, so that the sets $\Delta_\mu, \mu \in \mathbb{R}_+ \cup \{\infty\}$ partition $\mathbb{R}^2_+ \diagdown 0$, and are all the level curves of $\widetilde{\rho}$. By construction they are also the level curves in $\mathbb{R}^2_+ \diagdown 0$ of the function $\widetilde{\tau}(a,v) = \frac{v}{\beta(a)}$. Equivalently $\tau(x,v) = \frac{v}{\beta(x-v)}$ has the same level curves as $\rho(x,v)$ in $\mathcal{D}$. This means that τ and ρ define the same rationing methods through (19).■

8 Uniform gains, uniform losses

These two benchmark standard methods are not covered by Theorem 2 because they are resource monotonic, but not strictly so. We find that both are extendable to the bipartite context in *infinitely many ways*.

8.1 Uniform gains

The standard *uniform gains* method h^{ug} (Section 2) equalizes shares as much as possible under the constraint that no one gets more than her claim.

In the following result, it is convenient to consider problems in which all resource-types and all claims are strictly positive. No flow passes through a type with zero resource, or an agent with zero claim, so we can always delete such a type or agent.

Proposition 1 *Fix a strictly concave function W with domain $[0,\infty[$ and range within $[-\infty,+\infty[$. For any problem $(N,Q,G,x,r) \in \mathcal{P}$ such that $x,r \gg 0$ the flow*

$$\widetilde{\varphi} = \arg \max_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} W(\varphi_{ia}) \tag{21}$$

defines a rationing method $H^W \in \mathcal{H}(Edge-CSY, IMT)$ extending h^{ug} for standard problems.

Proof The function maximized in (21) is strictly concave on $\mathcal{F}(G,x,r)$, and as $x,r \gg 0$, the set $\mathcal{F}G,x,r)$ contains some flows φ such that $\varphi_{ia} > 0$ for all $ia \in G$. Therefore the objective function is not everywhere $-\infty$ and our method is well defined. Symmetry is clear, and continuity follows from Berge's Maximum Theorem as in the proof of Theorem 1.

For Edge-CSY we fix (G,x,r), an edge $ia \in G$, and write $H(G,x,r) = \varphi$. With the notation in the definition of the axiom, observe that if $\varphi' \in \mathcal{F}(G \setminus \{ia\}, x^H(-ia), r^H(-ia))$, then adding ia to G and φ_{ia} to φ' yields a flow (φ',φ_{ia}) in $\mathcal{F}(G,x,r)$. If the restriction of φ to $P^H(-ia)$ is not optimal in that problem, we can then construct a flow (φ',φ_{ia}) beating φ in P.

We fix again (G,x,r) and describe the Kuhn Tucker conditions characterizing the optimal solution φ of (21) with associated profile of shares y. If $y_i = 0$ for some agent, $r \gg 0$ implies $\varphi_{ia} = 0 < \varphi_{ja}$ for any $a \in f(i)$ and some $j \in g(a)$, so a transfer from φ_{ja} to φ_{ia} would yield a better flow. Thus the KT conditions:

for all i, j and $a \in f(i) \cap f(j)$

$$\begin{aligned} \{y_i &< x_i \text{ and } y_j < x_j\} \Rightarrow \varphi_{ia} = \varphi_{ja}; \\ \{y_i &< x_i \text{ and } y_j = x_j\} \Rightarrow \varphi_{ia} \geq \varphi_{ja} \end{aligned} \tag{22}$$

In particular our method is h^{ug} when there is a single type of resource.

For IMT, if in (G, x, r) the two types a, b have $g(a) = g(b)$, and the flow φ meets the above system, then merging the flows through a and b gives a new flow still meeting the KT conditions. We omit the straightforward details.■

We stress that two different functions W^1, W^2 in Proposition 1 yield typically different rationing methods. In the examples below, we choose $W^1(z) = -z^2$, and $W^2(z) = \ln(z)$, so that W^2 guarantees $\varphi_{ia} > 0$ for all $ia \in G$ whereas W^1 does not.

If G is complete, the profiles of shares y^1, y^2 coincide, but the flows φ^1, φ^2 may not. Assume for instance $N = \{1, 2\}, x = (1, 4), Q = \{a, b\}, r = (1, 3)$. Then $y^1 = y^2 = (1, 3)$, and the corresponding max- flow takes the form

$$\varphi_{1a} = z; \varphi_{1b} = 1 - z; \varphi_{2a} = 1 - z; \varphi_{2b} = 2 + z$$

for some $z \in [0, 1]$. Check that $\arg\max_z \{W^1(z) + 2W^1(1-z) + W^1(2+z)\} = \{0\}$, that is the single unit of type a goes to agent 2, who also gets 2 units of type b. On the other hand the optimal z for W^2 is $\frac{1}{2}(\sqrt{3} - 1)$, so agent 2 gets 0.63 units of type a and 2.37 units of type b.

If G is not complete, even the shares y^1, y^2 may differ. For instance we modify our earlier numerical example (Subsection 1.2) by keeping the same graph G, but with claims $x = (1, 1, 4)$ and resources $r = (1, 4)$ (see Figure 2).

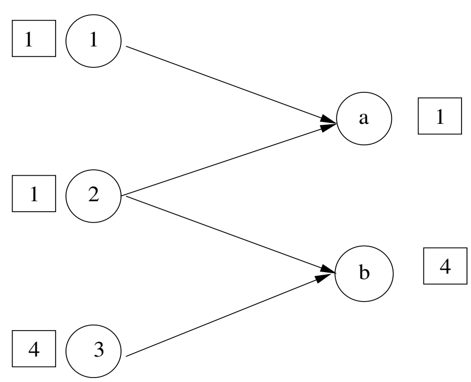

Figure 2: Example illustrating Uniform Gains

For any max-flow we have $\varphi_{2b} < \varphi_{3b}$, therefore (22) implies $\varphi_{2a} + \varphi_{2b} = 1$ for any choice of W. The max-flows take the form

$$\varphi_{1a} = z; \varphi_{2a} = 1 - z; \varphi_{2b} = z; \varphi_{3b} = 4 - z$$

so for the same functions W^1, W^2 we get $z^1 = 0$ and $z^2 > 0$.

8.2 Uniform losses

The standard *uniform losses* method h^{ul} equalizes losses as much as possible under the constraint that no one gets more than her claim.

Each one of the bipartite extensions of UL we construct is defined in two steps. The first step selects the profile of total shares y^*, the same for every extension of UL.

Fix a problem $(G, x, r) \in \mathcal{P}$ and let $\mathcal{Y}(G, x, r) = \{y \in \mathbb{R}^N_{++} |$for some $\varphi \in \mathcal{F}(G, x, r)$: $y_i = \varphi_{if(i)}$ for all $i\}$ be the set of feasible profiles of shares. We claim that there is a unique vector $y^* \in \mathcal{Y}(G, x, r)$ at which the profile of losses $x - y^*$ is *Lorenz optimal* in $\{x\} - \mathcal{Y}(G, x, r)$. To recall the meaning of Lorenz optimality, use the notation $\mathbb{R}^N \ni z \to \widetilde{z}$ for rearranging the coordinates of z in increasing order. Then y^* satisfies for all $k = 1, \cdots, n$

$$\sum_{j=1}^{k} \widetilde{(x - y^*)}_j \leq \sum_{j=1}^{k} \widetilde{(x - y)}_j \text{ for all } y \in \mathcal{Y}(G, x, r)$$

An equivalent definition of y^* is that, for any strictly concave function U, it maximizes the sum $\sum_{i \in N} U(x_i - y_i)$ over $\mathcal{Y}(G, x, r)$. Note that for a standard problem (x, t), $x - h^{ul}(x, t)$ is Lorenz optimal within all feasible profiles of losses $x - y$, so our first step guarantees a bipartite generalization of h^{ul}.

The claim follows from the representation of $\mathcal{Y}(G, x, r)$ as the core of a submodular cooperative game[9] in N, combined with the familiar fact that such a set has a Lorenz dominant element ([13]).

For the second step of our construction, we choose a strictly concave function V from $\mathbb{R}_+$ into itself, and select the flow $\varphi^* =^V H(G, x, r)$ as follows in the balanced problem (G, y^*, r):

$$\varphi^* = \arg \max_{\varphi \in \mathcal{F}(G, y^*, r)} \sum_{ia \in G} V(\varphi_{ia}) \tag{23}$$

(φ^* is well defined because the objective function is strictly concave and finite).

Proposition 2 *The rationing method ${}^V H$ defined by (23) is in $\mathcal{H}(Edge - CSY, IMT)$ and extends h^{ug} for standard problems.*

Observe that we can use a similar two-step construction to define bipartite methods that coincide with the standard uniform gains h^{ug} or the proportional h^{pro}. For the former we pick a Lorenz optimal element $\widetilde{y}$ inside $\mathcal{Y}(G, x, r)$ (it equalizes shares as much as possible given the constraints), and implement them by a flow maximizing $\sum_{ia \in G} W(\varphi_{ia})$ within $\mathcal{F}(G, \widetilde{y}, r)$ (where W is chosen as in Proposition 1). For the latter we choose $\overline{y}$ such that $(\frac{\overline{y_i}}{x_i})_{i \in N}$ is Lorenz optimal among all $(\frac{y_i}{x_i})_{i \in N}$, $y \in \mathcal{Y}(G, x, r)$ and use the same second step.

[9] The value of coalition $S \subseteq N$ is $v(S) = \min_{\varnothing \subseteq T \subseteq S} \{x_T + r_{f(S \setminus T)}\}$, and $y \in \mathcal{Y}(G, x, r) \Leftrightarrow y_S \leq v(S)$ for all S, with equality for $S = N$ (see [7]). Then $\{x\} - \mathcal{Y}(G, x, r)$ is the core of the *supermodular* game $w(S) = x_S - v(S)$.

But recall from the examples in subsection 1.2 that the corresponding bipartite rules cannot be even node consistent. In this sense the Uniform Losses method has a special status in our model.

Proof of Proposition 2 Our method is clearly symmetric. Continuity follows from applying Berge's Maximum Theorem twice, once to $(x,r) \to y^*$, then to $(y^*,r) \to \varphi^*$.

For Edge-CSY, we fix $(G,x,r) \in \mathcal{P}$, $ia \in G$ and $\varphi^* =^V H(G,x,r)$. With the notation in the previous proof, for any $y' \in \mathcal{Y}(G\setminus\{ia\}, x^H(-ia), r^H(-ia))$ check that the profile $y : y_i = y'_i + \varphi^*_{ia}, y_j = y'_j$ for $j \neq i$, is in $\mathcal{Y}(G,x,r)$. Moreover for any function V we have $\sum_{j\in N} V(x^H_j(-ia) - y'_j) = \sum_{j\in N} V(x_j - y_j)$. Therefore the Lorenz optimum in the reduced problem must be $y' : y'_i = y^*_i - \varphi^*_{ia}, y'_j = y_j$ for $j \neq i$. Finally the separability of the objective function (23) implies $\varphi'_e = \varphi^*_e$ for all $e \in G\diagdown\{ia\}$.

For IMT observe that if we merge two types a,b such that $g(a) = g(b)$, the set $\mathcal{Y}(G,x,r)$ of feasible total shares is unchanged, therefore the vector of shares y^* remains the same. The argument in the previous proof showing that H^W meets IMT applies to the solution ${}^V H$ of problem (23) as well.■

In the example of subsection 1.2, the uniform losses method selects the flow (1), and when we drop resource b, in the reduced problem it gives all of resource a to agent 1, as required by node-consistency.

In the example of Figure 2, the uniform losses method achieves the same loss $x_i - y_i = \frac{1}{3}$ for every agent by the flow

$$\varphi_{1a} = \frac{2}{3}; \varphi_{2a} = \frac{1}{3}; \varphi_{2b} = \frac{2}{3}; \varphi_{3b} = 3\frac{2}{3}$$

9 Some open questions

As discussed after Lemma 2, all reasonable symmetric standard rationing methods, including uniform gains, losses, and all loss calibrated methods, are monotonic with respect to claims (CM) and to resource (RM) (Definition 5). They are also cross monotonic (CRM) and meet Ranking (RK).

It is natural to ask if the multi-claim methods that we have indentified meet the corresponding properties, properly extended to the bipartite framework.

For CM: is $x_i \to \varphi_{ia}$ weakly increasing for all $a \in f(i)$?

For CRM: is $x_i \to \varphi_{ja}$ weakly decreasing for all $j \neq i$ and $a \in f(i) \cap f(j)$? What about $a \in f(j)\diagdown f(i)$?

For RM: is $r_a \to y_i$ weakly increasing for all $i \in g(a)$? Same question for $r_a \to \varphi_{ia}$? What about $r_a \to y_j, \varphi_{ja}$ for $j \notin g(a)$?

We can also extend RK and RK* by applying it to two agents with identical connections, which generate more questions of the same kind.

None of these questions has an easy answer. For the extensions of uniform gains and uniform losses, it is possible that the answer depends upon the choice of the concave functions W, V in Propositions 1 and 2.

10 Appendix

10.1 Canonical Decomposition

We show that any overdemanded problem $P \in \mathcal{P}$ can be uniquely decomposed into irreducible problems over a partition of agents and resources. Then we explain how a method defined only for irreducible problems is canonically extended into a full-fledged bipartite method. This construction is used in the proof of both Theorems 1 and 2, where several properties are proven first for irreducible problems, then extended to all overdemanded problems by means of the decomposition.

Lemma 4 *For any problem $P = (N, Q, G, x, r) \in \mathcal{P}$, there is an integer $K \geq 1$ and two partitions, $N = \cup_1^K N^k$, $Q = \cup_1^K Q^k$, such that:*

- $g(Q^1) = N^1; \cdots ; g(Q^k) \diagdown \{N^1 \cup \ldots \cup N^{k-1}\} = N^k$ for all $k, 2 \leq k \leq K$;
- for all $k, 1 \leq k \leq K$, the problem $P^k = (N^k, Q^k, G(N^k \times Q^k), x_{[N^k]}, r_{[Q^k]})$ is irreducible; and if $K > 1$, the problems P^k for $1 \leq k \leq K - 1$ are balanced.
- a flow φ in problem P is a max-flow if and only if it is the "union" of K max-flows φ^k, one in each subproblem P^k.

Proof sketch If P is irreducible only the coarsest partition can fit the bill. This is the only case where $K = 1$. If P is not irreducible, there is at least one "balanced" subset B of Q, i.e., $r_B = x_{g(B)}$. Any two balanced subsets either are disjoint or their intersection satisfies the same property. Thus the inclusion minimal balanced subsets are disjoint, and they are the first elements $Q^1, \cdots, Q^k$, of the partition of Q. The inductive contruction continues on the problem reduced to $N \diagdown g(Q^1 \cup \ldots \cup Q^k)$ and $Q \diagdown \{Q^1 \cup \ldots \cup Q^k\}$.■

The canonical partition is unique up to possibly relabeling the P^k: if the first step delivers several inclusion minimal balanced subsets, we can exchange them freely; similarly if $g(Q^k) \cap N^{k-1} = \varnothing$, we can exchange P^k and P^{k-1}.

Extending a bipartite rationing method from the set $\mathcal{P}^{ir}$ of irreducible problems to $\mathcal{P}$ is done in the following way.

Definition 6 *Given a method H^{ir} on $\mathcal{P}^{ir}$, its canonical extension H to $\mathcal{P}$ selects for every $P \in \mathcal{P}$ the max-flow $H(P) = \varphi$ that is the union of the max-flows $H^{ir}(P^k)$ for the decomposition above.*

Clearly the canonical extension of a method from $\mathcal{P}^{ir}$ to $\mathcal{P}$ does not depend on the labeling of the irreducible subproblems P^k of a given problem P.

Definition 7 *The method H^{ir} on $\mathcal{P}^{ir}$ is node consistent (resp. edge consistent) iff its canonical extension is.*

We cannot define consistency directly for methods on $\mathcal{P}^{ir}$, because the reduced problem of an irreducible one may not be irreducible. However for any method H^{ir} on $\mathcal{P}^{ir}$, its canonical extension is the only possible method on $\mathcal{P}$ that extends H^{ir} and is node/edge consistent, so this is the right definition. In particular if the method H on $\mathcal{P}$ is node-consistent, it is the canonical extension of its projection on $\mathcal{P}^{ir}$.

Note further that the canonical extension preserves symmetry, and if the method on $\mathcal{P}^{ir}$ meets IMT or IFM, so does its canonical extension. But continuity is not guaranteed, as it requires some conditions linking the solutions for irreducible problems of different sizes.

10.2 Proof of Lemma 3

i) Lower Composition: Fix a method $h \in \mathcal{H}^0_{[lc]}$ satisfying LC and $N = \{1, 2\}$. For any $x, y, z \in \mathbb{R}^2_+$ such that $0 \ll y + z \ll x$, the equations

$$\frac{y_1}{\beta(x_1 - y_1)} = \frac{y_2}{\beta(x_2 - y_2)}\ ;\ \frac{z_1}{\beta(x_1 - y_1 - z_1)} = \frac{z_2}{\beta(x_2 - y_2 - z_2)}$$

imply $y = h(x, t')$ for $t' = y_{12}$, and $z = h(x - y, t'')$ for $t'' = z_{12}$. Applying LC to x, t' and $t = t' + t''$ gives $h(x, t) = y + z \Leftrightarrow \frac{y_1+z_1}{\beta(x_1-y_1-z_1)} = \frac{y_2+z_2}{\beta(x_2-y_2-z_2)}$. Changing variables to y, z, b such that $b = x - y - z$, we have: for all $y, z, b \gg 0$

$$\{\frac{y_1}{\beta(z_1 + b_1)} = \frac{y_2}{\beta(z_2 + b_2)} \text{ and } \frac{z_1}{\beta(b_1)} = \frac{z_2}{\beta(b_2)}\} \Rightarrow \frac{y_1 + z_1}{\beta(b_1)} = \frac{y_2 + z_2}{\beta(b_2)}$$

Eliminating the variable y, this property becomes: for all $z, b \gg 0$

$$\frac{\beta(b_2)}{\beta(b_1)} = \frac{z_2}{z_1} \Rightarrow \frac{\beta(b_2)}{\beta(b_1)} = \frac{\beta(z_2 + b_2)}{\beta(z_1 + b_1)}$$

Fix b and set $\lambda = \frac{\beta(b_2)}{\beta(b_1)}$. Then we have $\frac{\beta(\lambda s+b_2)}{\beta(s+b_1)} = \lambda$ for all $s > 0$. Therefore

$$\frac{\beta(\lambda s + b_2) - \beta(b_2)}{\lambda s} = \frac{\beta(s + b_1) - \beta(b_1)}{s} \text{ for all } s > 0$$

As b_1, b_2 are arbitrary positive numbers, this implies that β is differentiable everywhere on $]0, \infty[$ (it is differentiable almost everywhere by monotonicity), and its derivative is constant.

ii) Upper Composition: Fix a method $h \in \mathcal{H}^0_{[lc]}$ satisfying UC, and $N = \{1, 2\}$. For any $x, y, z, 0 \ll z \ll y \ll x$, the equations

$$\frac{y_1}{\beta(x_1 - y_1)} = \frac{y_2}{\beta(x_2 - y_2)}\ ;\ \frac{z_1}{\beta(y_1 - z_1)} = \frac{z_2}{\beta(y_2 - z_2)}$$

imply $y = h(x, t)$ for $t = y_{12}$, and $z = h(y, t')$ for $t' = z_{12}$. Applying UC to x, t' and t gives $h(x, t') = z \Leftrightarrow \frac{z_1}{\beta(x_1-z_1)} = \frac{z_2}{\beta(x_2-z_2)}$. Changing variables to z, a, b such that $a = y - z, b = x - y$, we have: for all $z, a, b \gg 0$

$$\{\frac{z_1 + a_1}{\beta(b_1)} = \frac{z_2 + a_2}{\beta(b_2)} \text{ and } \frac{z_1}{\beta(a_1)} = \frac{z_2}{\beta(a_2)}\} \Rightarrow \frac{z_1}{\beta(a_1 + b_1)} = \frac{z_2}{\beta(a_2 + b_2)} \quad (24)$$

If h is not the proportional method, there is some $a \gg 0$ s.t. $\frac{\beta(a_2)}{\beta(a_1)} \neq \frac{a_2}{a_1}$. Label these numbers so that $\frac{a_2}{a_1} < \frac{\beta(a_2)}{\beta(a_1)}$, and set $\frac{\beta(a_2)}{\beta(a_1)} = \lambda$. For any $b \gg 0$ there exists

z satisfying the two equations on the left hand side of (24) if and only if there exists $z_1 > 0$ s.t. $\frac{\beta(b_2)}{\beta(b_1)} = \frac{\lambda z_1 + a_2}{z_1 + a_1}$, i.e., iff $\frac{\beta(b_2)}{\beta(b_1)} \in]\frac{a_2}{a_1}, \frac{\beta(a_2)}{\beta(a_1)}[$. Let $\Omega(a)$ be the set of those b: $\Omega(a)$ is an open set intersecting for every $K > 0$ the line $b_1 + b_2 = K$ in an non empty interval (because $\lim_{z \to 0} \beta(z) = 0$). In particular $0 \in \partial\Omega(a)$. Property (24) reads: for all $b \gg 0$

$$b \in \Omega(a) \;\Rightarrow\; \frac{\beta(a_2 + b_2)}{\beta(a_1 + b_1)} = \frac{\beta(a_2)}{\beta(a_1)}$$

For any b in the open set $\Omega(a)$, this implies that β is constant on a small open interval containing $a_1 + b_1$, as well as on one open interval around $a_2 + b_2$. As $0 \in \partial\Omega(a)$ and $\Omega(a)$ contains arbitrary large vectors, this implies that β is constant on $[a_i, \infty[, i = 1, 2$.

Set $m = \inf\{a_1 | \; \frac{\beta(a_2)}{\beta(a_1)} \neq \frac{a_2}{a_1}$ for some $a_2 > a_1\}$. We just showed that β is constant on $[m, \infty[$. The definition of β precludes $m = 0$, and we now have

$$\{\frac{\beta(\frac{m}{2})}{\beta(m)} = \frac{\frac{m}{2}}{m} \text{ and } \beta(m) = \beta(2m)\} \Rightarrow \frac{\beta(\frac{m}{2})}{\beta(2m)} \neq \frac{\frac{m}{2}}{2m}$$

the desired contradiction.

iii) Self-duality: Fix a self-dual method $h \in \mathcal{H}^0_{[lc]}$ and $N = \{1, 2\}$. For any x, y, t, if $y = h(x, t)$ then we also have $x - y = h(x, x_{12} - t)$. By (15) this gives, for all $x, y, 0 \ll y \ll x$,

$$\frac{y_1}{\beta(x_1 - y_1)} = \frac{y_2}{\beta(x_2 - y_2)} \Rightarrow \frac{x_1 - y_1}{\beta(y_1)} = \frac{x_2 - y_2}{\beta(y_2)}$$

Changing variables to $a = y, b = x - y$, we have

$$a_2\beta(b_1) = a_1\beta(b_2) \Rightarrow b_2\beta(a_1) = b_1\beta(a_2) \text{ for all } a, b \gg 0$$

Fix $a_2 = b_2 = 1$ and $a_1 = \frac{\beta(b_1)}{\beta(1)}$. The above equation gives $\beta(a_1) = b_1\beta(1)$. If $a_1 < b_1$ the two equations above give $\beta(b_1) = a_1\beta(1) < b_1\beta(1) = \beta(a_1)$, contradicting the fact that β is weakly increasing. Assuming $a_1 > b_1$ yields a similar contradiction, and we conclude $\frac{\beta(b_1)}{\beta(1)} = b_1$ for all $b_1 > 0$, as desired.$\blacksquare$

References

[1] Ahuja R.K., Magnati T.L., and Orlin J.B., (1993). "Network Flows: Theory, Algorithms and Applications," Prentice Hall.

[2] Aumann, R.J., and Maschler, M., (1985). "Game-theoretic analysis of a bankruptcy problem from the Talmud", *Journal of Economic Theory*, 36, 195-213.

[3] Banker, R., (1981). "Equity considerations in traditional full cost allocation practices:an axiomatic perspective", in *Joint cost Allocations,* S. Moriarty ed., Oklahoma City, University of Oklahoma Press, 110-130.

[4] Berge, C., (1963). *Topological Spaces,* London: Oliver and Boyd.

[5] Bogomolnaia A., Moulin H., (2004). "Random Matching Under Dichotomous Preferences," *Econometrica*, 72, 257-279.

[6] Bochet, O., İlkılıç, R. and Moulin, H., (2010) "Egalitarianism under earmark constraints," Submitted for publication.

[7] Bochet, O., İlkılıç, R., Moulin, H., and J. Sethuraman, (2010), "Clearing Supply and Demand under Bilateral Constraints," Submitted for publication.

[8] Chakraborty, T., and M. Kearns, (2008). "Bargaining solutions in a social network", in *Workshop on Internet and Network economics (WINE),* 2008, 548-555.

[9] Chakraborty, T., M. Kearns, and S. Khanna, (2009). "Network bargaining: algorithms and structural results", *10th ACM Conference on Electronic Commerce (EC09)*, July 6-10, 2009, Stanford, California.

[10] Chambers, C., (2004). "Consistency in the probabilistic assignment model", *Journal of Mathematical Economics,* 40, 953-62.

[11] Cook, K., and K. Hegtvedt, (1983). "Distributive justice, equity and equality", *Annual Reviews of Sociology,* 9, 217-241.

[12] Deutsch, M., (1975). Equity, equality and need: what determines which value will be used as the basis of distributive justice?, *Journal of Social Issues,* 3, 31, 137-149.

[13] Dutta B. and Ray D., (1989). "A Concept of Egalitarianism Under Participation Constraints," *Econometrica*, 57, 615-635.

[14] Kaminski, M., (2000), "Hydraulic rationing", *Mathematical Social Sciences*, 40, 2, 131-156.

[15] Kleinberg, J. and E. Tardos, (2008). "Balanced outcomes in social exchange networks", *40th ACM Symposium on Theory of Computing (STOC)*, May 17-20, 2008, Victoria, British Columbia.

[16] Maschler, M., (1990). "Consistency", in *Game Theory and Applications,* T. Ichiishi, A. Neyman, and Y. Tauman, eds, New York: Academic Press, 183-186.

[17] Moulin, H., (2000). "Priority Rules and Other Asymmetric Rationing Methods," *Econometrica*, 68, 3, 643–684.

[18] Moulin, H., (2002). "Axiomatic Cost and Surplus-Sharing," in *Handbook of Social Choice and Welfare*, K. Arrow, A, Sen and K. Suzumura eds, North-Holland.

[19] Moulin, H., (1987). "Equal or Proportional Division of a Surplus, and Other Methods", *International Journal of Game Theory*, 16, 3, 161–186

[20] O'Neill, B., (1982). "A problem of rights arbitration from the Talmud", *Mathematical Social Sciences,* 2, 345-371.

[21] Rescher, N., (1966). *Distributive Justice,* Indianapolis: Bobbs-Merill.

[22] Roth R., Sönmez, T., and Ünver, U., (2005). "Pairwise Kidney Exchange," *Journal of Economic Theory*, 125, 151-188.

[23] Thomson, W., (2003). "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems", *Mathematical Social Sciences,* 45, 3, 249-297.

[24] Thomson, W., (2005). "Consistent allocation rules", mimeo, University of Rochester.

[25] Young, H. P., (1987). "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research*, 12, 397-414.

[26] Young, H.P., (1987). "Distributive justice in taxation", *Journal of Economic Theory*, 48, 321-335.