Geometry of Complex Monge-Ampère Equations

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iii

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Abstract

The Kähler-Ricci flow is studied on compact Kähler manifolds with positive first Chern class, where it reduces to a parabolic complex Monge-Ampère equation. It is shown that the flow converges to a Kähler-Einstein metric if the curvature remains bounded along the flow, and if the manifold is stable in an algebro-geometric sense.

On a compact Calabi-Yau manifold there is a unique Ricci-flat Kähler metric in each Kähler cohomology class, produced by Yau solving a complex Monge-Ampère equation. The behaviour of these metrics when the class degenerates to the boundary of the Kähler cone is studied. The problem splits into two cases, according to whether the total volume goes to zero or not.

On a compact symplectic four-manifold Donaldson has proposed an analog of the complex Monge-Ampère equation, the Calabi-Yau equation. If solved, it would lead to new results in symplectic topology. We solve the equation when the manifold is nonnegatively curved, and reduce the general case to bounding an integral of a scalar function.

Contents

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Chapter 1

Introduction

We begin by summarizing the main results of the thesis. In section 1.2 we collect some basic facts in Kähler geometry, and in section 1.3 we prove Yau's estimates for complex Monge-Ampère equations.

1.1 Summary of results

This thesis consists of three main parts, all revolving around the theory of elliptic and parabolic complex Monge-Ampère equations and its geometric applications. We will begin with the results on the Kähler-Ricci flow on Fano manifolds which form the content of Chapter 2.

The Ricci flow is a nonlinear second order parabolic flow on Riemannian manifolds introduced by Hamilton in [Ha1]. On a compact complex manifold M if the initial metric is Kähler then the flow will preserve this property, and is thus called the *Kähler-Ricci flow*. If the first Chern class $c_1(M)$ is definite and the initial metric lies in the correct class, then the flow has a long time solution. When $c_1(M)$ is negative or zero the flow always converges smoothly exponentially fast to a Kähler-Einstein metric (with negative or zero scalar curvature). Thus, we will restrict our attention to compact Kähler manifolds with $c_1(M)$ positive, that is *Fano manifolds*. The (normalized) Kähler-Ricci flow equation then takes the form

$$(1.1) \qquad \frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t,$$

where ω_t is a family of Kähler metrics cohomologous to $c_1(M)$. It can also be rewritten as a parabolic complex Monge-Ampère equation. Our main result on the Kähler-Ricci flow is as follows.

Theorem 1.1.1. *Let M be a compact complex manifold with $c_1(M) > 0$. Assume that along the Kähler-Ricci flow (1.1) the sectional curvatures remain bounded*

$$(1.2) \qquad\qquad\qquad |\mathrm{Rm}_t| \le C.$$

Assume moreover that (M, K_M^{-1}) is K-polystable and asymptotically Chow semistable. Then the flow converges smoothly exponentially fast to a Kähler-Einstein metric on M.

This is in line with a long-standing conjecture of Yau [Y5] which says that a Kähler-Einstein metric should exist precisely when the manifold, polarized by the anticanonical bundle K_M^{-1}, is stable in a suitable algebro-geometric sense. The precise notion of stability involved is K-polystability, as introduced by Donaldson [Do2]. Chow semistability is another algebraic GIT notion which is likely to be implied by K-polystability (see [RT]). The assumption of bounded curvature is certainly very strong, but there are currently no known examples of Fano manifolds where it fails. Moreover it is known to hold on all Fano manifolds of complex dimension 2, and in higher dimension if the initial metric has nonnegative bisectional curvature.

Next, we will briefly discuss the results of Chapter 3. The main objects of study are Ricci-flat Kähler metrics on a compact projective Calabi-Yau manifold X. The solution of the Calabi conjecture given by Yau [Y1] guarantees the existence of a unique such metric in each given ample class. The set of all ample classes is an open convex cone inside the Néron-Severi space $N^1(X)_{\mathbb{R}}$, the *ample cone $\mathcal{K}_{NS}$*. As long as the Kähler class stays inside the cone, the corresponding Ricci-flat metrics vary smoothly, but they will degenerate when the class approaches the boundary of the cone. Several people have addressed the question of understanding this degeneration process and seeing what the limiting space looks like [Y4, Y5, McM, Wi2]. The problem splits naturally into two cases, according whether the limiting class α has positive volume (we call α nef and big) or zero volume (and α is nef and not big).

Our first result deals with the nef and big case.

Theorem 1.1.2. *Let X be a compact projective Calabi-Yau manifold, and let $\alpha \in N^1(X)_{\mathbb{R}}$ be a big and nef class that is not ample. Then there exist a proper analytic subvariety $E \subset X$, which is the null locus of α, and a smooth incomplete Ricci-flat Kähler metric ω_1 on $X \backslash E$ such that for any smooth path $\alpha_t \in \mathcal{K}_{NS}$ with $\alpha_1 = \alpha$, the Ricci-flat metrics $\omega_t \in \alpha_t$ converge to ω_1 in the C^∞ topology on compact sets of $X \backslash E$. Moreover ω_1 extends to a closed positive current with continuous potentials on the whole of X, that lies in α, and that is the pullback of a singular Ricci-flat Kähler metric on a Calabi-Yau model of X obtained from the contraction map of α.*

Incompleteness of the limit follows from a general diameter bound for Ricci-flat Kähler metrics that might be of independent interest (Theorem 3.3.1). There are many interesting concrete examples of our theorem, and we will examine a few of

them in Chapter 3. This theorem generalizes classical results in dimension 2 of Anderson [An], Bando-Kasue-Nakajima [BKN], Tian [Ti2] and Kobayashi-Todorov [KT].

We will now discuss the case when the volume of α is zero. A natural example of this is to consider an algebraic fiber space $f : X \to Y$ where Y is an algebraic variety of lower dimension, and let α be the pullback of an ample divisor on Y. This picture is conjecturally always true if α is a rational class. In this case we have the following result.

Theorem 1.1.3. *Let X be a compact projective Calabi-Yau manifold and let $f : X \to Y$ be an algebraic fiber space with Y an irreducible normal algebraic variety of lower dimension. Let ω_X be a Kähler form on X and α be the pullback of an ample divisor on Y. Then there exist a proper analytic subvariety $E \subset X$ and a smooth Kähler metric ω on $Y \backslash f(E)$, such that the the Ricci-flat metrics $\omega_t \in \alpha + t\omega_X$, $0 < t \leq 1$, converge to $f^*\omega$ as t goes to zero in the $C_{loc}^{1,\beta}$ topology of potentials on compact sets of $X \backslash E$, for any $0 < \beta < 1$. The metric ω satisfies*

$$\mathrm{Ric}(\omega) = \omega_{WP},$$

on $Y \backslash f(E)$, where ω_{WP} is a Weil-Petersson metric measuring the change of complex structures of the fibers. Moreover for any $y \in Y \backslash f(E)$ if we restrict to $X_y = f^{-1}(y)$, the metrics ω_t converge to zero in the C^1 topology of metrics, uniformly as y varies in a compact set of $Y \backslash f(E)$.

What this theorem says is that the Ricci-flat metrics collapse to a metric on the base of the fibration, and that the rescaled metrics on the fibers tend to be Ricci-flat. The metric on the base is not Ricci-flat but somehow remembers the fibration structure. This generalizes a result of Gross-Wilson [GW] where they proved this for an elliptically fibered $K3$ surface.

Finally, in Chapter 4, we consider the analog of complex Monge-Ampère equations on symplectic four-manifolds, which are called Calabi-Yau equations. This is part of a recent program of Donaldson [Do5], which if carried out would lead to many new and exciting results in symplectic geometry (see the survey [TW]). A necessary element of this program is to obtain estimates for the Calabi-Yau equation on symplectic four-manifolds with a compatible but non-integrable almost complex structure.

In [Do5], Donaldson made the following conjecture.

Conjecture 1.1.4. *Let (M, Ω) be a compact symplectic four-manifold equipped with an almost complex structure J tamed by Ω. Let σ be a smooth volume form on M. If $\tilde{\omega} \in [\Omega]$ is a symplectic form on M which is compatible with J and solves the Calabi-Yau equation*

$$(1.3) \qquad\qquad\qquad \tilde{\omega}^2 = \sigma,$$

then there are C^∞ a priori bounds on $\tilde{\omega}$ depending only on Ω, J and σ.

More precisely, we have the following. For each $k = 0, 1, 2, \ldots$, there exists a constant A_k depending smoothly on the data Ω, J and σ such that

$$\|\tilde{\omega}\|_{C^k(g_\Omega)} \leq A_k. \tag{1.4}$$

We now state our results, which hold for symplectic manifolds of any even dimension $2n$. Let us define a smooth real-valued function φ by

$$\frac{1}{2n}\tilde{\Delta}\varphi = 1 - \frac{\tilde{\omega}^{n-1} \wedge \Omega}{\tilde{\omega}^n}, \quad \sup_M \varphi = 0. \tag{1.5}$$

Then we have the following result.

Theorem 1.1.5. *Let $\alpha > 0$ be given. Let (M, Ω) be a compact symplectic $2n$-manifold equipped with an almost complex structure J tamed by Ω. Let σ be a smooth volume form on M. If $\tilde{\omega} \in [\Omega]$ is a symplectic form on M which is compatible with J and solves the Calabi-Yau equation*

$$\tilde{\omega}^n = \sigma, \tag{1.6}$$

there are C^∞ a priori bounds on $\tilde{\omega}$ depending only on Ω, J, σ, α and

$$I_\alpha(\varphi) := \int_M e^{-\alpha\varphi}\Omega^n,$$

for φ defined by (1.5).

The function φ is precisely the usual Kähler potential in the case that $\tilde{\omega}$ and Ω are Kähler forms. In this case it is known (see [Ti1, Hö]) that the quantity $I_\alpha(\varphi)$ is always uniformly bounded as long as α is sufficiently small (where the bounds depend only on M, Ω, J).

Associated to the fixed data Ω, J is a modified curvature tensor $\mathcal{R}(\Omega, J)$ which reduces to the usual bisectional curvature in the Kähler case. Our second result is as follows.

Theorem 1.1.6. *If $\mathcal{R}(g, J) \geq 0$ in the sense of Griffiths, Conjecture 4.1.1 holds, as well as its higher-dimensional analog.*

This shows that Conjecture 1.1.4 holds for small perturbations (Ω, J) of the standard Kähler structure of $\mathbb{CP}^2$.

1.2 Basic Kähler geometry

Let (M, J) be a compact complex manifold of complex dimension n. A Riemannian metric g is called Hermitian if it satisfies $g(JX, JY) = g(X, Y)$ for all tangent vectors X, Y. In this case we then define a real 2-form ω by the formula

$$\omega(X, Y) = g(JX, Y).$$

If ω is closed, we call g a *Kähler metric*. Since ω and g are equivalent data, we will often refer to ω as the Kähler metric, or Kähler form. It is of type $(1, 1)$, and if locally we write

$$g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j,$$

then we have

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where here and henceforth we are using the Einstein summation convention. The Riemannian volume form of g is equal to $\frac{\omega^n}{n!}$, and we will denote by V the volume of M

$$V = \int_M \frac{\omega^n}{n!}.$$

We will write Δ_ω for the Laplacian of g, which acts on a function F as

$$\Delta_\omega F = g^{i\bar{j}} \frac{\partial^2 F}{\partial z^i \partial \bar{z}^j}.$$

The Ricci curvature of ω is the tensor locally defined by

$$R_{i\bar{j}} = -\frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} \log \det(g),$$

and we associate to it the Ricci form

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

It is a closed real $(1, 1)$-form that represents the cohomology class $c_1(M) \in H^2(M, 2\pi\mathbb{Z})$. The scalar curvature of ω is denoted by

$$R = g^{i\bar{j}} R_{i\bar{j}}.$$

Notice that $\underline{R}$, the average of R, only depends on cohomological data, since

$$\underline{R} = \frac{1}{V} \int_M R \frac{\omega^n}{n!} = \frac{1}{V} \int_M \mathrm{Ric}(\omega) \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{n c_1(M) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

To the metric ω we can associate its Ricci potential f_ω, which is the real function defined by

(1.7) $$R - \underline{R} = \Delta_\omega f_\omega,$$

and

$$(1.8) \qquad \int_M (e^{f_\omega} - 1)\frac{\omega^n}{n!} = 0.$$

The space of Kähler potentials of the metric ω is the set of all smooth real functions φ such that $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a Kähler metric. Then we can define a real-valued functional F_ω^0 on the space of Kähler potentials by the formula

$$F_\omega^0(\varphi) = -\frac{1}{V} \int_0^1 \int_M \frac{\partial\varphi_t}{\partial t}\frac{\omega_{\varphi_t}^n}{n!},$$

where φ_t is any smooth path of Kähler potentials with $\varphi_0 = 0$ and $\varphi_1 = \varphi$ (for example one can take $\varphi_t = t\varphi$). It can be written also as

$$(1.9) \qquad F_\omega^0(\varphi) = J_\omega(\varphi) - \frac{1}{V} \int_M \varphi\frac{\omega^n}{n!},$$

where the functional J_ω is defined by

$$J_\omega(\varphi) = \frac{1}{V} \int_0^1 \int_M \frac{\partial\varphi_t}{\partial t}\left(\frac{\omega^n}{n!} - \frac{\omega_{\varphi_t}^n}{n!}\right),$$

and integration by parts shows that $J_\omega(\varphi) \geq 0$. Moreover F_ω^0 satisfies the following cocycle condition

$$(1.10) \qquad F_\omega^0(\varphi) = F_\omega^0(\psi) + F_{\omega_\psi}^0(\varphi - \psi),$$

for all Kähler potentials φ, ψ. Another useful functional is the Mabuchi energy $\mathcal{M}_\omega(\varphi)$, which is defined by

$$\mathcal{M}_\omega(\varphi) = -\frac{1}{V} \int_0^1 \int_M \frac{\partial\varphi_t}{\partial t}(R(\omega_{\varphi_t}) - \underline{R})\frac{\omega_{\varphi_t}^n}{n!},$$

where φ_t is any smooth path of Kähler potentials with $\varphi_0 = 0$ and $\varphi_1 = \varphi$. It satisfies the same cocycle condition as F_ω^0, namely

$$(1.11) \qquad \mathcal{M}_\omega(\varphi) = \mathcal{M}_\omega(\psi) + \mathcal{M}_{\omega_\psi}(\varphi - \psi).$$

We now assume that L is an ample line bundle over M, so that M is a projective algebraic manifold, and we assume that ω is cohomologous to $c_1(L)$. We now consider the space $H^0(M, L^m)$ of holomorphic sections of the mth tensor power of L, where $m \geq 1$. This is a vector space whose dimension N_m can be computed from the Riemann-Roch formula, when m is large

$$(1.12) \qquad N_m = \int_M \text{ch}(L^m) \wedge \text{Todd}(M) \approx Vm^n + \frac{RV}{2}m^{n-1} + O(m^{n-2}).$$

Let us fix h a Hermitian metric along the fibers of L with curvature equal to ω. This induces metrics h^m on the tensor powers L^m. For each given positive integer m we also fix $\{S_i\}$ a basis of $H^0(M, L^m)$ which is orthonormal with respect to the L^2 inner product defined by h^m, ω^n:

$$\int_M \langle S, T \rangle_{h^m} \frac{\omega^n}{n!}.$$

Then we can define the "density of states" function

$$(1.13) \qquad \rho_m(\omega) = \sum_{i=1}^{N_m} |S_i|_{h^m}^2.$$

It does not depend on the choice of orthonormal basis $\{S_i\}$ or on the choice of h, and so it is canonically attached to ω and J. Its name stems from the property that

$$(1.14) \qquad \int_M \rho_m(\omega) \frac{\omega^n}{n!} = N_m.$$

The Tian-Yau-Zelditch-Catlin expansion is the following

Theorem 1.2.1 (Zelditch [Ze], Catlin [Cat]). *When m is large we have an expansion*

$$(1.15) \qquad \rho_m(\omega) \approx m^n + a_1(\omega)m^{n-1} + a_2(\omega)m^{n-2} + \dots,$$

where $a_i(\omega)$ are smooth functions defined locally by ω, and the expansion is valid in any $C^k(\omega)$ norm. More precisely this means that given any $k, N \geq 1$ there is a constant C that depends only on k, N, ω such that

$$\left\| \rho_m(\omega) - m^n - \sum_{i=1}^{N} a_i(\omega)m^{n-i} \right\|_{C^k(\omega)} \leq Cm^{n-N-1},$$

for all $m \geq 1$.

Moreover Z. Lu [Lu] has computed that $a_1(\omega) = \frac{R}{2}$. The expansion (1.15) integrates term by term to the Riemann-Roch expansion (1.12).

1.3 Complex Monge-Ampère equations

In this section (M, ω) will be a closed Kähler manifold, with complex dimension n. Any Kähler metric $\tilde{\omega}$ cohomologous to ω can be written as $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some real-valued Kähler potential φ, that we can normalize so that

$$(1.16) \qquad \int_M \varphi\omega^n = 0.$$

We now consider the equation

$$(1.17) \qquad \tilde{\omega}^n = e^F \omega^n,$$

where F is a smooth function on M with

$$(1.18) \qquad \int_M (e^F - 1) \omega^n = 0.$$

If locally we write $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$, then (1.17) becomes

$$(1.19) \qquad \det \left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = e^F \det(g_{i\bar{j}}),$$

which is a *complex Monge-Ampère equation*. The celebrated solution of the Calabi conjecture by Yau [Y2] says that we can always solve (1.17):

Theorem 1.3.1. *Let (M, ω) be a closed n-dimensional Kähler manifold, and let F be a smooth real function on M that satisfies (1.18). Then there is a unique Kähler form $\tilde{\omega}$ on M which is cohomologous to ω and which solves (1.17).*

The uniqueness part was proved by Calabi in [Ca2]. To prove the existence of a solution, Yau derived a priori C^k estimates for $\tilde{\omega}$ and then applied the continuity method. The precise statement of the estimates is as follows:

Theorem 1.3.2. *Assume the setup of Theorem 1.3.1. Then there are constants A_k, $k = 0, 1, ...,$ that depend only on k, F, ω, such that*

$$(1.20) \qquad \|\tilde{\omega}\|_{C^k(\omega)} \leq A_k.$$

Notice that (1.20) with $k = 0$ implies that

$$(1.21) \qquad \tilde{\omega} \geq C^{-1} \omega,$$

for a uniform constant C (here and in the following, a uniform constant is a constant that depends only on F and ω). To see this, we first we choose local coordinates so that at one point $p \in M$ the metric ω is the identity and $\tilde{\omega}$ is diagonal with positive entries $\lambda_1, \ldots, \lambda_n$. Then (1.20) with $k = 0$ gives

$$\sum_{i=1}^n \lambda_i \leq n A_0.$$

For any fixed i we then have

$$\lambda_i = \frac{\prod_{j=1}^n \lambda_j}{\prod_{j \neq i} \lambda_j} \geq \frac{\prod_{j=1}^n \lambda_j}{(\sum_{j=1}^n \lambda_j)^{n-1}} \geq \frac{e^{\inf_M F}}{(n A_0)^{n-1}},$$

where in the last inequality we have used (1.17). This is precisely the estimate (1.21).

We will now give the proof of Theorem 1.3.2, referring the reader to Chapter 4 for some of the computations. Recall that we are writing $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ where φ has average zero with respect to the volume form ω^n. We will write $g, \tilde{g}$ for the Riemannian metrics associated to $\omega, \tilde{\omega}$, and $\Delta, \tilde{\Delta}$ for their corresponding scalar Laplace operators. It follows that the quantity $n + \Delta\varphi$ is equal to $\mathrm{tr}_g\tilde{g}$. We clearly have that $\|\tilde{\omega}\|_{C^0(\omega)} \leq \mathrm{tr}_g\tilde{g}$. Yau's estimates (1.20) are derived in four steps.

Step 1. The inequality

$$(1.22) \qquad \mathrm{tr}_g\tilde{g} \leq Ce^{A(\varphi - \inf_M \varphi)},$$

holds for uniform constants A, C.

Step 2. The Kähler potential φ satisfies

$$(1.23) \qquad \sup_M \varphi - \inf_M \varphi \leq C,$$

for a uniform C.

Step 3. If $\|\tilde{\omega}\|_{C^0(\omega)}$ is uniformly bounded, we have $\|\tilde{\omega}\|_{C^1(\omega)} \leq C$, for a uniform C.

Step 4. Given a Hölder bound $\|\tilde{\omega}\|_{C^\beta(\omega)} \leq C$ for some $\beta > 0$, we have, for each $k = 2, 3, \ldots,$, the estimates $\|\tilde{\omega}\|_{C^k(\omega)} \leq A_k$, for uniform A_k.

It is clear that proving these four steps will prove Theorem 1.3.2. To prove step 1, one first computes

$$\tilde{\Delta} \log \mathrm{tr}_g\tilde{g} \geq -C\mathrm{tr}_{\tilde{g}}g - C,$$

for a uniform constant C. This computation is performed in a more general setting in Chapter 4 (see Lemma 4.3.2). On the other hand

$$\tilde{\Delta}\varphi = n - \mathrm{tr}_{\tilde{g}}g,$$

and so if we pick A larger than $C + 1$ we get

$$\tilde{\Delta}(\log \mathrm{tr}_g\tilde{g} - A\varphi) \geq \mathrm{tr}_{\tilde{g}}g - C.$$

At the point $p \in M$ where $\log \mathrm{tr}_g\tilde{g} - A\varphi$ achieves it maximum, we get $\mathrm{tr}_{\tilde{g}}g(p) \leq C$. Diagonalizing g and $\tilde{g}$ as above, we can write this as

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} \leq C.$$

We then apply the inequality

$$\sum_{i=1}^{n} \lambda_i \le \left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)^{n-1} \prod_{i=1}^{n} \lambda_i,$$

which can be written intrinsically as

(1.24)
$$\mathrm{tr}_g \tilde{g} \le (\mathrm{tr}_{\tilde{g}} g)^{n-1} \frac{\det(\tilde{g})}{\det(g)}.$$

This together with the Monge-Ampère equation (1.19) gives

$$\mathrm{tr}_g \tilde{g}(p) \le C.$$

The maximum principle then yields (1.22).

To prove step 2, one employs a Moser iteration argument. For any $p > 1$ we compute

(1.25)
$$\begin{aligned}
\int_M |\nabla(\varphi|\varphi|^{\frac{p-2}{2}})|^2 \omega^n &= \frac{p^2}{4} \int_M |\varphi|^{p-2} |\nabla \varphi|^2 \omega^n \\
&= \frac{np^2}{4} \int_M |\varphi|^{p-2} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-1} \\
&\le \frac{np^2}{4} \int_M |\varphi|^{p-2} \partial \varphi \wedge \overline{\partial} \varphi \wedge \left(\sum_{i=0}^{n-1} \omega^{n-1-i} \wedge \tilde{\omega}^i\right) \\
&= -\frac{np^2}{4(p-1)} \int_M \varphi |\varphi|^{p-2} \partial \overline{\partial} \varphi \wedge \left(\sum_{i=0}^{n-1} \omega^{n-1-i} \wedge \tilde{\omega}^i\right) \\
&= \frac{np^2}{4(p-1)} \int_M \varphi |\varphi|^{p-2} (\omega - \tilde{\omega}) \wedge \left(\sum_{i=0}^{n-1} \omega^{n-1-i} \wedge \tilde{\omega}^i\right) \\
&= \frac{np^2}{4(p-1)} \int_M \varphi |\varphi|^{p-2} (\omega^n - \tilde{\omega}^n) \\
&\le Cp \int_M |\varphi|^{p-1} \omega^n,
\end{aligned}$$

where we used (1.17) in the last inequality. The Sobolev inequality for ω gives us, for $\beta = \frac{n}{n-1}$ and any smooth function f,

(1.26)
$$\left(\int_M |f|^{2\beta} \omega^n\right)^{\frac{1}{\beta}} \le C \left(\int_M |\nabla f|^2 \omega^n + \int_M f^2 \omega^n\right).$$

Applying this to $f = \varphi|\varphi|^{\frac{p-2}{2}}$ we get

$$
\begin{aligned}
\left(\int_M |\varphi|^{p\beta} \omega^n \right)^{\frac{1}{\beta}} &\leq C \left(\int_M |\nabla(\varphi|\varphi|^{\frac{p-2}{2}})|^2 \omega^n + \int_M |\varphi|^p \omega^n \right) \\
&\leq Cp \int_M |\varphi|^{p-1} \omega^n + C \int_M |\varphi|^p \omega^n \\
&\leq Cp \left(\int_M |\varphi|^p \omega^n \right)^{\frac{p-1}{p}} + C \int_M |\varphi|^p \omega^n \\
&\leq Cp \max \left(\int_M |\varphi|^p \omega^n, 1 \right).
\end{aligned}
$$

(1.27)

Raising to the power $\frac{1}{p}$ we get

$$
\max(\|\varphi\|_{p\beta}, 1) \leq C^{\frac{1}{p}} p^{\frac{1}{p}} \max(\|\varphi\|_p, 1),
$$

where $\| \ \|_q$ denotes the L^q norm with respect to ω^n. Successively replacing p by $p\beta$ and iterating we get

$$
\log \max(\|\varphi\|_{p\beta^k}, 1) \leq \log \max(\|\varphi\|_p, 1) + \frac{1}{p} \left(\sum_{i=0}^{k-1} \frac{1}{\beta^i} \right) (\log C + \log p) + \frac{1}{p} \left(\sum_{i=1}^{k-1} \frac{i}{\beta^i} \right) \log \beta.
$$

Setting $p = 2$ and letting $k \to \infty$ we finally get

(1.28)
$$
\|\varphi\|_{L^\infty} \leq C \left(\int_M \varphi^2 \omega^n \right)^{\frac{1}{2}} + C.
$$

Recall that the Poincaré inequality for ω says that for any smooth function f with $\int_M f \omega^n = 0$ we have

(1.29)
$$
\int_M f^2 \omega^n \leq C \int_M |\nabla f|^2 \omega^n.
$$

We use this together with (1.25) with $p = 2$ and with the Hölder inequality to get

(1.30)
$$
\int_M \varphi^2 \omega^n \leq C \int_M |\nabla \varphi|^2 \omega^n \leq C \int_M |\varphi| \omega^n \leq C \left(\int_M \varphi^2 \omega^n \right)^{\frac{1}{2}},
$$

which gives $\int_M \varphi^2 \omega^n \leq C$, and so with (1.28) this gives an L^∞ bound $|\varphi| \leq C$. Notice that since $\int_M \varphi \omega^n = 0$, it follows that φ must vanish somewhere, and so the oscillation of φ is controlled by its L^∞ norm. For future use we record here the estimate we have just proved, keeping track of the dependence of the constants.

Theorem 1.3.3 (Yau's L^∞ estimate). *Let (M, ω) be an n-dimensional compact Kähler manifold, and $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ be another Kähler form on M with $\int_M \varphi\omega^n = 0$. If $\tilde{\omega}$ satisfies (1.17), then we have*

$$\sup_M |\varphi| \leq C(\sup_M e^F + 1)^{(n+2)/2}(C_{\mathrm{Sob}} + 1)^{n/2}(C_{\mathrm{Poi}} + 1),$$

where C is a constant that depends only on n and the volume of ω, and C_{Sob} and C_{Poi} are upper bounds for the Sobolev and Poincaré constants of ω, as in (1.26) and (1.29).

To prove step 3 one first defines the quantity $S = |\nabla\tilde{g}|^2_{\tilde{g}}$, where ∇ is the covariant derivative associated to the metric g. Using φ we can write

$$S = \tilde{g}^{i\bar{p}}\tilde{g}^{q\bar{j}}\tilde{g}^{k\bar{r}}\varphi_{i\bar{j}k}\varphi_{\bar{p}q\bar{r}},$$

where again lower indices are covariant derivatives with respect to g. Since we assume that g and $\tilde{g}$ are uniformly equivalent in C^0, we only need to prove an upper bound for S. One computes

$$\tilde{\Delta}S \geq -CS - C.$$

Again, this is done in a more general setting in Chapter 4 (Lemma 4.4.5). On the other hand, again using that g and $\tilde{g}$ are equivalent, we also have

$$\tilde{\Delta}\mathrm{tr}_g\tilde{g} \geq \frac{1}{C}S - C,$$

see for example (4.56). One can then apply the maximum principle to $S + A\mathrm{tr}_g\tilde{g}$, where A is sufficiently large, to get the required estimate $S \leq C$.

Finally, step 4 follows from a bootstrapping argument. The arguments are purely local, so we can restrict to a small ball where $\omega = \sqrt{-1}\partial\bar{\partial}\psi$ for a local function ψ. Then differentiating the logarithm of (1.19) with respect to z^i we get

$$\tilde{\Delta}\left(\frac{\partial(\psi + \varphi)}{\partial z^i}\right) = \Delta\left(\frac{\partial\psi}{\partial z^i}\right) + \frac{\partial F}{\partial z^i}.$$

Since the operator $\tilde{\Delta}$ is uniformly elliptic, with coefficients uniformly bounded in $C^\beta(\omega)$, Schauder's estimates give a uniform $C^{3,\beta}(\omega)$ bound on φ. But this gives a uniform $C^{1,\beta}(\omega)$ bound on the coefficients of $\tilde{\Delta}$, and so bootstrapping gives all the higher order estimates. This completes the proof of Theorem 1.3.2.

The complex Monge-Ampère equation (1.17) can be thought as "nondegenerate". There are two ways to make the equation more degenerate. First, we can modify the right-hand side to

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F-\varphi}\omega^n,$$

so that now the determinant of the complex Hessian of φ is not controlled a priori. This is precisely the equation to construct Kähler-Einstein metrics with positive

Ricci curvature on Fano manifolds, and its parabolic analog is the Käher-Ricci flow. This will be the topic studied in Chapter 2.

Second, we could move the cohomology class of ω to approach the boundary of the Kähler cone. Since the limit cohomology class is not Kähler, it cannot be represented by any Kähler form. Thus the solutions of (1.17) must blow up when we reach the boundary of the cone, and we will address this phenomenon in Chapter 3.

Finally, in Chapter 4, we consider the same equation (1.17) but we don't insist that the complex structure be integrable. Thus instead of a Kähler manifold we are considering a symplectic manifold. The equation is still nondegenerate, but it is now a system of equation (rather than a scalar equation), and this makes the analysis more complicated.

Chapter 2

Kähler-Ricci flow on Fano manifolds

The Kähler-Ricci flow is a second order nonlinear parabolic flow of Kähler metrics. In section 2.1 we describe this flow and state our main convergence result (Theorem 2.1.1). We then recall some recent fundamental estimates of Perelman [Pe] that are used in the proof. In section 2.3 we define Chow semistability and, following Donaldson [Do3] and Zhang [ZhS], we show that it is equivalent to a lower bound of a suitable functional defined on the space of Kähler potentials (Proposition 2.3.1). Finally in section 2.4 we prove Theorem 2.1.1. The key new ideas are: first to use K-polystability to show that if the flow does not converge then the Mabuchi energy decays at least linearly at infinity. Second to show that the Mabuchi energy is well approximated by the functionals controlling asymptotic Chow semistability, so that their lower boundedness gives a contradiction. And third to prove the approximation result by showing that the so-called Tian-Yau-Zelditch-Catlin expansion holds uniformly along the flow (Proposition 2.4.1), a result that uses boundedness of the curvature.

   The results of this chapter can be found in [To3].

2.1   Kähler-Ricci flow

In this section we introduce the Kähler-Ricci flow, describe our main result, and state some estimates of Perelman that will be used extensively.

   In this chapter (M, J) denotes a compact complex manifold of complex dimension n and with positive first Chern class $c_1(M) > 0$. We will often drop the reference to the complex structure J. We fix a reference metric ω cohomologous to

$c_1(M)$. The (normalized) Kähler-Ricci flow starting at ω is the evolution equation

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t, \tag{2.1}$$

with $\omega_0 = \omega$. Since the right hand side of (2.1) is a closed real $(1,1)$-form, it follows that ω_t is Kähler (as long as it exists) and cohomologous to $c_1(M)$. The fixed points of the flow are Kähler-Einstein metrics with positive scalar curvature, that is metrics ω_{KE} that satisfy

$$\mathrm{Ric}(\omega_{KE}) = \omega_{KE}. \tag{2.2}$$

If such a metric exists then it is unique up to the action of $\mathrm{Aut}^0(M)$, the connected component of the identity of the biholomorphism group of M [BM]. In general there are obstructions to the existence of Kähler-Einstein metrics, and these fall into two categories: obstructions arising from $\mathrm{Aut}^0(M)$, such as Matsushima's Theorem [Ma] or the Futaki invariant [Fu], and obstructions arising from stability. A long-standing conjecture of Yau [Y5] says that a Kähler-Einstein metric should exist precisely when the manifold, polarized by the anticanonical bundle K_M^{-1}, is stable in a suitable algebro-geometric sense. The precise notion of stability involved is called *K-polystability*, has been introduced by Donaldson [Do2]. Yau's conjecture then states that the existence of a Kähler-Einstein metric on a Fano manifold M is equivalent to K-polystability of (M, K_M^{-1}). There has been much progress on the subject, see for example [Do1, Do4, Ti3] but the conjecture is still open in general. A natural approach to this conjecture is to show that the Kähler-Ricci flow (1.1) converges to a Kähler-Einstein metric. Since we know that the flow (1.1) exists for all time the issue is to show that stability implies convergence of the flow at infinity. Despite some recent powerful estimates of Perelman [Pe, ST], this seems to be out of reach at present. On the other hand some progress has been done under the assumption that the curvature remains bounded along the flow. In [PS2, PSSW2, Sz] it is shown that if this holds and if the manifold is stable in some different analytic ways, then the flow converges to a Kähler-Einstein metric. Our main result is the following (the definitions of K-polystability and Chow semistability are in section 2.3):

Theorem 2.1.1. *Let M be a compact complex manifold with $c_1(M) > 0$. Assume that along the Kähler-Ricci flow (1.1) the sectional curvatures remain bounded*

$$|\mathrm{Rm}_t| \le C. \tag{2.3}$$

Assume moreover that (M, K_M^{-1}) is K-polystable and asymptotically Chow semistable. Then the flow converges smoothly exponentially fast to a Kähler-Einstein metric on M.

We are going to rewrite the Kähler-Ricci flow (2.1) as a parabolic complex Monge-Ampère equation as follows: since the metrics ω_t are all cohomologous to $c_1(M)$, we can write $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$, where the normalization of the potentials φ_t will be specified presently. Then (1.1) is equivalent to the following parabolic complex Monge-Ampère equation

$$(2.4) \qquad \left(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t\right)^n = e^{f_\omega - \varphi_t + \dot{\varphi}_t}\omega^n,$$

where $\dot{\varphi}_t$ denotes $\partial\varphi_t/\partial t$, f_ω is the Ricci potential defined by (1.7), (1.8), and φ_0 is a constant. The choice of this constant matters: if φ_t is the solution of (2.4) with initial value the constant φ_0, then $\tilde{\varphi}_t = \varphi_t + (\tilde{\varphi}_0 - \varphi_0)e^t$ also solves (2.4) and has initial value the constant $\tilde{\varphi}_0$. This shows that there is at most one initial value that guarantees convergence of the flow as $t \to \infty$. Notice that taking $\partial\bar{\partial}$ of the logarithm of (2.4) shows that the Ricci potential f_{ω_t} of ω_t is equal to $-\dot{\varphi}_t$ plus a time-dependent constant. We now need the following estimates of Perelman (we refer to the exposition [ST] and to [Pe] for proofs).

Theorem 2.1.2 (Perelman). *Independently of the choice of φ_0, there is a constant C that depends only on ω such that for all $t \geq 0$ we have*

$$(2.5) \qquad |f_{\omega_t}| + |\nabla f_{\omega_t}|_t + \mathrm{diam}(M, \omega_t) + |R_t| \leq C,$$

where R_t denotes the scalar curvature of ω_t. Moreover given any $r_0 > 0$ there is a constant $\kappa > 0$ that depends only on r_0 and ω such that for all $t \geq 0$, all $p \in M$ and all $0 < r < r_0$ we have

$$(2.6) \qquad \int_{B_t(p,r)} \frac{\omega_t^n}{n!} \geq \kappa r^{2n},$$

where $B_t(p,r)$ is the geodesic ball in the metric ω_t centered at p of radius r.

We then set φ_0 equal to the constant

$$(2.7) \qquad \frac{1}{V}\int_M f_\omega \frac{\omega^n}{n!} + \int_0^\infty e^{-t}\left(\int_M |\nabla\dot{\varphi}_t|_t^2 \frac{\omega_t^n}{n!}\right) dt.$$

Perelman's estimate (2.5) shows that this is finite, and the discussion above shows that it is well-defined (since $\nabla\dot{\varphi}_t = \nabla\dot{\tilde{\varphi}}_t$). An easy computation (see [PSS]) shows that with this choice we get the uniform estimate

$$(2.8) \qquad |\dot{\varphi}_t| \leq C.$$

2.2 Stability of algebraic varieties

In this expository section we define several notions of stability for algebraic varieties. We start by defining the general notion of Geometric Invariant Theory (GIT) stability, and then specialize to Chow stability and K-stability for algebraic manifolds.

More details can be found in [Mu, MFK, Th, Wa].

Let V be an $(n+1)$-dimensional complex vector space, and let $G = SL(N+1, \mathbb{C})$ act on V through a linear representation $SL(N+1, \mathbb{C}) \to GL(n+1, \mathbb{C})$. This action induces an action of G on $\mathbb{P}^n = \mathbb{P}(V)$, and we will call $\pi : V \backslash \{0\} \to \mathbb{P}^n$ the canonical projection. Then a point $x \in \mathbb{P}^n$ is called *GIT semistable* if for some (and hence for any) $\hat{x} \in V$ with $\pi(\hat{x}) = x$, the closure of the G-orbit of $\hat{x}$ doesn't contain the origin. The point x is called *GIT polystable* if the G-orbit of $\hat{x}$ is closed in V, and it is called *GIT stable* if it is polystable and if its G-stabilizer is finite. Notice that the stability of a point x depends only on its G-orbit.

We now fix $\| \cdot \|$ a $U(N+1)$-invariant metric on V, fix a lift $\hat{x}$ as above, and consider the function on G given by

$$F : g \mapsto \log \frac{\|g \cdot \hat{x}\|^2}{\|\hat{x}\|^2}.$$

Then we have the fundamental

Theorem 2.2.1 (Kempf-Ness [KeN]). *The point x is GIT semistable if and only if F is bounded below on G. It is GIT polystable if and only if F achieves a global minimum on G. And it is GIT stable if and only if F is bounded below and proper on G.*

We now apply this general theory to a concrete situation. We assume that we have a projective n-dimensional manifold M embedded in projective space $\mathbb{P}^N$ as a subvariety of degree d. We denote by $\mathbb{G}$ the Grassmanian $\mathrm{Gr}(N-n-1, N)$ of linear $(N-n-1)$-dimensional subspaces inside $\mathbb{P}^N$. The subset

$$Z_M = \{\Lambda \in \mathbb{G} \mid \Lambda \cap M \neq \emptyset\}$$

is a divisor in $\mathbb{G}$ of degree d. Since the Picard group of $\mathbb{G}$ is generated by the hyperplane bundle $\mathcal{O}_{\mathbb{G}}(1)$, it follows that $Z_M = \{s_M = 0\}$ for a holomorphic section $s_M \in H^0(\mathbb{G}, \mathcal{O}(d))$. The section s_M is unique up to multiplication by a nonzero constant, and we get a well defined point

$$\mathrm{Chow}(M) = [s_M] \in \mathbb{P}H^0(\mathbb{G}, \mathcal{O}(d)) = \mathbb{P}^m.$$

The group $G = SL(N+1, \mathbb{C})$ acts naturally on $\mathbb{P}^m$, and we can then define Chow (semi-, poly-)stability of M by requiring GIT (semi-, poly-)stability of $\mathrm{Chow}(M)$.

Let now M be a compact complex manifold equipped with an ample line bundle L. For all m sufficiently large, holomorphic sections of L^m give projective embeddings in bigger and bigger projective spaces $\mathbb{P}^{N_m-1}$. Here the number N_m can be computed from the Riemann-Roch formula (1.12). To any such embedding there is an associated Chow point $\mathrm{Chow}_m(M)$. For any fixed m all the possible embeddings

into $\mathbb{P}^{N_m-1}$ are parametrized by $G = SL(N_m, \mathbb{C})$, and changing the embedding by $g \in G$ just corresponds to letting g act on $\text{Chow}_m(M)$. We will then say that (M, L^m) is Chow (semi-, poly-)stable if $\text{Chow}_m(M)$ is (semi-, poly-)stable. Finally, we say that (M, L) is asymptotically Chow (semi-, poly-)stable if for all m sufficiently large (M, L^m) is (semi-, poly-)stable.

We now recall the Mumford numerical criterion for stability. Let's go back to the setup of the abstract GIT stability. Let $\rho : \mathbb{C}^* \to G$ be a 1-parameter subgroup (1-PSG) of G, i.e. an algebraic group homomorphism. We will say that ρ is nontrivial if it is nonconstant. For any $x \in \mathbb{P}^n$, we can take the limit $x_0 = \lim_{t \to 0} \rho(t)x$ and get a point $x_0 \in \mathbb{P}^n$ which is fixed by the action of the 1-PSG ρ. This means that ρ acts on the tautological line $\mathcal{O}_{x_0}(-1)$, and it thus has a weight $\mu \in \mathbb{Z}$. Then we have

Theorem 2.2.2 (Mumford's numerical criterion)**.** *The point x is GIT stable if and only if $\mu < 0$ for all nontrivial 1-PSG ρ. The point x is GIT semistable if and only if $\mu \leq 0$ for all nontrivial 1-PSG ρ.*

Let now (M, L) be a polarized manifold as above, fix m large so that L^m embeds M in $\mathbb{P}^{N_m-1}$, and let ρ be a 1-PSG of $G = SL(N_m, \mathbb{C})$ as before. We can then use the action of ρ to move M around inside $\mathbb{P}^{N_m-1}$ by projective transformations, and we can take the flat limit M_0 of $\rho(t)(M)$ as $t \to 0$. We thus get a family $\mathcal{M} \to \mathbb{C}$ with all fibers isomorphic to M except possibly the fiber M_0 over 0, which might be highly singular. Moreover the hyperplane bundle over $\mathbb{P}^{N_m-1}$ induces a line bundle $\mathcal{L}$ over $\mathcal{M}$ which restricts to L^m to the generic fiber. The 1-PSG then induces a $\mathbb{C}^*$ action on the pair $(\mathcal{M}, \mathcal{L})$ which makes the map $\mathcal{M} \to \mathbb{C}$ equivariant, and which induces a $\mathbb{C}^*$ action on the central fiber $(M_0, \mathcal{L}|_{M_0})$, and the weight μ can be calculated from it. This motivated Donaldson to pose the following

Definition 2.2.3. *A test configuration for (M, L) of order m is a $\mathbb{C}^*$-equivariant flat family $(\mathcal{M}, \mathcal{L})$ over $\mathbb{C}$, with $\mathcal{L}$ ample on all fibers, such that the generic fiber is isomorphic to (M, L^m). The test configuration is called a product configuration if $M_0 \cong M$, and a trivial configuration if moreover $\mathbb{C}^*$ acts trivially on $M_0 \cong M$.*

We have just seen that a 1-PSG gives rise to a test configuration, and the converse is true as well [RT]. One can then rephrase Chow stability for (M, L^m) by saying that for any nontrivial test configuration of order m, the numerical invariant μ is negative. Donaldson then defined a different notion of stability, by replacing μ with the so-called Futaki invariant F_1. If the central fiber M_0 is smooth then this is just the classical Futaki invariant of the holomorphic vector field generating the $\mathbb{C}^*$ action on M_0.

Definition 2.2.4. *A polarized manifold (M, L) is called K-semistable if for any m large and for any nontrivial test configuration for (M, L) of order m the Futaki invariant F_1 is nonpositive. It is called K-stable if F_1 is negative, and K-polystable if F_1 is nonpositive and zero only on product configurations.*

As an aside, we remark that changing the weight from μ to F_1 just amounts to changing the linearized line bundle over the Hilbert scheme in the definition of GIT stability. But unfortunately the line bundle that corresponds to K-stability is not ample over the Hilbert scheme. This means that K-stability is not a *bona fide* algebraic stability notion.

2.3 Chow semistability

In this section we link Chow semistability to a certain functional on the space of Kähler potentials. The results in this section follow from work of Donaldson [Do3] and S. Zhang [ZhS].

From now on M will be a Fano manifold, polarized by the anticanonical bundle K_M^{-1}. For each m sufficiently large the line bundle K_M^{-m} is very ample, and so choosing a basis $\{S_i\}$ of holomorphic sections in $H^0(M, K_M^{-m})$ gives an embedding of M inside $\mathbb{P}^{N_m-1} = \mathbb{P}H^0(M, K_M^{-m})^*$. Associated to this embedding there is a point $\text{Chow}_m(M)$ in the Chow variety of cycles in $\mathbb{P}^{N_m-1}$ of dimension n and degree $d = Vm^n n!$. If we let $\mathbb{G}$ be the Grassmannian of $N_m - n - 2$-planes in $\mathbb{P}^{N_m-1}$ and if we call $W = H^0(\mathbb{G}, \mathcal{O}(d))$, then the Chow variety sits inside the projective space $\mathbb{P}(W^*)$. As we have seen, the Kempf-Ness theorem [KeN] says that Chow semistability of (M, K_M^{-m}) is equivalent to the fact that the function

$$(2.9) \qquad \tau \mapsto \log \frac{\|\tau \cdot \text{Chow}_m(M)\|^2}{\|\text{Chow}_m(M)\|^2}$$

is bounded below on $SL(N_m, \mathbb{C})$. Here $\|\cdot\|$ is any norm on the vector space W which is invariant under $SU(N_m)$.

We now fix h a metric on K_M^{-1} with curvature equal to ω, and for each m we also fix $\{S_i\}$ a basis of $H^0(M, K_M^{-m})$ which is orthonormal with respect to the L^2 inner product defined by h^m, ω^n. Given a matrix $\tau \in GL(N_m, \mathbb{C})$ we define the corresponding "algebraic Kähler potential" by

$$\varphi_\tau = \frac{1}{m} \log \frac{\sum_i |\sum_j \tau_{ij} S_j|_{h^m}^2}{\sum_i |S_i|_{h^m}^2}.$$

This has the following interpretation. We use the sections $\{S_i\}$ to embed the manifold M inside $\mathbb{P}H^0(M, K_M^{-m}) = \mathbb{P}^{N_m-1}$. This carries a natural Kähler form ω_{FS}, the Fubini-Study form associated to the L^2 inner product of h^m, ω^n. If we let τ act on $\mathbb{P}^{N_m-1}$ via the natural action, then on M we have

$$(2.10) \qquad \tau^* \omega_{FS} = \omega_{FS} + m\sqrt{-1}\partial\bar{\partial}\varphi_\tau,$$

so φ_τ is a Kähler potential for $\frac{\omega_{FS}}{m}$. On the other hand, we also have that

$$(2.11) \qquad \omega = \frac{\omega_{FS}}{m} - \frac{1}{m}\sqrt{-1}\partial\bar{\partial}\log\rho_m(\omega),$$

and so the function

$$\psi_\tau = \frac{1}{m} \log \sum_i |\sum_j \tau_{ij} S_j|_{h^m}^2 = \varphi_\tau + \frac{1}{m} \log \rho_m(\omega)$$

is a Kähler potential for ω. Now if we go back to (2.9) and we choose the norm $\|\cdot\|$ suitably (see [PS1]) then a theorem of Zhang [ZhS] (see also [Pa, PS1]) gives that

$$(2.12) \qquad F^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau) = -\frac{1}{Vm(n+1)} \log \frac{\|\tau \cdot \mathrm{Chow}_m(M)\|^2}{\|\mathrm{Chow}_m(M)\|^2},$$

for all $\tau \in SL(N_m, \mathbb{C})$. We now introduce a slight variant of F^0_ω, following Donaldson [Do3]. Given a Kähler potential φ we let $h_\varphi = he^{-\varphi}$, which is a metric on K_M^{-1} with curvature equal to ω_φ. The L^2 inner product on $H^0(M, K_M^{-m})$ defined by h_φ^m, ω_φ^n, can be represented as a positive definite Hermitian matrix, with respect to the fixed basis $\{S_i\}$. Explicitly, this means that we set

$$(2.13) \qquad H_{i\bar{j},\varphi} = \int_M \langle S_i, S_j \rangle_{h_\varphi^m} \frac{\omega_\varphi^n}{n!}.$$

We then let

$$c_\varphi = \log|\det H_{i\bar{j},\varphi}|.$$

Notice that changing the basis $\{S_i\}$ does not affect c_φ, which depends only on φ and the choice of h. Also c_φ changes smoothly if φ does. We then define the functional

$$\tilde{\mathcal{L}}_m(\varphi) = \frac{c_\varphi}{N_m} - mF^0_\omega(\varphi).$$

If φ_t is a smooth path of Kähler potentials then the variation of $\tilde{\mathcal{L}}_m$ can be computed as follows: since $c_t = c_{\varphi_t}$ is independent of the choice of S_i we can pick them so that for a fixed time t we have $H_{i\bar{j},t} = \lambda_i^2 \delta_{ij}$ where the numbers λ_i are real and nonzero. Then the holomorphic sections $T_i = \frac{S_i}{\lambda_i}$ are orthonormal with respect to h_t^m, ω_t^n, and we have

$$\begin{aligned}
\frac{\partial}{\partial t} c_t &= H_t^{i\bar{j}} \frac{\partial}{\partial t} H_{i\bar{j},t} = \sum_i \frac{1}{\lambda_i^2} \int_M |S_i|_{h^m}^2 \frac{\partial}{\partial t}\left(e^{-m\varphi_t} \frac{\omega_t^n}{n!}\right) \\
&= \sum_i \frac{1}{\lambda_i^2} \int_M |S_i|_{h_t^m}^2 (-m\dot{\varphi}_t + \Delta_t \dot{\varphi}_t) \frac{\omega_t^n}{n!} \\
(2.14) \qquad &= \sum_i \int_M |T_i|_{h_t^m}^2 (-m\dot{\varphi}_t + \Delta_t \dot{\varphi}_t) \frac{\omega_t^n}{n!} \\
&= \int_M \rho_m(\omega_t)(-m\dot{\varphi}_t + \Delta_t \dot{\varphi}_t) \frac{\omega_t^n}{n!} \\
&= \int_M \dot{\varphi}_t (\Delta_t \rho_m(\omega_t) - m\rho_m(\omega_t)) \frac{\omega_t^n}{n!}.
\end{aligned}$$

So we get (cfr. Lemma 2 in [Do3])

$$(2.15) \qquad \frac{\partial}{\partial t}\tilde{\mathcal{L}}_m(\varphi_t) = \frac{1}{N_m}\int_M \dot{\varphi}_t\left(\Delta_t\rho_m(\omega_t) - m\rho_m(\omega_t) + \frac{mN_m}{V}\right)\frac{\omega_t^n}{n!}.$$

We then have the following

Proposition 2.3.1. *The pair (M, K_M^{-m}) is Chow semistable if and only if there exists a constant C, that might depend on m, such that*

$$(2.16) \qquad \tilde{\mathcal{L}}_m(\varphi) \geq -C,$$

for all Kähler potentials φ.

Proof. First we prove that Chow semistability implies the lower boundedness of $\tilde{\mathcal{L}}_m$. For each potential φ set $h_\varphi = he^{-\varphi}$ and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. Choose $\{S_i(\varphi)\}$ a basis of $H^0(M, K_M^{-m})$ which is orthonormal with respect to the L^2 inner product defined by h_φ^m, ω_φ^n. Then we can write

$$S_i(\varphi) = \sum_j \tau_{ij}S_j,$$

for some matrix $\tau = (\tau_{ij}) \in GL(N_m, \mathbb{C})$ that depends on φ. From the definition we get

$$c_\varphi = -2\log|\det\tau|,$$

and so

$$(2.17) \qquad \tilde{\mathcal{L}}_m(\varphi) = -\frac{2}{N_m}\log|\det\tau| - mF_\omega^0(\varphi).$$

We now observe that from the cocycle formula (1.10) we have

$$\begin{aligned}(2.18) \qquad F_\omega^0(\varphi) - F_\omega^0(\psi_\tau) &= -F_{\omega_\varphi}^0(\psi_\tau - \varphi)\\ &= -J_{\omega_\varphi}(\psi_\tau - \varphi) + \frac{1}{V}\int_M(\psi_\tau - \varphi)\frac{\omega_\varphi^n}{n!}\\ &\leq \frac{1}{V}\int_M(\psi_\tau - \varphi)\frac{\omega_\varphi^n}{n!},\end{aligned}$$

where we have also used the fact that $J_{\omega_\varphi} \geq 0$. On the other hand we have

$$\int_M e^{m(\psi_\tau - \varphi)}\frac{\omega_\varphi^n}{n!} = \sum_i\int_M |S_i(\varphi)|_{h^m}^2 e^{-m\varphi}\frac{\omega_\varphi^n}{n!} = N_m,$$

and so by Jensen's inequality

$$\frac{m}{V}\int_M(\psi_\tau - \varphi)\frac{\omega_\varphi^n}{n!} \leq \log(N_m/V).$$

Together with (2.18) this gives

(2.19)
$$F^0_\omega(\varphi) - F^0_\omega(\psi_\tau) \le \frac{\log(N_m/V)}{m}.$$

Then the cocycle formula (1.10) gives

(2.20)
$$F^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau) = F^0_\omega(\psi_\tau) + F^0_{\frac{\omega_{FS}}{m}}\left(-\frac{1}{m}\log\rho_m(\omega)\right),$$

and this together with (2.19) gives

(2.21)
$$-mF^0_\omega(\varphi) \ge -mF^0_\omega(\psi_\tau) - C \ge -mF^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau) - C.$$

This and (2.17) give

(2.22)
$$\tilde{\mathcal{L}}_m(\varphi) \ge -\frac{2}{N_m}\log|\det\tau| - mF^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau) - C.$$

We now set
$$\tilde{\tau} = (\det\tau)^{-\frac{1}{N_m}}\tau,$$

so now $\tilde{\tau} \in SL(N_m, \mathbb{C})$ and we notice that
$$\varphi_{\tilde{\tau}} = \varphi_\tau - \frac{2}{mN_m}\log|\det\tau|,$$

and so

(2.23)
$$-\frac{2}{N_m}\log|\det\tau| - mF^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau) = -mF^0_{\frac{\omega_{FS}}{m}}(\varphi_{\tilde{\tau}}) \ge -C,$$

by Chow semistability and (2.12). Combining (2.22) with (2.23) finally gives

(2.24)
$$\tilde{\mathcal{L}}_m(\varphi) \ge -C.$$

To show the other implication we assume that (2.24) holds and we let τ be any matrix in $SL(N_m, \mathbb{C})$. By (2.12) it is enough to prove that the function
$$-mF^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau)$$

has a uniform lower bound independently of τ. By (2.20) we have
$$-mF^0_{\frac{\omega_{FS}}{m}}(\varphi_\tau) \ge -mF^0_\omega(\psi_\tau) - C,$$

and we have
$$\tilde{\mathcal{L}}_m(\psi_\tau) = \frac{c_{\psi_\tau}}{N_m} - mF^0_\omega(\psi_\tau),$$

so we are reduced to showing that c_{ψ_τ} is bounded above independently of τ. Notice that in (2.13) if we use the basis $\tilde{S}_i = \sum_j \tau_{ij} S_j$ instead of S_i we get a different matrix $\tilde{H}_{i\bar{j},\psi_\tau}$ but its log determinant is the same. Using the definitions we have that

$$\omega_{\psi_\tau} = \frac{\tau^* \omega_{FS}}{m},$$

and

$$\tilde{H}_{i\bar{j},\psi_\tau} = \int_M \frac{\langle \tilde{S}_i, \tilde{S}_j \rangle_{h^m}}{\sum_k |\sum_l \tau_{kl} S_l|^2_{h^m}} \cdot \frac{(\tau^* \omega_{FS})^n}{m^n n!},$$

so using the arithmetic-geometric mean inequality we get

$$\frac{c_{\psi_\tau}}{N_m} \leq \log\left(\frac{1}{N_m} \int_M \frac{\sum_i |\tilde{S}_i|^2_{h^m}}{\sum_i |\tilde{S}_i|^2_{h^m}} \cdot \frac{(\tau^* \omega_{FS})^n}{m^n n!} \right),$$

and since the integral above is just the volume of M in the metric $\frac{\tau^* \omega_{FS}}{m}$, this is bounded independent of τ. $\qquad\square$

2.4   Convergence of the flow

In this section we prove Theorem 2.1.1 by relating the behavior of the functionals $\tilde{\mathcal{L}}_m$ to the Mabuchi energy.

Proof of Theorem 2.1.1. First of all we use Perelman's estimate (2.6): this together with (2.3), Perelman's diameter bound (2.5) and Theorem 4.7 of [CGT] gives a uniform lower bound for the injectivity radius of (M, ω_t) independent of t. Then Hamilton's compactness theorem [Ha2] gives that for any sequence $t_i \to \infty$ we can find a subsequence (still denoted t_i), a Kähler structure $(\omega_\infty, J_\infty)$ on the differentiable manifold M and diffeomorphisms $F_i : M \to M$ such that $\omega_i = F_i^* \omega_{t_i} \to \omega_\infty$ and $J_i = F_i^{-1}{}_* \circ J \circ F_{i*} \to J_\infty$ smoothly. We will denote by ∂_i (resp. ∂_∞) the ∂-operators of J_i (resp. J_∞). An argument of Šešum-Tian (see [ST] or [PSSW2] p. 662) shows that $(\omega_\infty, J_\infty)$ is a Kähler-Ricci soliton, and so it satisfies

(2.25) $$\mathrm{Ric}(\omega_\infty) = \omega_\infty + \sqrt{-1}\partial_\infty \bar{\partial}_\infty \psi,$$

for a smooth function ψ whose gradient is a J_∞-holomorphic vector field. Such a function ψ is only defined up to addition of a constant, but we can choose it by requiring that

$$\int_M (e^\psi - 1)\frac{\omega_\infty^n}{n!} = 0.$$

Since along the flow we have that

$$\mathrm{Ric}(\omega_t) = \omega_t + \sqrt{-1}\partial\bar{\partial} f_t,$$

where f_t is the Ricci potential of ω_t, it follows that the functions $F_i^* f_{t_i}$ will converge smoothly to ψ. In fact $F_i^* \partial \bar{\partial} f_{t_i} = \partial_i \bar{\partial}_i F_i^* f_{t_i}$ converges smoothly to $\partial_\infty \bar{\partial}_\infty \psi$, and the statement follows because of the normalizations we chose. Notice that f_t is equal to $\dot{\varphi}_t$ up to a constant.

We let $\mathcal{M}_\omega(\varphi_t)$ be the Mabuchi energy, normalized so that $\mathcal{M}_\omega(\varphi_0) = 0$. Recall that the variation of the Mabuchi energy is

$$\frac{\partial}{\partial t} \mathcal{M}_\omega(\varphi_t) = -\frac{1}{V} \int_M \dot{\varphi}_t (R_t - n) \frac{\omega_t^n}{n!},$$

while the variation of $\tilde{\mathcal{L}}_m$ was computed in (2.15) to be

$$\frac{\partial}{\partial t} \tilde{\mathcal{L}}_m(\varphi_t) = \frac{1}{N_m} \int_M \dot{\varphi}_t \left(\Delta_t \rho_m(\omega_t) - m\rho_m(\omega_t) + \frac{mN_m}{V} \right) \frac{\omega_t^n}{n!}.$$

For a fixed metric ω the Tian-Yau-Zelditch-Catlin expansion (1.15) says that as $m \to \infty$ we have

$$\rho_m(\omega) \approx m^n + \frac{R}{2} m^{n-1} + O(m^{n-2}),$$

Recalling that by Riemann-Roch (1.12) we also have

$$N_m \approx V m^n + \frac{nV}{2} m^{n-1} + O(m^{n-2}),$$

we get that for a fixed metric ω

$$\Delta \rho_m(\omega) - m\rho_m(\omega) + \frac{mN_m}{V} \approx \frac{m^n}{2}(n - R) + O(m^{n-1}).$$

We claim that this still holds uniformly along the flow.

Proposition 2.4.1. *Given any k, m_0 and $\varepsilon > 0$ there exist an $m \geq m_0$ and a $t_0 > 0$ such that for all $t \geq t_0$ we have*

$$\frac{1}{m^{n-1}} \left\| \rho_m(\omega_t) - m^n - \frac{R_t}{2} m^{n-1} \right\|_{C^k(\omega_t)} \leq \varepsilon.$$

The proof of this proposition is postponed. As above applying this with $k = 2$ and m_0, ε to be specified later, we get

$$(2.26) \qquad \frac{1}{m^n} |\Delta_t \rho_m(\omega_t)| \leq \frac{\varepsilon}{m} + \frac{|\Delta_t R_t|}{2m},$$

while Riemann-Roch implies that

$$(2.27) \qquad \left| \frac{N_m}{V m^{n-1}} - \left(m + \frac{n}{2} \right) \right| \leq \frac{C_0}{m}.$$

Proposition 2.4.1 also gives that

$$(2.28) \qquad \left| \frac{\rho_m(\omega_t)}{m^{n-1}} - m - \frac{R_t}{2} \right| \leq \varepsilon.$$

Putting together (2.26), (2.27) and (2.28) gives

$$\left| \frac{1}{m^n} \left(\Delta_t \rho_m(\omega_t) - m\rho_m(\omega_t) + \frac{mN_m}{V} \right) - \frac{1}{2}(n - R_t) \right| \leq 2\varepsilon + \frac{C_0}{m} + \frac{|\Delta_t R_t|}{2m}.$$

From the boundedness of curvature and Shi's estimates [Shi], it follows that for all $t \geq 0$ we have

$$|\Delta_t R_t| \leq C_1,$$

for a uniform constant C_1.

Using now the fact that $|\dot{\varphi}_t| \leq C_2$, which is just (2.8), and Riemann-Roch, we get

$$\left| \frac{\partial}{\partial t} \left(\tilde{\mathcal{L}}_m - \frac{\mathcal{M}_\omega}{2} \right)(\varphi_t) \right| \leq 2(2\varepsilon + C_3/m)\frac{1}{V} \int_M |\dot{\varphi}_t| \frac{\omega_t^n}{n!} \leq C_2(2\varepsilon + C_3/m).$$

In particular given any $\varepsilon_1 > 0$ we can fix ε and m_0 so that

$$C_2(2\varepsilon + C_3/m_0) \leq \varepsilon_1,$$

and moreover (M, K_M^{-m}) is Chow semistable for all $m \geq m_0$. Then Proposition 2.4.1 with the above arguments gives an $m \geq m_0$ and a t_0 such that for all $t \geq t_0$

$$\left| \frac{\partial}{\partial t} \left(\tilde{\mathcal{L}}_m - \frac{\mathcal{M}_\omega}{2} \right)(\varphi_t) \right| \leq \varepsilon_1.$$

Integrating this, we get that for all $t \geq t_0$ we have

$$(2.29) \qquad \tilde{\mathcal{L}}_m(\varphi_t) \leq \frac{\mathcal{M}_\omega(\varphi_t)}{2} + \varepsilon_1 t + C.$$

We now claim that either M already admits a Kähler-Einstein metric, or there is a constant $\gamma > 0$ such that

$$(2.30) \qquad \frac{\partial}{\partial t} \mathcal{M}_\omega(\varphi_t) \leq -\gamma,$$

for all t sufficiently large. In fact, if the above estimate fails, then we can find a sequence of times $t_i \to \infty$ such that

$$\frac{\partial}{\partial t} \mathcal{M}_\omega(\varphi_{t_i}) > -\frac{1}{i}.$$

Since

$$\frac{\partial}{\partial t}\mathcal{M}_\omega(\varphi_{t_i}) = -\frac{1}{V}\int_M |\nabla\dot\varphi_{t_i}|^2_{t_i}\frac{\omega_{t_i}^n}{n!},$$

we get

(2.31) $$\frac{1}{V}\int_M |\nabla\dot\varphi_{t_i}|^2_{t_i}\frac{\omega_{t_i}^n}{n!} < \frac{1}{i}.$$

By passing to a subsequence, we may assume that there are diffeomorphisms F_i : $M \to M$ such that $\omega_i = F_i^*\omega_{t_i} \to \omega_\infty$ a Kähler-Ricci soliton as above. Then we have

(2.32) $$\int_M |\nabla\dot\varphi_{t_i}|^2_{t_i}\frac{\omega_{t_i}^n}{n!} = \int_M |\nabla(F_i^* f_{t_i})|^2_{t_i}\frac{F_i^*\omega_{t_i}^n}{n!} \to \int_M |\nabla\psi|^2_\infty\frac{\omega_\infty^n}{n!},$$

so by (2.31) we see that the Ricci potential ψ of ω_∞ must be constant, and so ω_∞ is a Kähler-Einstein metric on J_∞ a complex structure on M of which J is a small deformation. Notice that since ω_∞ is Kähler-Einstein its cohomology class is $c_1(M, J_\infty)$. Moreover we have that the Chern classes $c_1(M, J_i) \to c_1(M, J_\infty)$ and since they are integral classes, we must have $c_1(M, J_i) = c_1(M, J_\infty)$ for all i large. So we can assume that the canonical bundles $K_{M,i}$ are all isomorphic to $K_{M,\infty}$ as complex line bundles, but with different holomorphic structures. So $(M, J_i, K_{M,i}^{-m})$ is a small deformation of $(M, J_\infty, K_{M,\infty}^{-m})$. Then the fact that M is K-polystable together with Theorem 2 of [Sz] shows that M admits a Kähler-Einstein metric. The claim is proved.

Now we assume that M does not admit a Kähler-Einstein metric, so that (2.30) holds. We now pick $\varepsilon_1 < \gamma/2$, and consequently get an m such that (2.29) holds. But we are also assuming that m is large enough, so that (M, K_M^{-m}) is Chow semistable and so by Proposition 2.3.1 we have that (2.16) holds. We can integrate (2.30), which holds for all t large, and get

(2.33) $$\mathcal{M}_\omega(\varphi_t) \le -\gamma t + C.$$

and this together with (2.29), (2.16) gives

$$-C \le \tilde{\mathcal{L}}_m(\varphi_t) \le \frac{\mathcal{M}_\omega(\varphi_t)}{2} + \varepsilon_1 t + C \le -(\gamma/2 - \varepsilon_1)t + C,$$

for all t large, which is absurd. Hence M must admit a Kähler-Einstein metric. Once we know this, results of Perelman-Tian-Zhu [TiZhu] and Phong-Song-Sturm-Weinkove [PSSW1] imply that the flow converges exponentially fast. In fact, we can avoid the analysis of Perelman-Tian-Zhu in our case: the theorem in [Sz] that we used constructs a Kähler-Einstein metric $g_{KE,i}$ on (M, J_i) for i large as a small C^∞ perturbation of a Kähler-Einstein metric $g_{KE,\infty}$ on (M, J_∞) (here we use the notation g instead of ω to emphasize that we are considering the Riemannian metrics).

In particular the Gromov-Hausdorff distance of $g_{KE,i}$ to $g_{KE,\infty}$ goes to zero as i goes to infinity. But the metrics $(F_i^{-1})^* g_{KE,i}$ are then Kähler-Einstein on (M, J) and by the Bando-Mabuchi uniqueness theorem [BM] they must be all isometric to a fixed Kähler-Einstein metric g_{KE} on (M, J). Since their Gromov-Hausdorff distance to $g_{KE,\infty}$ is arbitrarily small, it follows that the Gromov-Hausdorff distance between g_{KE} and $g_{KE,\infty}$ is zero, and so they are isometric. By Matsushima's theorem [Ma] the space of holomorphic vector fields of (M, J) is the complexification of the space of Killing vector fields of g_{KE}, but this is the same as the space of Killing vector fields of $g_{KE,\infty}$. It follows that (M, J) and (M, J_∞) have the same dimension of holomorphic vector fields. By the argument in the proof of Theorem 5 in [Sz] this implies that J must be biholomorphic to J_∞. So we have shown that there is a sequence of times t_i and diffeomorphisms F_i such that the metrics $F_i^* \omega_{t_i}$ converge smoothly to a Kähler-Einstein metric on (M, J). Then by a theorem of Bando-Mabuchi [BM] the Mabuchi energy $\mathcal{M}_\omega$ has a lower bound, and the arguments in section 2 of [PS2] show that

$$\frac{\partial}{\partial t} \mathcal{M}_\omega(\varphi_t) \to 0,$$

as $t \to \infty$. This together with the above arguments imply that given any sequence $t_i \to \infty$ we can find a subsequence, still denoted t_i, and diffeomorphisms F_i such that $F_i^* \omega_{t_i}$ converges smoothly to a Kähler-Einstein metric on (M, J). A contradiction argument then implies that the flow converges modulo diffeomorphisms: there exists ω_∞ a Kähler-Einstein metric on (M, J) and diffeomorphisms $F_t : M \to M$ such that $F_t^* \omega_t$ conveges smoothly to ω_∞. Then [PSSW1] shows that the original flow ω_t converges to a Kähler-Einstein metric exponentially fast. $\qquad \square$

Proof of Proposition 2.4.1. If the conclusion is not true, then there are a k, m_0 and $\varepsilon_0 > 0$ such that for all $m \geq m_0$ and $i \geq 1$ there is a $t_i \geq i$ such that

$$\frac{1}{m^{n-1}} \left\| \rho_m(\omega_{t_i}) - m^n - \frac{R_{t_i}}{2} m^{n-1} \right\|_{C^k(\omega_{t_i})} \geq \varepsilon_0.$$

Up to a subsequence, we may assume that $t_i \to \infty$ and that there are diffeomorphisms $F_i : M \to M$ such that $\omega_i = F_i^* \omega_{t_i}$ and $J_i = F_{i*}^{-1} \circ J \circ F_{i*}$ converge smoothly to some limit ω_∞ and J_∞ respectively. We remark that while the complex structures J_i are all biholomorphic to each other, they might not be biholomorphic to J_∞. Notice also that the C^k norms involved are invariant under diffeomorphisms, and that $F_i^* R_{t_i} = R_i$, $F_i^* \rho_m(\omega_{t_i}) = \rho_m(\omega_i)$. The Tian-Yau-Zelditch-Catlin expansion applied to ω_∞ gives that there exists a uniform constant C such that for all m we have

$$\frac{1}{m^{n-1}} \left\| \rho_m(\omega_\infty) - m^n - \frac{R_\infty}{2} m^{n-1} \right\|_{C^k(\omega_\infty)} \leq \frac{C}{m}.$$

Moreover since ω_i converges smoothly to ω_∞, it follows that the C^k norms they

define are uniformly equivalent, so we will also have

$$\frac{1}{m^{n-1}} \left\| \rho_m(\omega_\infty) - m^n - \frac{R_\infty}{2} m^{n-1} \right\|_{C^k(\omega_i)} \leq \frac{C}{m}.$$

Then we get

$$\varepsilon_0 \leq \frac{1}{m^{n-1}} \left\| \rho_m(\omega_i) - m^n - \frac{R_i}{2} m^{n-1} \right\|_{C^k(\omega_i)}$$

$$\leq \frac{1}{2} \| R_i - R_\infty \|_{C^k(\omega_i)} + \frac{1}{m^{n-1}} \| \rho_m(\omega_i) - \rho_m(\omega_\infty) \|_{C^k(\omega_i)} + \frac{C}{m}.$$

We now fix $m \geq m_0$ such that $C/m \leq \varepsilon_0/4$. Since ω_i converges smoothly to ω_∞, when i is sufficiently large we will have

$$\frac{1}{2} \| R_i - R_\infty \|_{C^k(\omega_i)} \leq \frac{\varepsilon_0}{4}.$$

We now claim that when i is large we will also have

$$\frac{1}{m^{n-1}} \| \rho_m(\omega_i) - \rho_m(\omega_\infty) \|_{C^k(\omega_i)} \leq \frac{\varepsilon_0}{4},$$

which will give a contradiction. In fact we will show that, for m fixed as above, the function $\rho_m(\omega_i)$ converges smoothly to $\rho_m(\omega_\infty)$ as i goes to infinity. This can be done in several ways, for example using the implicit function theorem, or the L^2 estimates for the $\bar\partial$ operator. We choose the first way because it is easier, though the second way gives more precise estimates. As remarked earlier we can assume that the canonical bundles $K_{M,i}$ are all isomorphic to $K_{M,\infty}$ as complex line bundles, but with different holomorphic structures. Fix h_∞ a metric on $K_{M,\infty}^{-1}$ with curvature ω_∞, and perturb it to a family of metrics h_i on $K_{M,i}^{-1}$ with curvature ω_i that converge smoothly to h_∞. Given S a holomorphic section of $K_{M,\infty}^{-m}$ we wish to perturb S to a family S_i of holomorphic sections of $K_{M,i}^{-m}$ that converge smoothly to S. Once this is done, it is clear that $\rho_m(\omega_i)$ converges smoothly to $\rho_m(\omega_\infty)$, and we are done. Since m can be assumed to be large, by Riemann-Roch we can write the dimension of $H^0(M, K_{M,\infty}^{-m})$ as an integral over M of Chern forms of ω_∞:

$$\dim H^0(M, K_{M,\infty}^{-m}) = \int_M \mathrm{ch}(K_M^{-m}) \wedge \mathrm{Todd}(M, \omega_\infty, J_\infty),$$

and we can do the same for ω_i, J_i. But since ω_i and J_i converge smoothly to ω_∞ and J_∞, it follows that the Riemann-Roch integrals are equal, and so the dimension of $H^0(M, K_{M,\infty}^{-m})$ is the same as N_m. This means that we have a sequence of elliptic operators, $\Box_{\bar\partial_i}$, acting on $\Gamma(M, K_M^{-m})$ (more precisely on Sobolev $W^{r,2}$ sections of this line bundle with $r \geq n+k+1$ to make them C^k) which converge smoothly to

$\Box_{\bar{\partial}_\infty}$ (which acts on the same space) and such that the dimension of their kernel is the same as in the limit. Then Lemma 4.3 in [Ko], which is a simple consequence of the implicit function theorem, ensures that any element in the kernel of $\Box_{\bar{\partial}_\infty}$ can be smoothly deformed to a sequence of elements in the kernel of $\Box_{\bar{\partial}_i}$. $\qquad\Box$

Chapter 3

Degenerations of Calabi-Yau metrics

In this chapter we study the way in which Ricci-flat Kähler metrics degenerate when the Kähler class approaches the boundary of the Kähler cone. In section 3.1 we provide some background and state the main results, Theorems 3.1.1 and 3.1.2. In section 3.2 we collect some facts from algebraic geometry. In section 3.3 we prove a diameter bound for Calabi-Yau metrics (Theorem 3.3.1). In section 3.4 we prove Theorem 3.1.1 and in section 3.5 we prove Theorem 3.1.2. The key points of the proof of Theorem 3.1.1 are: first, algebraic geometry provides a smooth nonnegative reference $(1,1)$ form in the limit class. The diameter bound then gives a uniform Sobolev inequality which can be used in a Moser-type iteration argument to prove an L^∞ estimate. Then a trick of Tsuji can be used for the C^2 estimates, and higher order estimates are standard. As for Theorem 3.1.2, the L^∞ bound was proved in [DP, EGZ2]. We then use the fibration structure and a Schwarz Lemma type computation to prove a Laplacian bound. A modification of Calabi's C^3 estimate then gives the fiberwise collapse, and these estimates are then used to prove the convergence results in Theorem 3.1.2. Finally in section 3.6 we give some explicit examples where our theorems apply.

The esults of this chapter are contained in [To2], except for the results in section 3.5, which are new.

3.1   Degenerations of Calabi-Yau metrics

Einstein metrics, namely metrics with constant Ricci curvature, have been an important subject of study in the field of differential geometry since the early days. The solution of the Calabi Conjecture given by Yau [Y1] in 1976 provided a very powerful existence theorem for Kähler-Einstein metrics with negative or zero Ricci curvature (the negative case was also done independently by Aubin [Au]). This produced a

number of nonhomogeneous examples of Ricci-flat manifolds. These spaces have been named Calabi-Yau manifolds by the physicists in the Eighties, and have been throughly studied in several different areas of mathematics and physics. Prompted by the physical intuition of mirror symmetry, mathematicians have studied the ways in which Calabi-Yau manifolds can degenerate when they are moving in families. In general both the complex and symplectic (Kähler) structure are changing, and the behaviour is not well understood. In this thesis we will consider the case when the complex structure is fixed, and so we will be looking at a single compact projective Calabi-Yau manifold. The Kähler class is then allowed to vary inside the ample cone. As long as the class stays inside the cone, the corresponding Ricci-flat metrics vary smoothly, but they will degenerate when the class approaches the boundary of the cone. We will try to understand this degeneration process and see what the limiting space looks like.

To introduce our results, let us fix some notation first. Let X be a compact projective Calabi-Yau manifold, of complex dimension n. This is by definition a projective manifold such that $c_1(X) = 0$ in $H^2(X, \mathbb{R})$. The real Néron-Severi space is by definition

$$N^1(X)_{\mathbb{R}} = (H^2(X, \mathbb{Z})_{free} \cap H^{1,1}(X)) \otimes \mathbb{R} = N^1(X)_{\mathbb{Z}} \otimes \mathbb{R},$$

and we assume that

$$\dim N^1(X)_{\mathbb{R}} = \rho(X) > 1.$$

This cohomology space contains $\mathcal{K}_{NS}$ the ample cone, which is open. Its closure $\overline{\mathcal{K}}_{NS}$ is the nef cone. Fix a nonzero class $\alpha \in \overline{\mathcal{K}}_{NS} \backslash \mathcal{K}_{NS}$, which exists precisely when $\rho(X) > 1$, and a smooth path $\alpha_t : [0, 1] \to \overline{\mathcal{K}}_{NS}$ such that $\alpha_t \in \mathcal{K}_{NS}$ for $t < 1$ and $\alpha_1 = \alpha$. For any $t < 1$ Yau's Theorem [Y2] gives us a unique Ricci-flat Kähler metric $\omega_t \in \alpha_t$. Fixing a smooth path of reference metrics in α_t, it can be verified that the Ricci-flat metrics ω_t vary smoothly, as long as $t < 1$. We have the following very natural

Question 1: *What is the behaviour of the metrics ω_t as $t \to 1$?*

This question has a long history: it is a special case of a problem by Yau [Y4, Y5], where the complex structure is also allowed to vary; it has been stated explicitly in this form by McMullen [McM] and Wilson [Wi2]. Physicists have also looked at this question, roughly predicting the behaviour that we will describe in Theorem 3.1.1 (see e.g., [HW]). One of the reasons that makes this question interesting is that the Ricci-flat metrics are not known explicitly, except in very few cases.

A nef class $\alpha \in N^1(X)_{\mathbb{R}}$ is called *big* if $\alpha^n > 0$. Our first main theorem addresses Question 1 in this case (see section 3.2 for definitions).

Theorem 3.1.1. *Let X be a compact projective Calabi-Yau manifold, and let $\alpha \in N^1(X)_{\mathbb{R}}$ be a big and nef class that is not ample. Then there exist a proper analytic*

subvariety $E \subset X$, which is the null locus of α, and a smooth incomplete Ricci-flat Kähler metric ω_1 on $X \backslash E$ such that for any smooth path $\alpha_t \in \mathcal{K}_{NS}$ with $\alpha_1 = \alpha$, the Ricci-flat metrics $\omega_t \in \alpha_t$ converge to ω_1 in the C^∞ topology on compact sets of $X \backslash E$. Moreover ω_1 extends to a closed positive current with continuous potentials on the whole of X, which lies in α, and which is the pullback of a singular Ricci-flat Kähler metric on a Calabi-Yau model of X obtained from the contraction map of α.

There are many interesting concrete examples of our theorem, and we will examine a few of them in section 3.6. Roughly speaking, the case when α is nef and big corresponds to a "non-collapsing" sequence of metrics, meaning that the Gromov-Hausdorff limit has the same dimension. We will now discuss Question 1 in the "collapsing" case, when the volume of α is zero. A natural example of this is to consider an algebraic fiber space $f : X \to Y$ where Y is an algebraic variety of lower dimension, and let α be the pullback of an ample divisor on Y. This picture is conjecturally always true if α is a rational class. In this case we have the following result, which is a combination of Theorems 3.5.2, 3.5.3 and 3.5.4.

Theorem 3.1.2. *Let X be a compact projective Calabi-Yau manifold and let $f : X \to Y$ be an algebraic fiber space with Y an irreducible normal algebraic variety of lower dimension. Let ω_X be a Kähler form on X and α be the pullback of an ample divisor on Y. Then there exist a proper analytic subvariety $E \subset X$ and a smooth Kähler metric ω on $Y \backslash f(E)$, such that the the Ricci-flat metrics $\omega_t \in \alpha + t\omega_X$, $0 < t \leq 1$, converge to ω as t goes to zero in the $C_{loc}^{1,\beta}$ topology of potentials on compact sets of $X \backslash E$, for any $0 < \beta < 1$. The metric ω satisfies*

$$\mathrm{Ric}(\omega) = \omega_{WP},$$

on $Y \backslash f(E)$, where ω_{WP} is a Weil-Petersson metric measuring the change of complex structures of the fibers. Moreover for any $y \in Y \backslash f(E)$ if we restrict to $X_y = f^{-1}(y)$, the metrics ω_t converge to zero in the C^1 topology of metrics, uniformly as y varies in a compact set of $Y \backslash f(E)$.

3.2 Some facts from algebraic geometry

In this section we will review some definitions and results from algebraic geometry, mainly from Mori's Program, that will be used in the proof.

Let X be a compact Calabi-Yau n-fold, that is a compact Kähler manifold of dimension n and such that $c_1(X) = 0$ in $H^2(X, \mathbb{R})$. We don't insist that X be simply connected. Notice that it follows that $aK_X \cong \mathcal{O}_X$ for some integer $a > 0$: in fact by Theorem 1 in [Be] a finite unramified $a : 1$ cover of X, $p : \tilde{X} \to X$, has trivial canonical bundle. But we have that $p^*K_X \cong K_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$ and then Lemma 16.2 in [BHPV] implies that $aK_X \cong \mathcal{O}_X$. This can be rewritten as $K_X \sim_{\mathbb{Q}} 0$ where

$\sim_{\mathbb{Q}}$ indicates $\mathbb{Q}$-linear equivalence of Cartier $\mathbb{Q}$-divisors. For the rest of this section we will assume that X is projective.

Definition 3.2.1. *A projective variety X has canonical singularities if it is normal, if rK_X is Cartier for some $r \geq 1$ and if there exists a resolution $f : Y \to X$ such that*

$$rK_Y = f^*(rK_X) + \sum_i a_i E_i,$$

where E_i ranges over all exceptional prime divisors of f, and $a_i \geq 0$.

Definition 3.2.2 (Wilson [Wi1]). *A* Calabi-Yau model Y *is a normal projective variety with canonical singularities and such that $K_Y \sim_{\mathbb{Q}} 0$.*

Let L be a nef line bundle on X, and let $\kappa(X, L)$ be its Iitaka dimension, that is

$$\kappa(X, L) = m \quad \Longleftrightarrow \quad h^0(X, kL) \sim k^m \text{ for all } k \text{ large enough}$$

and $\kappa(X, L) = -\infty$ if kL has no sections for all $k \geq 0$. We call $\nu(X, L)$ its numerical dimension, that is the largest nonnegative integer m such that there exists an $m-$cycle V such that $(L^m \cdot V) > 0$. It is always true that

$$\kappa(X, L) \leq \nu(X, L) \leq n.$$

Definition 3.2.3. *If $\kappa(X, L) = \nu(X, L)$ we say that L is* good *(or abundant). If the complete linear system $|kL|$ is base-point-free for some $k \geq 1$ we say L is* semiample.

When $|kL|$ is base-point-free, we get a morphism $\Phi_{|kL|} : X \to \mathbb{P}H^0(X, kL)^*$ that satisfies $kL = \Phi^*_{|kL|}\mathcal{O}(1)$. Notice that if L is *big*, that is $\kappa(X, L) = n$, then it is automatically good. The following is an immediate consequence of the base-point-free Theorem (Theorem 6.1.11 in [KMM]).

Theorem 3.2.4 (Kawamata). *Assume X is a projective Calabi-Yau. If L is good then it is semiample.*

The next theorem is classical (see Theorem 2.1.27 in [La]).

Theorem 3.2.5 (Iitaka). *Let L be semiample. Then there exists a surjective morphism $f : X \to Y$ where Y is a normal irreducible variety of dimension $\kappa(X, L)$, and we have $f_*\mathcal{O}_X = \mathcal{O}_Y$, and $L = f^*A$ for some ample line bundle A on Y. In fact $f = \Phi_{|kL|}$ for all k sufficiently divisible.*

We'll call f the *contraction map* of L. If $\dim(Y) = \kappa(X, L) < n$, we will also call $f : X \to Y$ an *algebraic fiber space*. There is a version of the base-point-free Theorem for Cartier $\mathbb{R}$-divisors, essentially due to Shokurov [Sho]. If D is a Cartier $\mathbb{R}$-divisor on X we say that D is semiample if there exist Y a normal irreducible projective variety, $f : X \to Y$ a surjective morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$, and A an ample $\mathbb{R}$-divisor on Y such that $D \sim_{\mathbb{R}} f^*A$. Again we will call f the contraction map of D. Then the following holds (Theorem 7.1 in [HM]):

Theorem 3.2.6. *Assume X is a projective Calabi-Yau. If D is a Cartier $\mathbb{R}$-divisor which is nef and big, then it is semiample.*

The contraction map of D is in fact also the contraction map of a suitable nef and big line bundle L (see the proof of Proposition 3.4.1). We also have the following theorem (Theorem 5.7 in [Ka1] or Theorem 1.9 in [Ka2]).

Theorem 3.2.7 (Kawamata)**.** *Assume X is a projective Calabi-Yau. Then the subcone of $\overline{\mathcal{K}}_{NS}$ given by nef and big classes is locally rational polyhedral.*

If D is a nef and big $\mathbb{R}$-divisor, then we can define its augmented base locus $\mathbf{B}_+(D)$ to be the intersection of the support of E, for all effective Cartier $\mathbb{R}$-divisor such that $D = A + E$ for some ample $\mathbb{R}$-divisor A (see [ELMNP]). Thanks to Kodaira's Lemma (Example 2.2.23 in [La]) there is always such a decomposition, and so $\mathbf{B}_+(D)$ is a proper subvariety of X. The key result that we need is Corollary 5.6 of [ELMNP2], which says that $\mathbf{B}_+(D)$ is equal to the null locus of D, that is the union of all positive-dimensional subvarieties $V \subset X$ such that $(D^{\dim V} \cdot V) = 0$.

Finally let us state a well-known conjecture (see 10.3 of Peternell's lectures in [MP]).

Conjecture 3.2.8. *Assume X is a projective Calabi-Yau. If L is a nef line bundle, then L is semiample.*

If L is effective, this conjecture follows from the log abundance conjecture. Indeed for any small rational $\varepsilon > 0$, the pair $(X, \varepsilon L)$ is klt, and the log abundance conjecture would imply that $K_X + \varepsilon L \sim_{\mathbb{Q}} \varepsilon L$ is semiample.

Notice that when X is a surface, Conjecture 3.2.8 holds: in fact if L is nef and non trivial, then $H^2(X, L) = H^0(X, K_X - L) = 0$ and by Riemann-Roch

$$\dim H^0(X, L) \geq 2 + \frac{1}{2} L \cdot L \geq 2,$$

thus L is effective. Then we can apply the log abundance theorem for surfaces (see e.g., [FM]) and get the result.

3.3 Diameter bound for Calabi-Yau metrics

In this section we will prove a uniform diameter bound for Ricci-flat Kähler metrics.

Let X be a compact Calabi-Yau n-fold, that is a compact Kähler manifold with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$. Thanks to Yau's Theorem [Y2] there is a unique Ricci-flat Kähler metric in each Kähler class. Let ω_0 be one of these metrics, which will be considered a reference metric. If ω is another Ricci-flat Kähler metric on X, we

would like to get a uniform bound for its diameter $\text{diam}(X, \omega)$. Without any further assumption on ω this is impossible: one can take X to be an elliptic curve and ω a flat metric with very large volume (and diameter). But if the cohomology class of ω lies in a fixed compact set in $H^2(X, \mathbb{R})$ then we can bound the diameter, and the bound does not degenerate as the class approaches the boundary of the Kähler cone. The result is as follows.

Theorem 3.3.1. *Let (X, ω_0) be a compact n-dimensional Ricci-flat Kähler manifold and let ω be another Ricci-flat Kähler metric such that*

$$(3.1) \qquad \int_X \omega_0^{n-1} \wedge \omega \le C_0,$$

for some constant C_0. Then the diameter of (X, ω) is bounded above by a constant that depends only on n, C_0, ω_0.

To prove Theorem 3.3.1 we need a lemma, which appears as Lemma 1.3 in [DPS].

Lemma 3.3.2. *In the above situation there exists a constant C_1 that depends only on n, C_0, ω_0, such that given any $\delta > 0$ there exists an open set $U_\delta \subset X$ such that its volume with respect to ω_0 is at least $\int_X \omega_0^n - \delta$, and any two points in U_δ can be joined by a path in X with ω-length less than $C_1 \delta^{-1/2}$.*

Proof. First notice that (3.1) gives a uniform L^1 bound on ω. Up to covering X by finitely many charts, we may assume that $X = K$ is a compact convex set in $\mathbb{C}^n$, and we will denote by g_E the Euclidean metric on K. If $x_1, x_2 \in K$, we denote by $[x_1, x_2]$ the segment joining them in K, and we compute the average of the length square of $[x_1, x_2]$ with respect to ω, when the endpoints vary. We will denote by g the Riemannian metric associated to ω. Using Fubini's Theorem and the Cauchy-Schwarz inequality we get

$$\int_{K \times K} \left(\int_0^1 \sqrt{g_{(1-s)x_1 + sx_2}(x_2 - x_1, x_2 - x_1)} ds \right)^2 dx_1 dx_2$$

$$\le \|x_2 - x_1\|_{g_E}^2 \int_0^1 \int_{K \times K} |\omega_{(1-s)x_1 + sx_2}| dx_1 dx_2 ds$$

$$(3.2) \qquad \le \text{diam}_{g_E}^2(K) 2^{2n} \left(\int_0^{\frac{1}{2}} \int_{K \times K} |\omega_{y + sx_2}| dy dx_2 ds \right.$$

$$+ \left. \int_{\frac{1}{2}}^1 \int_{K \times K} |\omega_{(1-s)x_1 + y}| dy dx_1 ds \right)$$

$$\le \text{diam}_{g_E}^2(K) 2^{2n} \text{Vol}_{g_E}(K) \|\omega\|_{L^1(K)} \le C_1,$$

where C_1 is a uniform constant, we changed variable $y = (1-s)x_1$ if $s \le \frac{1}{2}$ and $y = sx_2$ when $s \ge \frac{1}{2}$ and integrated first with respect to y. Then the set S of pairs

$(x_1, x_2) \in K \times K$ such that the length of $[x_1, x_2]$ with respect to ω is more than $(C_1/\delta)^{1/2}$ has Euclidean measure less than or equal δ: otherwise

$$\int_{K \times K} \left(\int_0^1 \sqrt{g_{(1-s)x_1 + sx_2}(x_2 - x_1, x_2 - x_1)} ds \right)^2 dx_1 dx_2$$

$$\geq \int_S \left(\int_0^1 \sqrt{g_{(1-s)x_1 + sx_2}(x_2 - x_1, x_2 - x_1)} ds \right)^2 dx_1 dx_2 \geq \frac{C_1}{\delta} \mathrm{Vol}_{g_E}(S)$$

which is more than C_1, and this contradicts (3.2). If $x_1 \in K$ we let $S(x_1)$ to be the set of the $x_2 \in K$ such that $(x_1, x_2) \in S$, and we let Q to be the set of the $x_1 \in K$ such that $\mathrm{Vol}_{g_E}(S(x_1)) \geq \frac{1}{2}\mathrm{Vol}_{g_E}(K)$ and R to be the set of $(x_1, x_2) \in S$ such that $x_1 \in Q$. Then by Fubini's Theorem

$$\delta \geq \mathrm{Vol}_{g_E}(R) = \int_R dx_2 dx_1 = \int_Q \left(\int_{S(x_1)} dx_2 \right) dx_1 \geq \frac{1}{2}\mathrm{Vol}_{g_E}(K)\mathrm{Vol}_{g_E}(Q),$$

and so $\mathrm{Vol}_{g_E}(Q) \leq \frac{2\delta}{\mathrm{Vol}_{g_E}(K)}$. We let $U_\delta = K \backslash Q$. Then U_δ is open and if $x_1, x_2 \in U_\delta$ then $\mathrm{Vol}_{g_E}(S(x_i)) < \frac{1}{2}\mathrm{Vol}_{g_E}(K)$, for $i = 1, 2$. Hence

$$\mathrm{Vol}_{g_E}((K \backslash S(x_1)) \cap (K \backslash S(x_2))) > 0,$$

and so this set is nonempty. If y belongs to it, then (x_1, y) and (x_2, y) are not in S, which means that the lengths with respect to ω of the segments $[x_1, y]$ and $[y, x_2]$ are both less than $(C_1/\delta)^{1/2}$. Concatenating these two segments we get a path from x_1 to x_2 with length less than $2(C_1/\delta)^{1/2}$. We also have that

$$\mathrm{Vol}_{\omega_0}(Q) \leq C_2\mathrm{Vol}_{g_E}(Q) \leq \frac{2C_2\delta}{\mathrm{Vol}_{g_E}(K)}.$$

Up to adjusting the constants, this is what we want. $\qquad\square$

Proof of Theorem 3.3.1. Choose $\delta \leq \min(C_1^2, 1/2 \int_X \omega_0^n)$, and pick any $p \in U_\delta$. If we denote the metric ball of ω centered at p and with radius r by $B(p, r)$, then we get that $U_\delta \subset B(p, C_2)$, where $C_2 = C_1\delta^{-1/2} \geq 1$. Then

$$\int_{B(p, C_2)} \omega_0^n \geq \int_{U_\delta} \omega_0^n \geq \frac{1}{2}\int_X \omega_0^n.$$

Moreover we have

$$\sqrt{-1}\partial\bar{\partial}\log \frac{\omega^n}{\omega_0^n} = \mathrm{Ric}(\omega_0) - \mathrm{Ric}(\omega) = 0,$$

which implies that $\omega^n = B\omega_0^n$ where B is the constant

$$(3.3) \qquad\qquad B = \frac{\int_X \omega^n}{\int_X \omega_0^n}.$$

So we get that

$$(3.4) \qquad \int_{B(p,C_2)} \omega^n \geq BC_3,$$

for some constant $C_3 > 0$ independent of ω. Since $\mathrm{Ric}(\omega) = 0$, the Bishop volume comparison Theorem and (3.4) give that

$$(3.5) \qquad \int_{B(p,1)} \omega^n \geq \frac{\int_{B(p,C_2)} \omega^n}{C_2^{2n}} \geq BC_4 > 0.$$

The following lemma is due to Yau (see e.g., Theorem I.4.1 in [SY]).

Lemma 3.3.3. *Let (M^{2n}, g) be a closed Riemannian manifold with $\mathrm{Ric}(g) \geq 0$, let $p \in M$ and $1 < R < \mathrm{diam}(X, g)$. Then*

$$\frac{R-1}{4n} \leq \frac{\mathrm{Vol}(B(p, 2(R+1)))}{\mathrm{Vol}(B(p,1))}.$$

Proof. Choose $x_0 \in \partial B(p, R)$, so that $d(x_0, p) = R$, and denote by $\rho(x) = d(x, x_0)$. The Laplacian comparison theorem gives $\Delta \rho^2 \leq 4n$ in the sense of distributions. Let $\varphi(x) = \psi(\rho(x))$ where

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq R-1, \\ \frac{1}{2}(R+1-t) & \text{if } R-1 < t < R+1, \\ 0 & \text{if } t \geq R+1. \end{cases}$$

Then φ is a nonnegative Lipschitz function supported in $B(x_0, R+1)$, and we have that

$$\int_M \varphi \Delta \rho^2 dV_g = -\int_{B(x_0, R+1)} \nabla \varphi \cdot \nabla \rho^2 dV_g = -2\int_{B(x_0, R+1)} \rho |\nabla \rho|^2 \psi'(\rho(x)) dV_g$$

$$= \int_{B(x_0, R+1) \setminus B(x_0, R-1)} \rho dV_g$$

$$\geq (R-1)\mathrm{Vol}(B(x_0, R+1) \setminus B(x_0, R-1)),$$

and also

$$\int_M \varphi \Delta \rho^2 dV_g \leq 4n \int_{B(x_0, R+1)} \varphi dV_g \leq 4n \mathrm{Vol}(B(x_0, R+1)).$$

Notice that $B(p, 1) \subset B(x_0, R+1) \setminus B(x_0, R-1)$ and so the previous two equations give

$$(R-1)\mathrm{Vol}(B(p,1)) \leq 4n\mathrm{Vol}(B(x_0, R+1)).$$

The conclusion follows from the fact that $B(x_0, R+1) \subset B(p, 2(R+1))$. $\qquad \square$

Lemma 3.3.3 gives that for any $1 < R < \operatorname{diam}(X, \omega)$ we have

$$(3.6) \qquad \frac{R-1}{4n} \leq \frac{\int_{B(p, 2(R+1))} \omega^n}{\int_{B(p,1)} \omega^n}.$$

Choosing $R = \operatorname{diam}(X, \omega) - 1$ and using (3.5), (3.3) we get

$$\operatorname{diam}(X, \omega) \leq 2 + \frac{4n}{BC_4} \int_X \omega^n = 2 + \frac{4n}{C_4} \int_X \omega_0^n,$$

which is bounded independent of ω. This completes the proof of Theorem 3.3.1 (a somewhat similar argument can be found in [Pn]). $\qquad \square$

3.4 Limits of Ricci-flat metrics: the noncollapsing case

In this section we will prove Theorem 3.1.1. The idea is to carefully set up a family of complex Monge-Ampère equations that degenerate in the limit, and prove estimates for the solutions that are uniform outside a subvariety.

We begin with the following result.

Proposition 3.4.1. *Let X be a projective Calabi-Yau n-fold, and $\alpha \in N^1(X)_{\mathbb{R}}$ a big and nef class that is not ample. Then there exists $\omega \in \alpha$ a smooth real $(1,1)$ form that is* pointwise *nonnegative and which is Kähler outside a proper analytic subvariety of X. Moreover if $\alpha_t : [0,1] \to \overline{\mathcal{K}}_{NS}$ is a smooth path such that $\alpha_t \in \mathcal{K}_{NS}$ for $t < 1$ and $\alpha_1 = \alpha$, then we can find a continuous family of Kähler forms $\beta_t \in \alpha_t$, $t < 1$, such that $\beta_t \to \omega$ in the C^∞ topology as t approaches 1.*

Proof. Let's assume first that that $\alpha = c_1(L)$ for some line bundle L, which is equivalent to requiring that $\alpha \in N^1(X)_{\mathbb{Z}}$. Now L is nef and big and so Theorem 3.2.4 implies that L is semiample, so there exists some $k \geq 1$ such that kL is globally generated. This gives a morphism $f : X \to \mathbb{P}^N$ such that $f^*\mathcal{O}(1) = kL$. If we let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^N$, then $\omega = \frac{f^*\omega_{FS}}{k}$ is a pointwise nonnegative smooth real $(1,1)$ form in the class α. Moreover ω is Kähler outside the exceptional set of f, which is a proper subvariety of X. If $\alpha \in N^1(X)_{\mathbb{Q}}$, then $k\alpha \in N^1(X)_{\mathbb{Z}}$ for some integer $k \geq 1$, and we can proceed as above. If finally $\alpha \in N^1(X)_{\mathbb{R}}$ then by Theorem 3.2.7 we know that the subcone of nef and big classes is locally rational polyhedral. Hence α lies on a face of this cone which is cut out by linear equations with rational coefficients. It follows that rational points on this face are dense, and it is then possible to write α as a linear combination of classes in $N^1(X)_{\mathbb{Q}}$ which are nef and big, with nonnegative coefficients. It is now clear that we can represent α by a smooth nonnegative form ω. Notice that all of these classes give the same contraction map $f : X \to Y$, because they lie on the same face. This map is then

also the contraction map of α, and ω is again Kähler outside the exceptional set of f.

We choose a ball $\mathcal{U}$ in $N^1(X)_{\mathbb{R}}$ centered at α, such that $\mathcal{K}_{NS} \cap \mathcal{U}$ is defined by $\{\Phi_\beta > 0\}_{1 \leq \beta \leq k}$ where the Φ_β are linear forms with rational coefficients. Since the big cone is open, up to shrinking $\mathcal{U}$ we may also assume that all the classes in $\partial \mathcal{K}_{NS} \cap \mathcal{U}$ are big. We may add some more linear forms to the Φ_β, until they define a strongly convex rational polyhedral cone C which is contained in $\overline{\mathcal{K}}_{NS} \cap \mathcal{U}$. We can then write

$$C = \left\{ \sum_{i=1}^{\ell} a_i \gamma_i \ \middle| \ a_i \geq 0 \right\},$$

where the γ_i are nef and big classes in $\mathcal{U}$. We claim that, when t is bigger than some $t_0 < 1$, it is possible to write the path α_t as $\sum_i a_i(t)\gamma_i$ where the functions $a_i(t)$ are continuous and nonnegative. Assume first that the cone C is simplicial, which means that the γ_i are linearly independent. Then the path α_t enters and eventually stays in C, and so it can be expressed uniquely as

$$(3.7) \qquad \alpha_t = \sum_{i=1}^{\ell} a_i(t)\gamma_i,$$

where the $a_i(t)$ are smooth and nonnegative, $t_0 \leq t \leq 1$. If on the other hand C is not simplicial, it can be written as a finite union of simplicial subcones that intersect only along faces, and that are spanned by some linearly independent subsets of the γ_i. On any time interval when α_t belongs to the interior of a simplicial cone, the coefficients $a_i(t)$ in (3.7) vary smoothly, and on a common face of two simplicial cones the coefficients agree, hence the $a_i(t)$ vary continuously when $t_0 \leq t < 1$. Moreover since we only have finitely many simplicial subcones, we see that as $t \to 1$ the $a_i(t)$ converge to the coefficients of α_1 in any of the simplicial cones that contain it, and so the $a_i(t)$ are continuous on the whole interval $t_0 \leq t \leq 1$.

By the first part of the proof we know that we can choose $\delta_i \in \gamma_i$ a smooth nonnegative representative, for all i. Choose a smooth function $\varepsilon(t) : [t_0, 1] \to \mathbb{R}$ that is positive on $[t_0, 1)$ and $\varepsilon(1) = 0$, and that is small enough so that the classes $\tilde{\alpha}_t = \alpha_t - \varepsilon(t)\alpha_{t_0}$ are ample for all $t_0 \leq t < 1$. Then the new path $\tilde{\alpha}_t$ is also converging to α as $t \to 1$, and by the previous claim we can write

$$\tilde{\alpha}_t = \sum_{i=1}^{\ell} \tilde{a}_i(t)\gamma_i,$$

where $\tilde{a}_i(t)$ is a continuous nonnegative function, for all i. Then the smooth $(1,1)$ forms

$$\tilde{\beta}_t := \sum_{i=1}^{\ell} \tilde{a}_i(t)\delta_i$$

are nonnegative representatives of $\tilde{\alpha}_t$ that vary continuously in t. When t approaches 1, the forms $\tilde{\beta}_t$ converge in the C^∞ topology to a smooth nonnegative form $\tilde{\omega}$ representing α. If χ is a Kähler form in α_{t_0}, then the forms $\beta_t = \tilde{\beta}_t + \varepsilon(t)\chi$ defined on $[t_0, 1)$ are Kähler, represent α_t and converge to $\tilde{\omega}$ as $t \to 1$. Up to replacing ω by $\tilde{\omega}$, this gives the desired family of forms on $[t_0, 1)$. It is very easy to extend the family β_t on the whole $[0, 1)$, and since we're not going to use this, we leave the proof to the reader. $\qquad \square$

Of course, a similar statement holds if we are given a sequence of ample classes α_i converging to α, instead of a path.

Let us now recall some notation and facts from analytic geometry. If X is any complex manifold and ω is a Hermitian form on X, we'll denote by $PSH(X, \omega)$ the set of all upper semicontinuous (usc) functions $\varphi : X \to [-\infty, +\infty)$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a positive current. In the case when (X, ω) is Kähler, then all Kähler potentials for ω belong to $PSH(X, \omega)$. A fundamental result by Bedford-Taylor [BT] says that the Monge-Ampère operator $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n$ is well defined whenever $\varphi \in PSH(X, \omega)$ is locally bounded. Let's also recall the definition of a singular Kähler metric [EGZ1] on a (possibly singular) algebraic variety X. This is given by specifying its Kähler potentials on an open cover (U_i) of X, that are usc functions $\varphi_i : U_i \to [-\infty, +\infty)$ with the following property: φ_i extends to a plurisubharmonic function on an open set $V_i \subset \mathbb{C}^m$ where $U_i \subset V_i$ is a local embedding. We refer the reader to section 7 of [EGZ1] for the definition of a singular Ricci-flat Kähler metric and for a proof that they always exist on Calabi-Yau models. With these facts in mind, we can now give the

Proof of Theorem 3.1.1. Proposition 3.4.1 gives us $\omega \in \alpha$ a smooth nonnegative representative, and $\beta_t \in \alpha_t$ continuously varying Kähler forms, when $t < 1$, such that $\beta_t \to \omega$ as $t \to 1$. As in the proof of Proposition 3.4.1, there is a contraction map $f : X \to Y$ such that Y is a normal irreducible projective variety, f is birational and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Moreover ω is the pullback of a (singular) Kähler metric on Y, and it is Kähler outside the exceptional set of f. Then setting $D_0 = 0$ as Cartier divisors on Y, we have $aK_X = f^*D_0$ for some integer $a > 0$, so

$$f_*(aK_X) = D_0 = 0$$

holds as Weil divisors, but since f is birational we also have $f_*(aK_X) = aK_Y$ (as Weil divisors), hence aK_Y is Cartier and is equal to zero. So we have $f^*K_Y = K_X$ as $\mathbb{Q}$-divisors, which implies that Y has at most canonical singularities and is a Calabi-Yau model (see also Corollary 1.5 of [Ka1]).

Denote by Ω the smooth volume form on X given by

$$\Omega = \frac{\omega_0^n}{\int_X \omega_0^n},$$

which satisfies $\int_X \Omega = 1$. We can write

$$\Omega = F\omega^n,$$

where $F \in L^1(\omega^n)$, $F > 0$. The following argument to show that actually $F \in L^p(\omega^n)$ for some $p > 1$ is similar to Lemma 3.2 in [EGZ1]. First of all $1/F$ is smooth, nonnegative, and vanishes precisely on the exceptional set of f. Fixing local coordinates (z^i) on a polydisc $D \subset X$ and a local embedding $G : f(D) \to \mathbb{C}^m$, we see that $1/F$ is comparable to

$$\left| \frac{\partial G}{\partial z^1} \wedge \cdots \wedge \frac{\partial G}{\partial z^n} \right|^2$$

on D. But this is in turn comparable to

$$\sum_{i=1}^r |g_i|^2,$$

where the g_i are holomorphic functions on D, and so $F^\varepsilon \in L^1(D, \Omega)$ for some small $\varepsilon > 0$ that depends on the vanishing orders of the g_i. Then

$$(3.8) \qquad \int_D F^{1+\varepsilon}\omega^n = \int_D F^\varepsilon \Omega < \infty.$$

The compactness of X gives $F \in L^{1+\varepsilon}(\omega^n)$, and so we can apply Theorem 2.1 and Proposition 3.1 of [EGZ1] or Theorem 1.1 in [Zh1] (which rely on the seminal work of Kołodziej [Koł]) to get a unique bounded $\varphi \in PSH(X, \omega)$ such that

$$(3.9) \qquad (\omega + \sqrt{-1}\partial\bar\partial\varphi)^n = \alpha^n \Omega,$$

and $\sup_X \varphi = 0$. We then embed Y into projective space and extend ω to a Kähler form in a neighborhood of Y as in Proposition 3.3 of [DPa] or in Theorem 4 of [De]. Composing the embedding with f we get a morphism which is birational with the image, with connected fibers, and we can then apply Theorem 1.1 in [Zh1] (see also [Zh2] and Remark 5.2 in [DZ]) and get that φ is continuous. Moreover we can see that φ descends to a function on Y: if V is a fiber of f, the restriction of φ to V is a plurisubharmonic function, because $\omega|_V = 0$. Desingularizing V and applying the maximum principle we see that $\varphi|_V$ has to be constant, and so φ descends to Y. Since ω by construction is the pullback of a (singular) Kähler form on Y, we see that $\omega + \sqrt{-1}\partial\bar\partial\varphi$ is a singular Ricci-flat metric on Y, in the terminology of [EGZ1]. On X, the closed positive current $\omega_1 = \omega + \sqrt{-1}\partial\bar\partial\varphi$ clearly lies in the class α and has continuous potentials. Intuitively, our goal is to get estimates in the open set where ω is positive. This can be done rigorously in the following way, which was first used by H. Tsuji [Ts]. Since α is nef and big, by Kodaira's lemma (Example

2.2.23 in [La]) there exists E an effective Cartier $\mathbb{R}$-divisor such that for all $\varepsilon > 0$ small enough, $\alpha - \varepsilon E = \kappa_\varepsilon$ is Kähler. We'll show that φ is smooth on $X \backslash E$, and so ω_1 is a smooth Ricci-flat metric there, and that the Ricci-flat metrics ω_t converge to ω_1 in the C^∞ topology on compact sets of $X \backslash E$. Notice that the metric ω_1 on $X \backslash E$ cannot be complete, since its diameter is finite by Theorem 3.3.1. Our argument is very similar to the proof of Theorem 3.5 in [EGZ1] (see also [Y2]). Once this is proved, we can repeat the argument for any other E given by Kodaira's lemma, and by uniqueness we see that ω_1 is smooth off E', the intersection of the supports of all such E. But this is equal to the augmented base locus $\mathbf{B}_+(\alpha)$, just from its definition, and we have already mentioned the fact that this is equal to the null locus of α.

We now fix once and for all an $\varepsilon > 0$ small enough so that Kodaira's lemma holds. First of all notice that the classes $\alpha_t - \varepsilon E = \kappa_\varepsilon^t$ are all Kähler when t is close to 1. Choose a Kähler form $\chi_\varepsilon \in \kappa_\varepsilon$, let $\sigma \in H^0(X, \mathcal{O}_X(E))$ be the canonical section, and fix a Hermitian metric $|\cdot|$ on E such that the following Poicaré-Lelong equation holds

$$(3.10) \qquad \omega - \varepsilon[E] = \chi_\varepsilon - \varepsilon\sqrt{-1}\partial\overline{\partial}\log|\sigma|,$$

where $[E]$ denotes the current of integration on E. Then we have

$$\beta_t - \varepsilon[E] = \chi_\varepsilon + (\beta_t - \omega) - \varepsilon\sqrt{-1}\partial\overline{\partial}\log|\sigma|,$$

and $\chi_\varepsilon^t = \chi_\varepsilon + (\beta_t - \omega)$ is Kähler for t close to 1. There are smooth functions φ_t solutions of

$$(3.11) \qquad \omega_t^n = (\beta_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = \alpha_t^n\Omega,$$

where the positive constants α_t^n approach α^n as t goes to 1, and $\sup_X \varphi_t = 0$. We now derive a uniform L^∞ estimate for φ_t. Since the Ricci-flat metrics ω_t have a uniform upper bound on the diameter by Theorem 3.3.1 and a uniform positive lower bound for the volume $\int_X \omega_t^n$, classical results of Croke [Cr], Li [Li] and Li-Yau [LY] give uniform upper bounds for the Sobolev and Poincaré constants of ω_t. We temporarily modify the normalization of φ_t by requiring that $\int_X \varphi_t\omega_t^n = 0$ and we're going to show that $|\varphi_t| \leq C$. This will then hold for the original φ_t as well, with perhaps a bigger constant. We employ a Moser iteration argument in the following

way, inspired by [Y2]. For any $p > 1$ we compute

(3.12)
$$
\begin{aligned}
\int_X |\nabla(\varphi_t |\varphi_t|^{\frac{p-2}{2}})|^2_{\omega_t} \omega_t^n &= \frac{p^2}{4} \int_X |\varphi_t|^{p-2} |\nabla \varphi_t|^2_{\omega_t} \omega_t^n \\
&= \frac{np^2}{4} \int_X |\varphi_t|^{p-2} \partial \varphi_t \wedge \overline{\partial} \varphi_t \wedge \omega_t^{n-1} \\
&\leq \frac{np^2}{4} \int_X |\varphi_t|^{p-2} \partial \varphi_t \wedge \overline{\partial} \varphi_t \wedge \left(\sum_{i=0}^{n-1} \omega_t^{n-1-i} \wedge \beta_t^i \right) \\
&= -\frac{np^2}{4(p-1)} \int_X \varphi_t |\varphi_t|^{p-2} \partial \overline{\partial} \varphi_t \wedge \left(\sum_{i=0}^{n-1} \omega_t^{n-1-i} \wedge \beta_t^i \right) \\
&= \frac{np^2}{4(p-1)} \int_X \varphi_t |\varphi_t|^{p-2} (\beta_t - \omega_t) \wedge \left(\sum_{i=0}^{n-1} \omega_t^{n-1-i} \wedge \beta_t^i \right) \\
&= \frac{np^2}{4(p-1)} \int_X \varphi_t |\varphi_t|^{p-2} (\beta_t^n - \omega_t^n) \\
&\leq Cp \int_X |\varphi_t|^{p-1} \omega_t^n,
\end{aligned}
$$

where we used (3.11) in the last inequality. Using the uniform Sobolev and Poincaré inequalities for ω_t and iterating in the same way as in (1.25), (1.27), (1.28), (1.30) we get the required L^∞ bound $|\varphi_t| \leq C$. Notice that such a bound is also proved in a more general setting in [EGZ1, Zh1], using sophisticated tools from pluripotential theory.

Outside E we now have

$$
\beta_t = \chi_\varepsilon^t - \varepsilon \sqrt{-1} \partial \overline{\partial} \log |\sigma|,
$$

so that the functions $\psi_t = \varphi_t - \varepsilon \log |\sigma|$ solve

(3.13)
$$
(\chi_\varepsilon^t + \sqrt{-1} \partial \overline{\partial} \psi_t)^n = \alpha_t^n \Omega = e^{F_\varepsilon^t} (\chi_\varepsilon^t)^n
$$

there, for some appropriate smooth functions F_ε^t, defined on the whole of X. As t approaches 1, the Kähler forms χ_ε^t are uniformly bounded in the smooth topology (with eigenvalues bounded away from 0 uniformly), and so are the functions F_ε^t. Yau's second order estimates [Y2] for the Monge-Ampère equation (3.13) give

(3.14) $\Delta_t'(e^{-A\psi_t}(n + \Delta_t \psi_t)) \geq e^{-A\psi_t} \left(-C_1 - C_2(n + \Delta_t \psi_t) + (n + \Delta_t \psi_t)^{\frac{n}{n-1}} \right),$

where A, C_1 and C_2 are uniform positive constants, Δ_t is the Laplacian of χ_ε^t and Δ_t' is the Laplacian of $\chi_\varepsilon^t + \sqrt{-1} \partial \overline{\partial} \psi_t$. Now notice that on $X \backslash E$ we have

$$
e^{-A\psi_t}(n + \Delta_t \psi_t) = |\sigma|^{A\varepsilon} e^{-A\varphi_t}(n + \Delta_t \varphi_t - \varepsilon \Delta_t \log |\sigma|),
$$

and

$$|\Delta_t \log |\sigma|| \leq C,$$

for some uniform constant C. Hence the function $e^{-A\psi_t}(n + \Delta_t \psi_t)$ goes to zero when we approach E, and so its maximum will be attained. The maximum principle applied to (3.14) then gives

$$n + \Delta_t \psi_t \leq Ce^{A(\psi_t - \inf_{X \setminus E} \psi_t)},$$

on the whole of $X \setminus E$. But noticing that $\inf_{X \setminus E} \psi_t \geq \inf_X \varphi_t - C$ for a uniform constant C, and recalling that $|\varphi_t| \leq C_0$, we get

$$n + \Delta_t \varphi_t \leq C + n + \Delta_t \psi_t \leq C(1 + |\sigma|^{-A\varepsilon}).$$

This gives uniform interior C^2 estimates of φ_t and ψ_t on compact sets of $X \setminus E$. Then the Harnack estimate of Evans-Krylov gives uniform $C^{2,\gamma}$ estimates, for some $0 < \gamma < 1$, and a standard bootstrapping argument gives uniform $C^{k,\gamma}$ estimates for all $k \geq 2$, on compact sets of $X \setminus E$, independent of $t < 1$. Thus the family (φ_t) is precompact $C^{k,\gamma'}(X \setminus E)$ for any $0 < \gamma' < \gamma$, and any limit point ψ belongs to $PSH(X \setminus E, \omega)$, it satisfies

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = \alpha^n \Omega$$

on $X \setminus E$, and is bounded near E. Hence ψ extends to a bounded function in $PSH(X, \omega)$ and the above Monge-Ampère equation holds on X because the Borel measure $(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n$ doesn't charge the analytic set E. Then by the uniqueness part of Theorem 2.1 of [EGZ1], we must have $\psi = \varphi$. This implies that $\varphi_t \to \varphi$ in C^∞ on compact sets of $X \setminus E$, and that φ is smooth there. $\qquad\square$

Let us briefly compare this with some previous results. Using the diameter bound (Theorem 3.3.1), we can apply the Bishop volume comparison Theorem and get that for any point $p \in X$ and any $r > 0$, $t < 1$

$$(3.15) \qquad \int_{B_t(p,r)} \omega_t^n \geq r^{2n} \frac{\int_X \omega_t^n}{\text{diam}(X, \omega_t)^{2n}} \geq cr^{2n},$$

where $c > 0$ is a uniform constant. A well-known computation in Chern-Weil theory gives

$$\frac{1}{n(n-1)} \int_X \|\text{Rm}_t\|_t^2 \omega_t^n = \int_X c_2(X, \omega_t) \wedge \omega_t^{n-2} = c_2(X) \cdot \alpha_t^{n-2} \leq C,$$

where Rm_t is the Riemann curvature tensor of ω_t and $c_2(X, \omega_t)$ is the second Chern form of ω_t. If $n = 2$ we can thus apply Theorem C of [An], Theorem 5.5 of [BKN] or Proposition 3.2 of [Ti2] and get that a subsequence of (X, ω_t) converges to an Einstein orbifold with isolated singularities in the Gromov-Hausdorff topology, and

also in the C^∞ topology on compact sets outside the orbifold points. If $n > 2$ these theorems require a uniform bound on

$$\int_X \|\mathrm{Rm}_t\|_t^n \omega_t^n,$$

which in general can not be expressed in terms of topological data as above. Instead when $n > 2$ we apply a general theorem of Gromov [Gr] that says that any sequence of compact Riemannian manifolds of dimension $2n$ with diameter bounded above and Ricci curvature bounded below, has a subsequence that converges in the Gromov-Hausdorff topology to a compact length space. Thus a subsequence of (X, ω_t) converges to a compact metric space Y, and Theorem 1.15 in [CCT] says that Y is a complex manifold outside a rectifiable set $R \subset Y$ of real Hausdorff codimension at least 4. Moreover their Theorem 9.1 gives supporting evidence that R should in fact be a complex subvariety of Y.

On the other hand our Theorem 3.1.1 gives the convergence of the whole sequence of metrics, and not just of a subsequence, and the limit metric is uniquely determined by the class α. When $n > 2$ the convergence we get is stronger than Gromov-Hausdorff convergence, but it only happens outside the singular set E. Also we see precisely what the limit space Y is, namely the Calabi-Yau model of X obtained from the contraction map of α. It has canonical singularities, so its singular set is a subvariety of complex codimension at least 2, and when $n = 2$ canonical singularities are precisely rational double points, that are of orbifold type. We will discuss the case $n = 2$ with more details in section 3.6.

3.5 Limits of Ricci-flat metrics: the collapsing case

In this section we will prove Theorem 3.1.2.

Let X be a projective manifold with $c_1(X) = 0$, that is a Calabi-Yau manifold, and we will fix ω_X a Kähler form. We assume that we have a class $\alpha \in \partial \mathcal{K}_{NS}$ which is nef but not big, and moreover that it is a rational class. This means that a multiple of α is equal to $c_1(L)$ for some nef line bundle L over X, with Iitaka dimension $\kappa(X, L) = m < n$. If Conjecture 3.2.8 holds, then L is semiample, and so there is an algebraic fiber space $f : X \to Y$, with Y an m-dimensional normal variety, and $\alpha = f^* A_\alpha$ for some ample divisor A_α on Y. From now on we will assume that L is semiample, and so we have an algebraic fiber space $f : X \to Y$. We can then find proper subvarieties $S \subset X$ and $f(S) \subset Y$ such that Y is smooth away from $f(S)$, and $f : X \backslash S \to Y \backslash f(S)$ is a smooth submersion. Lemma 10.6 in [I] shows that a generic fiber of f is also a Calabi-Yau manifold. We will denote by ω_0 the pullback of the Fubini-Study metric on X, and by ω_Y the pullback to Y. Then ω_0 is a smooth nonnegative form representing α, while ω_Y is a smooth Kähler

metric on the regular part of Y, and $\omega_0 = f^*\omega_Y$ on $X\backslash S$. Moreover on X we have

$$\omega_0^k \wedge \omega_X^{n-k} = 0,$$

for $m + 1 \leq k \leq n$, and

$$(3.16) \qquad \omega_0^m \wedge \omega_X^{n-m} = H\omega_X^n,$$

where the smooth non-negative function H vanishes precisely on S and is such that $H^{-\gamma}$ is in L^1 for some small $\gamma > 0$. This is because H is locally comparable to the modulus square of the Jacobian of f, which is locally given as the sum of the modulus square of holomorphic functions that are not identically zero. In particular it follows that

$$\int_X \omega_0^m \wedge \omega_X^{n-m} > 0.$$

In this setting we look at the Kähler forms $\omega_t = \omega_0 + t\omega_X$ for $0 < t \leq 1$, and we call $\tilde{\omega}_t$ the unique Ricci-flat Kähler metric cohomologous to ω_t for $t > 0$, whose existence is guaranteed by Yau's theorem [Y2]. Explicitly, if we denote by χ the unique Ricci-flat Kähler metric cohomologous to ω_1 then Yau's theorem guarantees the existence of smooth functions φ_t for $0 < t \leq 1$ so that $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$, $\sup_X \varphi_t = 0$ and

$$(3.17) \qquad (\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = a_t\chi^n,$$

where

$$a_t = \frac{\int_X \omega_t^n}{\int_X \omega_1^n}.$$

Notice that as t approaches zero, the constants a_t behave like

$$(3.18) \qquad \binom{n}{m}\frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_1^n}t^{n-m} + O(t^{n-m+1}).$$

We can define a smooth function E by the relation $\chi^n = e^E\omega_X^n$, and then we can write (3.17) as

$$(3.19) \qquad (\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = c_t t^{n-m}e^E\omega_X^n,$$

where the constant c_t is bounded away from zero and infinity as t goes to zero. Equation (3.19) has been studied for example in [KTi] where a uniform L^∞ bound on φ_t was conjectured. This was then proved independently by Demailly and Pali [DP] and by Eyssidieux, Guedj and Zeriahi [EGZ2]:

Theorem 3.5.1 ([DP, EGZ2]). *There is a constant C that depends only on X, E, ω_X, ω_0 such that for all $0 < t \leq 1$ we have*

$$(3.20) \qquad \|\varphi_t\|_{L^\infty} \leq C.$$

Our goal is to show higher order estimates for φ_t which are uniform on compact sets of $X \backslash S$. Notice that since

$$0 < \text{tr}_{\omega_X} \tilde{\omega}_t = \text{tr}_{\omega_X} \omega_t + \Delta_{\omega_X} \varphi_t,$$

and since $\text{tr}_{\omega_X} \omega_t$ is uniformly bounded, we always have a uniform lower bound for $\Delta_{\omega_X} \varphi_t$.

The following are the main results of this section, and together they imply Theorem 3.1.2. The estimates that follow contain the function $|\sigma|_h^\lambda$. This is a nonnegative function on Y, which will be defined later on, and whose zero set is a subvariety that contains $f(S)$. Moreover, we will have freedom in choosing σ, and the common zero locus of all the possible choices is precisely $f(S)$.

Theorem 3.5.2 (Laplacian bound). *There are constants A, B, C that depend only on the fixed data, so that on $X \backslash S$ and for any $0 < t \leq 1$ we have*

$$(3.21) \qquad \frac{t}{Ce^{Ae^{B|\sigma|_h^{-\lambda}}}} \omega_X \leq \tilde{\omega}_t \leq Ce^{Ae^{B|\sigma|_h^{-\lambda}}} \omega_X.$$

In particular the Laplacian $\Delta_{\omega_X} \varphi_t$ is bounded uniformly on compact sets of $X \backslash S$, independent of t.

Theorem 3.5.3 (Fiberwise collapse). *Given any $y \in Y \backslash f(S)$ denote by X_y the fiber $f^{-1}(y)$, by ω_y the Kähler form $\omega_X|_{X_y}$ and by $\tilde{\omega}_y$ the restriction of the Ricci-flat metric $\tilde{\omega}_t|_{X_y}$. Then there are constants A, B, C that only depend on the fixed data, so that on the fiber X_y and any $0 < t \leq 1$ we have*

$$(3.22) \qquad \frac{t}{Ce^{Ae^{B|\sigma(y)|_h^{-\lambda}}}} \omega_y \leq \tilde{\omega}_y \leq tCe^{Ae^{B|\sigma(y)|_h^{-\lambda}}} \omega_y,$$

$$(3.23) \qquad |\nabla \tilde{\omega}_y|_{\omega_y}^2 \leq t^{1/2} Ce^{Ae^{B|\sigma(y)|_h^{-\lambda}}},$$

where ∇ is the covariant derivative of ω_y. In particular the metrics $\tilde{\omega}_y$ converge to zero in $C^1(\omega_y)$ as t approaches zero, uniformly as y varies in a compact set of $Y \backslash f(S)$.

Theorem 3.5.4. *As $t \to 0$ the Ricci-flat metrics $\tilde{\omega}_t$ on $X \backslash S$ converge to a smooth Kähler metric ω on $Y \backslash f(S)$, in the $C^{1,\beta}_{loc}$ topology of Kähler potentials for any $0 < \beta < 1$. The metric ω satisfies*

$$\text{Ric}(\omega) = \omega_{WP},$$

on $Y \backslash f(S)$, where ω_{WP} is the pullback of the Weil-Petersson metric from the moduli space of the Calabi-Yau fibers, and it measures the change of complex structures of the fibers.

To prove Theorems 3.5.2, 3.5.3 and 3.5.4 we need a few lemmas.

Lemma 3.5.5. *There is a uniform constant C so that for all $0 < t \leq 1$ we have*

$$\text{(3.24)} \qquad tr_{\tilde{\omega}_t}\omega_0 \leq C.$$

Proof. Recall that we are assuming that $\omega_0 = cf^*\omega_{FS}$ where $f : X \to \mathbb{P}^N$ is a holomorphic map. We can then use the Chern-Lu formula that appears in Yau's Schwarz lemma computation ([Y3], [To1]) and get

$$\Delta_{\tilde{\omega}_t} \log \text{tr}_{\tilde{\omega}_t}\omega_0 \geq -A\text{tr}_{\tilde{\omega}_t}\omega_0,$$

for a uniform constant A. Noticing that

$$\Delta_{\tilde{\omega}_t}\varphi_t = n - \text{tr}_{\tilde{\omega}_t}\omega_t \leq n - \text{tr}_{\tilde{\omega}_t}\omega_0,$$

we see that

$$\text{(3.25)} \qquad \Delta_{\tilde{\omega}_t}(\log \text{tr}_{\tilde{\omega}_t}\omega_0 - (A+1)\varphi_t) \geq \text{tr}_{\tilde{\omega}_t}\omega_0 - n(A+1).$$

Then the maximum principle applied to (3.25), together with the estimate (3.20), gives (3.24). $\qquad \square$

The next lemma, which gives a Sobolev constant bound, is essentially due to Michael and Simon [MS].

Lemma 3.5.6. *There is a uniform constant C so that for any $0 < t \leq 1$, for any $y \in Y\backslash f(S)$ and for any $u \in C^\infty(X_y)$ we have*

$$\text{(3.26)} \qquad \left(\int_{X_y} |u|^{\frac{2(n-m)}{n-m-1}}\omega_y^{n-m}\right)^{\frac{n-m-1}{n-m}} \leq C\int_{X_y}(|\nabla u|^2_{\omega_y} + |u|^2)\omega_y^{n-m}.$$

Proof. For any $y \in Y\backslash f(S)$ the fiber X_y is a smooth $(n-m)$-dimensional complex submanifold of X. Since X is Kähler, it follows that X_y is a minimal submanifold, and so it has vanishing mean curvature vector. We then use the Nash embedding theorem to isometrically embed (X, ω_X) into Euclidean space, and so we have an isometric embedding $X \to \mathbb{R}^N$. The length of the mean curvature vector of the composite isometric embedding $X_y \to X \to \mathbb{R}^N$ is then uniformly bounded independent of y, since it depends only on the second fundamental form of $X \to \mathbb{R}^N$. Then (3.26) follows from the uniform Sobolev inequality of [MS]. Notice that they prove an L^1 Sobolev inequality, but this implies the stated L^2 Sobolev inequality thanks to the Hölder inequality. $\qquad \square$

One can easily avoid the Nash embedding theorem by using a partition of unity to reduce directly to the Euclidean case, but the above proof is perhaps cleaner.

We note here that the volume of X_y with respect to ω_y, $\int_{X_y} \omega_y^{n-m}$, is a homological constant independent of $y \in Y\backslash f(S)$, up to scaling ω_X we may assume that it is equal to 1. The next step is to prove a diameter bound for ω_y:

Lemma 3.5.7. *There is a uniform constant C so that for any $0 < t \le 1$, for any $y \in Y \backslash f(S)$ we have*

$$(3.27) \qquad \qquad \mathrm{diam}(X_y, \omega_y) \le C.$$

Proof. As above we embed (X, ω_X) isometrically into $\mathbb{R}^N$ and we get that the length of the mean curvature vector of the composite isometric embedding $X_y \to X \to \mathbb{R}^N$ is then uniformly bounded independent of y. We can then apply Theorem 1.1 of [Tp] and get the required diameter bound.

Alternatively, first one observes that (3.26) implies that there are uniform constants r_0, κ so that that geodesic balls in X_y of radius $r < r_0$ have volume at least $\kappa r^{2(n-m)}$ (Lemma 3.2 in [He]). Since the total volume of X_y is constant equal to 1, an elementary argument gives the required diameter bound. $\qquad \square$

The next step is to prove a Poincaré inequality for the restricted metric ω_y. This time the constant will not be uniformly bounded, but it will blow up like a power of $\frac{1}{H}$. To this end, we first estimate the Ricci curvature of ω_y. Fix a point $y \in Y \backslash f(S)$ and choose local coordinates $z^1, \ldots, z^{n-m}$ on the fiber X_y, which extend locally to coordinates in a ball in X. Then pick local coordinates $w^{n-m+1}, \ldots, w^n$ near $y \in Y \backslash f(S)$, so that $z^1, \ldots, z^{n-m}, z^{n-m+1} = f^*(w^{n-m+1}), \ldots, z^n = f^*(w^n)$ give local holomorphic coordinates on X. We can also assume that at the point y the metric ω_Y is the identity. At any fixed point of X_y we then have

(3.28)

$$\mathrm{Ric}(\omega_y) = -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_y^{n-m}}{dz^1 \wedge \cdots \wedge d\bar{z}^{n-m}} = -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_X^{n-m} \wedge \omega_0^m}{dz^1 \wedge \cdots \wedge d\bar{z}^n}$$

$$= -\sqrt{-1} \partial \bar{\partial} \log H - \sqrt{-1} \partial \bar{\partial} \log \frac{\omega_X^n}{dz^1 \wedge \cdots \wedge d\bar{z}^n} \ge -\frac{\sqrt{-1} \partial \bar{\partial} H}{H} + \mathrm{Ric}(\omega_X)|_{X_y}$$

$$\ge -\left(\frac{C}{H} + C\right) \omega_y \ge -\frac{C}{H} \omega_y.$$

Now if we fix an ample line bundle L_Y cohomologous to A_α, some high power of it has a holomorphic section σ whose zero set contains the subvariety $f(S)$. Up to changing ω_Y with a multiple of it, we will assume that σ is a section of L_Y. We then fix h a Hermitian metric on L_Y with curvature ω_Y, and consider the function on Y given by $|\sigma|_h^2$. Then on Y we have the inequality

$$(3.29) \qquad \qquad |\sigma(y)|_h^\lambda \le C \inf_{X_y} H,$$

where λ, C are constants depending only on the fixed data. This is because the function H is locally a sum of squares of holomorphic functions (the minors of the Jacobian of f) which have well-defined vanishing orders on S, and $f^*(|\sigma|_h^2)$ also vanishes on S by construction. Combining (3.28) and (3.29) we see that the Ricci

curvature of ω_y is bounded below by $-C|\sigma|_h^{-\lambda}$. Since the diameter of ω_y is bounded by Lemma 3.5.7, a theorem of Li-Yau [LY] then shows that the Poincaré constant of ω_y is bounded above by $Ce^{B|\sigma|_h^{-\lambda}}$. This proves the following

Lemma 3.5.8. *There are uniform constants λ, B, C so that for any $0 < t \leq 1$, for any $y \in Y \backslash f(S)$ and for any $u \in C^\infty(X_y)$ with $\int_{X_y} u\omega_y^{n-m} = 0$ we have*

$$(3.30) \qquad \int_{X_y} |u|^2 \omega_y^{n-m} \leq Ce^{B|\sigma|_h^{-\lambda}} \int_{X_y} |\nabla u|^2_{\omega_y} \omega_y^{n-m}.$$

We now let $\tilde{\omega}_y$ be the restriction $\tilde{\omega}_t|_{X_y}$. We have the following estimate for the volume form of $\tilde{\omega}_y$ on X_y:

$$(3.31) \qquad \begin{aligned} \frac{\tilde{\omega}_y^{n-m}}{\omega_y^{n-m}} &= \frac{\tilde{\omega}_t^{n-m} \wedge \omega_0^m}{\omega_X^{n-m} \wedge \omega_0^m} = \frac{\tilde{\omega}_t^{n-m} \wedge \omega_0^m}{\tilde{\omega}_t^n} \cdot \frac{\tilde{\omega}_t^n}{H\omega_X^n} \leq \left(\frac{\tilde{\omega}_t^{n-1} \wedge \omega_0}{\tilde{\omega}_t^n}\right)^m \frac{c_t t^{n-m} e^E}{H} \\ &= (\operatorname{tr}_{\tilde{\omega}_t} \omega_0)^m \frac{c_t t^{n-m} e^E}{H} \leq \frac{Ct^{n-m}}{|\sigma|_h^{\lambda}}. \end{aligned}$$

Notice that when we restrict to X_y we have

$$\tilde{\omega}_y = (\omega_0 + t\omega_X + \sqrt{-1}\partial\bar{\partial}\varphi_t)|_{X_y} = t\omega_y + (\sqrt{-1}\partial\bar{\partial}\varphi_t)|_{X_y}.$$

It is convenient to define a function $\underline{\varphi_t}$ on $Y \backslash f(S)$ by

$$\underline{\varphi_t}(y) = \int_{X_y} \varphi_t \omega_y^{n-m}.$$

This is just the "integration along the fibers" of φ_t, and we will also denote by $\underline{\varphi_t}$ its pullback to $X \backslash S$ via f. We also define a function on $X \backslash S$ by

$$\psi = \frac{1}{t}\left(\varphi_t - \underline{\varphi_t}\right),$$

so that we have $\int_{X_y} \psi\omega_y^{n-m} = 0$ and

$$(3.32) \qquad (\omega_y + \sqrt{-1}\partial\bar{\partial}\psi)^{n-m} = \frac{\tilde{\omega}_y^{n-m}}{t^{n-m}} \leq \frac{C}{|\sigma|_h^{\lambda}}\omega_y^{n-m}.$$

We can then apply Yau's L^∞ estimate Theorem 1.3.3 to the inequality (3.32). Since the volume of X_y is constant equal to 1, the Sobolev constant of ω_y is uniformly bounded (Lemma 3.5.6) and the Poincaré constant is controlled by Lemma 3.5.8, we get

$$(3.33) \qquad \sup_{X_y} \left|\varphi_t - \underline{\varphi_t}\right| = t\sup_{X_y} |\psi| \leq tCe^{B|\sigma(y)|_h^{-\lambda}}.$$

Recall that from (3.20) we have a uniform bound for the oscillation of φ_t.

Proof of Theorem 3.5.2. First we will show the right-hand side inequality in (3.21). We will apply the maximum principle to the quantity

$$K = e^{-B|\sigma|_h^{-\lambda}} \left(\log \mathrm{tr}_{\omega_X} \tilde{\omega}_t - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) \right),$$

where A is a suitably chosen uniform large constant. The maximum of K on $X \backslash S$ is obviously achieved, and we will show that in fact $K \le C$ for a uniform constant C. This together with (3.33) will show that on $X \backslash S$ we have

(3.34) $$\Delta_{\omega_X} \varphi_t = \mathrm{tr}_{\omega_X} \tilde{\omega}_t - \mathrm{tr}_{\omega_X} \omega_0 - nt \le \mathrm{tr}_{\omega_X} \tilde{\omega}_t \le C e^{C e^{B|\sigma|_h^{-\lambda}}},$$

which is half of (3.21) To do this, we first compute as in Lemma 4.3.2

$$\Delta_{\tilde{\omega}_t} \log \mathrm{tr}_{\omega_X} \tilde{\omega}_t \ge -C \mathrm{tr}_{\tilde{\omega}_t} \omega_X - C,$$

for a uniform constant C. On the other hand

$$\Delta_{\tilde{\omega}_t} \varphi_t \le n - t \cdot \mathrm{tr}_{\tilde{\omega}_t} \omega_X,$$

and so if A is large enough we get

$$\Delta_{\tilde{\omega}_t} \left(\log \mathrm{tr}_{\omega_X} \tilde{\omega}_t - \frac{A}{t} \varphi_t \right) \ge \mathrm{tr}_{\tilde{\omega}_t} \omega_X - \frac{C}{t}.$$

Since f is locally a submersion on $X \backslash S$, the fiber integration formula

$$\partial \bar{\partial} \underline{\varphi_t} = f_*(\partial \bar{\partial} \varphi_t \wedge \omega_X^{n-m})$$

holds. So we can compute that

(3.35) $$\begin{aligned} \Delta_{\tilde{\omega}_t} \underline{\varphi_t} &= \mathrm{tr}_{\tilde{\omega}_t} f_*(\sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \omega_X^{n-m}) = \mathrm{tr}_{\tilde{\omega}_t} f_*((\tilde{\omega}_t - \omega_t) \wedge \omega_X^{n-m}) \\ &\ge -\mathrm{tr}_{\tilde{\omega}_t} f_*(\omega_t \wedge \omega_X^{n-m}) = -\mathrm{tr}_{\tilde{\omega}_t} f_*(f^* \omega_Y \wedge \omega_X^{n-m}) - t \mathrm{tr}_{\tilde{\omega}_t} f_*(\omega_X^{n-m+1}) \\ &= -\mathrm{tr}_{\tilde{\omega}_t} \omega_0 - t \mathrm{tr}_{\tilde{\omega}_t} f_*(\omega_X^{n-m+1}). \end{aligned}$$

On $Y \backslash f(S)$ the Kähler form $f_*(\omega_X^{n-m+1})$ can be estimated by

(3.36) $$\begin{aligned} f_*(\omega_X^{n-m+1}) &\le \frac{\omega_Y^{m-1} \wedge f_*(\omega_X^{n-m+1})}{\omega_Y^m} \omega_Y = \frac{f_*(\omega_0^{m-1} \wedge \omega_X^{n-m+1})}{\omega_Y^m} \omega_Y \\ &\le C \frac{f_*(\omega_X^n)}{\omega_Y^m} \omega_Y = C \frac{f_*(H^{-1} \omega_0^m \wedge \omega_X^{n-m})}{\omega_Y^m} \omega_Y \\ &\le C |\sigma|_h^{-\lambda} \frac{f_*(\omega_0^m \wedge \omega_X^{n-m})}{\omega_Y^m} \omega_Y = C |\sigma|_h^{-\lambda} \omega_Y. \end{aligned}$$

and so using (3.24) we get

$$\Delta_{\tilde{\omega}_t}\underline{\varphi_t} \geq -C - tC|\sigma|_h^{-\lambda}.$$

It follows that

$$(3.37) \qquad \Delta_{\tilde{\omega}_t}\left(\log\operatorname{tr}_{\omega_X}\tilde{\omega}_t - \frac{A}{t}(\varphi_t - \underline{\varphi_t})\right) \geq \operatorname{tr}_{\tilde{\omega}_t}\omega_X - \frac{C}{t} - C|\sigma|_h^{-\lambda}.$$

Next we compute on $Y\backslash f(S)$

$$(3.38) \qquad \begin{aligned} \Delta_{\tilde{\omega}_t}|\sigma|_h^\lambda &= (\lambda/2)^2|\sigma|_h^\lambda|\nabla\log|\sigma|_h^2|_{\tilde{\omega}_t}^2 + \lambda/2|\sigma|_h^\lambda\operatorname{tr}_{\tilde{\omega}_t}\sqrt{-1}\partial\bar{\partial}\log|\sigma|_h^2 \\ &= (\lambda/2)^2|\sigma|_h^{\lambda-4}|\nabla|\sigma|_h^2|_{\tilde{\omega}_t}^2 - \lambda/2|\sigma|_h^\lambda\operatorname{tr}_{\tilde{\omega}_t}\omega_0 \geq -C|\sigma|_h^\lambda, \end{aligned}$$

$$|\nabla|\sigma|_h^2|_{\tilde{\omega}_t}^2 = |\sigma|_h^2|\nabla\sigma|_{h,\tilde{\omega}_t}^2 \leq C|\sigma|_h^2\operatorname{tr}_{\tilde{\omega}_t}\omega_Y \leq C|\sigma|_h^2,$$

$$(3.39) \qquad |\nabla|\sigma|_h^\lambda|_{\tilde{\omega}_t}^2 = (\lambda/2)^2|\sigma|_h^{2\lambda-4}|\nabla|\sigma|_h^2|_{\tilde{\omega}_t}^2 \leq C|\sigma|_h^{2\lambda-2},$$

$$(3.40) \qquad \begin{aligned} \Delta_{\tilde{\omega}_t}|\sigma|_h^\lambda &= (\lambda/2)^2|\sigma|_h^{\lambda-4}|\nabla|\sigma|_h^2|_{\tilde{\omega}_t}^2 - \lambda/2|\sigma|_h^\lambda\operatorname{tr}_{\tilde{\omega}_t}\omega_0 \\ &\leq C|\sigma|_h^{\lambda-2}. \end{aligned}$$

Using (3.37) we then compute

$$(3.41) \qquad \begin{aligned} \Delta_{\tilde{\omega}_t}K &\geq e^{-B|\sigma|_h^{-\lambda}}\left(\operatorname{tr}_{\tilde{\omega}_t}\omega_X - \frac{C}{t} - C|\sigma|_h^{-\lambda}\right) \\ &\quad + \left(\log\operatorname{tr}_{\omega_X}\tilde{\omega}_t - \frac{A}{t}(\varphi_t - \underline{\varphi_t})\right)\Delta_{\tilde{\omega}_t}\left(e^{-B|\sigma|_h^{-\lambda}}\right) \\ &\quad + 2e^{B|\sigma|_h^{-\lambda}}\operatorname{Re}\langle\nabla K, \nabla e^{-B|\sigma|_h^{-\lambda}}\rangle_{\tilde{\omega}_t} \\ &\quad - 2\left(\log\operatorname{tr}_{\omega_X}\tilde{\omega}_t - \frac{A}{t}(\varphi_t - \underline{\varphi_t})\right)e^{B|\sigma|_h^{-\lambda}}|\nabla e^{-B|\sigma|_h^{-\lambda}}|_{\tilde{\omega}_t}^2. \end{aligned}$$

Using (3.38), (3.39) and $|\sigma|_h^\lambda \leq C$, the second term in (3.41) can be estimated as follows

$$(3.42) \qquad \begin{aligned} \Delta_{\tilde{\omega}_t}\left(e^{-B|\sigma|_h^{-\lambda}}\right) &= \frac{Be^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{2\lambda}}\Delta_{\tilde{\omega}_t}|\sigma|_h^\lambda + \frac{B^2e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{4\lambda}}|\nabla|\sigma|_h^\lambda|_{\tilde{\omega}_t}^2 \\ &\quad - \frac{2Be^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{3\lambda}}|\nabla|\sigma|_h^\lambda|_{\tilde{\omega}_t}^2 \\ &\geq -C\frac{e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^\lambda} - C\frac{e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{\lambda+2}} \\ &\geq -C\frac{e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{\lambda+2}}. \end{aligned}$$

At the maximum of K we may assume that $K \geq 0$, otherwise we have nothing to prove. Hence we can use (3.33) to estimate

(3.43)
$$\left(\log \operatorname{tr}_{\omega_X} \tilde{\omega}_t - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) \right) \Delta_{\tilde{\omega}_t} \left(e^{-B|\sigma|_h^{-\lambda}} \right) \geq -C \frac{e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{\lambda+2}} \log \operatorname{tr}_{\omega_X} \tilde{\omega}_t - \frac{C}{|\sigma|_h^{\lambda+2}}.$$

The fourth term in (3.41) can be estimated using (3.39)

(3.44)
$$|\nabla e^{-B|\sigma|_h^{-\lambda}}|_{\tilde{\omega}_t}^2 = \frac{B^2 e^{-2B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{4\lambda}} |\nabla |\sigma|_h^{\lambda}|_{\tilde{\omega}_t}^2 \leq \frac{Ce^{-2B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{2\lambda+2}},$$

(3.45)
$$-2 \left(\log \operatorname{tr}_{\omega_X} \tilde{\omega}_t - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) \right) e^{B|\sigma|_h^{-\lambda}} |\nabla e^{-B|\sigma|_h^{-\lambda}}|_{\tilde{\omega}_t}^2$$
$$\geq -C \frac{e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{2\lambda+2}} \log \operatorname{tr}_{\omega_X} \tilde{\omega}_t - \frac{C}{|\sigma|_h^{2\lambda+2}}.$$

Plugging (3.43) and (3.45) in (3.41), at the maximum point of K we get

$$0 \geq \operatorname{tr}_{\tilde{\omega}_t} \omega_X - \frac{C}{t} - \frac{C}{|\sigma|_h^{\lambda}} - \frac{C}{|\sigma|_h^{2\lambda+2}} \log \operatorname{tr}_{\omega_X} \tilde{\omega}_t - C \frac{e^{B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{2\lambda+2}}.$$

From (1.24) we see that

$$\operatorname{tr}_{\omega_X} \tilde{\omega}_t \leq C t^{n-m} (\operatorname{tr}_{\tilde{\omega}_t} \omega_X)^{n-1} \leq C (\operatorname{tr}_{\tilde{\omega}_t} \omega_X)^{n-1},$$

and using this and the inequalities $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ and $(\log x^{n-1})^2 \leq x + C$ we get

$$\operatorname{tr}_{\tilde{\omega}_t} \omega_X \leq \frac{C}{t} + \frac{C}{|\sigma|_h^{\lambda}} + \frac{C}{|\sigma|_h^{4\lambda+4}} + C \frac{e^{B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{2\lambda+2}} + \frac{1}{2} \operatorname{tr}_{\tilde{\omega}_t} \omega_X,$$

whence

$$\operatorname{tr}_{\tilde{\omega}_t} \omega_X \leq \frac{C}{t} + C e^{C|\sigma|_h^{-\lambda}}.$$

At the same point we then get

$$\operatorname{tr}_{\tilde{\omega}_t} \omega_t = \operatorname{tr}_{\tilde{\omega}_t}(\omega_0 + t\omega_X) \leq C + tC e^{C|\sigma|_h^{-\lambda}}.$$

and using (1.24) we get

(3.46)
$$\operatorname{tr}_{\omega_t} \tilde{\omega}_t \leq (\operatorname{tr}_{\tilde{\omega}_t} \omega_t)^{n-1} \frac{\tilde{\omega}_t^n}{\omega_t^n} \leq \left(C + tC e^{C|\sigma|_h^{-\lambda}} \right)^{n-1} \frac{\tilde{\omega}_t^n}{\omega_t^n}.$$

We now use (3.16), (3.19) and (3.29) to get

(3.47)
$$\frac{\tilde{\omega}_t^n}{\omega_t^n} \leq \frac{C t^{n-m} \omega_X^n}{\omega_0^m \wedge (t\omega_X)^{n-m}} = \frac{C}{H} \leq \frac{C}{|\sigma|_h^{\lambda}}.$$

Combining (3.46) and (3.47) we get

$$\mathrm{tr}_{\omega_t}\tilde{\omega}_t \leq Ce^{C|\sigma|_h^{-\lambda}},$$

for some uniform constant C. But we also have $\omega_t = \omega_0 + t\omega_X \leq C\omega_X$ and so we get

$$\mathrm{tr}_{\omega_X}\tilde{\omega}_t \leq Ce^{C|\sigma|_h^{-\lambda}}.$$

Using (3.33) again, this implies that at the maximum of K we have

$$K \leq C + e^{-B|\sigma|_h^{-\lambda}}\log(Ce^{C|\sigma|_h^{-\lambda}}) \leq C.$$

We now show the left-hand side inequality in (3.21). To this extent we apply the maximum principle to the quantity

$$K_1 = e^{-B|\sigma|_h^{-\lambda}}\left(\log(t\cdot\mathrm{tr}_{\tilde{\omega}_t}\omega_X) - \frac{A}{t}(\varphi_t - \underline{\varphi_t})\right),$$

where A is a suitably chosen uniform large constant. The maximum of K_1 on $X\backslash S$ is obviously achieved, and we will show that in fact $K_1 \leq C$ for a uniform constant C. This together with (3.33) will show that on $X\backslash S$ we have

(3.48) $$\mathrm{tr}_{\tilde{\omega}_t}\omega_X \leq \frac{C}{t}e^{Ce^{B|\sigma|_h^{-\lambda}}},$$

which is the other half of (3.21). To prove that $K_1 \leq C$ we use the maximum principle and, as in (3.41), we compute

(3.49)
$$\begin{aligned}
\Delta_{\tilde{\omega}_t}K_1 &\geq e^{-B|\sigma|_h^{-\lambda}}\left(\mathrm{tr}_{\tilde{\omega}_t}\omega_X - \frac{C}{t} - C|\sigma|_h^{-\lambda}\right)\\
&+ \left(\log(t\cdot\mathrm{tr}_{\tilde{\omega}_t}\omega_X) - \frac{A}{t}(\varphi_t - \underline{\varphi_t})\right)\Delta_{\tilde{\omega}_t}\left(e^{-B|\sigma|_h^{-\lambda}}\right)\\
&+ 2e^{B|\sigma|_h^{-\lambda}}\mathrm{Re}\langle\nabla K_1, \nabla e^{-B|\sigma|_h^{-\lambda}}\rangle_{\tilde{\omega}_t}\\
&- 2\left(\log(t\cdot\mathrm{tr}_{\tilde{\omega}_t}\omega_X) - \frac{A}{t}(\varphi_t - \underline{\varphi_t})\right)e^{B|\sigma|_h^{-\lambda}}|\nabla e^{-B|\sigma|_h^{-\lambda}}|^2_{\tilde{\omega}_t}.
\end{aligned}$$

We estimate this in the same way as before and get

(3.50)
$$\begin{aligned}
\Delta_{\tilde{\omega}_t}K_1 &\geq e^{-B|\sigma|_h^{-\lambda}}\left(\mathrm{tr}_{\tilde{\omega}_t}\omega_X - \frac{C}{t} - C|\sigma|_h^{-\lambda}\right)\\
&- C\frac{e^{-B|\sigma|_h^{-\lambda}}}{|\sigma|_h^{2\lambda+2}}\log(t\cdot\mathrm{tr}_{\tilde{\omega}_t}\omega_X) - \frac{C}{|\sigma|_h^{2\lambda+2}}\\
&+ 2e^{B|\sigma|_h^{-\lambda}}\mathrm{Re}\langle\nabla K_1, \nabla e^{-B|\sigma|_h^{-\lambda}}\rangle_{\tilde{\omega}_t}.
\end{aligned}$$

At the maximum of K_1 we get

$$0 \geq \mathrm{tr}_{\tilde{\omega}_t}\omega_X - \frac{C}{t} - \frac{C}{|\sigma|_h^\lambda} - \frac{C}{|\sigma|_h^{2\lambda+2}}\log(t \cdot \mathrm{tr}_{\tilde{\omega}_t}\omega_X) - Ce^{C|\sigma|_h^{-\lambda}},$$

and using the inequalities $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ and $(\log x)^2 \leq x + C$ we get

$$\mathrm{tr}_{\tilde{\omega}_t}\omega_X \leq \frac{C}{t} + \frac{C}{|\sigma|_h^\lambda} + \frac{C}{|\sigma|_h^{4\lambda+4}} + Ce^{C|\sigma|_h^{-\lambda}} + \frac{1}{2}\mathrm{tr}_{\tilde{\omega}_t}\omega_X,$$

whence

$$t \cdot \mathrm{tr}_{\tilde{\omega}_t}\omega_X \leq C + tCe^{C|\sigma|_h^{-\lambda}} \leq Ce^{C|\sigma|_h^{-\lambda}},$$

and so at that point

$$K_1 \leq C + e^{-B|\sigma|_h^{-\lambda}}\log(Ce^{C|\sigma|_h^{-\lambda}}) \leq C,$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 3.5.3. We will first show (3.22), which is an easy consequence of (3.21). The left-hand side follows immediately from (3.21), which implies

$$(3.51) \qquad\qquad \mathrm{tr}_{\tilde{\omega}_y}\omega_y \leq \frac{C}{t}e^{Ce^{B|\sigma|_h^{-\lambda}}}.$$

Then (1.24) and (3.31) give

$$(3.52) \qquad \mathrm{tr}_{\omega_y}\tilde{\omega}_y \leq (\mathrm{tr}_{\tilde{\omega}_y}\omega_y)^{n-m-1}\frac{\tilde{\omega}_y^{n-m}}{\omega_y^{n-m}} \leq t\frac{Ce^{Ce^{B|\sigma|_h^{-\lambda}}}}{|\sigma|_h^\lambda} \leq tCe^{Ce^{B|\sigma|_h^{-\lambda}}},$$

which proves (3.22).

Next, we show (3.23). Recall from (3.34) and (3.48) that on $X\backslash S$ we have

$$(3.53) \qquad\qquad \mathrm{tr}_{\omega_X}\tilde{\omega}_t \leq Ce^{C_0 e^{B|\sigma|_h^{-\lambda}}}.$$

$$(3.54) \qquad\qquad \mathrm{tr}_{\tilde{\omega}_t}\omega_X \leq \frac{C}{t}e^{C_0 e^{B|\sigma|_h^{-\lambda}}},$$

for uniform constants B, C, C_0. We apply the maximum principle to the quantity

$$K_2 = e^{-Ae^{B|\sigma|_h^{-\lambda}}}\left(S + C\frac{e^{3C_0 e^{B|\sigma|_h^{-\lambda}}}}{t^{5/2}}\mathrm{tr}_{\omega_X}\tilde{\omega}_t\right),$$

for suitable constants A, C, where the quantity S is the same quantity as in section 1.3:

$$S = |\nabla\tilde{\omega}_t|_{\tilde{\omega}_t}^2,$$

where ∇ is the covariant derivative associated to the metric ω_X. Using φ_t we can write

$$S = \tilde{g}_t^{i\bar{p}}\tilde{g}_t^{q\bar{j}}\tilde{g}_t^{k\bar{r}}\varphi_{i\bar{j}k}\varphi_{\bar{p}q\bar{r}},$$

where again lower indices are covariant derivatives with respect to ω_X. We are going to show that $K_2 \leq \frac{C}{t^{5/2}}$, and using (3.53) this implies that

$$(3.55) \qquad\qquad S \leq \frac{Ce^{Ae^{B|\sigma|_h^{-\lambda}}}}{t^{5/2}}.$$

We now use (3.52), which says that on X_y we have

$$(3.56) \qquad\qquad \mathrm{tr}_{\omega_y}\tilde{\omega}_y \leq tCe^{C_0 e^{B|\sigma|_h^{-\lambda}}},$$

At any given point of X_y we can assume that ω_X is the identity and $\tilde{\omega}_t$ is diagonal with positive entries λ_i, $1 \leq i \leq n$, so that the first $n - m$ directions are tangent to the fiber X_y. Then (3.56) gives that

$$\lambda_i \leq tCe^{C_0 e^{B|\sigma|_h^{-\lambda}}},$$

for $1 \leq i \leq n - m$. Then using (3.55) we see that

$$\sum_{i=1}^{n-m}\frac{1}{\lambda_i\lambda_j\lambda_k}|\varphi_{i\bar{j}k}|^2 \leq \sum_{i=1}^{n}\frac{1}{\lambda_i\lambda_j\lambda_k}|\varphi_{i\bar{j}k}|^2 = S \leq \frac{Ce^{Ae^{B|\sigma|_h^{-\lambda}}}}{t^{5/2}},$$

and using (3.56) we get

$$|\nabla\tilde{\omega}_y|^2_{\omega_y} = \sum_{i=1}^{n-m}|\varphi_{i\bar{j}k}|^2 \leq t^{1/2}Ce^{2Ae^{B|\sigma|_h^{-\lambda}}},$$

provided we choose $A \geq 4C_0$, and this is (3.23).

We now prove that $K_2 \leq \frac{C}{t^{5/2}}$. To simplify the computation, we will use the notation

$$F(x) = e^{xe^{B|\sigma|_h^{-\lambda}}},$$

where x is a real number, and we note here that F is increasing. The starting point is the formula for $\Delta_{\tilde{\omega}_t}S$ which is done in a more general setting in Chapter 4 (Lemma 4.4.5). With the notation there, we can write

$$S = \sum_{i,j,k}|a_{jk}^i|^2.$$

We then choose local unitary frames $\{\theta^1,\ldots,\theta^n\}$ for ω_X and $\{\tilde{\theta}^1,\ldots,\tilde{\theta}^n\}$ for $\tilde{\omega}_t$, and write

$$\tilde{\theta}^i = \sum_j a_j^i\theta^j,$$

$$\theta^i = \sum_j b^i_j \tilde{\theta}^j,$$

for some local matrices of functions a^i_j, b^i_j. Notice that at any given point we can choose the frames and arrange that

(3.57) $$a^i_j = \sqrt{\lambda_i} \delta^i_j,$$

(3.58) $$b^i_j = \frac{1}{\sqrt{\lambda_i}} \delta^i_j.$$

Then in our case (4.63) reads

$$\Delta_{\tilde{\omega}_t} S \;\geq\; 2\mathrm{Re}\Big(\overline{a^i_{k\ell}}\Big(b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}} a^i_{rp} a^r_j - a^i_j b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}} a^r_{kp} b^m_r$$

$$- a^i_j b^m_k \overline{b^s_p} R^j_{mq\overline{s}} a^r_{\ell p} b^q_r + a^i_j b^m_k b^q_\ell \overline{b^s_p} b^u_p R^j_{mq\overline{s},u} \Big)\Big),$$

where we are summing over all indices, $R^j_{mq\overline{s}}$ represents the curvature of ω_X and $R^j_{mq\overline{s},u}$ its covariant derivative (with respect to ω_X). Since these are fixed tensors, we can use the Cauchy-Schwarz inequality and (3.57), (3.58) to estimate the first term on the right hand side of (3.59) by

$$\left| 2\mathrm{Re}\left(\overline{a^i_{k\ell}} b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}} a^i_{rp} a^r_j \right) \right| \leq C \sum_{i,k,\ell,r,p} |a^i_{k\ell} a^i_{rp}| \sqrt{\frac{\lambda_r}{\lambda_k \lambda_\ell \lambda_p}}$$

$$\leq C \left(\sum_j \lambda_j \right)^{\frac{1}{2}} \left(\sum_q \frac{1}{\lambda_q} \right)^{\frac{3}{2}} \sum_{k,\ell,r,p} \left(\sum_i |a^i_{k\ell}|^2 \right)^{\frac{1}{2}} \left(\sum_i |a^i_{rp}|^2 \right)^{\frac{1}{2}}$$

$$= C(\mathrm{tr}_{\omega_X} \tilde{\omega}_t)^{\frac{1}{2}} (\mathrm{tr}_{\tilde{\omega}_t} \omega_X)^{\frac{3}{2}} \left(\sum_{i,k,\ell} |a^i_{k\ell}|^2 \right)^{\frac{1}{2}} \left(\sum_{i,r,p} |a^i_{rp}|^2 \right)^{\frac{1}{2}}$$

$$= CS(\mathrm{tr}_{\omega_X} \tilde{\omega}_t)^{\frac{1}{2}} (\mathrm{tr}_{\tilde{\omega}_t} \omega_X)^{\frac{3}{2}}.$$

The second and third term in (3.59) are estimated similarly, while the fourth term can be bounded by

$$\left| 2\mathrm{Re}\left(\overline{a^i_{k\ell}} a^i_j b^m_k b^q_\ell \overline{b^s_p} b^u_p R^j_{mq\overline{s},u} \right) \right| \leq C \sum_{i,k,\ell,p} |a^i_{k\ell}| \sqrt{\frac{\lambda_i}{\lambda_k \lambda_\ell \lambda_p^2}}$$

$$\leq C \left(\sum_j \lambda_j \right)^{\frac{1}{2}} \left(\sum_q \frac{1}{\lambda_q} \right)^2 \sum_{i,k,\ell} |a^i_{k\ell}|$$

$$\leq C\sqrt{S}(\mathrm{tr}_{\omega_X} \tilde{\omega}_t)^{\frac{1}{2}} (\mathrm{tr}_{\tilde{\omega}_t} \omega_X)^2.$$

Overall we can estimate

$$(3.59) \qquad \Delta_{\tilde\omega_t} S \geq -CS(\mathrm{tr}_{\tilde\omega_t}\omega_X)^{3/2}(\mathrm{tr}_{\omega_X}\tilde\omega_t)^{1/2} - C\sqrt{S}(\mathrm{tr}_{\tilde\omega_t}\omega_X)^2(\mathrm{tr}_{\omega_X}\tilde\omega_t)^{1/2}.$$

On the other hand from (4.56) we see that

$$
\begin{aligned}
\Delta_{\tilde\omega_t}\mathrm{tr}_{\omega_X}\tilde\omega_t &= a^i_{k\ell}\overline{a^i_{p\ell}}a^k_j\overline{a^p_j} + \overline{a^i_j}a^i_r b^q_\ell \overline{b^s_\ell} R^r_{jq\overline{s}} \\
&\geq \sum_{i,j,\ell}|a^i_{j\ell}|^2 \lambda_j - C\sum_{i,\ell}\frac{\lambda_i}{\lambda_\ell} \\
(3.60) \qquad\qquad &\geq \left(\sum_k \frac{1}{\lambda_k}\right)^{-1}\sum_{i,j,\ell}|a^i_{j\ell}|^2 - C\left(\sum_k \lambda_k\right)\left(\sum_k \frac{1}{\lambda_k}\right) \\
&= \frac{S}{\mathrm{tr}_{\tilde\omega_t}\omega_X} - C(\mathrm{tr}_{\tilde\omega_t}\omega_X)(\mathrm{tr}_{\omega_X}\tilde\omega_t).
\end{aligned}
$$

We now insert (3.53), (3.54) in (3.59), (3.60) and get

$$(3.61) \qquad\qquad \Delta_{\tilde\omega_t} S \geq -\frac{CF(2C_0)}{t^{3/2}}S - \frac{CF(5C_0/2)}{t^2}\sqrt{S},$$

$$\Delta_{\tilde\omega_t}\mathrm{tr}_{\omega_X}\tilde\omega_t \geq \frac{tF(-C_0)}{C}S - \frac{CF(2C_0)}{t}.$$

We then compute

$$(3.62)$$
$$
\begin{aligned}
\Delta_{\tilde\omega_t}\left(\frac{F(3C_0)}{t^{5/2}}\mathrm{tr}_{\omega_X}\tilde\omega_t\right) &\geq \frac{F(2C_0)}{Ct^{3/2}}S - \frac{CF(5C_0)}{t^{7/2}} + \frac{2}{t^{5/2}}\mathrm{Re}\langle\nabla F(3C_0), \nabla\mathrm{tr}_{\omega_X}\tilde\omega_t\rangle_{\tilde\omega_t} \\
&\quad + \frac{1}{t^{5/2}}(\mathrm{tr}_{\omega_X}\tilde\omega_t)\Delta_{\tilde\omega_t}F(3C_0),
\end{aligned}
$$

and estimate

$$\mathrm{Re}\langle\nabla F(3C_0), \nabla\mathrm{tr}_{\omega_X}\tilde\omega_t\rangle_{\tilde\omega_t} \geq -|\nabla F(3C_0)|_{\tilde\omega_t}|\nabla\mathrm{tr}_{\omega_X}\tilde\omega_t|_{\tilde\omega_t}.$$

Using (4.57) we see that

$$|\nabla\mathrm{tr}_{\omega_X}\tilde\omega_t|_{\tilde\omega_t} \leq \sqrt{S}(\mathrm{tr}_{\omega_X}\tilde\omega_t).$$

On the other hand a direct computation using (3.38), (3.39) and (3.40) shows that there is a constant C such that for any real number x we have

$$|\nabla F(x)|_{\tilde\omega_t} \leq CF(x+1),$$

$$|\Delta_{\tilde\omega_t}F(x)| \leq CF(x+1),$$

and so we have

$$
\begin{aligned}
(3.63) \qquad \Delta_{\tilde{\omega}_t}\left(\frac{F(3C_0)}{t^{5/2}}\mathrm{tr}_{\omega_X}\tilde{\omega}_t\right) &\geq \frac{F(2C_0)}{Ct^{3/2}}S - \frac{CF(5C_0)}{t^{7/2}} - \frac{CF(5C_0)}{t^{5/2}}\sqrt{S} \\
&\quad - \frac{CF(5C_0)}{t^{5/2}}.
\end{aligned}
$$

This and (3.61) give

$$
\begin{aligned}
(3.64) \qquad \Delta_{\tilde{\omega}_t}\left(S + \frac{CF(3C_0)}{t^{5/2}}\mathrm{tr}_{\omega_X}\tilde{\omega}_t\right) &\geq \frac{F(2C_0)}{t^{3/2}}S - \frac{CF(5C_0/2)}{t^2}\sqrt{S} \\
&\quad - \frac{CF(5C_0)}{t^{7/2}} - \frac{CF(5C_0)}{t^{5/2}}\sqrt{S} - \frac{CF(5C_0)}{t^{5/2}} \\
&\geq \frac{F(2C_0)}{t^{3/2}}S - \frac{CF(5C_0)}{t^{7/2}} - \frac{CF(5C_0)}{t^{5/2}}\sqrt{S},
\end{aligned}
$$

and

$$
\begin{aligned}
(3.65) \qquad \Delta_{\tilde{\omega}_t}K_2 &\geq F(-A)\left(\frac{F(2C_0)}{t^{3/2}}S - \frac{CF(5C_0)}{t^{7/2}} - \frac{CF(5C_0)}{t^{5/2}}\sqrt{S}\right. \\
&\quad \left. - CF(1)S - \frac{CF(4C_0+1)}{t^{5/2}}\right) + 2F(A)\mathrm{Re}\langle\nabla K_2, \nabla F(-A)\rangle_{\tilde{\omega}_t} \\
&\geq F(-A)\left(\frac{F(2C_0)}{Ct^{3/2}}S - \frac{CF(5C_0)}{t^{7/2}} - \frac{CF(5C_0)}{t^{5/2}}\sqrt{S}\right) \\
&\quad + 2F(A)\mathrm{Re}\langle\nabla K_2, \nabla F(-A)\rangle_{\tilde{\omega}_t}.
\end{aligned}
$$

At the maximum of K_2 we then get

$$
S \leq \frac{CF(3C_0)}{t}\sqrt{S} + \frac{CF(3C_0)}{t^2},
$$

which implies that

$$
S \leq \frac{CF(3C_0)}{t^2},
$$

and so

$$
K_2 = F(-A)\left(S + \frac{CF(3C_0)}{t^{5/2}}\mathrm{tr}_{\omega_X}\tilde{\omega}_t\right) \leq F(-A)\frac{CF(4C_0)}{t^{5/2}} \leq \frac{C}{t^{5/2}},
$$

because we chose $A \geq 4C_0$. $\qquad\square$

We now explain the meaning of the Weil-Petersson metric, following the discussion in [SoT2]. Fix a Ricci-flat Kähler metric χ on X cohomologous to $\omega_1 = \omega_0 + \omega_X$ and call $\Omega = \chi^n$ its volume form. The generic fiber X_y of f is an $(n-m)$-dimensional Calabi-Yau manifold, and it is naturally equipped with the Kähler form $\omega_y = \omega_X|_{X_y}$.

Recall that the volume of X_y is a homological constant independent of y, and that we assume that it is equal to 1. Since $c_1(X_y) = 0$, there is a smooth function F_y such that $\mathrm{Ric}(\omega_y) = \sqrt{-1}\partial\bar\partial F_y$ and $\int_{X_y}(e^{F_y} - 1)\omega_y^{n-m} = 0$. The functions F_y vary smoothly in y, since so do the Kähler forms ω_y. By Yau's theorem there is a unique Ricci-flat Kähler metric $\omega_{SF,y}$ on X_y cohomologous to ω_y, given by the solution of

$$(3.66) \qquad \omega_{SF,y}^{n-m} = e^{F_y}\omega_y^{n-m}.$$

If we write $\omega_{SF,y} = \omega_y + \sqrt{-1}\partial\bar\partial\zeta_y$, the functions ζ_y vary smoothly in y and so they define a smooth function ζ on $X\backslash S$. We then define a real closed $(1,1)$-form ω_{SF} on $X\backslash S$ by $\omega_{SF} = \omega_X + \sqrt{-1}\partial\bar\partial\zeta$, and call it the semi-flat form. Notice that ω_{SF} is not necessarily nonnegative (it is Kähler only in the fiber directions), but on $X\backslash S$ the (n,n)-form $\omega_{SF}^{n-m} \wedge \omega_0^m$ is strictly positive, and so we can define a smooth positive function F on $X\backslash S$ by

$$(3.67) \qquad F = \frac{\Omega}{\omega_{SF}^{n-m} \wedge \omega_0^m}.$$

We claim that F is actually constant on each fiber X_y, and so it is the pullback of a function on $Y\backslash f(S)$. To see this, fix a point $y \in Y\backslash f(S)$ and choose local coordinates $z^1,\ldots,z^{n-m}$ on the fiber X_y, which extend locally to coordinates in a ball in X. Then take local coordinates $w^{n-m+1},\ldots,w^n$ near $y \in Y\backslash f(S)$, so that $z^1,\ldots,z^{n-m},z^{n-m+1} = f^*(w^{n-m+1}),\ldots,z^n = f^*(w^n)$ give local holomorphic coordinates on X. In these coordinates write

$$\omega_0 = \sqrt{-1}\sum_{i,j=n-m+1}^{n} g_{i\bar j}^0 dz^i \wedge d\bar z^j,$$

$$\omega_{SF,y} = \sqrt{-1}\sum_{i,j=1}^{n-m} g_{i\bar j}^{SF} dz^i \wedge d\bar z^j,$$

$$\Omega = G(\sqrt{-1})^n dz^1 \wedge \cdots \wedge d\bar z^n.$$

Then locally

$$F = \frac{G}{\det(g_{i\bar j}^0)\det(g_{i\bar j}^{SF})},$$

and so on the fiber X_y we have

$$\sqrt{-1}\partial\bar\partial \log F = -\mathrm{Ric}(\chi) + \mathrm{Ric}(\omega_{SF,y}) = 0,$$

because ω_0 is the pullback of a metric from Y, and so F is indeed constant on X_y. Moreover, it is easy to check [SoT2, Lemma 3.3] that on $Y\backslash f(S)$ we have

$$F = \frac{f_*\Omega}{\omega_Y^m},$$

and so

$$\int_Y F\omega_Y^m = \int_X \Omega = \int_X \omega_1^n$$

is finite. In fact there is a positive ε so that $\int_Y F^{1+\varepsilon}\omega_Y^m$ [SoT2, Proposition 3.2]. Then we apply [SoT2, Theorem 3.2], which relies on the seminal work of Kołodziej [Koł], to solve (uniquely) the complex Monge-Ampère equation

$$(3.68) \qquad (\omega_Y + \sqrt{-1}\partial\overline{\partial}\psi)^m = \frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_1^n} F\omega_Y^m,$$

with $\psi \in L^\infty(Y)$ and moreover ψ is smooth on $Y\backslash f(S)$ (the proof of this follows the arguments of Yau in [Y2]). We will call $\omega = \omega_Y + \sqrt{-1}\partial\overline{\partial}\psi$ the Kähler metric on $Y\backslash f(S)$ that we've just constructed. Its Ricci curvature is the Weil-Petersson metric that we are about to define. Recall that the fibers X_y have torsion canonical bundle, so that there is a number k such that $K_{X_y}^{\otimes k}$ is trivial for all $y \in Y\backslash f(S)$. The Weil-Petersson metric is a smooth nonnegative $(1,1)$-form on $Y\backslash f(S)$ defined as the curvature form of a Hermitian pseudometric on the relative canonical line bundle $f_*(\Omega_{X/Y}^{n-m})^{\otimes k}$: if Ψ_y is a local nonzero holomorphic section of $f_*(\Omega_{X/Y}^{n-m})^{\otimes k}$, which means that Ψ_y is a nonzero holomorphic k-pluricanonical form on X_y that varies holomorphically in y, then we let its length be

$$|\Psi_y|_{h_{WP}}^2 = \int_{X_y} (\Psi_y \wedge \overline{\Psi_y})^{\frac{1}{k}}.$$

For $k > 1$ this is not a Hermitian metric, but just a norm. The Weil-Petersson metric ω_{WP} on $Y\backslash f(S)$ is just formally the curvature of h_{WP}, that is locally we set

$$\omega_{WP} = -\sqrt{-1}\partial\overline{\partial}\log|\Psi_y|_{h_{WP}}^2,$$

and this is well-defined because the bundle $K_{X_y}^{\otimes k}$ is trivial. It is a classical fact (see e.g., [FS]) that ω_{WP} is pointwise nonnegative.

Proposition 3.5.9 (cfr. [SoT2]). *On $Y\backslash f(S)$ we have*

$$(3.69) \qquad \qquad \text{Ric}(\omega) = \omega_{WP}.$$

Proof. Differentiating (3.68) we see that

$$\text{Ric}(\omega) = \text{Ric}(\omega_Y) - \sqrt{-1}\partial\overline{\partial}\log F.$$

If we fix $y \in Y\backslash f(S)$ and choose Ψ a local never vanishing holomorphic section of $f_*(\Omega_{X/Y}^{n-m})^{\otimes k}$, then we can define a local function $u = \frac{(\Psi\wedge\overline{\Psi})^{1/k}}{\omega_{SF}^{n-m}}$ on $X\backslash S$, which is constant on each fiber X_y. Since $\int_{X_y}\omega_{SF}^{n-m} = 1$, we see that

$$-\sqrt{-1}\partial\overline{\partial}\log u = \omega_{WP}.$$

Then

$$(3.70) \qquad \mathrm{Ric}(\omega) = \mathrm{Ric}(\omega_Y) - \sqrt{-1}\partial\bar{\partial}\log\frac{u\Omega}{(\Psi \wedge \overline{\Psi})^{\frac{1}{k}} \wedge \omega_0^m}.$$

Picking local coordinates z^i as above, and writing

$$\Psi = K[(\sqrt{-1})^{n-m}dz^1 \wedge \cdots \wedge dz^{n-m}]^{\otimes k},$$

$$\omega_0 = \sqrt{-1} \sum_{i,j=n-m+1}^{n} g_{i\bar{j}}^0 dz^i \wedge d\bar{z}^j,$$

$$\Omega = G(\sqrt{-1})^n dz^1 \wedge \cdots \wedge d\bar{z}^n,$$

we see that

$$\frac{u\Omega}{(\Psi \wedge \overline{\Psi})^{\frac{1}{k}} \wedge \omega_0^m} = \frac{uG}{|K|^{\frac{2}{k}}\det(g_{i\bar{j}}^0)},$$

and since K is holomorphic and Ω is Ricci-flat we see that

$$-\sqrt{-1}\partial\bar{\partial}\log\frac{uG}{|K|^{\frac{2}{k}}\det(g_{i\bar{j}}^0)} = \omega_{WP} - \mathrm{Ric}(\omega_Y),$$

which together with (3.70) gives (3.69). □

With these preparations, we can now show Theorem 3.5.4, which can be recast as follows

Theorem 3.5.10. *Consider the Ricci-flat metrics $\tilde{\omega}_t$ on X, which can be written as $\tilde{\omega}_t = \omega_0 + t\omega_X + \sqrt{-1}\partial\bar{\partial}\varphi_t$. As $t \to 0$ we have that $\varphi_t \to \psi$ in the $C_{loc}^{1,\beta}$ topology on $X\backslash S$, for any $0 < \beta < 1$, and so $\tilde{\omega}_t$ converges in this topology to ω, which satisfies (3.69).*

Proof. We first prove that $\tilde{\omega}_t$ converges to ω in the weak topology of currents. Since the cohomology class of $\tilde{\omega}_t$ is bounded, weak compactness of currents implies that from any sequence $t_i \to 0$ we can extract a subsequence so that $\tilde{\omega}_{t_i}$ converges weakly to a limit closed positive $(1,1)$-current $\hat{\omega}$, which a priori depends on the sequence. If we write $\hat{\omega} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}$, it follows that $\varphi_{t_i} \to \hat{\varphi}$ in L^1, and from the bound (3.20) we infer that $\hat{\varphi}$ is in L^∞. Moreover restricting $\hat{\omega}$ to any smooth fiber X_y we see that

$$\sqrt{-1}\partial\bar{\partial}\hat{\varphi}|_{X_y} \geq 0,$$

and the maximum principle implies that $\hat{\varphi}$ is constant on each fiber, and so descends to a bounded function $\hat{\varphi}$ on $Y\backslash f(S)$. We will show that $\hat{\varphi}$ satisfies the same equation (3.68) as ψ, and so by uniqueness $\hat{\varphi} = \psi$. To this end we first fix a compact set

$K \subset Y \backslash f(S)$, and we will show that $\hat{\varphi}$ satisfies (3.68) on K. Since K is arbitrary, this will prove the Theorem.

We then fix η a smooth function with support contained in K, and we will also denote by η its pullback to X via f. Recall that we have called χ the Ricci-flat metric in the class $[\omega_1]$, and $\Omega = \chi^n$. Then from the Monge-Ampère equation (3.17) we have

$$(3.71) \qquad \int_X \eta \Omega = \frac{1}{a_t} \int_X \eta(\omega_0 + t\omega_X + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n,$$

where the constants a_t are equal to

$$\frac{\int_X \omega_t^n}{\int_X \omega_1^n},$$

and behave like (3.18). We can also write

$$(3.72) \qquad \int_X \eta \Omega = \int_X \eta F \omega_{SF}^{n-m} \wedge \omega_0^m.$$

We are now going to estimate $\frac{1}{a_t} \int_X \eta(\omega_0 + t\omega_X + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n$. We have

$$\frac{1}{a_t} \int_X \eta(\omega_0 + t\omega_X + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = \frac{1}{a_t} \int_X \eta \left((\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t}) + (t\omega_X + \sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t})) \right)^n$$

$$= \frac{1}{a_t} \int_X \eta \sum_{k=0}^n \binom{n}{k} (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k \wedge (t\omega_X + \sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^{n-k}$$

First of all observe that the form $\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t}$ is the pullback of a form on Y, and it can be wedged with itself at most m times, so all terms in the sum with $k > m$ are zero. Next, we claim that all the terms with $k < m$ go to zero as $t \to 0$. To see this, start by observing that on the compact set K the estimate (3.53) gives a constant C (that depends on K) such that

$$(3.73) \qquad -C\omega_X \leq \sqrt{-1}\partial\bar{\partial}\varphi_t \leq C\omega_X.$$

Moreover from the equation

$$\partial\bar{\partial}\underline{\varphi_t} = f_*(\partial\bar{\partial}\varphi_t \wedge \omega_X^{n-m})$$

together with (3.73), (3.36), we see that on $f(K)$ we have

$$(3.74) \qquad -C\omega_Y \leq \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t} \leq C\omega_Y.$$

We also need to use (3.33) which on K gives

$$(3.75) \qquad \sup_K |\varphi_t - \underline{\varphi_t}| \leq Ct.$$

Then any term with $k < m$ is equal to

$$\frac{\binom{n}{k}}{a_t} \int_X \eta(\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k \wedge (t\omega_X + \sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^{n-k},$$

and it can be expanded into

$$\frac{\binom{n}{k}}{a_t} \sum_{i=0}^{n-k} \binom{n-k}{i} \int_X \eta(\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k \wedge (t\omega_X)^{n-k-i} \wedge (\sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^i.$$

On K the $(1,1)$-form $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$ is bounded by (3.74). Since $a_t = O(t^{n-m})$ from (3.18), we see that the term in this sum with $i = 0$ goes to zero. Any term with $i > 0$ is comparable to

(3.76)
$$\frac{1}{t^{n-m}} \int_X (\varphi_t - \underline{\varphi_t})\sqrt{-1}\partial\bar{\partial}\eta \wedge (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k \wedge (t\omega_X)^{n-k-i} \wedge (\sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^{i-1}.$$

Notice that all the $(1,1)$-forms appearing inside the integral are bounded by (3.73), (3.74), and that the function $\varphi_t - \underline{\varphi_t}$ is $O(t)$ by (3.75). On K the estimate (3.22) gives

(3.77) $$-Ct\omega_y \leq (\sqrt{-1}\partial\bar{\partial}\varphi_t)|_{X_y} = (\sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))|_{X_y} \leq Ct\omega_y.$$

The form $\sqrt{-1}\partial\bar{\partial}\eta \wedge (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k$ is the pullback of a form from Y, and so we can use (3.77) to estimate

$$\left| \frac{\sqrt{-1}\partial\bar{\partial}\eta \wedge (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k \wedge (t\omega_X)^{n-k-i} \wedge (\sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^{i-1}}{\omega_X^n} \right| \leq Ct^{n-m},$$

and so the term (3.76) goes to zero. This proves our claim.

We are then left with only the term with $k = m$, which is

$$\frac{1}{a_t} \int_X \eta \binom{n}{m} (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^m \wedge (t\omega_X + \sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^{n-m},$$

and if we expand the term $(t\omega_X + \sqrt{-1}\partial\bar{\partial}(\varphi_t - \underline{\varphi_t}))^{n-m}$, we get

$$\frac{1}{a_t} \int_X \eta \binom{n}{m} (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^m \wedge (t\omega_X)^{n-m} + \frac{1}{a_t} \int_X \sqrt{-1}\partial\bar{\partial}\eta \wedge (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^m \wedge \ldots,$$

and the second term is zero because $\partial\bar{\partial}\eta$ is the pullback of a form from the base. We are then left with the term

(3.78) $$\frac{1}{a_t} \int_X \eta \binom{n}{m} (\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^m \wedge (t\omega_X)^{n-m},$$

which we need to further estimate. Using (3.73) we see that, up to taking a further subsequence, the functions φ_{t_i} converge to $\hat{\varphi}$ in the $C^{1,\beta}(K)$ topology, and (3.75) implies that the functions $\underline{\varphi_{t_i}}$ also converge to $\hat{\varphi}$ uniformly. We can then rewrite (3.78) as

$$\frac{t^{n-m}\binom{n}{m}}{a_t}\int_X \eta(\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^m \wedge \omega_X^{n-m}.$$

Using (3.18) we see that as t goes to zero the coefficient $\frac{t^{n-m}\binom{n}{m}}{a_t}$ converges to

$$\frac{\int_X \omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}}.$$

On the other hand we have

$$\int_X \eta(\omega_0 + \sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^m \wedge \omega_X^{n-m} = \sum_{k=0}^{m}\binom{m}{k}\int_X \eta\omega_0^{m-k} \wedge (\sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^k \wedge \omega_X^{n-m}.$$

The term with $k=0$ is independent of t, while any term with $k>0$ can be written as

$$(3.79)\qquad \int_X \underline{\varphi_t}\sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_0^{m-k} \wedge (\sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^{k-1} \wedge \omega_X^{n-m}.$$

The (n,n)-form $\sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_0^{m-k} \wedge (\sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^{k-1} \wedge \omega_X^{n-m}$ is supported in K and is uniformly bounded by (3.74), and the functions $\underline{\varphi_{t_i}}$ converge uniformly to $\hat{\varphi}$, and so along the sequence t_i the term (3.79) has the same limit as

$$\int_X \hat{\varphi}\sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_0^{m-k} \wedge (\sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^{k-1} \wedge \omega_X^{n-m}.$$

But this is equal to

$$\int_X \underline{\varphi_t}\sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_0^{m-k} \wedge (\sqrt{-1}\partial\bar{\partial}\underline{\varphi_t})^{k-2} \wedge \sqrt{-1}\partial\bar{\partial}\hat{\varphi} \wedge \omega_X^{n-m},$$

and repeating the same argument $k-1$ times we see that along the sequence t_i the term (3.79) converges to

$$\int_X \eta\omega_0^{m-k} \wedge (\sqrt{-1}\partial\bar{\partial}\hat{\varphi})^k \wedge \omega_X^{n-m}.$$

It follows that along the sequence t_i the term (3.78) converges to

$$\frac{\int_X \omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}}\int_X \eta(\omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi})^m \wedge \omega_X^{n-m},$$

and using (3.71), (3.72) we get

$$\int_X \eta F \omega_{SF}^{n-m} \wedge \omega_0^m = \frac{\int_X \omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \int_X \eta(\omega_0 + \sqrt{-1}\partial\bar{\partial}\hat{\varphi})^m \wedge \omega_X^{n-m}.$$

We then integrate first along the fibers and get

$$\int_Y \eta F \omega_Y^m \left(\int_{X_y} \omega_{SF,y}^{n-m} \right) = \frac{\int_X \omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \int_Y \eta(\omega_Y + \sqrt{-1}\partial\bar{\partial}\hat{\varphi})^m \left(\int_{X_y} \omega_y^{n-m} \right),$$

and since ω_y is cohomologous to $\omega_{SF,y}$, we get

$$\int_Y \eta F \omega_Y^m = \frac{\int_X \omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \int_Y \eta(\omega_Y + \sqrt{-1}\partial\bar{\partial}\hat{\varphi})^m,$$

which is just the weak form of (3.68). This shows that any weak limit $\hat{\omega}$ of $\tilde{\omega}_t$ as $t \to 0$ satisfies (3.68) weakly, and we have already remarked that we can write $\hat{\omega} = \omega_Y + \sqrt{-1}\partial\bar{\partial}\hat{\varphi}$ with $\hat{\varphi}$ in L^∞. By Kołodziej's uniqueness of L^∞ weak solutions of (3.68) (see [SoT2, Theorem 3.2]), we must have $\hat{\varphi} = \psi$, and so the whole sequence $\tilde{\omega}_t$ converges weakly to ω as $t \to 0$. Then the bound (3.21) implies that φ_t actually converges to ψ in the $C_{loc}^{1,\beta}$ topology on $X \backslash S$. $\qquad \square$

3.6 Examples

In this section we will give some examples where Theorems 3.1.1 and 3.1.2 apply. The constructions are well-known and come from algebraic geometry.

Let's look at the case $n = 2$ first, the case $n = 1$ being trivial. The only projective Calabi-Yau surfaces are tori, bi-elliptic, Enriques and $K3$ surfaces. If X is a torus and L is a nef and big line bundle on X, then L is ample, and so Theorem 3.1.1 is vacuous in this case. Similarly if X is bi-elliptic, then X is a finite unramified quotient of a torus, so a nef and big line bundle on X pulls back to a nef and big line bundle on a torus. But this must be ample, and so the original line bundle is ample too (Corollary 1.2.28 in [La]) and Theorem 3.1.1 is again empty. If X is an Enriques surface, then X is an unramified $2:1$ quotient of a $K3$ surface, so the study of Ricci-flat metrics on X is reduced to the case of a $K3$ surface. Finally let's see that there exist projective $K3$s that admit a nef and big line bundle that is not ample, to which Theorem 3.1.1 applies. For example let Y be the quotient surface T/i where T is the standard torus $\mathbb{C}^2/\mathbb{Z}^4$ and i is induced by the involution $i(z, w) = (-z, -w)$ of $\mathbb{C}^2$. The surface Y has 16 singular points, that are rational double points, and is a Calabi-Yau model. Blowing up these 16 points gives a smooth projective $K3$ surface X (called a Kummer surface), and we can take L to be the pullback of any ample divisor on Y. The set E, being equal to the null locus of L, is readily seen to be

the union of the 16 exceptional divisors, that are (-2)-curves. Then Theorem 3.1.1 applies, and the limit of smooth Ricci-flat metrics on X with classes approaching $c_1(L)$ is the pullback of the unique Ricci-flat (actually flat) orbifold Kähler metric on Y in the given class. This originally appeared as Theorem 8 in [KT]. Now we show that conversely all examples of Theorem 3.1.1 on $K3$ surfaces with $\alpha = c_1(L)$ are of the form $f : X \to Y$ where Y is an orbifold $K3$ surface, $kL = f^*A$, for some $k \geq 1$ and some A ample divisor on Y. Let X be a projective $K3$ surface and L a nef and big line bundle on X. By Theorem 3.2.4 we know that some power kL is globally generated, and we might as well assume that $k = 1$. Then the contraction map f of L contracts an irreducible curve C to a point if and only if $C \cdot L = 0$. But since $L \cdot L > 0$, the Hodge Index theorem implies that $C \cdot C < 0$. The long exact sequence in cohomology associated to the sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

gives that $H^1(X, \mathcal{O}(-C)) = 0$. Serre duality on the other hand gives $H^2(X, \mathcal{O}(C)) = H^0(X, \mathcal{O}(-C)) = 0$, and $H^1(X, \mathcal{O}(C)) = H^1(X, \mathcal{O}(-C)) = 0$. Riemann-Roch then gives

$$\dim H^0(X, \mathcal{O}(C)) = 2 + \frac{1}{2} C \cdot C,$$

which implies that $C \cdot C$ must be even. But since $\pi(C) = \frac{C \cdot C}{2} + 1$, the virtual genus of C, is nonnegative, we see that $C \cdot C = -2$. This implies that $\pi(C) = 0$ and so C is a smooth rational curve with self-intersection -2. Then the point $f(C)$ is a rational double point, and so $Y = f(X)$ is an orbifold $K3$ surface. Notice that Ricci-flat orbifold metrics on Y exist by [Y2, KoR].

Now we turn to examples in dimension 3. The first one is known as *conifold* in the physics literature [GMS], and is described in detail in section 1.2 of [Ro], for example. Roughly speaking, it is a 3 dimensional Calabi-Yau model Y that sits in $\mathbb{P}^4$ as a nodal quintic. It has 16 singular points, that are nodes and not of orbifold type. Moreover there exists a small resolution $f : X \to Y$, that is a birational morphism with X a smooth Calabi-Yau threefold, that is an isomorphism outside the preimages of the nodes, which are 16 rational curves. If L is the pullback of any ample divisor on Y, then L is nef and big on X, and the limit of smooth Ricci-flat metrics on X with classes approaching $c_1(L)$ is the pullback of the unique singular Ricci-flat metric on Y, which exists by [EGZ1]. The convergence is smooth on compact sets outside the union of the 16 exceptional curves (which is clearly equal to the null locus of L). There are also other 3 dimensional examples where the singularities of Y are not isolated: one of these is described in Example 4.6 in [Wi1], and Y has a curve C of singularities. Blowing up C gives a Calabi-Yau threefold X; if L is the pullback of any ample divisor on Y, then the null locus of L is the exceptional divisor S which is a smooth surface ruled over C. Again our Theorem 3.1.1 applies, and the convergence is smooth off S.

We now discuss examples of Theorem 3.1.2. The easiest example is a complex torus X fibering over another torus Y of lower dimension. The fibers are also tori and they are all biholomorphic. In this case Ricci-flat metrics are just flat, and if the volume of the fibers is shrunk to zero the flat metrics on X obviously converge to the flat metric on Y. This is of course compatible with Theorem 3.1.2, because in this case the Weil-Petersson metric is identically zero.

To see a more interesting example, let X be an elliptically fibered $K3$ surface, so X comes equipped with a morphism $f : X \to \mathbb{P}^1$ with generic fibers elliptic curves. Then the pullback of an ample line bundle on $\mathbb{P}^1$ gives a nef line bundle L on X with Iitaka dimension 1. In the case when all the singular fibers of f are of Kodaira type I_1, Gross-Wilson have shown in [GW] that sequences of Ricci-flat metrics on X whose class approaches $c_1(L)$ converge in C^∞ on compact sets of the complement of the singular fibers to the pullback of a Kähler metric on $\mathbb{P}^1$ (minus the 24 points which correspond to the singular fibers). Their argument relies on explicit model metrics that are almost Ricci-flat, and so it is not well-suited to generalization to higher dimensions. More recently Song-Tian [SoT1] gave a more direct proof of the result of Gross-Wilson (with a weaker convergence) and they noticed that the limit metric has Ricci curvature equal to the Weil-Petersson metric. Our Theorem 3.1.2 applies in this example, as well as in higher dimensions, although the convergence that we prove is weaker than C^∞. We conjecture that C^∞ convergence always holds.

If we go back to the example of the torus, we can now choose a class on the boundary of the Kähler cone of X which is not rational (i.e. it is not $c_1(L)$ for any line bundle L over X). In this case there won't be a fibration structure, but rather a foliation : the limit cohomology class can be represented by a Hermitian matrix which is nonnegative definite, and its kernel gives an irrational foliation of the torus. In this case the Ricci-flat metrics converge to a smooth nonnegative form, which is positive definite in the directions transversal to the foliation. Theorem 3.1.2 does not apply here, and it would be very interesting to prove that a similar behaviour occurs in general. We still expect the Ricci-flat metrics to converge smoothly on compact sets outside a subvariety E to a limit nonnegative form ω, whose determinant vanishes identically. The kernel of ω would then define a complex foliation with singularities on $X \backslash E$, whose leaves might be dense in X. The leaves of the foliation are always complex submanifolds, but they might not vary holomorphically and the rank of the foliation might change on different open sets: an example of McMullen [McM] on a nonalgebraic $K3$ surface shows that the Ricci-flat metrics can converge smoothly to zero on an open set of X.

Notice that if the curvature is uniformly bounded, then a result of Ruan [Rua] implies that this picture is basically true and moreover that the foliation is holomorphic, so that its rank is constant on a Zariski open set. In McMullen's example the curvature blows up, and the resulting foliation is not holomorphic, thus showing that Ruan's result doesn't hold if the curvature is unbounded.

Chapter 4

Symplectic Calabi-Yau equation

In this chapter we prove some estimates for the Calabi-Yau equation on a symplectic manifold. In section 4.1 we provide the necessary background and state our main results Theorems 4.1.3 and 4.1.4. In section 4.2 we explain the formalism of moving frames and the canonical connection on almost-Hermitian manifolds. In section 4.3 we estimate a metric solving the Calabi-Yau equation in terms of a scalar function. In section 4.4 we estimate the first derivatives of the metric in terms of the metric itself. Theorems 4.1.3 and 4.1.4 are proved in section 4.5 and 4.6 respectively. The key new ideas are: to use the canonical connection instead of the Levi-Civita connection, and to use moving frames instead of normal coordinates, to perform the computations needed to apply the maximum principle. And to use two new Moser-iteration type arguments to prove the main theorems.

The results of this chapter are joint work with B. Weinkove and S.-T. Yau and can be found in [TWY]. A survey of these and related topics is [TW].

4.1 Symplectic Calabi-Yau equation

Calabi's conjecture [Ca1], proved thirty years ago by Yau [Y2], states that any representative of the first Chern class of a compact Kähler manifold (M, ω) can be uniquely represented as the Ricci curvature of a Kähler metric in a fixed cohomology class. This can be restated in terms of volume forms as follows. For any volume form σ satisfying $\int_M \sigma = \int_M \omega^n$, there exists a unique Kähler form $\tilde{\omega}$ in $[\omega]$ solving

$$(4.1) \qquad\qquad\qquad \tilde{\omega}^n = \sigma,$$

where n is the complex dimension of the manifold. We call (4.1) the Calabi-Yau equation.

Recently, Donaldson [Do5] has described how the Calabi-Yau theory could be generalized in a natural way in the setting of two-forms on four-manifolds. His program, if carried out, would lead to many new and exciting results in symplectic

geometry. A necessary element of this program is to obtain estimates for the Calabi-Yau equation on symplectic four-manifolds with a compatible but non-integrable almost complex structure. In this chapter we will make some progress towards Donaldson's program by showing that the estimates for (4.1) can be reduced to an integral bound of the potential function, and that all the estimates indeed hold under a curvature assumption.

Before stating the results precisely, we will recall some basic terminology. An almost-Kähler manifold is a symplectic manifold (M, ω) together with a compatible almost complex structure J, meaning that ω and J satisfy the two conditions

$$(4.2) \qquad \omega(X, JX) \ > \ 0, \quad \text{for all } X \neq 0$$
$$(4.3) \qquad \omega(JX, JY) \ = \ \omega(X, Y), \quad \text{for all } X, Y.$$

Associated to this data is a Riemannian metric g given by $g(X, Y) = \omega(X, JY)$. We call ω an almost-Kähler form, and g an almost-Kähler metric. On the other hand, if the first condition (4.2) holds, but not necessarily the second (4.3), then we say that ω tames J. In this case, we can still define a Riemannian metric g by

$$g(X, Y) = \frac{1}{2} \left(\omega(X, JY) + \omega(Y, JX) \right).$$

Observe that g is an almost-Hermitian metric, meaning that $g(JX, JY) = g(X, Y)$ for all vectors X and Y.

In [Do5], Donaldson made the following conjecture.

Conjecture 4.1.1. *Let (M, Ω) be a compact symplectic four-manifold equipped with an almost complex structure J tamed by Ω. Let σ be a smooth volume form on M. If $\tilde{\omega} \in [\Omega]$ is a symplectic form on M which is compatible with J and solves the Calabi-Yau equation*

$$(4.4) \qquad \qquad \tilde{\omega}^2 = \sigma,$$

then there are C^∞ a priori bounds on $\tilde{\omega}$ depending only on Ω, J and σ.

More precisely, we have the following. For each $k = 0, 1, 2, \ldots$, there exists a constant A_k depending smoothly on the data Ω, J and σ such that

$$(4.5) \qquad \qquad \|\tilde{\omega}\|_{C^k(g_\Omega)} \leq A_k.$$

If this conjecture were to hold, it would imply, by the arguments of [Do5] (see also the description in [TW]), the following result.

Conjecture 4.1.2. *Let M be a compact 4-manifold with $b^+(M) = 1$ and let J be an almost complex structure on M. If there exists a symplectic form on M taming J then there exists a symplectic form compatible with J.*

Moreover, Conjecture 4.1.1 would also imply a Calabi-Yau theorem on almost-Kähler 4-manifolds (M, ω) with $b^+(M) = 1$: given a normalized volume form σ there would exist a unique almost-Kähler form $\tilde{\omega} \in [\omega]$ satisfying $\tilde{\omega}^2 = \sigma$. For other applications of Conjecture 4.1.1, and to see how it relates to Donaldson's broader program, see [Do5] and [TW].

We now state our results. Our first result says that, in any dimension, all the *a priori* bounds for Conjecture 4.1.1 can be reduced to an integral bound of a scalar potential function. Namely, given any symplectic form Ω and almost-Kähler form $\tilde{\omega}$ with $[\tilde{\omega}] = [\Omega]$, define a smooth real-valued function φ by

$$(4.6) \qquad \frac{1}{2n} \tilde{\Delta} \varphi = 1 - \frac{\tilde{\omega}^{n-1} \wedge \Omega}{\tilde{\omega}^n}, \quad \sup_M \varphi = 0,$$

where $\tilde{\Delta}$ is the usual Laplacian on functions associated to the almost-Kähler metric $\tilde{g}$. Then we have the following result.

Theorem 4.1.3. *Let $\alpha > 0$ be given. Let M be a compact $2n$-manifold equipped with an almost complex structure J and a taming symplectic form Ω. Let σ be a smooth volume form on M with $\int_M \sigma = \int_M \Omega^n$. Then if $\tilde{\omega}$ is an almost-Kähler form with $[\tilde{\omega}] = [\Omega]$ and solving the Calabi-Yau equation*

$$(4.7) \qquad \tilde{\omega}^n = \sigma,$$

there are C^∞ a priori bounds on $\tilde{\omega}$ depending only on Ω, J, σ, α and

$$I_\alpha(\varphi) := \int_M e^{-\alpha\varphi} \Omega^n,$$

for φ defined by (4.6).

We make a few remarks. The function φ is precisely the usual Kähler potential in the case that $\tilde{\omega}$ and Ω are Kähler forms, and it coincides with the 'almost-Kähler potential' φ_1 in the terminology of [We2] if they are both almost-Kähler. A general result in Kähler geometry [Hö, Ti1], which is independent of the Calabi-Yau equation, says the quantity $I_\alpha(\varphi)$ is always uniformly bounded if Ω and $\tilde{\omega}$ are Kähler, as long as α is sufficiently small (where the bounds depend only on M, Ω, J and α). Indeed, the supremum of all such α so that this quantity can be bounded independent of $\tilde{\omega} \in [\Omega]$ is known as the alpha-invariant and has been much studied [Ti1, TiY]. In particular this gives a different proof of the C^0 estimate of Yau's theorem in the Kähler case.

As remarked in [Do5], Conjecture 4.1.2 is false in dimensions six or higher. The deformation argument used to infer it from the first conjecture crucially uses four dimensions. It is still possible, as far as we know, for Conjecture 4.1.1 to hold in all dimensions. However it is also quite possible that a four dimensional argument will be needed to remove the dependence on $I_\alpha(\varphi)$ in Theorem 4.1.3.

Now let g be the almost-Hermitian metric associated to Ω and J. There exists a *canonical connection* ∇ associated to (M, J, g). This differs from the Levi-Civita connection, and it is described in section 4.2. Under a positivity condition on the curvature of this connection, we can solve Donaldson's conjecture. More precisely, define a tensor

$$\mathcal{R}_{i\bar{j}k\bar{\ell}}(g, J) = R^j_{ik\bar{\ell}} + 4N^r_{\ell\bar{j}}\overline{N^i_{\bar{r}\bar{k}}},$$

where $R^j_{ik\bar{\ell}}$ is the (1,1) part of the curvature of ∇ and N represents the Nijenhuis tensor (for precise definitions, see section 4.2). We write $\mathcal{R} \geq 0$ if the tensor $\mathcal{R}_{i\bar{j}k\bar{\ell}}$ is nonnegative in the Griffiths sense: $\mathcal{R}_{i\bar{j}k\bar{\ell}}X^i\overline{X^j}Y^k\overline{Y^\ell} \geq 0$ for all (1,0) vectors X and Y.

Theorem 4.1.4. *If $\mathcal{R}(g, J) \geq 0$, Conjecture 4.1.1 holds.*

In fact under this condition we can prove Conjecture 4.1.1 in any dimension $2n$. Note that if g were Kähler and the bisectional curvature of g positive, then we would have $\mathcal{R} > 0$. Hence the condition holds on $\mathbb{CP}^n$ if the pair (Ω, J) is not too far from the Fubini-Study symplectic form paired with the standard complex structure.

It will be convenient to reformulate Donaldson's conjecture as follows. Let $\tilde{g}$ be an almost-Kähler metric with Kähler form $\tilde{\omega}$ satisfying (4.1). Write $\sigma/n! = e^F dV_g$ where dV_g is the volume form associated to g and F is a smooth function on M. Then (4.1) can be written locally as

$$(4.8) \qquad\qquad \det \tilde{g} = e^{2F} \det g,$$

Finding bounds on $\tilde{g}$ depending only on g, J and F is equivalent to solving the conjecture.

4.2 Almost-Hermitian manifolds and the canonical connection

In this section, we give some background on almost-Hermitian manifolds, almost- and quasi-Kähler manifolds, the canonical connection and its torsion and curvature.

4.2.1 Almost-Hermitian metrics and connections

Let M be a manifold of dimension $2n$ with an almost complex structure J and a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y),$$

for all tangent vectors X and Y. We say that (M, J, g) is an *almost-Hermitian manifold*.

Write $T_p^{\mathbb{R}} M$ for the (real) tangent space of M at a point p. In the following we will drop the subscript p. Denote the complexified tangent space by $T^{\mathbb{C}} M = T^{\mathbb{R}} M \otimes \mathbb{C}$. Extending g and J linearly to $T^{\mathbb{C}} M$, we see that the complexified tangent space can be decomposed as

$$T^{\mathbb{C}} M = T' M \oplus T'' M,$$

where $T' M$ and $T'' M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Extending J to forms, we can uniquely decompose m-forms into (p, q)-forms for each p, q with $p + q = m$.

Choose a local unitary frame $\{e_1, \ldots, e_n\}$ for $T' M$ with respect to the Hermitian inner product induced from g, and let $\{\theta^1, \ldots, \theta^n\}$ be a dual coframe. The metric g can be written as

$$g = \theta^i \otimes \overline{\theta^i} + \overline{\theta^i} \otimes \theta^i.$$

Remark 4.2.1 Here and henceforth, we sum over pairs of repeated indices. Such pairs of indices can appear either as one raised and one lowered or, otherwise, one barred and one unbarred.

Let ∇ be an affine connection on $T^{\mathbb{R}} M$, which we extend linearly to $T^{\mathbb{C}} M$. We say that ∇ is an *almost-Hermitian connection* if

$$\nabla J = \nabla g = 0.$$

This is equivalent to a connection on the principal $U(n)$-bundle of unitary frames over M. Since every principal bundle has a connection (see e.g., [KN]), we see that:

Lemma 4.2.1. *Almost-Hermitian connections always exist on almost-Hermitian manifolds.*

From now on, assume that ∇ is almost-Hermitian. Observe that for $i = 1, \ldots, n$,

$$J(\nabla e_i) = \sqrt{-1} \nabla e_i,$$

and hence $\nabla e_i \in T' M \otimes (T^{\mathbb{C}}(M))^*$. Then locally there exists a matrix of complex valued 1-forms $\{\theta_i^j\}$, called the *connection 1-forms*, such that

$$\nabla e_i = \theta_i^j e_j.$$

Applying ∇ to $g(e_i, \overline{e_j})$ and using the condition $\nabla g = 0$ we see that $\{\theta_i^j\}$ satisfies the skew-Hermitian property

$$\theta_i^j + \overline{\theta_j^i} = 0.$$

Now define the *torsion* $\Theta = (\Theta^1, \ldots, \Theta^n)$ of ∇ by

(4.9) $$d\theta^i = -\theta_j^i \wedge \theta^j + \Theta^i, \qquad \text{for } i = 1, \ldots, n.$$

Notice that the Θ^i are 2-forms. Equation (4.9) is known as the *first structure equation*. Define the *curvature* $\Psi = \{\Psi^i_j\}$ of ∇ by

$$(4.10) \qquad\qquad d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Psi^i_j.$$

Note that $\{\Psi^i_j\}$ is a skew-Hermitian matrix of 2-forms. Equation (4.10) is known as the *second structure equation*.

4.2.2 The canonical connection

We have the following lemma (see e.g., [Ga, KoS]).

Lemma 4.2.2. *There exists a unique almost-Hermitian connection ∇ on (M, J, g) whose torsion Θ has everywhere vanishing $(1, 1)$ part.*

Such a connection is known as the *second canonical connection* and was first introduced by Ehresmann and Libermann in [EL]. It is also sometimes referred to as the *Chern connection*, since when J is integrable it coincides with the connection defined in [Ch]. We will call it simply the *canonical connection*.

Define functions T^i_{jk} and $N^i_{\bar{j}\,\bar{k}}$ by

$$\begin{aligned}
(\Theta^i)^{(2,0)} &= T^i_{jk}\theta^j \wedge \theta^k \\
(\Theta^i)^{(0,2)} &= N^i_{\bar{j}\,\bar{k}}\overline{\theta^j} \wedge \overline{\theta^k},
\end{aligned}$$

with $T^i_{jk} = -T^i_{kj}$ and $N^i_{\bar{j}\,\bar{k}} = -N^i_{\bar{k}\,\bar{j}}$.

Lemma 4.2.3. *The (0,2) part of the torsion is independent of the choice of metric.*

Indeed $(\Theta^i)^{(0,2)}$ can be regarded as the Nijenhuis tensor of J. For a proof of this lemma, see section 4.3.

Let's consider now the real $(1, 1)$ form

$$\omega = \sqrt{-1}\theta^i \wedge \overline{\theta^i}.$$

We say that (M, J, g) is *almost-Kähler* if $d\omega = 0$, and that it is *quasi-Kähler* if $(d\omega)^{(1,2)} = 0$. An almost-Kähler or quasi-Kähler manifold with J integrable is a Kähler manifold. Observe from the first structure equation,

$$\begin{aligned}
d\omega &= \sqrt{-1}(\Theta^i \wedge \overline{\theta^i} - \theta^i \wedge \overline{\Theta^i}) \\
&= \sqrt{-1}(N^i_{\bar{j}\,\bar{k}}\overline{\theta^i} \wedge \overline{\theta^j} \wedge \overline{\theta^k} - \overline{N^i_{\bar{j}\,\bar{k}}}\theta^i \wedge \theta^j \wedge \theta^k \\
&\qquad + T^i_{jk}\overline{\theta^i} \wedge \theta^j \wedge \theta^k - \overline{T^i_{jk}}\theta^i \wedge \overline{\theta^j} \wedge \overline{\theta^k}).
\end{aligned}$$

Thus we have the following alternative definitions using the torsion of the canonical connection.

Lemma 4.2.4. *An almost-Hermitian manifold* (M, J, g) *is almost-Kähler if and only if*

$$T^i_{jk} = 0,$$

(4.11)
$$N_{\bar{i}\bar{j}\bar{k}} + N_{\bar{j}\bar{k}\bar{i}} + N_{\bar{k}\bar{i}\bar{j}} = 0,$$

where $N_{\bar{i}\bar{j}\bar{k}} = N^i_{\bar{j}\bar{k}}$, *and is quasi-Kähler if and only if*

$$T^i_{jk} = 0.$$

In particular on a quasi-Kähler manifold the torsion of the canonical connection has only a $(0, 2)$ component

$$\Theta^i = N^i_{\bar{j}\bar{k}}\overline{\theta^j} \wedge \overline{\theta^k}.$$

4.2.3 Curvature identities

Let (M, J, g) be an almost-Hermitian manifold and let ∇ be the canonical connection with torsion Θ and curvature Ψ. Define $R^j_{ik\bar{\ell}}$, $K^i_{jk\ell}$ and $K^i_{j\bar{k}\bar{\ell}}$ by

$$
\begin{aligned}
(\Psi^j_i)^{(1,1)} &= R^j_{ik\bar{\ell}}\theta^k \wedge \overline{\theta^\ell} \\
(\Psi^i_j)^{(2,0)} &= K^i_{jk\ell}\theta^k \wedge \theta^\ell \\
(\Psi^i_j)^{(0,2)} &= K^i_{j\bar{k}\bar{\ell}}\overline{\theta^k} \wedge \overline{\theta^\ell},
\end{aligned}
$$

with $K^i_{jk\ell} = -K^i_{j\ell k}$ and $K^i_{j\bar{k}\bar{\ell}} = -K^i_{j\bar{\ell}\bar{k}}$. We define the *Ricci curvature* and *scalar curvature* of the canonical connection to be the tensors $R_{k\bar{\ell}} = R^i_{ik\bar{\ell}}$ and $R = R_{k\bar{k}}$ respectively.

We will now derive some curvature identities. Applying the exterior derivative to the first and second structure equations, we obtain the *first Bianchi identity*,

(4.12)
$$d\Theta^i = \Psi^i_j \wedge \theta^j - \theta^i_j \wedge \Theta^j,$$

and *second Bianchi identity*,

(4.13)
$$d\Psi^i_j = \Psi^i_k \wedge \theta^k_j - \theta^i_k \wedge \Psi^k_j.$$

Let us rewrite these. First, define $T^i_{jk,p}$, $T^i_{jk,\bar{p}}$ by

(4.14)
$$dT^i_{jk} + \theta^i_p T^p_{jk} - T^i_{pk}\theta^p_j - T^i_{jp}\theta^p_k = T^i_{jk,p}\theta^p + T^i_{jk,\bar{p}}\overline{\theta^p},$$

and $N^i_{\bar{j}\bar{k},p}$ and $N^i_{\bar{j}\bar{k},\bar{p}}$ by

(4.15)
$$dN^i_{\bar{j}\bar{k}} + \theta^i_p N^p_{\bar{j}\bar{k}} - N^i_{\bar{p}\bar{k}}\overline{\theta^p_j} - N^i_{\bar{j}\bar{p}}\overline{\theta^p_k} = N^i_{\bar{j}\bar{k},p}\theta^p + N^i_{\bar{j}\bar{k},\bar{p}}\overline{\theta^p}.$$

Then the first Bianchi identity can be written as

$$dT^i_{jk} \wedge \theta^j \wedge \theta^k - T^i_{jk}\theta^j_p \wedge \theta^p \wedge \theta^k + T^i_{jk}\Theta^j \wedge \theta^k + T^i_{jk}\theta^j \wedge \theta^k_p \wedge \theta^p$$

$$-T^i_{jk}\theta^j \wedge \Theta^k + dN^i_{\bar{j}\,\bar{k}} \wedge \overline{\theta^j} \wedge \overline{\theta^k} - N^i_{\bar{j}\,\bar{k}}\overline{\theta^j_p} \wedge \overline{\theta^p} \wedge \overline{\theta^k}$$

$$+N^i_{\bar{j}\,\bar{k}}\overline{\theta^j} \wedge \overline{\theta^k_p} \wedge \overline{\theta^p} + N^i_{\bar{j}\,\bar{k}}\overline{\Theta^j} \wedge \overline{\theta^k} - N^i_{\bar{j}\,\bar{k}}\overline{\theta^j} \wedge \overline{\Theta^k}$$

$$= K^i_{jk\ell}\theta^k \wedge \theta^\ell \wedge \theta^j + R^i_{jk\bar{\ell}}\theta^k \wedge \overline{\theta^\ell} \wedge \theta^j + K^i_{j\bar{k}\,\bar{\ell}}\overline{\theta^k} \wedge \overline{\theta^\ell} \wedge \theta^j$$

$$-T^j_{k\ell}\theta^i_j \wedge \theta^k \wedge \theta^\ell - N^j_{\bar{k}\,\bar{\ell}}\theta^i_j \wedge \overline{\theta^k} \wedge \overline{\theta^\ell}.$$

After substituting from (4.14) and (4.15), and comparing bidegrees, we arrive at the following four identities:

$$(T^i_{jk,\ell} + 2T^i_{pj}T^p_{k\ell} - K^i_{jk\ell})\theta^j \wedge \theta^k \wedge \theta^\ell \;=\; 0$$

$$(T^i_{jk,\bar{\ell}} + 2\overline{N^p_{\bar{j}\,\bar{k}}}N^i_{\bar{p}\,\ell} - R^i_{jk\bar{\ell}})\theta^j \wedge \theta^k \wedge \overline{\theta^\ell} \;=\; 0$$

$$(2T^i_{pj}N^p_{\bar{k}\,\bar{\ell}} + N^i_{\bar{k}\,\bar{\ell},j} - K^i_{j\bar{k}\,\bar{\ell}})\theta^j \wedge \overline{\theta^k} \wedge \overline{\theta^\ell} \;=\; 0$$

$$(N^i_{\bar{j}\,\bar{k},\bar{\ell}} + 2N^i_{\bar{p}\,\bar{j}}\overline{T^p_{k\ell}})\overline{\theta^j} \wedge \overline{\theta^k} \wedge \overline{\theta^\ell} \;=\; 0,$$

which are equivalent to:

$$(4.16) \quad T^i_{jk,\ell} + T^i_{k\ell,j} + T^i_{\ell j,k} + 2T^i_{pj}T^p_{k\ell} + 2T^i_{pk}T^p_{\ell j} + 2T^i_{p\ell}T^p_{jk} \;=\; K^i_{jk\ell} + K^i_{k\ell j} + K^i_{\ell jk}$$

$$(4.17) \qquad\qquad\qquad\qquad\qquad 2T^i_{jk,\bar{\ell}} + 4\overline{N^p_{\bar{j}\,\bar{k}}}N^i_{\bar{p}\,\ell} \;=\; R^i_{jk\bar{\ell}} - R^i_{kj\bar{\ell}}$$

$$(4.18) \qquad\qquad\qquad\qquad\qquad 2T^i_{pj}N^p_{\bar{k}\,\bar{\ell}} + N^i_{\bar{k}\,\bar{\ell},j} \;=\; K^i_{j\bar{k}\,\bar{\ell}}$$

$$(4.19) \quad N^i_{\bar{j}\,\bar{k},\bar{\ell}} + N^i_{\bar{k}\,\bar{\ell},\bar{j}} + N^i_{\bar{\ell}\,\bar{j},\bar{k}} + 2N^i_{\bar{p}\,\bar{j}}\overline{T^p_{k\ell}} + 2N^i_{\bar{p}\,\bar{k}}\overline{T^p_{\ell j}} + 2N^i_{\bar{p}\,\bar{\ell}}\overline{T^p_{jk}} \;=\; 0.$$

By a similar reasoning, we obtain the following from the second Bianchi identity:

$$(K^i_{jk\ell,p} + 2T^q_{k\ell}K^i_{jqp} - R^i_{jk\bar{q}}\overline{N^q_{\bar{\ell}\,\bar{p}}})\theta^k \wedge \theta^\ell \wedge \theta^p \;=\; 0$$

$$(K^i_{jk\ell,\bar{p}} - R^i_{jk\bar{p},\ell} + R^i_{jq\bar{p}}T^q_{k\ell} + 2K^i_{j\bar{q}\,\bar{p}}\overline{N^q_{\bar{k}\,\bar{\ell}}})\theta^k \wedge \theta^\ell \wedge \overline{\theta^p} \;=\; 0$$

$$(R^i_{jk\bar{\ell},\bar{p}} + K^i_{j\bar{\ell}\,\bar{p},k} + 2K^i_{jqk}N^q_{\bar{\ell}\,\bar{p}} - R^i_{jk\bar{q}}\overline{T^q_{\ell p}})\theta^k \wedge \overline{\theta^\ell} \wedge \overline{\theta^p} \;=\; 0$$

$$(K^i_{j\bar{k}\,\bar{\ell},\bar{p}} + R^i_{jq\bar{k}}N^q_{\bar{\ell}\,\bar{p}} + 2K^i_{j\bar{q}\,\bar{p}}\overline{T^q_{k\ell}})\overline{\theta^k} \wedge \overline{\theta^\ell} \wedge \overline{\theta^p} \;=\; 0,$$

where $K^i_{jk\ell,p}$, $K^i_{jk\ell,\bar{p}}$ etc. are defined in the obvious way. The above four identities

can be rewritten as

$$K^i_{jk\ell,p} + K^i_{j\ell p,k} + K^i_{jpk,\ell} + 2T^q_{k\ell}K^i_{jqp} + 2T^q_{\ell p}K^i_{jqk}$$

(4.20)
$$+ 2T^q_{pk}K^i_{jq\ell} - R^i_{jk\bar{q}}\overline{N^q_{\ell\,\bar{p}}} - R^i_{j\ell\bar{q}}\overline{N^q_{\bar{p}\,k}} - R^i_{jp\bar{q}}\overline{N^q_{\bar{k}\,\ell}} = 0$$

(4.21)
$$2K^i_{jk\ell,\bar{p}} - R^i_{jk\bar{p},\ell} + R^i_{j\ell\bar{p},k} + 2R^i_{jq\bar{p}}T^q_{k\ell} + 4K^i_{j\bar{q}\,\bar{p}}\overline{N^q_{\bar{k}\,\ell}} = 0$$

(4.22)
$$R^i_{jk\bar{\ell},\bar{p}} - R^i_{jk\bar{p},\bar{\ell}} + 2K^i_{j\bar{\ell}\,\bar{p},k} + 4K^i_{jqk}N^q_{\ell\,\bar{p}} - 2R^i_{jk\bar{q}}\overline{T^q_{\ell p}} = 0$$

$$K^i_{j\bar{k}\,\bar{\ell},\bar{p}} + K^i_{j\bar{\ell}\,\bar{p},\bar{k}} + K^i_{j\bar{p}\,\bar{k},\bar{\ell}} + R^i_{jq\bar{k}}N^q_{\ell\,\bar{p}} + R^i_{jq\bar{\ell}}N^q_{\bar{p}\,k} + R^i_{jq\bar{p}}N^q_{\bar{k}\,\ell}$$

(4.23)
$$+ 2K^i_{j\bar{q}\,\bar{p}}\overline{T^q_{k\ell}} + 2K^i_{j\bar{q}\,\bar{k}}\overline{T^q_{\ell p}} + 2K^i_{j\bar{q}\,\bar{\ell}}\overline{T^q_{pk}} = 0.$$

Now assume that (M, J, g) is quasi-Kähler, so that the (2,0) part of the torsion vanishes. Then (4.17), (4.18) and (4.21) above simplify to

(4.24)
$$4\overline{N^p_{\bar{j}\,k}}N^i_{\bar{p}\,\ell} = R^i_{jk\bar{\ell}} - R^i_{kj\bar{\ell}},$$

(4.25)
$$N^i_{\bar{k}\,\bar{\ell},j} = K^i_{j\bar{k}\,\bar{\ell}},$$

(4.26)
$$2K^i_{jk\ell,\bar{p}} + 4K^i_{j\bar{q}\,\bar{p}}\overline{N^q_{\bar{k}\,\ell}} = R^i_{jk\bar{p},\ell} - R^i_{j\ell\bar{p},k}.$$

Recall that the curvature matrix $\{\Psi^i_j\}$ is skew-Hermitian, hence

(4.27)
$$K^i_{jk\ell} = \overline{K^j_{i\ell\,\bar{k}}}, \quad R^i_{jk\bar{\ell}} = \overline{R^j_{i\ell\bar{k}}}.$$

From this we compute

$$
\begin{aligned}
R^i_{jk\bar{\ell}} &= R^i_{kj\bar{\ell}} + 4N^i_{\bar{p}\,\bar{\ell}}\overline{N^p_{\bar{j}\,k}} \\
&= \overline{R^k_{i\ell\bar{j}}} + 4N^i_{\bar{p}\,\bar{\ell}}\overline{N^p_{\bar{j}\,k}} \\
&= \overline{R^k_{\ell i\bar{j}}} + 4N^i_{\bar{p}\,\bar{\ell}}\overline{N^p_{\bar{j}\,k}} + 4N^p_{\bar{i}\,\bar{\ell}}\overline{N^k_{\bar{p}\,\bar{j}}} \\
&= R^\ell_{kj\bar{i}} + 4N^i_{\bar{p}\,\bar{\ell}}\overline{N^p_{\bar{j}\,k}} + 4N^p_{\bar{i}\,\bar{\ell}}\overline{N^k_{\bar{p}\,\bar{j}}},
\end{aligned}
$$

(4.28)

giving us the following formula for the Ricci curvature

(4.29)
$$R_{k\bar{\ell}} = R^i_{ik\bar{\ell}} = R^\ell_{ki\bar{i}} + 4N^i_{\bar{p}\,\bar{\ell}}\overline{N^p_{\bar{i}\,k}} + 4N^p_{\bar{i}\,\bar{\ell}}\overline{N^k_{\bar{p}\,i}}.$$

4.2.4 The canonical Laplacian

Suppose that (M, J, g) is almost-Hermitian and let ∇ be its canonical connection. Let f be a function on M. We define the *canonical Laplacian* Δ of f by

$$\Delta f = \sum_i \left((\nabla \nabla f)(e_i, \overline{e_i}) + (\nabla \nabla f)(\overline{e_i}, e_i) \right).$$

This expression is independent of the choice of unitary frame.

Define f_i and $f_{\bar{i}}$ by

$$(4.30) \qquad\qquad df = f_i \theta^i + f_{\bar{i}} \overline{\theta^i}.$$

Writing ∂f and $\overline{\partial} f$ for the (1,0) and (0,1) parts of df respectively we see that $\partial f = f_i \theta^i$ and $\overline{\partial} f = f_{\bar{i}} \overline{\theta^i}$. Applying the exterior derivative to (4.30) and using the first structure equation we obtain

$$
\begin{aligned}
0 &= df_i \wedge \theta^i - f_i \theta^i_j \wedge \theta^j + f_i \Theta^i + df_{\bar{i}} \wedge \theta^i - f_{\bar{i}} \overline{\theta^i_j} \wedge \overline{\theta^j} + f_{\bar{i}} \overline{\Theta^i} \\
(4.31) \qquad &= (df_i - f_j \theta^j_i) \wedge \theta^i + (df_{\bar{i}} - f_{\bar{j}} \overline{\theta^j_i}) \wedge \overline{\theta^i} + f_i \Theta^i + f_{\bar{i}} \overline{\Theta^i}.
\end{aligned}
$$

Define f_{ik}, $f_{i\bar{k}}$, $f_{\bar{i}k}$ and $f_{\bar{i}\bar{k}}$ by

$$
\begin{aligned}
df_i - f_j \theta^j_i &= f_{ik} \theta^k + f_{i\bar{k}} \overline{\theta^k} \\
df_{\bar{i}} - f_{\bar{j}} \overline{\theta^j_i} &= f_{\bar{i}k} \theta^k + f_{\bar{i}\bar{k}} \overline{\theta^k}.
\end{aligned}
$$

Taking the (1,1) part of (4.31) we see that

$$f_{i\bar{k}} \overline{\theta^k} \wedge \theta^i + f_{\bar{i}k} \theta^k \wedge \overline{\theta^i} = 0,$$

and hence

$$f_{i\bar{k}} = f_{\bar{k}i}.$$

Now calculate

$$
\begin{aligned}
\nabla \nabla f &= \nabla(f_i \theta^i + f_{\bar{i}} \overline{\theta^i}) \\
&= df_i \otimes \theta^i - f_i \theta^i_j \otimes \theta^j + df_{\bar{i}} \otimes \overline{\theta^i} - f_{\bar{i}} \overline{\theta^i_j} \otimes \overline{\theta^j} \\
&= (f_{ij} \theta^j + f_{i\bar{j}} \overline{\theta^j}) \otimes \theta^i + (f_{\bar{i}j} \theta^j + f_{\bar{i}\bar{j}} \overline{\theta^j}) \otimes \overline{\theta^i}.
\end{aligned}
$$

Hence

$$(4.32) \qquad\qquad \Delta f = f_{i\bar{i}} + f_{\bar{i}i} = 2 f_{i\bar{i}}.$$

There are other ways of writing Δf.

Lemma 4.2.5.

$$(4.33) \qquad \Delta f = -2\sum_i (d\partial f)^{(1,1)}(e_i, \overline{e_i})$$

$$(4.34) \qquad = 2\sum_i (d\overline{\partial}f)^{(1,1)}(e_i, \overline{e_i})$$

$$(4.35) \qquad = -\sqrt{-1}\sum_i (d(Jdf))^{(1,1)}(e_i, \overline{e_i}),$$

where J acts on a 1-form α by $(J\alpha)(X) = \alpha(J(X))$ for a vector X.

Proof. Calculate

$$\begin{aligned} d\partial f &= d(f_i\theta^i) \\ &= (f_{ik}\theta^k + f_{i\overline{k}}\overline{\theta^k} + f_j\theta_i^j) \wedge \theta^i - f_i\theta_j^i \wedge \theta^j + f_i\Theta^i \\ (4.36) \qquad &= f_{ik}\theta^k \wedge \theta^i + f_{i\overline{k}}\overline{\theta^k} \wedge \theta^i + f_i\Theta^i. \end{aligned}$$

Hence

$$(4.37) \qquad (d\partial f)^{(1,1)} = -f_{i\overline{k}}\theta^i \wedge \overline{\theta^k},$$

and (4.33) follows from (4.32). For (4.34), just observe that $\partial = d - \overline{\partial}$ and $d^2 = 0$. For (4.35), recall that $J\theta^i = -\sqrt{-1}\theta^i$. Then

$$\begin{aligned} d(Jdf) &= d(J(f_i\theta^i + f_{\overline{i}}\overline{\theta^i})) \\ &= -\sqrt{-1}d(f_i\theta^i - f_{\overline{i}}\overline{\theta^i}) \\ &= -\sqrt{-1}d(\partial f - \overline{\partial}f) \\ &= -2\sqrt{-1}d\partial f, \end{aligned}$$

and this completes the proof. □

Finally we have the following lemma.

Lemma 4.2.6. *If the metric g is quasi-Kähler then the canonical Laplacian is equal to the usual Laplacian of the Levi-Civita connection of g.*

Proof. In fact, the Laplacian of the Levi-Civita connection applied to a function f is given by the trace of the map $F : TM \to TM$ defined by

$$F(X) = \nabla_X(\mathrm{grad}_g f) + \tau(\mathrm{grad}_g f, X),$$

where ∇ is the canonical connection and τ is its torsion (see for example [KN] p.282). But if g is quasi-Kähler τ is just the Nijenhuis tensor, which maps $T''M \otimes T''M \to T'M$ and so the second term above has trace zero.

□

4.3   Estimate of the metric

In this section we will prove an estimate on an almost-Kähler metric $\tilde{g}$ solving (4.8), in terms of the potential function φ. Recall that φ is defined by (4.6), which can be rewritten as

$$(4.38) \qquad\qquad \tilde{\Delta}\varphi = 2n - \operatorname{tr}_{\tilde{g}}g,$$

since

$$(4.39) \qquad\qquad \operatorname{tr}_{\tilde{g}}g = 2n\frac{\tilde{\omega}^{n-1} \wedge \Omega}{\tilde{\omega}^n}.$$

To see (4.39), observe that

$$g_{ij} = \frac{1}{2}\left(\Omega_{ik}J_j{}^k + \Omega_{jk}J_i{}^k\right),$$

and so we have

$$\operatorname{tr}_{\tilde{g}}g = \tilde{g}^{ij}g_{ij} = \tilde{J}^{ik}\Omega_{ik},$$

where $\tilde{J}^{ik} = \tilde{g}^{il}J_l{}^k$. Working in a coordinate system in which $\tilde{\omega} = 2(dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n})$ and $\tilde{g}_{ij} = \delta_{ij}$ at a fixed point p in M we see that

$$\tilde{J}^{ik}\Omega_{ik} = 2n\frac{\tilde{\omega}^{n-1} \wedge \Omega}{\tilde{\omega}^n},$$

as required.

The estimate we wish to prove in this section is:

Theorem 4.3.1. *Let $\tilde{g}$ be an almost-Kähler metric solving the Calabi-Yau equation (4.8), where g is an almost-Hermitian metric. Then there exist constants C and A depending only on J, R, the lower bound of $\mathcal{R}_{i\bar{j}k\bar{l}}$, $\sup|F|$ and the lower bound of ΔF such that*

$$\operatorname{tr}_g\tilde{g} \le Ce^{A(\varphi-\inf_M \varphi)}.$$

We introduce some notation. Let (M, J) be an almost complex manifold with two almost-Hermitian metrics g and $\tilde{g}$. Let θ^i and $\tilde{\theta}^i$ be local unitary coframes for g and $\tilde{g}$ respectively. Denote by ∇ and $\tilde{\nabla}$ the associated canonical connections. We will use $\tilde{\Theta}$, $\tilde{\Psi}$ etc. to denote the torsion, curvature and so on with respect to $\tilde{\nabla}$. Define local matrices (a_j^i) and (b_j^i) by

$$(4.40) \qquad\qquad \tilde{\theta}^i \;=\; a_j^i\theta^j$$
$$(4.41) \qquad\qquad \theta^j \;=\; b_i^j\tilde{\theta}^i,$$

so that $a_j^i b_i^k = \delta_j^k$. Define a function u by

$$u = a_j^i\overline{a_j^i} = \frac{1}{2}\operatorname{tr}_g\tilde{g}.$$

We will prove Theorem 4.3.1 using the maximum principle. The key result which we need for this is the following lemma.

Lemma 4.3.2. *Suppose that g is almost-Hermitian and $\tilde{g}$ is quasi-Kähler and solves the Calabi-Yau equation (4.8). Then*

$$(4.42) \qquad \tilde{\Delta} \log u \geq \frac{1}{u}\left(\Delta F - 2R + 8N^{\ell}_{\bar{p}i}\overline{N^{p}_{\bar{\ell}\bar{i}}} + 2\overline{a^{p}_{i}}a^{p}_{j}b^{k}_{q}\overline{b^{\ell}_{q}}\mathcal{R}_{i\bar{j}k\bar{\ell}} \right),$$

where $\mathcal{R}_{i\bar{j}k\bar{\ell}} = R^{j}_{ik\bar{\ell}} + 4N^{r}_{\bar{\ell}\bar{j}}\overline{N^{i}_{\bar{r}\bar{k}}}$.

To prove this lemma, we use some general identities (Lemmas 4.3.3 and 4.3.4 below) which are independent of the Calabi-Yau equation.

First, differentiating (4.40) and using the first structure equations we obtain

$$-\tilde{\theta}^{i}_{k} \wedge \tilde{\theta}^{k} + \tilde{\Theta}^{i} \quad = \quad da^{i}_{j} \wedge \theta^{j} - a^{i}_{j}\theta^{j}_{k} \wedge \theta^{k} + a^{i}_{j}\Theta^{j}.$$

Using (4.41) and rearranging, we have

$$(4.43) \qquad (b^{j}_{k}da^{i}_{j} - a^{i}_{j}b^{\ell}_{k}\theta^{j}_{\ell} + \tilde{\theta}^{i}_{k}) \wedge \tilde{\theta}^{k} \quad = \quad \tilde{\Theta}^{i} - a^{i}_{j}\Theta^{j}.$$

Taking the $(0,2)$ part of this equation, we see that

$$(4.44) \qquad \tilde{N}^{i}_{\bar{j}\,\bar{k}} = \overline{b^{r}_{j}b^{s}_{k}}a^{i}_{t}N^{t}_{\bar{r}\,\bar{s}},$$

which shows that the $(0,2)$ part of the torsion is independent of the choice of the metric (thus giving the proof of Lemma 4.2.3).

By the definition of the canonical connection, the right hand side of (4.43) has no (1,1)-part. Hence there exist functions $a^{i}_{k\ell}$ such that

$$(4.45) \qquad b^{j}_{k}da^{i}_{j} - a^{i}_{j}b^{\ell}_{k}\theta^{j}_{\ell} + \tilde{\theta}^{i}_{k} = a^{i}_{k\ell}\tilde{\theta}^{\ell},$$

which can be rewritten as

$$(4.46) \qquad da^{i}_{m} - a^{i}_{j}\theta^{j}_{m} + a^{k}_{m}\tilde{\theta}^{i}_{k} = a^{i}_{k\ell}a^{k}_{m}\tilde{\theta}^{\ell}.$$

Note that $a^{i}_{k\ell}\tilde{e}_{i}\tilde{\theta}^{k}\tilde{\theta}^{\ell}$ can be interpreted as the difference of the two connections $\tilde{\nabla} - \nabla$. Also, if g and $\tilde{g}$ are quasi-Kähler, from (4.43) we see that we have $a^{i}_{k\ell} = a^{i}_{\ell k}$. We will now calculate a formula for $\tilde{\Delta}u$.

Lemma 4.3.3. *For g and $\tilde{g}$ almost-Hermitian metrics, and a^{i}_{j}, $a^{i}_{k\ell}$, b^{i}_{j} as defined above, we have*

$$\frac{1}{2}\tilde{\Delta}u = a^{i}_{k\ell}\overline{a^{i}_{p\ell}}a^{k}_{j}\overline{a^{p}_{j}} - \overline{a^{i}_{j}}a^{k}_{j}\tilde{R}^{i}_{k\ell\bar{\ell}} + \overline{a^{i}_{j}}a^{i}_{r}b^{q}_{\ell}\overline{b^{s}_{\ell}}R^{r}_{jq\bar{s}}.$$

Proof. Applying the exterior derivative to (4.46), using the first and second structure equations and simplifying, we have

$$- a^{i}_{j}\Psi^{j}_{m} + a^{k}_{j\ell}a^{j}_{m}\tilde{\theta}^{\ell} \wedge \tilde{\theta}^{i}_{k} + a^{k}_{m}\tilde{\Psi}^{i}_{k} = a^{k}_{m}da^{i}_{k\ell} \wedge \tilde{\theta}^{\ell}$$
$$- a^{i}_{k\ell}a^{j}_{m}\tilde{\theta}^{k}_{j} \wedge \tilde{\theta}^{\ell} + a^{i}_{k\ell}a^{k}_{jp}a^{j}_{m}\tilde{\theta}^{p} \wedge \tilde{\theta}^{\ell} - a^{i}_{k\ell}a^{k}_{m}\tilde{\theta}^{\ell}_{j} \wedge \tilde{\theta}^{j} + a^{i}_{k\ell}a^{k}_{m}\tilde{\Theta}^{\ell}.$$

Multiplying by b_r^m and rearranging, we obtain

$$(da_{r\ell}^i + a_{k\ell}^i a_{rj}^k \tilde\theta^j + a_{r\ell}^k \tilde\theta_k^i - a_{k\ell}^i \tilde\theta_r^k - a_{rj}^i \tilde\theta_\ell^j) \wedge \tilde\theta^\ell$$

(4.47)
$$= -b_r^m \Psi_m^j a_j^i + \tilde\Psi_r^i - a_{r\ell}^i \tilde\Theta^\ell.$$

Define $a_{r\ell p}^i$ and $a_{r\ell\overline{p}}^i$ by

(4.48) $$da_{r\ell}^i + a_{k\ell}^i a_{rj}^k \tilde\theta^j + a_{r\ell}^k \tilde\theta_k^i - a_{k\ell}^i \tilde\theta_r^k - a_{rj}^i \tilde\theta_\ell^j = a_{r\ell p}^i \tilde\theta^p + a_{r\ell\overline{p}}^i \overline{\tilde\theta^p}.$$

Then taking the (1,1) part of (4.47) we see that

(4.49) $$a_{r\ell\overline{p}}^i \overline{\tilde\theta^p} \wedge \tilde\theta^\ell = (-\tilde R_{r\ell\overline{p}}^i + a_j^i b_r^m b_\ell^q \overline{b_p^s} R_{mq\overline{s}}^j)\overline{\tilde\theta^p} \wedge \tilde\theta^\ell,$$

where we recall that by definition

$$\begin{aligned}(\tilde\Psi_r^i)^{(1,1)} &= -\tilde R_{r\ell\overline{p}}^i \overline{\tilde\theta^p} \wedge \tilde\theta^\ell\\ (\Psi_m^j)^{(1,1)} &= -R_{mq\overline{s}}^j \overline{\theta^s} \wedge \theta^q.\end{aligned}$$

Note that

(4.50) $$du = \overline{a_j^i} da_j^i + a_j^i d\overline{a_j^i}.$$

Then we see that from (4.46),

$$\begin{aligned}du &= \overline{a_j^i}(a_{k\ell}^i a_j^k \tilde\theta^\ell + a_m^i \theta_j^m - a_j^k \tilde\theta_k^i) + a_j^i(\overline{a_{k\ell}^i a_j^k \tilde\theta^\ell} + \overline{a_m^i \theta_j^m} - \overline{a_j^k \tilde\theta_k^i}).\end{aligned}$$

(4.51) $$= \overline{a_j^i} a_{k\ell}^i a_j^k \tilde\theta^\ell + a_j^i \overline{a_{k\ell}^i a_j^k \tilde\theta^\ell}.$$

Hence $\partial u = \overline{a_j^i} a_{k\ell}^i a_j^k \tilde\theta^\ell$. Applying the exterior derivative to this and substituting from (4.46), (4.48) and (4.49) we have,

$$\begin{aligned}d\partial u &= a_{k\ell}^i \overline{a_j^i} a_{pq}^k a_j^p \tilde\theta^q \wedge \tilde\theta^\ell + a_{k\ell}^i a_j^k \overline{a_{pq}^i a_j^p \tilde\theta^q} \wedge \tilde\theta^\ell\\ &\quad + \overline{a_j^i} a_j^k (a_{k\ell p}^i \tilde\theta^p + a_{k\ell\overline{p}}^i \overline{\tilde\theta^p} - a_{r\ell}^i a_{kp}^r \tilde\theta^p) \wedge \tilde\theta^\ell + \overline{a_j^i} a_j^k a_{k\ell}^i \tilde\Theta^\ell\\ &= a_{k\ell}^i a_j^k \overline{a_{pq}^i a_j^p \tilde\theta^q} \wedge \tilde\theta^\ell + \overline{a_j^i} a_j^k a_{k\ell p}^i \tilde\theta^p \wedge \tilde\theta^\ell\\ &\quad + \overline{a_j^i} a_j^k (-\tilde R_{k\ell\overline{p}}^i + a_r^i b_k^m b_\ell^q \overline{b_p^s} R_{mq\overline{s}}^r)\overline{\tilde\theta^p} \wedge \tilde\theta^\ell + \overline{a_j^i} a_j^k a_{k\ell}^i \tilde\Theta^\ell.\end{aligned}$$

Hence

$$(d\partial u)^{(1,1)} = a_{k\ell}^i a_j^k \overline{a_{pq}^i a_j^p \tilde\theta^q} \wedge \tilde\theta^\ell - \overline{a_j^i} a_j^k \tilde R_{k\ell\overline{p}}^i \overline{\tilde\theta^p} \wedge \tilde\theta^\ell + \overline{a_j^i} a_r^i b_\ell^q \overline{b_p^s} R_{jq\overline{s}}^r \overline{\tilde\theta^p} \wedge \tilde\theta^\ell.$$

Then from the definition of the canonical Laplacian, we have proved the lemma. $\square$

Now let $\nu = \det(a_i^j)$ and set $v = |\nu|^2 = \nu\overline\nu$, which is the ratio of the volume forms of $\tilde g$ and g. We have the following lemma.

Lemma 4.3.4. *For g and $\tilde{g}$ almost-Hermitian metrics, and v as above, the following identities hold.*

(i) $(d\partial \log v)^{(1,1)} = -R_{k\bar{\ell}}\theta^k \wedge \overline{\theta^\ell} + \tilde{R}_{k\bar{\ell}}a_i^k\overline{a_j^\ell}\theta^i \wedge \overline{\theta^j}$

(ii) $\Delta \log v = 2R - 2\tilde{R}_{k\bar{\ell}}a_i^k\overline{a_i^\ell}$.

Proof. This proof is essentially contained in [GH], but we include it here for the reader's convenience. Write ν_j^i for the (i,j)th cofactor of the matrix (a_i^j), so that $\nu_j^i = \nu b_j^i$. Then

$$d\nu = \nu_j^i da_i^j.$$

From (4.46) we have

$$da_m^i - a_j^i\theta_m^j + a_m^k\tilde{\theta}_k^i = a_{k\ell}^ia_m^ka_r^\ell\theta^r.$$

Hence

$$
\begin{aligned}
d\nu &= \nu_j^i(a_{pq}^ja_i^pa_k^q\theta^k + a_k^j\theta_i^k - a_i^k\tilde{\theta}_k^j)\\
&= \nu_k\theta^k + \nu(\theta_i^i - \tilde{\theta}_i^i),
\end{aligned}
$$

(4.52)

for $\nu_k = \nu_j^ia_{pq}^ja_i^pa_k^q$. Now

$$
\begin{aligned}
dv &= \overline{\nu}d\nu + \nu d\overline{\nu}\\
&= \overline{\nu}(\nu_k\theta^k + \nu(\theta_i^i - \tilde{\theta}_i^i)) + \nu(\overline{\nu_k}\overline{\theta^k} + \overline{\nu}(\overline{\theta_i^i} - \overline{\tilde{\theta}_i^i}))\\
&= \overline{\nu}\nu_k\theta^k + \nu\overline{\nu_k}\overline{\theta^k}.
\end{aligned}
$$

Hence $\partial v = \overline{\nu}\nu_k\theta^k$. Define v_k and $v_{\overline{k}}$ by $dv = v_k\theta^k + v_{\overline{k}}\overline{\theta^k}$. Then $v_k = \overline{\nu}\nu_k$. Applying the exterior derivative to (4.52) and using the second structure equation we have

$$
\begin{aligned}
0 &= d(\nu_k\theta^k) + d\nu \wedge (\theta_i^i - \tilde{\theta}_i^i) + \nu d(\theta_i^i - \tilde{\theta}_i^i)\\
&= d(\nu_k\theta^k) + \nu_k\theta^k \wedge (\theta_i^i - \tilde{\theta}_i^i) + \nu(\Psi_i^i - \tilde{\Psi}_i^i).
\end{aligned}
$$

Multiplying by $\overline{\nu}$ and using (4.52) again we have

$$
\begin{aligned}
0 &= \overline{\nu}d(\nu_k\theta^k) + \nu_k\theta^k \wedge (\overline{\nu_\ell}\overline{\theta^\ell} - d\overline{\nu}) + v(\Psi_i^i - \tilde{\Psi}_i^i)\\
&= d(\overline{\nu}\nu_k\theta^k) + \nu_k\overline{\nu_\ell}\theta^k \wedge \overline{\theta^\ell} + v(\Psi_i^i - \tilde{\Psi}_i^i).
\end{aligned}
$$

Consider the $(1,1)$ part

$$
\begin{aligned}
(d\partial v)^{(1,1)} &= -\nu_k\overline{\nu_\ell}\theta^k \wedge \overline{\theta^\ell} - v(\Psi_i^i - \tilde{\Psi}_i^i)^{(1,1)}\\
&= -\frac{v_k\overline{v_\ell}}{v}\theta^k \wedge \overline{\theta^\ell} - vR_{k\bar{\ell}}\theta^k \wedge \overline{\theta^\ell} + v\tilde{R}_{k\bar{\ell}}a_i^k\overline{a_j^\ell}\theta^i \wedge \overline{\theta^j}.
\end{aligned}
$$

(4.53)

We also have

$$d\partial \log v = \frac{d\partial v}{v} + \frac{\partial v \wedge \overline{\partial}v}{v^2},$$

which combines with (4.53) to give (i). From the definition of the canonical Laplacian we immediately obtain (ii). $\qquad\square$

Using Lemmas 4.3.3 and 4.3.4 we can now prove Lemma 4.3.2.

Proof of Lemma 4.3.2. Recall that the Calabi-Yau equation (4.8) is

$$(4.54) \qquad \det \tilde{g} = e^{2F} \det g,$$

for smooth F, where g is almost-Hermitian and $\tilde{g}$ is almost-Kähler. Note that this equation can be rewritten in terms of v as

$$(4.55) \qquad \log v = F.$$

First we prove the following equality, which will also be useful in its own right later:

$$(4.56) \qquad \tilde{\Delta} u = 2 a^i_{k\ell} \overline{a^i_{p\ell}} a^k_j \overline{a^p_j} + \Delta F - 2R + 8 N^\ell_{\bar{p}\bar{i}} \overline{N^p_{\ell\bar{i}}} + 2 \overline{a^p_i} a^p_j b^k_q \overline{b^\ell_q} \mathcal{R}_{i\bar{j}k\bar{\ell}}.$$

From Lemma 4.3.3, Lemma 4.3.4 and the identity (4.29),

$$
\begin{aligned}
\tilde{\Delta} u &= 2 \left(a^i_{k\ell} \overline{a^i_{p\ell}} a^k_j \overline{a^p_j} + \overline{a^i_j} a^i_r b^q_\ell \overline{b^s_\ell} R^r_{jq\bar{s}} \right) \\
&\quad + \Delta F - 2R + 8 \overline{a^i_j} a^k_j \left(\tilde{N}^\ell_{\bar{p}\bar{i}} \overline{\tilde{N}^p_{\ell\bar{k}}} + \tilde{N}^p_{\ell\bar{i}} \overline{\tilde{N}^k_{\bar{p}\bar{\ell}}} \right).
\end{aligned}
$$

Using (4.44), we have

$$\overline{a^i_j} a^k_j \left(\tilde{N}^\ell_{\bar{p}\bar{i}} \overline{\tilde{N}^p_{\ell\bar{k}}} + \tilde{N}^p_{\ell\bar{i}} \overline{\tilde{N}^k_{\bar{p}\bar{\ell}}} \right) = N^\ell_{\bar{p}\bar{i}} \overline{N^p_{\ell\bar{i}}} + \overline{a^k_s} a^k_j \overline{b^t_\ell} b^r_\ell N^p_{\bar{t}\bar{j}} \overline{N^s_{\bar{p}\bar{r}}},$$

giving (4.56).

To obtain (4.42), we compute,

$$
\begin{aligned}
\tilde{\Delta} \log u &= \frac{1}{u} \left(\tilde{\Delta} u - \frac{|du|^2_{\tilde{g}}}{u} \right) \\
&= \frac{1}{u} \left(2 a^i_{k\ell} \overline{a^i_{p\ell}} a^k_j \overline{a^p_j} + 8 N^\ell_{\bar{p}\bar{i}} \overline{N^p_{\ell\bar{i}}} + 2 \overline{a^p_i} a^p_j b^k_q \overline{b^\ell_q} \mathcal{R}_{i\bar{j}k\bar{\ell}} \right. \\
&\quad \left. + \Delta F - 2R - \frac{|du|^2_{\tilde{g}}}{u} \right).
\end{aligned}
$$

To complete the proof of Lemma 4.3.2, it remains to prove the inequality

$$(4.57) \qquad |du|^2_{\tilde{g}} \le 2 u a^i_{k\ell} \overline{a^i_{p\ell}} a^k_j \overline{a^p_j}.$$

From (4.51) we have

$$|du|^2_{\tilde{g}} = 2 u_\ell \overline{u_\ell},$$

where $u_\ell = \overline{a_j^i} a_{k\ell}^i a_j^k = \overline{a_j^i} B_{\ell j}^i$, where $B_{\ell j}^i = a_{k\ell}^i a_j^k$. Then using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
u_\ell \overline{u_\ell} &= \sum_{i,j,\ell,p,q} \overline{a_j^i} B_{\ell j}^i a_q^p \overline{B_{\ell q}^p} \\
&\leq \sum_{i,j,p,q} \left(\sum_\ell |\overline{a_j^i} B_{\ell j}^i|^2 \right)^{1/2} \left(\sum_\ell |a_q^p \overline{B_{\ell q}^p}|^2 \right)^{1/2} \\
&= \left(\sum_{i,j} \left(\sum_\ell |a_j^i|^2 |B_{\ell j}^i|^2 \right)^{1/2} \right)^2 \\
&= \left(\sum_{i,j} |a_j^i| \left(\sum_\ell |B_{\ell j}^i|^2 \right)^{1/2} \right)^2 \\
&\leq \left(\sum_{i,j} |a_j^i|^2 \right) \left(\sum_{i,j,\ell} |B_{\ell j}^i|^2 \right) \\
&= u a_{k\ell}^i \overline{a_{p\ell}^i} a_j^k \overline{a_j^p},
\end{aligned}
$$

which gives (4.57). □

Finally, we can give the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1.   Note that from the Calabi-Yau equation and the arithmetic-geometric means inequality, $u = \frac{1}{2} \mathrm{tr}_g \tilde{g}$ is bounded below away from zero by a positive constant depending only on $\inf F$. Hence there exists a constant C' depending only on J, $\inf F$, ΔF and R such that

(4.58) $$ \left| \frac{1}{u} \left(\Delta F - 2R + 8 N_{\overline{p}i}^\ell \overline{N_{\overline{\ell}i}^p} \right) \right| \leq C'. $$

Choose A' sufficiently large such that

$$ \mathcal{R}_{i\overline{j}k\overline{\ell}} + A' \delta_{ij} \delta_{k\ell} \geq 0, $$

where the nonnegativity is in the Griffiths sense (as in the paragraph immediately before the statement of Theorem 4.1.4). Then

(4.59) $$ \frac{1}{u} 2 \overline{a_i^p} a_j^p b_q^k \overline{b_q^\ell} \mathcal{R}_{i\overline{j}k\overline{\ell}} \geq -\frac{1}{u} 2 A' \overline{a_i^p} a_i^p b_q^k \overline{b_q^k} = -A' \mathrm{tr}_{\tilde{g}} g. $$

Combining (4.58) and (4.59) with Lemma 4.3.2 we obtain

$$ \tilde{\Delta} \log u \geq -C' - A' \mathrm{tr}_{\tilde{g}} g, $$

We apply the maximum principle to $(\log u - 2A'\varphi)$. Suppose that the maximum of this quantity is achieved at a point x_0. Then at this point, using (4.38),

$$0 \geq \tilde{\Delta}(\log u - 2A'\varphi) \geq -C' + A'\mathrm{tr}_{\tilde{g}}g - 4A'n.$$

Hence

$$(\mathrm{tr}_{\tilde{g}}g)(x_0) \leq \frac{4A'n + C'}{A'}.$$

Using the inequality

$$\frac{\sum_{i=1}^{n}\lambda_i}{\prod_{i=1}^{n}\lambda_i} = \sum_{i=1}^{n}\frac{1}{\lambda_1\cdots\widehat{\lambda_i}\cdots\lambda_n} \leq \frac{1}{(n-1)!}\left(\sum_{i=1}^{n}\frac{1}{\lambda_i}\right)^{n-1},$$

which holds for any set of real numbers $\lambda_i > 0$, we see that

(4.60)
$$\frac{\mathrm{tr}_g\tilde{g}}{2}\sqrt{\frac{\det g}{\det \tilde{g}}} \leq \frac{1}{(n-1)!}\left(\frac{\mathrm{tr}_{\tilde{g}}g}{2}\right)^{n-1}.$$

Hence, using the Calabi-Yau equation again, u can be bounded from above in terms of $\mathrm{tr}_{\tilde{g}}g$ and $\sup F$, and so we obtain an estimate

$$u(x_0) \leq C''.$$

It follows that for any $x \in M$,

$$\log u(x) - 2A'\varphi(x) \leq \log C'' - 2A'\inf_M \varphi,$$

and, after exponentiating, this proves the theorem. $\square$

Remark 4.3.2 Notice that if we assume $\mathcal{R}(g,J) > 0$ in Theorem 4.3.1 then we easily obtain a uniform bound $u \leq C$. To see this, note that by assumption there exists a positive constant $\hat{A}$ so that

$$\frac{1}{u}2\overline{a_i^p}a_j^p b_q^k \overline{b_q^\ell}\mathcal{R}_{i\bar{j}k\bar{\ell}} \geq \frac{1}{u}2\hat{A}\,\overline{a_i^p}a_i^p b_q^k \overline{b_q^k} \geq \hat{A}\mathrm{tr}_{\tilde{g}}g.$$

Combining this inequality with Lemma 4.3.2 and (4.58) we obtain

$$\tilde{\Delta}\log u \geq -C' + \hat{A}\mathrm{tr}_{\tilde{g}}g.$$

Applying the maximum principle to $\log u$, we see that at the maximum of $\log u$, the quantity $\mathrm{tr}_{\tilde{g}}g$ is bounded from above and hence so is $\log u$. Thus we obtain a uniform bound $u \leq C$.

4.4 First derivative estimate of $\tilde{g}$

In this section we give an estimate on the derivative of an almost-Kähler metric $\tilde{g}$ solving the Calabi-Yau equation (4.8). This is a generalization of the third order estimate of [Y2] (see also [PSS] for a succinct proof of the parabolic version of this estimate). Define

$$S = \frac{1}{4} |\nabla \tilde{g}|^2_{\tilde{g}},$$

where ∇ is the canonical connection associated to g, J. Then we have the following theorem.

Theorem 4.4.1. *Let $\tilde{g}$ be a solution of (4.8) and suppose that there exists a constant K such that*

$$\sup_M (\mathrm{tr}_g \tilde{g}) \leq K.$$

Then there exists a constant C_0 depending only on g, J, F and K such that

$$S \leq C_0.$$

Before we prove this theorem, we will need a number of lemmas.

Lemma 4.4.2. *S can be written as*

(4.61)
$$S = a^i_{k\ell} \overline{a^i_{k\ell}},$$

where $a^i_{k\ell}$ is defined by

(4.62)
$$da^i_m - a^i_j \theta^j_m + a^k_m \tilde{\theta}^i_k = a^i_{k\ell} a^k_m \tilde{\theta}^\ell.$$

Proof. To see (4.61) we calculate as follows:

$$
\begin{aligned}
\nabla\left(\tilde{\theta}^i \otimes \overline{\tilde{\theta}^i}\right) &= \nabla(a^i_j \theta^j) \otimes \overline{\tilde{\theta}^i} + \tilde{\theta}^i \otimes \nabla(\overline{a^i_j \theta^j}) \\
&= (da^i_j) b^j_k \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^i} - a^i_j \theta^j_k b^k_\ell \otimes \tilde{\theta}^\ell \otimes \overline{\tilde{\theta}^i} \\
&\quad + \overline{(da^i_j) b^j_k} \otimes \tilde{\theta}^i \otimes \overline{\tilde{\theta}^k} - \overline{a^i_j b^k_\ell \theta^j_k} \otimes \tilde{\theta}^i \otimes \overline{\tilde{\theta}^\ell} \\
&= (da^\ell_j - a^\ell_r \theta^r_j) b^j_k \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^\ell} + \overline{(da^k_j - a^k_r \theta^r_j) b^j_\ell} \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^\ell} \\
&= (a^\ell_{rs} a^r_j \tilde{\theta}^s - a^r_j \tilde{\theta}^\ell_r) b^j_k \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^\ell} \\
&\quad + \overline{(a^k_{rs} a^r_j \tilde{\theta}^s - a^r_j \tilde{\theta}^k_r) b^j_\ell} \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^\ell} \\
&= a^\ell_{ks} \tilde{\theta}^s \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^\ell} + \overline{a^k_{\ell s} \tilde{\theta}^s} \otimes \tilde{\theta}^k \otimes \overline{\tilde{\theta}^\ell}.
\end{aligned}
$$

Then since $\tilde{g} = \tilde{\theta}^i \otimes \overline{\tilde{\theta}^i} + \overline{\tilde{\theta}^i} \otimes \tilde{\theta}^i$, (4.61) follows immediately. $\square$

The following lemma gives a general formula for the Laplacian of S.

Lemma 4.4.3. *We have*

$$
\frac{1}{2}\tilde{\Delta}S = |a^i_{k\ell p} - a^i_{r\ell}a^r_{kp}|^2_{\tilde{g}} + |a^i_{k\ell\overline{p}}|^2_{\tilde{g}} + \overline{a^i_{k\ell}}a^i_{r\ell}\tilde{R}^r_{kp\overline{p}} + \overline{a^i_{k\ell}}a^i_{kj}\tilde{R}^j_{\ell p\overline{p}} - \overline{a^i_{k\ell}}a^r_{k\ell}\tilde{R}^i_{rp\overline{p}}
$$

$$
+ 2\mathrm{Re}\Bigg(\overline{a^i_{k\ell}}\bigg(b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}}a^i_{rp}a^r_j - a^i_j b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}}a^r_{kp}b^m_r
$$

$$
- a^i_j b^m_k \overline{b^s_p} R^j_{mq\overline{s}}a^r_{\ell p}b^q_r + a^i_j b^m_k b^q_\ell \overline{b^s_p}b^u_p R^j_{mq\overline{s},u} - \tilde{R}_{k\bar{i},\ell}
$$

$$
+ 4\tilde{N}^p_{\overline{q}\,\bar{i},\ell}\overline{\tilde{N}^q_{\overline{p}\,k}} + 4\tilde{N}^p_{\overline{q}\,\bar{i}}\overline{\tilde{N}^q_{\overline{p}\,k,\ell}} + 4\tilde{N}^p_{\overline{q}\,\bar{i},\ell}\overline{\tilde{N}^k_{\overline{p}\,q}} + 4\tilde{N}^p_{\overline{q}\,\bar{i}}\overline{\tilde{N}^k_{\overline{p}\,q,\ell}}
$$

$$
(4.63)\qquad + 4\tilde{N}^i_{\overline{q}\,\overline{p},k}\overline{\tilde{N}^q_{\overline{p}\,\ell}} + 2\overline{\tilde{N}^k_{\overline{\ell}\,\overline{p},ip}}\bigg)\Bigg).
$$

Proof. First, recall from (4.47) and (4.48) that $a^i_{r\ell p}$ and $a^i_{r\ell\overline{p}}$ are defined by

$$
(4.64)\qquad da^i_{r\ell} + a^i_{k\ell}a^k_{rj}\tilde{\theta}^j + a^k_{r\ell}\tilde{\theta}^i_k - a^i_{k\ell}\tilde{\theta}^k_r - a^i_{rj}\tilde{\theta}^j_\ell = a^i_{r\ell p}\tilde{\theta}^p + a^i_{r\ell\overline{p}}\overline{\tilde{\theta}^p},
$$

and that

$$
(4.65)\qquad (a^i_{r\ell p}\tilde{\theta}^p + a^i_{r\ell\overline{p}}\overline{\tilde{\theta}^p}) \wedge \tilde{\theta}^\ell = -b^m_r \Psi^j_m a^i_j + \tilde{\Psi}^i_r - a^i_{r\ell}\tilde{\Theta}^\ell.
$$

Define functions $a^i_{r\ell p,q}, a^i_{r\ell p,\overline{q}}, a^i_{r\ell\overline{p},q}$, and $a^i_{r\ell\overline{p},\overline{q}}$ by the formulas

$$
(4.66)\qquad da^i_{r\ell p} + a^k_{r\ell p}\tilde{\theta}^i_k - a^i_{r\ell q}\tilde{\theta}^q_p - a^i_{k\ell p}\tilde{\theta}^k_r - a^i_{rjp}\tilde{\theta}^j_\ell = a^i_{r\ell p,q}\tilde{\theta}^q + a^i_{r\ell p,\overline{q}}\overline{\tilde{\theta}^q},
$$

$$
(4.67)\qquad da^i_{r\ell\overline{p}} + a^k_{r\ell\overline{p}}\tilde{\theta}^i_k - a^i_{r\ell\overline{q}}\overline{\tilde{\theta}^q_p} - a^i_{k\ell\overline{p}}\tilde{\theta}^k_r - a^i_{rj\overline{p}}\tilde{\theta}^j_\ell = a^i_{r\ell\overline{p},q}\tilde{\theta}^q + a^i_{r\ell\overline{p},\overline{q}}\overline{\tilde{\theta}^q}.
$$

Applying the exterior derivative to (4.64), using the last two definitions, and canceling many terms we get

$$
a^i_{r\ell p,q}\tilde{\theta}^q \wedge \tilde{\theta}^p + a^i_{r\ell p,\overline{q}}\overline{\tilde{\theta}^q} \wedge \tilde{\theta}^p + a^i_{r l\overline{p},q}\tilde{\theta} \wedge \overline{\tilde{\theta}^p} + a^i_{r l\overline{p},\overline{q}}\overline{\tilde{\theta}^q} \wedge \overline{\tilde{\theta}^p}
$$

$$
+ a^i_{r\ell p}\tilde{\Theta}^p + a^i_{r\ell\overline{p}}\overline{\tilde{\Theta}^p}
$$

$$
= -a^k_{rp}a^i_{s\ell}a^s_{kt}\tilde{\theta}^t \wedge \tilde{\theta}^p + a^k_{rp}a^i_{k\ell s}\tilde{\theta}^s \wedge \tilde{\theta}^p + a^k_{rp}a^i_{k\ell\overline{s}}\overline{\tilde{\theta}^s} \wedge \tilde{\theta}^p
$$

$$
- a^i_{k\ell}a^k_{sp}a^s_{rt}\tilde{\theta}^t \wedge \tilde{\theta}^p + a^i_{k\ell}a^k_{rpt}\tilde{\theta}^t \wedge \tilde{\theta}^p + a^i_{k\ell}a^k_{rp\overline{t}}\overline{\tilde{\theta}^t} \wedge \tilde{\theta}^p
$$

$$
(4.68)\qquad + a^i_{k\ell}a^k_{rp}\tilde{\Theta}^p - a^i_{k\ell}\tilde{\Psi}^k_r - a^i_{rp}\tilde{\Psi}^p_\ell + a^k_{r\ell}\tilde{\Psi}^i_k,
$$

which will be useful later. To calculate the canonical Laplacian of S with respect to $\tilde{g}$, first compute

$$
\partial S = \overline{a^i_{k\ell}}\partial a^i_{k\ell} + a^i_{k\ell}\overline{\overline{\partial}a^i_{k\ell}}
$$

$$
= \big(\overline{a^i_{k\ell}}a^i_{k\ell p} + a^i_{k\ell}\overline{a^i_{k\ell\overline{p}}} - \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kp}\big)\tilde{\theta}^p.
$$

Then compute

$$
\begin{aligned}
d(\partial S) &= \big(a^i_{k\ell p}\overline{a^i_{k\ell q}\tilde{\theta}^q} + a^i_{k\ell p}\overline{a^i_{k\ell \overline{q}}}\tilde{\theta}^q - a^i_{k\ell p}\overline{a^i_{r\ell}a^r_{kq}\tilde{\theta}^q} + \overline{a^i_{k\ell}}a^i_{k\ell p,q}\tilde{\theta}^q \\
&\quad + \overline{a^i_{k\ell}}a^i_{k\ell p,\overline{q}}\tilde{\theta}^q + \overline{a^i_{k\ell\overline{p}}}a^i_{k\ell q}\tilde{\theta}^q + \overline{a^i_{k\ell\overline{p}}}a^i_{k\ell\overline{q}}\tilde{\theta}^q - \overline{a^i_{k\ell\overline{p}}}a^r_{r\ell}a^r_{kq}\tilde{\theta}^q \\
&\quad + a^i_{k\ell}\overline{a^i_{k\ell\overline{p},q}\tilde{\theta}^q} + a^i_{k\ell}\overline{a^i_{k\ell\overline{p},\overline{q}}}\tilde{\theta}^q - \overline{a^i_{k\ell q}}a^r_{r\ell}a^r_{kp}\tilde{\theta}^q - \overline{a^i_{k\ell\overline{q}}}a^r_{r\ell}a^r_{kp}\tilde{\theta}^q \\
&\quad + a^i_{r\ell}a^r_{kp}\overline{a^j_{j\ell}a^j_{kq}\tilde{\theta}^q} - \overline{a^i_{k\ell}}a^r_{kp}a^i_{r\ell q}\tilde{\theta}^q - \overline{a^i_{k\ell}}a^r_{kp}a^i_{r\ell\overline{q}}\tilde{\theta}^q \\
&\quad + \overline{a^i_{k\ell}}a^r_{kp}a^j_{j\ell}a^j_{rq}\tilde{\theta}^q - \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kpq}\tilde{\theta}^q - \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kp\overline{q}}\tilde{\theta}^q \\
&\quad + \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{jp}a^j_{kq}\tilde{\theta}^q\big) \wedge \tilde{\theta}^p + \big(\overline{a^i_{k\ell}}a^i_{k\ell p} + a^i_{k\ell}\overline{a^i_{k\ell\overline{p}}} - \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kp}\big)\tilde{\Theta}^p,
\end{aligned}
$$
(4.69)

and hence

$$
\begin{aligned}
(d(\partial S))^{(1,1)} &= \big(a^i_{k\ell p}\overline{a^i_{k\ell q}} - a^i_{k\ell p}\overline{a^i_{r\ell}a^r_{kq}} + \overline{a^i_{k\ell}}a^i_{k\ell p,\overline{q}} + \overline{a^i_{k\ell\overline{p}}}a^i_{k\ell\overline{q}} \\
&\quad + a^i_{k\ell}\overline{a^i_{k\ell\overline{p},q}} - \overline{a^i_{k\ell q}}a^i_{r\ell}a^r_{kp} + a^i_{r\ell}a^r_{kp}\overline{a^j_{j\ell}a^j_{kq}} \\
&\quad - \overline{a^i_{k\ell}}a^r_{kp}a^i_{r\ell\overline{q}} - \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kp\overline{q}}\big)\overline{\tilde{\theta}^q} \wedge \tilde{\theta}^p.
\end{aligned}
$$
(4.70)

Then taking the (1,1) part of (4.68) we see that

$$
\begin{aligned}
a^i_{k\ell p,\overline{q}}\overline{\tilde{\theta}^q} \wedge \tilde{\theta}^p &= \big(a^i_{k\ell\overline{q},p} + a^i_{r\ell\overline{q}}a^r_{kp} + a^i_{r\ell}a^r_{kp\overline{q}} \\
&\quad + a^i_{r\ell}\tilde{R}^r_{kp\overline{q}} + a^i_{kj}\tilde{R}^j_{\ell p\overline{q}} - a^r_{k\ell}\tilde{R}^i_{rp\overline{q}}\big)\overline{\tilde{\theta}^q} \wedge \tilde{\theta}^p.
\end{aligned}
$$
(4.71)

Multiplying (4.71) by $\overline{a^i_{k\ell}}$, substituting into (4.70) and using the formula for the Laplacian, we obtain

$$
\begin{aligned}
\tfrac{1}{2}\tilde{\Delta}S &= a^i_{k\ell p}\overline{a^i_{k\ell p}} - a^i_{k\ell p}\overline{a^i_{r\ell}a^r_{kp}} + \overline{a^i_{k\ell\overline{p}}}a^i_{k\ell\overline{p}} - a^i_{r\ell}a^r_{kp}\overline{a^i_{k\ell p}} \\
&\quad + a^i_{r\ell}a^r_{kp}\overline{a^i_{s\ell}a^s_{kp}} - \overline{a^i_{k\ell}}a^r_{kp}a^i_{r\ell\overline{p}} - \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kp\overline{p}} \\
&\quad + \overline{a^i_{k\ell}}a^i_{k\ell\overline{p},p} + \overline{a^i_{k\ell}}a^r_{kp}a^i_{r\ell\overline{p}} + \overline{a^i_{k\ell}}a^i_{r\ell}a^r_{kp\overline{p}} \\
&\quad + \overline{a^i_{k\ell}}a^i_{r\ell}\tilde{R}^r_{kp\overline{p}} + \overline{a^i_{k\ell}}a^i_{kj}\tilde{R}^j_{\ell p\overline{p}} - \overline{a^i_{k\ell}}a^r_{k\ell}\tilde{R}^i_{rp\overline{p}} + a^i_{k\ell}\overline{a^i_{k\ell\overline{p},p}} \\
&= a^i_{k\ell p}\overline{a^i_{k\ell p}} + \overline{a^i_{k\ell\overline{p}}}a^i_{k\ell\overline{p}} - 2\mathrm{Re}(a^i_{k\ell p}\overline{a^i_{r\ell}a^r_{kp}}) \\
&\quad + a^i_{r\ell}a^r_{kp}\overline{a^i_{s\ell}a^s_{kp}} + \overline{a^i_{k\ell}}a^i_{r\ell}\tilde{R}^r_{kp\overline{p}} + \overline{a^i_{k\ell}}a^i_{kj}\tilde{R}^j_{\ell p\overline{p}} - \overline{a^i_{k\ell}}a^r_{k\ell}\tilde{R}^i_{rp\overline{p}} \\
&\quad + 2\mathrm{Re}(\overline{a^i_{k\ell}}a^i_{k\ell\overline{p},p}).
\end{aligned}
$$

Completing the square, we obtain

$$
\begin{aligned}
\tfrac{1}{2}\tilde{\Delta}S &= \big|a^i_{k\ell p} - a^i_{r\ell}a^r_{kp}\big|^2_{\tilde{g}} + |a^i_{k\ell\overline{p}}|^2_{\tilde{g}} + \overline{a^i_{k\ell}}a^i_{r\ell}\tilde{R}^r_{kp\overline{p}} + \overline{a^i_{k\ell}}a^i_{kj}\tilde{R}^j_{\ell p\overline{p}} \\
&\quad - \overline{a^i_{k\ell}}a^r_{k\ell}\tilde{R}^i_{rp\overline{p}} + 2\mathrm{Re}(\overline{a^i_{k\ell}}a^i_{k\ell\overline{p},p}).
\end{aligned}
$$
(4.72)

To calculate the last term, take the $(1,1)$ part of (4.65) to obtain

$$(4.73) \qquad a^i_{k\ell\overline{p}} = a^i_j b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}} - \tilde{R}^i_{k\ell\overline{p}}.$$

Now recall from (4.46) that

$$(4.74) \qquad da^i_m - a^i_j \theta^j_m + a^k_m \tilde{\theta}^i_k = a^i_{k\ell} a^k_m \tilde{\theta}^\ell.$$

Similarly we have

$$(4.75) \qquad db^j_k + b^r_k \theta^j_r - b^j_i \tilde{\theta}^i_k = -b^j_i a^i_{k\ell} \tilde{\theta}^\ell.$$

Taking the exterior derivative of (4.73), using (4.66), (4.67), (4.74) and (4.75) we get

$$\begin{aligned} a^i_{k\ell\overline{p},t}\tilde{\theta}^t + a^i_{k\ell\overline{p},\overline{t}}\overline{\tilde{\theta}^t} =\ & a^i_j b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s},u}\theta^u + a^i_j b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s},\overline{u}}\overline{\theta^u} \\ & - \tilde{R}^i_{k\ell\overline{p},t}\tilde{\theta}^t - \tilde{R}^i_{k\ell\overline{p},\overline{t}}\overline{\tilde{\theta}^t} + b^m_k b^q_\ell \overline{b^s_p} a^r_j a^i_{rt} R^j_{mq\overline{s}}\tilde{\theta}^t \\ & - b^m_r b^q_\ell \overline{b^s_p} a^i_j a^r_{kt} R^j_{mq\overline{s}}\tilde{\theta}^t - b^m_k b^r_\ell \overline{b^s_p} a^i_j a^r_{\ell t} R^j_{mq\overline{s}}\tilde{\theta}^t \\ & - b^m_k b^q_\ell \overline{b^s_r} a^i_j \overline{a^r_{pt}} R^j_{mq\overline{s}}\overline{\tilde{\theta}^t}, \end{aligned}$$

$$(4.76)$$

whose $(1,0)$ part gives

$$\begin{aligned} a^i_{k\ell\overline{p},t} =\ & b^m_k b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}} a^i_{rt} a^r_j - a^i_j b^q_\ell \overline{b^s_p} R^j_{mq\overline{s}} a^r_{kt} b^m_r - a^i_j b^m_k \overline{b^s_p} R^j_{mq\overline{s}} a^r_{\ell t} b^q_r \\ & + a^i_j b^m_k b^q_\ell \overline{b^s_p} b^u_t R^j_{mq\overline{s},u} - \tilde{R}^i_{k\ell\overline{p},t}. \end{aligned}$$

$$(4.77)$$

Now from (4.25),(4.26) and (4.29)

$$\begin{aligned} \tilde{R}^i_{k\ell\overline{p},p} =\ & \tilde{R}^i_{kp\overline{p},\ell} - 2\tilde{K}^i_{kp\ell,\overline{p}} - 4\tilde{K}^i_{k\overline{q}\,\overline{p}}\overline{\tilde{N}^q_{\overline{p}\,\overline{\ell}}} \\ =\ & \tilde{R}_{k\overline{i},\ell} - 4\tilde{N}^p_{\overline{q}\,\overline{i},\ell}\overline{\tilde{N}^q_{\overline{p}\,\overline{k}}} - 4\tilde{N}^p_{\overline{q}\,\overline{i}}\overline{\tilde{N}^q_{\overline{p}\,\overline{k},\ell}} - 4\tilde{N}^p_{\overline{q}\,\overline{i},\ell}\overline{\tilde{N}^k_{\overline{p}\,\overline{q}}} \\ & - 4\tilde{N}^p_{\overline{q}\,\overline{i}}\overline{\tilde{N}^k_{\overline{p}\,\overline{q},\ell}} - 2\tilde{K}^i_{kp\ell,\overline{p}} - 4\tilde{N}^i_{\overline{q}\,\overline{p},k}\overline{\tilde{N}^q_{\overline{p}\,\overline{\ell}}}, \end{aligned}$$

$$(4.78)$$

and using (4.25) again we see that

$$(4.79) \qquad \tilde{K}^i_{kp\ell,\overline{p}} = \overline{\tilde{N}^k_{\ell\,\overline{p},ip}}.$$

Combining (4.72), (4.77), (4.78) and (4.79) gives (4.63). $\qquad\square$

To deal with the terms involving derivatives of $\tilde{N}^i_{\overline{j}\,\overline{k}}$ in (4.63) we need another lemma.

Lemma 4.4.4. *We have*

(i) $\tilde{N}^i_{\bar{j}\,\bar{k},m} = \overline{b^r_j b^s_k} b^\ell_m a^i_t N^t_{\bar{r}\,\bar{s},\ell} + \overline{b^r_j b^s_k} a^\ell_t N^t_{\bar{r}\,\bar{s}} a^i_{\ell m}$

(ii) $\tilde{N}^i_{\bar{j}\,\bar{k},\overline{m}} = \overline{b^r_j b^s_k b^\ell_m} a^i_t N^t_{\bar{r}\,\bar{s},\bar{\ell}} - \overline{b^r_\ell b^s_k} a^i_t N^t_{\bar{r}\,\bar{s}} \overline{a^\ell_{jm}} - \overline{b^r_j b^s_\ell} a^i_t N^t_{\bar{r}\,\bar{s}} \overline{a^\ell_{km}}$

(iii) $\left| a^i_{k\ell} \tilde{N}^k_{\bar{\ell}\,\bar{p},ip} \right|_{\tilde{g}} \le C(S+1) + \dfrac{1}{2}\left| a^i_{k\ell p} - a^i_{r\ell} a^r_{kp} \right|^2_{\tilde{g}},$

for a constant C depending only on g, J, $\sup_M \operatorname{tr}_g \tilde{g}$ and $\sup_M \operatorname{tr}_{\tilde{g}} g$.

Proof. Recall from (4.44) that we have

$$\tilde{N}^i_{\bar{j}\,\bar{k}} = \overline{b^r_j b^s_k} a^i_t N^t_{\bar{r}\,\bar{s}}.$$

Applying the exterior derivative to this and using (4.74), (4.75) and (4.15) we obtain

$$\tilde{N}^i_{\bar{j}\,\bar{k},m}\tilde{\theta}^m + \tilde{N}^i_{\bar{j}\,\bar{k},\overline{m}}\overline{\tilde{\theta}^m}$$

$$= \overline{b^r_j b^s_k} b^p_m a^i_t N^t_{\bar{r}\,\bar{s},p}\tilde{\theta}^m + \overline{b^r_j b^s_k b^p_m} a^i_t N^t_{\bar{r}\,\bar{s},\bar{p}}\overline{\tilde{\theta}^m} + a^i_{\ell m}\overline{b^r_j b^s_k}a^\ell_t N^t_{\bar{r}\,\bar{s}}\tilde{\theta}^m$$

(4.80)
$$\quad - \overline{a^\ell_{jm} b^r_\ell b^s_k} a^i_t N^t_{\bar{r}\,\bar{s}}\overline{\tilde{\theta}^m} - \overline{a^\ell_{km} b^r_j b^s_\ell} a^i_t N^t_{\bar{r}\,\bar{s}}\overline{\tilde{\theta}^m}.$$

Equating the $(1,0)$ and $(0,1)$ parts of (4.80) gives (i) and (ii). For (iii), apply the exterior derivative to (i) and substitute from (4.64) to get

$$\tilde{N}^i_{\bar{j}\,\bar{k},mp} = \overline{b^r_j b^s_k}b^\ell_m b^q_p a^i_t N^t_{\bar{r}\,\bar{s},\ell q} + \overline{b^r_j b^s_k}b^q_p a^\ell_t a^i_{\ell m} N^t_{\bar{r}\,\bar{s},q} - \overline{b^r_j b^s_k}b^\ell_q a^i_t a^q_{mp} N^t_{\bar{r}\,\bar{s},\ell}$$

(4.81)
$$\quad + \overline{b^r_j b^s_k}b^\ell_m a^q_t a^i_{qp} N^t_{\bar{r}\,\bar{s},\ell} + \overline{b^r_j b^s_k}a^\ell_t N^t_{\bar{r}\,\bar{s}}a^i_{\ell mp}.$$

The only term that is not comparable to $\sqrt{S}$ is the last one. To deal with this we first compute, using (4.24), (4.28) and (4.11)

$$\tilde{R}^i_{jk\bar{\ell}} = \tilde{R}^\ell_{kj\bar{i}} + 4\tilde{N}^i_{\bar{p}\,\bar{\ell}}\overline{\tilde{N}^p_{\bar{j}\,\bar{k}}} + 4\tilde{N}^p_{\bar{i}\,\bar{\ell}}\overline{\tilde{N}^k_{\bar{p}\,\bar{j}}}$$

$$= \tilde{R}^\ell_{jk\bar{i}} + 4\tilde{N}^\ell_{\bar{p}\,\bar{i}}\overline{\tilde{N}^p_{\bar{k}\,\bar{j}}} + 4\tilde{N}^i_{\bar{p}\,\bar{\ell}}\overline{\tilde{N}^p_{\bar{j}\,\bar{k}}} + 4\tilde{N}^p_{\bar{i}\,\bar{\ell}}\overline{\tilde{N}^k_{\bar{p}\,\bar{j}}}$$

$$= \tilde{R}^\ell_{jk\bar{i}} + 4\tilde{N}_{\bar{\ell}\,\bar{p}\,\bar{i}}\overline{\tilde{N}_{\bar{p}\,\bar{k}\,\bar{j}}} + 4\tilde{N}_{\bar{i}\,\bar{\ell}\,\bar{p}}\overline{\tilde{N}_{\bar{p}\,\bar{k}\,\bar{j}}} + 4\tilde{N}_{\bar{p}\,\bar{i}\,\bar{\ell}}\overline{\tilde{N}_{\bar{p}\,\bar{k}\,\bar{j}}} + 4\tilde{N}_{\bar{p}\,\bar{i}\,\bar{\ell}}\overline{\tilde{N}_{\bar{j}\,\bar{p}\,\bar{k}}}$$

$$= \tilde{R}^\ell_{jk\bar{i}} + 4\tilde{N}^p_{\bar{i}\,\bar{\ell}}\overline{\tilde{N}^j_{\bar{p}\,\bar{k}}},$$

and use this, (4.25), (4.26) and (4.27) to compute

$$
\begin{aligned}
2\overline{\tilde{N}^k_{\bar{\ell}\,\bar{p},ip}} &= 2\tilde{K}^i_{kp\ell,\bar{p}} \\
&= 4\overline{\tilde{K}^k_{iqp}\tilde{N}^q_{\bar{p}\,\bar{\ell}}} + \tilde{R}^i_{kp\bar{p},\ell} - \tilde{R}^i_{k\ell\bar{p},p} \\
&= 4\overline{\tilde{K}^k_{iqp}\tilde{N}^q_{\bar{p}\,\bar{\ell}}} + \tilde{R}^p_{kp\bar{i},\ell} - \tilde{R}^p_{k\ell\bar{i},p} + 4\tilde{N}^q_{i\bar{p},\ell}\overline{\tilde{N}^k_{\bar{q}\bar{p}}} + 4\tilde{N}^q_{i\bar{p}}\overline{\tilde{N}^k_{\bar{q}\bar{p},\bar{\ell}}} \\
&\quad - 4\tilde{N}^q_{i\bar{p},p}\overline{\tilde{N}^k_{\bar{q}\bar{\ell}}} - 4\tilde{N}^q_{i\bar{p}}\overline{\tilde{N}^k_{\bar{q}\bar{\ell},\bar{p}}} \\
&= 2\overline{\tilde{N}^k_{\bar{\ell}\,\bar{p},pi}} + 4\tilde{N}^i_{\bar{p}\,\bar{q},k}\overline{\tilde{N}^q_{\bar{p}\,\bar{\ell}}} - 4\tilde{N}^p_{i\bar{q},k}\overline{\tilde{N}^q_{\bar{p}\,\bar{\ell}}} + 4\tilde{N}^q_{i\bar{p},\ell}\overline{\tilde{N}^k_{\bar{q}\bar{p}}} + 4\tilde{N}^q_{i\bar{p}}\overline{\tilde{N}^k_{\bar{q}\bar{p},\bar{\ell}}} \\
&\quad - 4\tilde{N}^q_{i\bar{p},p}\overline{\tilde{N}^k_{\bar{q}\bar{\ell}}} - 4\tilde{N}^q_{i\bar{p}}\overline{\tilde{N}^k_{\bar{q}\bar{\ell},\bar{p}}}.
\end{aligned}
$$

(4.82)

This means that, up to an error comparable to $\sqrt{S}$, we can interchange the last two covariant derivatives on $\tilde{N}$. Finally recall from (4.43) that

$$
a^i_{k\ell}\tilde{\theta}^\ell \wedge \tilde{\theta}^k = a^i_j T^j_{pq}b^p_k b^q_\ell \tilde{\theta}^\ell \wedge \tilde{\theta}^k,
$$

and so

(4.83)
$$
a^i_{k\ell} = a^i_{\ell k} + 2a^i_j b^p_k b^q_\ell T^j_{pq}.
$$

From (4.82), (4.81), (4.44) and (4.83),

$$
\begin{aligned}
\left| a^i_{k\ell}\tilde{N}^k_{\bar{\ell}\,\bar{p},ip} \right|_{\tilde{g}} &\le C(S+1) + \left| a^i_{k\ell}\tilde{N}^k_{\bar{\ell}\,\bar{p},pi} \right|_{\tilde{g}} \\
&\le C(S+1) + \left| a^i_{k\ell}\tilde{N}^q_{\bar{\ell}\,\bar{p}}a^k_{qpi} \right|_{\tilde{g}} \\
&\le C(S+1) + \left| a^i_{k\ell}\tilde{N}^q_{\bar{\ell}\,\bar{p}}(a^k_{qpi} - a^k_{rp}a^r_{qi}) \right|_{\tilde{g}} + \left| a^i_{k\ell}a^k_{rp}a^r_{qi}\tilde{N}^q_{\bar{\ell}\,\bar{p}} \right|_{\tilde{g}} \\
&\le C(S+1) + \frac{1}{2}\left| a^k_{qpi} - a^k_{rp}a^r_{qi} \right|^2_{\tilde{g}} + \left| a^i_{k\ell}a^k_{rp}a^r_{qi}\tilde{N}^q_{\bar{\ell}\,\bar{p}} \right|_{\tilde{g}} \\
&\le C(S+1) + \frac{1}{2}\left| a^k_{qpi} - a^k_{rp}a^r_{qi} \right|^2_{\tilde{g}} + \left| a^i_{k\ell}a^k_{rp}a^r_{iq}\tilde{N}^q_{\bar{\ell}\,\bar{p}} \right|_{\tilde{g}},
\end{aligned}
$$

where the constant C differs from line to line, and where we have used the inequality

$$
2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2,
$$

for any $\varepsilon > 0$ and any real numbers a and b. Finally, using (4.11) we can see that the term $a^i_{k\ell}a^k_{rp}a^r_{iq}\tilde{N}_{\bar{q}\,\bar{\ell}\,\bar{p}}$ vanishes:

$$
\begin{aligned}
a^i_{k\ell}a^k_{rp}a^r_{iq}\tilde{N}_{\bar{q}\,\bar{\ell}\,\bar{p}} &= \frac{1}{3}(a^i_{k\ell}a^k_{rp}a^r_{iq} + a^i_{kp}a^k_{rq}a^r_{i\ell} + a^i_{kq}a^k_{r\ell}a^r_{ip})\tilde{N}_{\bar{q}\,\bar{\ell}\,\bar{p}} \\
&= \frac{1}{3}a^i_{k\ell}a^k_{rp}a^r_{iq}(\tilde{N}_{\bar{q}\,\bar{\ell}\,\bar{p}} + \tilde{N}_{\bar{p}\,\bar{q}\,\bar{\ell}} + \tilde{N}_{\bar{\ell}\,\bar{p}\,\bar{q}}) = 0,
\end{aligned}
$$

and this completes the proof. $\qquad\square$

We can now prove the following lemma.

Lemma 4.4.5. *Let $\tilde{g}$ be an almost-Kähler metric solving the Calabi-Yau equation (4.8) and suppose that there exists a constant K such that*

$$\sup_M (\operatorname{tr}_g \tilde{g}) \leq K.$$

Then there exist constants C_1, C_2 depending only on g, J, F and K such that

(4.84) $$\tilde{\Delta} S \geq -C_1 S - C_2.$$

Proof. By assumption, the a^i_j and b^i_j are uniformly bounded. From (4.55) and (4.37) we have

$$(d\partial \log v)^{(1,1)} = -F_{p\bar{q}} \theta^p \wedge \overline{\theta^q}.$$

Then from Lemma 4.3.4, we have

$$\tilde{R}_{k\bar{\ell}} = -F_{p\bar{q}} b^p_k \overline{b^q_\ell} + R_{p\bar{q}} b^p_k \overline{b^q_\ell}.$$

It follows that $|\tilde{R}_{k\bar{\ell}}|^2_{\tilde{g}} \leq C$ and $|\tilde{R}_{k\bar{\ell},p}|^2_{\tilde{g}} \leq C(S+1)$, for a constant C depending only on g, J, F and K. Then the inequality (4.84) follows from Lemma 4.4.3 and Lemma 4.4.4. $\square$

Finally, we complete the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. Following [Y2] we apply the maximum principle to $S + C'u$, for a constant C' to be determined later. Note that from (4.56), we have

$$\tilde{\Delta} u \geq C_3^{-1} S - C_4,$$

for positive constants C_3 and C_4 depending only on g, J, F and K. Choose $C' = C_3(C_1 + 1)$ then from Lemma 4.4.5 we see that

$$\tilde{\Delta}(S + C'u) \geq S - C_2 - C'C_4,$$

and then by the maximum principle S is bounded from above by $C_0 = C_2 + C'C_4 + C'K$. $\square$

4.5 Proof of Theorem 4.1.3

Let $\tilde{g}$ solve the Calabi-Yau equation (4.8). We will write dV_g for the volume form associated to the metric g. We have the following lemma.

Lemma 4.5.1. *For every $\alpha > 0$ there exists a constant C depending only on (M, J, g), F and α such that*

$$-\inf_M \varphi \leq C + \log \left(\int_M e^{-\alpha\varphi} dV_g \right)^{1/\alpha}.$$

Proof. Let $\delta > 0$ be a small constant. In the following C will denote a uniform constant, depending only on δ and the fixed data, which may change from line to line. Define $w = e^{-B\varphi}$ for $B = \frac{1}{1-\delta}A$, where A is the constant in Theorem 4.3.1. Write $\gamma = 1 - \delta > 0$. Notice that for any smooth function f we have that

$$(4.85) \qquad |df|^2_g \leq \operatorname{tr}_g \tilde{g} |df|^2_{\tilde{g}}.$$

For $p \geq 1$, from Theorem 4.3.1 and the Calabi-Yau equation,

$$
\begin{aligned}
\int_M |dw^{p/2}|^2_g dV_g &\leq -C\int_M (\operatorname{tr}_g \tilde{g}) de^{-\frac{Bp\varphi}{2}} \wedge J de^{-\frac{Bp\varphi}{2}} \wedge \tilde{\omega}^{n-1} \\
&\leq -Cp^2 e^{-B\gamma \inf_M \varphi} \int_M e^{-B(p-\gamma)\varphi} d\varphi \wedge J d\varphi \wedge \tilde{\omega}^{n-1} \\
&= C\frac{p^2}{p-\gamma}\|w\|^\gamma_{C^0} \int_M d\left(e^{-B(p-\gamma)\varphi}\right) \wedge J d\varphi \wedge \tilde{\omega}^{n-1} \\
&\leq Cp\|w\|^\gamma_{C^0} \int_M w^{p-\gamma} \tilde{\Delta}\varphi\, \tilde{\omega}^n \\
&\leq Cp\|w\|^\gamma_{C^0} \int_M w^{p-\gamma} dV_g,
\end{aligned}
$$

using (4.85) and the fact that $\tilde{\Delta}\varphi \leq 2n$ from (4.38). The Sobolev inequality gives us, for $\beta = \frac{n}{n-1}$, and any $f \in C^\infty(M)$,

$$\left(\int_M f^{2\beta} dV_g\right)^{1/\beta} \leq C\left(\int_M |df|^2_g dV_g + \int_M f^2 dV_g\right).$$

Applying this to $f = w^{p/2}$, we obtain

$$
\begin{aligned}
\left(\int_M w^{p\beta} dV_g\right)^{1/\beta} &\leq C\left(\int_M |dw^{p/2}|^2_g dV_g + \int_M w^p dV_g\right) \\
&\leq Cp\|w\|^\gamma_{C^0} \int_M w^{p-\gamma} dV_g.
\end{aligned}
$$

Raising to the power $1/p$ we have

$$(4.86) \qquad \|w\|_{p\beta} \leq C^{1/p} p^{1/p} \|w\|^{\gamma/p}_{C^0} \|w\|^{(p-\gamma)/p}_{p-\gamma},$$

where $\| \ \|_q$ denotes the L^q norm with respect to dV_g (we also allow $0 < q < 1$, defined in the obvious way). By the same iteration as in [We1] we replace p with $p\beta + \gamma$ in (4.86) to obtain for $k = 1, 2, \dots$,

$$(4.87) \qquad \|w\|_{p_k\beta} \leq C(k)\|w\|^{1-a(k)}_{C^0} \|w\|^{a(k)}_{p-\gamma},$$

where

$$
\begin{aligned}
p_k &= p\beta^k + \gamma(1 + \beta + \beta^2 + \cdots \beta^{k-1}) \\
C(k) &= C^{(1+\beta+\cdots+\beta^k)/p_k} p_0^{\beta^k/p_k} p_1^{\beta^{k-1}/p_k} \cdots p_k^{1/p_k} \\
a(k) &= \frac{(p-\gamma)\beta^k}{p_k}.
\end{aligned}
$$

Set $p = 1$. There exists $\ell = \ell(n) > 0$ such that $\beta^k \le p_k \le \beta^{\ell+k}$. Then $a(k) \to a \in (0,1)$ as $k \to \infty$. Moreover,

$$
C(k) \le C^{(1+\beta+\cdots+\beta^k)/\beta^k} \beta^\ell \beta^{(\ell+1)/\beta} \cdots \beta^{(\ell+k)/\beta^k},
$$

and so

$$
\log C(k) \le K_1 \log C + \log \beta \sum_{i=0}^{k} \frac{\ell + i}{\beta^i} \le K_2,
$$

for some uniform constants $K_1 = K_1(n)$ and $K_2 = K_2(n, C)$. Hence, letting $k \to \infty$ in (4.87) and recalling that $p = 1$, we have

$$
\|w\|_{C^0} \le C \|w\|_\delta.
$$

Choosing δ sufficiently small completes the proof of the lemma. $\square$

We can now give the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. From this lemma, Theorem 4.3.1 and Theorem 4.4.1 we have the estimate

$$
\|\tilde{g}\|_{C^1} \le C,
$$

where C depends on $\Omega, J, \sigma, \alpha$ and $I_\alpha(\varphi)$. It remains to prove the higher order estimates. Following [We2], define a 1-form a by the equations

$$
\tilde{\omega} = \Omega - \frac{1}{2} d(Jd\varphi) + da,
$$

and $d_{\tilde{g}}^* a = 0$, where $d_{\tilde{g}}^*$ is the formal adjoint of d associated to $\tilde{g}$. Note that a is defined only up to the addition of a harmonic 1-form. From the definition of φ it follows that $da \wedge \tilde{\omega}^{n-1} = 0$. Let's call $\mathcal{P} : \Lambda^2(M) \to \Lambda^2(M)$ the map that associates to a 2-form γ its $(2,0) + (0,2)$ part, so that

$$
\mathcal{P}\gamma(X, Y) = \frac{1}{2}(\gamma(X, Y) - \gamma(JX, JY)).
$$

Since $\tilde{\omega}$ is compatible with J we have $\mathcal{P}\tilde{\omega} = 0$, but in general $\mathcal{P}\Omega \ne 0$. Now set $f = \varphi$ in (4.31) and take the $(2,0)$ part to get

$$
\varphi_{ij}\theta^j \wedge \theta^i + \varphi_{\bar{k}}\overline{N_{\bar{j}\bar{i}}^k}\theta^j \wedge \theta^i + \varphi_k T_{ji}^k \theta^j \wedge \theta^i = 0.
$$

Applying $\mathcal{P}$ to (4.36),

$$
\begin{aligned}
\mathcal{P}d(Jd\varphi) &= 2\sqrt{-1}\mathcal{P}d\partial\varphi \\
&= 2\sqrt{-1}\left(\varphi_{ij}\theta^j \wedge \theta^i + \varphi_k T^k_{ji}\theta^j \wedge \theta^i + \varphi_k N^k_{\bar{j}\,\bar{i}}\overline{\theta^j} \wedge \overline{\theta^i}\right) \\
&= 2\sqrt{-1}\left(\varphi_k N^k_{\bar{j}\,\bar{i}}\overline{\theta^j} \wedge \overline{\theta^i} - \varphi_{\bar{k}}\overline{N^k_{\bar{j}\,\bar{i}}}\theta^j \wedge \theta^i\right),
\end{aligned}
$$

which involves only one derivative of φ. Now the 1-form a satisfies the following system

$$
\text{(4.88)} \qquad
\begin{cases}
da \wedge \tilde{\omega}^{n-1} &= 0 \\
\mathcal{P}da &= -\mathcal{P}\Omega + \sqrt{-1}\left(\varphi_k N^k_{\bar{j}\,\bar{i}}\overline{\theta^j} \wedge \overline{\theta^i} - \varphi_{\bar{k}}\overline{N^k_{\bar{j}\,\bar{i}}}\theta^j \wedge \theta^i\right) \\
d^*_{\tilde{g}}a &= 0,
\end{cases}
$$

which is elliptic (its symbol is injective, although not invertible if $n > 2$).

Note that the kernel of (4.88) consists of the harmonic 1-forms. Indeed, $da \wedge \tilde{\omega}^{n-1} = 0$ and $\mathcal{P}da = 0$ together imply that $*da = -c_n\tilde{\omega}^{n-2} \wedge da$ for some universal constant c_n. Then if a is in the kernel of (4.88), we have $\|da\|^2_{L^2(\tilde{g})} = 0$ after integrating by parts. Since $d^*_{\tilde{g}}a = 0$, we see that a is harmonic with respect to $\tilde{g}$.

Fix any $0 < \beta < 1$. Since $\tilde{g}$ is uniformly bounded in C^β, we can apply the elliptic Schauder estimates to (4.38) to get a bound

$$
\|\varphi\|_{C^{2+\beta}} \leq C\|\tilde{\Delta}\varphi\|_{C^\beta} \leq C.
$$

Here there should be a term like $\|\varphi\|_{C^\beta}$ on the right hand side, but since (4.38) and (4.88) depend only on the gradient of φ, we are free to add a constant to φ so that it is perpendicular to the kernel of $\tilde{\Delta}$. Hence $\|\varphi\|_{C^{2+\beta}} \leq C$ and so the right hand side of (4.88) is bounded in $C^{1+\beta}$, and the coefficients of the system have a C^β bound, so assuming that a is orthogonal to the harmonic 1-forms, the elliptic estimates applied to (4.88) give $C^{2+\beta}$ bounds on a. By differentiating the Calabi-Yau equation in a direction $\partial/\partial x^i$ we obtain

$$
\text{(4.89)} \qquad \tilde{\Delta}(\partial_i\varphi) + \{\text{lower order terms}\} = 2\partial_i F + g^{pq}\partial_i g_{pq},
$$

where the lower order terms may contain up to two derivatives of φ or a, and so are bounded in C^β. Applying the Schauder estimates again we get $\|\varphi\|_{C^{3+\beta}} \leq C$, and using (4.88) again we get $\|a\|_{C^{3+\beta}} \leq C$. Now a bootstrapping argument using (4.89) and (4.88) gives the required higher order estimates. This completes the proof of Theorem 4.1.3. $\qquad\square$

4.6 Proof of Theorem 4.1.4

As before, let $\tilde{g}$ be an almost-Kähler metric solving (4.8). Let g be an almost-Hermitian metric with the property that $\mathcal{R}(g, J) \geq 0$.

Proof of Theorem 4.1.4. By the argument of the last section, it suffices to prove a uniform upper bound for $u = \frac{1}{2}\mathrm{tr}_g \tilde{g}$. From Lemma 4.3.2, we have

$$\tilde{\Delta} u \geq -C,$$

for a constant C depending only on the fixed data. We claim that this is enough to bound u uniformly from above. Indeed, for $p > 0$,

$$
\begin{aligned}
\int_M |du^{p/2}|_g^2 dV_g &\leq -C \int_M u du^{p/2} \wedge J du^{p/2} \wedge \tilde{\omega}^{n-1} \\
&= -Cp^2 \int_M u^{p-1} du \wedge J du \wedge \tilde{\omega}^{n-1} \\
&= -Cp \int_M d(u^p) \wedge J du \wedge \tilde{\omega}^{n-1} \\
&= Cp \int_M u^p d(J du) \wedge \tilde{\omega}^{n-1} \\
&= -Cp \int_M u^p (\tilde{\Delta} u) \tilde{\omega}^n \\
&\leq Cp \int_M u^p dV_g.
\end{aligned}
$$

Hence

$$\int_M |du^{p/2}|_g^2 dV_g \leq Cp \int_M u^p dV_g.$$

Then from the Sobolev inequality, we obtain

$$\|u\|_{L^{p\beta}} \leq C^{1/p} p^{1/p} \|u\|_{L^p},$$

for $\beta = \frac{n}{n-1}$. Replacing p with $p\beta$, iterating, and then setting $p = \frac{1}{n-1}$ we obtain

$$\|u\|_{C^0} \leq C \|u\|_{L^{\frac{1}{n-1}}}.$$

Using (4.60) and the Calabi-Yau equation (4.1) we can bound

$$\|u\|_{L^{\frac{1}{n-1}}} \leq C \|\mathrm{tr}_{\tilde{g}} g\|_{L^1}.$$

But this last quantity is bounded, because from (4.39) and the Calabi-Yau equation (4.1),

$$\int_M \mathrm{tr}_{\tilde{g}} g \, dV_g \leq C \int_M \frac{\tilde{\omega}^{n-1} \wedge \Omega}{\tilde{\omega}^n} \Omega^n \leq C \int_M \tilde{\omega}^{n-1} \wedge \Omega = C[\Omega]^n.$$

This completes the proof of Theorem 4.1.4. $\qquad\qquad\square$

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