Vertex-Disjoint Paths in Graphs

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Abstract

Let $n_1, n_2, \ldots, n_k$ be integers of at least two. Johansson gave a minimum degree condition for a graph of order exactly $n_1 + n_2 + \cdots + n_k$ to contain k vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$, respectively. In this paper, we extend Johansson's result to a corresponding packing problem as follows. Let G be a connected graph of order *at least* $n_1 + n_2 + \cdots + n_k$. Under this notation, we show that if the minimum degree sum of three independent vertices in G is at least

$$3\left(\left\lfloor\frac{n_1}{2}\right\rfloor + \left\lfloor\frac{n_2}{2}\right\rfloor + \cdots + \left\lfloor\frac{n_k}{2}\right\rfloor\right),$$

then G contains k vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$, respectively, or else $n_1 = n_2 = \cdots = n_k = 3$, or $k = 2$ and $n_1 = n_2 =$ odd. The graphs in the exceptional cases are completely characterized. In particular, these graphs have more than $n_1 + n_2 + \cdots + n_k$ vertices.

1 Introduction

We consider only undirected graphs without loops or multiple edges.

Johansson[4] gave an El-Zahar type condition (see [1]) for a graph to be partitioned into paths with prescribed lengths.

Theorem A (Johansson[4]) *Suppose that G is a connected graph with $|G| = n_1 + n_2 + \cdots + n_k$, where $n_1, n_2, \ldots, n_k$ are integers of at least two.*

If every vertex of G has degree at least

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_k}{2} \right\rfloor,$$

then G contains vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that $|P_i| = n_i$ for $1 \le i \le k$. □

Note that it is assumed in Theorem A that every n_i is greater than or equal to two. Also there exist several examples showing that the assumption is necessary. (See the exceptional graphs in Theorem 1.) However, it is quite natural to consider the cases where some n_i's are one. If we employ the same degree condition as in Theorem A, then any additional n_i's with $n_i = 1$ does not affect the degree condition (because $\lfloor 1/2 \rfloor = 0$). Thus such a problem is equivalent to the packing problem corresponding to Johansson's result. Moreover, we consider a degree sum condition instead of the minimum degree condition.

For a graph G and a positive integer k, we define $\sigma_k(G)$ to denote the minimum degree sum of k independent vertices in G. (If G does not contain an independent set of k vertices, then we define $\sigma_k(G) = \infty$.)

In order to state our main result, we need some graph theoretical notations. Let H_1 and H_2 be graphs. The graph $H_1 \cup H_2$ is the vertex-disjoint union of H_1 and H_2. The graph $H_1 + H_2$ is the join of H_1 and H_2, which is obtained from $H_1 \cup H_2$ by joining every pair of vertices x in H_1 and y in H_2. For a positive integer m, the vertex-disjoint union of m copies of a graph H is denoted by mH.

Theorem 1 *Suppose that G is a connected graph with $|G| \ge n_1 + n_2 + \cdots + n_k$, where $n_1, n_2, \ldots, n_k$ are integers of at least two. If*

$$\sigma_3(G) \ \ge \ 3\left(\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_k}{2} \right\rfloor\right),$$

then G contains vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that $|P_i| = n_i$ for $1 \le i \le k$, or else

(1) *$n_1 = n_2 = \cdots = n_k = 3$, and for some integers a, b and c with $a \equiv b \equiv c \equiv 2 \pmod 3$ and $a+b+c = 3k$, G is the join $K_1+(K_a \cup K_b \cup K_c)$,*

(2) *$k = 2$ and $n_1 = n_2 \equiv 1 \pmod 2$, and for some integer m with $m \ge 3$, G is the join $K_1 + mK_{n_1-1}$, or*

(3) *$n_1 = n_2 = \cdots = n_k = 3$, and for some integer m with $m \ge k+1$, G is the join of mK_2 and a graph on $k-1$ vertices.*

Since these exceptional graphs have more than $n_1+n_2+\cdots+n_k$ vertices, it is immediate to obtain the following corollary, which is a σ_3 version of Johansson's theorem, and of course implies Johansson's theorem.

Corollary 2 *Suppose that G is a connected graph with $|G| = n_1 + n_2 + \cdots + n_k$, where $n_1, n_2, \ldots, n_k$ are integers of at least two. If*

$$\sigma_3(G) \geq 3\left(\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_k}{2} \right\rfloor\right),$$

then G contains vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that $|P_i| = n_i$ for $1 \leq i \leq k$. □

2 Lemmas Concerning Long Paths

In this section, we prove several lemmas which we use in the proof of Theorem 1.

We use the following notation and terminology. A path P in a graph G is said to be *dominating* if $E(G - V(P)) = \emptyset$. A path is said to be *one-way maximal* if one of the endvertex called the *terminal* has no neighbor outside the path.

Lemma 3 *Let G be a connected graph without hamiltonian paths. Suppose that P is a longest path in G and Q is a one-way maximal path in $G - V(P)$. If P is not dominating, then $|P| + |Q| \geq \sigma_3(G) + 1$.*

Proof. Let u and v be the endvertices of P, and w be the terminal of Q. Suppose that P is oriented from u to v. For a vertex $x \in V(P)$, x^+ and x^- denote the successor and the predecessor of x, respectively. For $X \subset V(P)$, we write X^+ for $\{x^+ \mid x \in X\}$.

Note that $N_G(u) \subset V(P)$ and $N_G(v) \subset V(P)$ since P is a longest path, and that $N_G(w) \subset V(P) \cup V(Q)$ since Q is oneway maximal. We claim that $N_G(u)^-$, $N_G(v)^+$ and $N_G(w)$ are mutually disjoint subsets of $V(P) \cup V(Q) - \{w\}$. If $N_G(u)^- \cap N_G(w) \neq \emptyset$ or $N_G(v)^+ \cap N_G(w) \neq \emptyset$, then we can easily find a path through $V(P) \cup \{w\}$, which contradicts the maximality of P. If $x \in N_G(u)^- \cap N_G(v)^+$, then we have a cycle $C = ux^+Pvx^-Pu$ consisting of $|P| - 1$ vertices. By the assumption that P is not dominating, there exists a component H' of $G - V(P)$ with $|H'| \geq 2$. Since G is connected, we can take a path containing at least two vertices of H' and all vertices of C, which is longer than P, a contradiction.

By the maximality of P, u, v and w are mutually nonadjacent, and hence

$$\begin{aligned}
\sigma_3(G) &\leq \deg_G(u) + \deg_G(v) + \deg_G(w) \\
&= |N_G(u)^-| + |N_G(v)^+| + |N_G(w)| \\
&\leq |P| + |Q| - 1.
\end{aligned}$$

Thus we have proved that $|P| + |Q| \geq \sigma_3(G) + 1$. □

Lemma 4 *Let G be a connected graph without hamiltonian paths, and P be a longest path in G. Then $|P| \geq \frac{2}{3}\sigma_3(G) + 1$.*

Proof. If P is not dominating, then let Q be a one-way maximal path in $G - V(P)$ such that the starting vertex has a neighbor in P. Then, since P is a longest path, it follows that $|P| \geq 2|Q| + 1$. Also by Lemma 3, we have $|P| + |Q| \geq \sigma_3(G) + 1$. Hence $|P| \geq \frac{2}{3}|P| + \frac{1}{3}(2|Q| + 1) \geq \frac{2}{3}\sigma_3(G) + 1$.

Suppose that P is dominating. Let u and v be the endvertices of P, and let w be a vertex not in P. Then by the maximality of P, it is not difficult to see that $|P| \geq \deg_G(u) + \deg_G(v) + 1$, and that $|P| \geq 2\deg_G(w) + 1$. Since u, v and w are mutually nonadjacent, $3|P| \geq 2(\deg_G(u) + \deg_G(v) + 1) + (2\deg_G(w) + 1) \geq 2\sigma_3(G) + 3$, and the desired inequality follows. □

Note that the minimum degree version of Lemmas 3 and 4 was proved in [3] (Lemma 4). In order to prove Theorem 1, we need the characterization of the extremal structure of Lemma 3.

Lemma 5 *Let G be a connected graph without hamiltonian paths, P a longest path in G, and Q a longest path in $G - V(P)$. If $|Q| \geq 2$ and $|P| + |Q| = \sigma_3(G) + 1$, then one of the followings holds:*

(5a) $G = K_1 + (K_a \cup K_b \cup K_c)$ *for some integers a, b and c with $2 \leq a \leq b \leq c$,*

(5b) $G = K_1 + tK_a$ *for some integers $a \geq 2$ and $t \geq 4$, or*

(5c) $G \subset K_s + tK_2$ *for some integers s and t with $t \geq s + 2 \geq 4$.*

Proof. Let u and v be the endvertices of P, and w an endvertex of Q. We use the same notations as in the proof of Lemma 3. Let H be the component of $G - V(P)$ containing the path Q.

By the argument in the proof of Lemma 3, the equality $|P| + |Q| = \sigma_3(G) + 1$ implies that $\deg_G(u) + \deg_G(v) + \deg_G(w) = \sigma_3(G)$ and

(1) $V(P) \cup V(Q) - \{w\}$ is the disjoint union of $N_G(u)^-$, $N_G(v)^+$ and $N_G(w)$.

In particular, w is adjacent to all vertices in Q. Thus Q is a spanning path of H, and every vertex in H can play the same role as w. Consequently, we can observe that

(2) H is complete, and for every vertex $z \in V(H)$,

$$N_G(z) \cap V(P) = N_G(w) \cap V(P) = V(P) - (N_G(u)^- \cup N_G(v)^+).$$

Let $X = N_G(w) \cap V(P)$, and let $u_1, u_2, \ldots, u_s$ be the vertices in X along the orientation of P. We prove the following claim:

(3) If u is adjacent to a vertex x in $u_1^+ P v$, then $x \in X$.

Suppose not. Let $x \in N_G(u)$ be a vertex in $V(u_1^+ P v) - X$ such that xPv is as short as possible. By (1), x is contained in exactly one of $N_G(u)^-$, $N_G(v)^+$ and $N_G(w)$. By the definition of x, we have $x \notin N_G(w)$. If $x \in N_G(v)^+$, then we can find a path $u_1^- P u x P v x^- P u_1 w$ which is longer than P, a contradiction. Hence $x \in N_G(u)^-$, i.e., $x^+ \in N_G(u)$. By the minimality of xPv, x^+ is contained in X. Then there exists a path $vPx^+ w u_1 P u x P u_1^+$ longer than P, a contradicion. Hence we have proved (3).

By symmetry, the following statement holds:

(3') If v is adjacent to a vertex x in uPu_s^-, then $x \in X$.

Next we prove the following two claims:

(4) $|u_i^+ P u_{i+1}^-| = 2$ for every i with $1 \le i \le s-1$.

(5) $N_G(u) = V(u^+ P u_1) \cup X$ and $N_G(v) = V(u_s P v^-) \cup X$.

If $|u_i^+ P u_{i+1}^-| \ge 3$, then u_i^{++} is contained in none of the sets $N_G(u)^-$, $N_G(v)^+$ and $N_G(w)$, contradicting (1). If $|u_i^+ P u_{i+1}^-| = 1$, then since $|Q| = |H| \ge 2$, taking a vertex $w' \in V(H) - \{w\}$, we can find a path $uPu_i w w' u_{i+1} P v$ which is longer than P, a contradiction. Thus (4) has been proved.

By (3') and (4), any vertex x in $V(uPu_1^-) \cup X^-$ is contained in neither $N_G(v)^+$ nor $N_G(w)$. Hence by (1), we have $x \in N_G(u)^-$. This shows $N_G(u) = V(u^+ P u_1) \cup X$. By symmetry, it follows that $N_G(v) = V(u_s P v^-) \cup X$.

Now we mention the following claim:

(6) In every component H' of $G - V(P)$, there exists a path Q' with $|Q'| = |Q|$.

Let Q' be a longest path in H'. Since Q is longest in $G - V(P)$, we have $|Q'| \le |Q|$. Since P is not dominating, by applying Lemma 3 to Q', we have $|P| + |Q'| \ge \sigma_3(G) + 1$. This implies $|Q'| \ge |Q|$. Thus $|Q'| = |Q|$ holds.

This claim implies that every component H' of $G - V(P)$ can play the same role as H, and hence by (2), (4), and (5), we have the following:

(7) Every component of $G - V(P)$ is a complete graph of same order, and for every vertex $z \in V(G) - V(P)$, $N_G(z) \cap V(P) = X$.

Thus in particular, every component of $G - V(P)$ is a component of $G - X$. In addition, we can prove the following:

(8) Every component of $P - X$ is a component of $G - X$.

This is because, if there is an edge joining different components of $P - X$, then we can easily find a path longer than P.

Let $a = |H|$, $b = |uPu_1^-|$ and $c = |u_s^+Pv|$. We may assume that $b \leq c$. Also, by the maximality of P, we have $a \leq b$. Then $|P| + |Q| = a + b + c + 3s - 2$, which is equal to $\sigma_3(G) + 1$ by the assumption. Thus we have

(9) $\sigma_3(G) = a + b + c + 3s - 3$.

Let h be the number of components in $G - V(P)$. If $s = 1$ and $h = 1$, then G is a subgraph of $K_1 + (K_a \cup K_b \cup K_c)$. However, G cannot be a proper subgraph since $\sigma_3(G) = a + b + c$ by (9). Thus (a) follows. If $s = 1$ and $h \geq 2$, then taking a vertex w' in $G - V(P) - V(H)$, we have $\sigma_3(G) \leq \deg_G(u) + \deg_G(w) + \deg_G(w') = 2a + b$. By (9), $a + b + c \leq \sigma_3(G) \leq 2a + b$, implying $c \leq a$. Since we assumed $a \leq b \leq c$, we have $a = b = c$. Thus G is a subgraph of $K_1 + (h + 2)K_a$. Considering the condition of $\sigma_3(G)$, we conclude that G is isomorphic to $K_1 + (h + 2)K_a$, and (b) follows.

If $s \geq 2$, then $\sigma_3(G) \leq \deg_G(u) + \deg_G(w) + \deg_G(u_1^+) \leq (b + s - 1) + (a + s - 1) + (s + 1) = a + b + 3s - 1$. Then by (9), we have $c \leq 2$. Thus we have $a = b = c = 2$. This implies that G is a subgraph of $K_s + (h + 2)K_2$, and (c) follows. This completes the proof of Lemma 5. □

Lemma 6 *Let $d = \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_k}{2} \rfloor$ with $n_i \geq 2$, $1 \leq i \leq k$. Suppose P and Q are vertex-disjoint paths with $|P| \geq 2d + 1$ and $|P| + |Q| \geq 3d + 1$. Then, one can divide these paths to obtain vertex-disjoint paths of order $n_1, n_2, \ldots, n_k$, unless $|P| + |Q| = 3d + 1$ and*

(6a) *$n_1 = n_2 = \cdots = n_k = 3$ (hence $d = k$) and $|P| \equiv |Q| \equiv 2 \pmod 3$, or*

(6b) *$k = 2$, $n_1 = n_2 \equiv 1 \pmod 2$ (hence $d = n_1 - 1$), $|P| = 2n_1 - 1$ and $|Q| = n_1 - 1$.*

Proof. We may assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. Let r be the number of odd integers among $n_1, n_2, \ldots, n_k$. Then, it follows that $n_1 + n_2 + \cdots + n_k = 2d + r$. When $k = 1$, since $|P| \geq 2d + 1 \geq n_1$, we can obtain a path of order n_1 from P. Hence we assume $k \geq 2$. Then, since $n_i \geq 2$ for each $i \geq 3$, we have $d \geq \frac{n_1 + n_2 - 2}{2} + (k - 2) \geq n_2 + k - 3$, and hence

(10) $n_2 \leq d - k + 3$, in which equality holds if and only if $n_1 = n_2 \equiv 1 \pmod 2$ and $n_i \leq 3$ for $i \geq 3$.

Now, we shall remove paths of order $n_1, n_2, \ldots$ from P and Q. We assume that we can remove these paths up to n_j, and cannot take a path of order n_{j+1}. Note that we can always remove a path of order n_1, since $|P| \geq 2d + 1 \geq n_1$. Thus $j \geq 1$. Then, there remain at most two paths of order less than n_{j+1}. Hence,

$$\begin{aligned} |P| + |Q| &\leq n_1 + n_2 + \cdots + n_j + 2(n_{j+1} - 1) \\ &\leq n_1 + n_2 + \cdots + n_k + n_{j+1} - 2 \\ &\leq (2d + r) + (d - k + 3) - 2 \\ &= 3d + 1 + r - k \\ &\leq 3d + 1, \end{aligned}$$

where equality must hold, since $|P| + |Q| \geq 3d + 1$. In particular, we have $|P| + |Q| = 3d + 1$. By the equality for the last inequality, we have $r = k$, namely, all n_i $(1 \leq i \leq k)$ are odd. By the equality of the second inequality, we have $j + 1 = k$. By the equality of the third inequality, we have $n_2 = n_k = d - k + 3$, and hence equality in (10) holds. If $k = 2$, then by (10), we have $n_1 = n_2 \equiv 1 \pmod 2$. Further by the equality of the first inequality, we have $|P| = n_1 + n_2 - 1 = 2n_1 - 1$ and $|Q| = n_1 - 1$. Thus (b) follows. If $k \geq 3$, then by (10) again, we have $n_1 = n_2 = \cdots = n_k = 3$. Further by the equality of the first inequality, each of P and Q consists of $2 \pmod 3$ vertices. Thus (a) follows. □

3 Proof of Theorem 1

We use the following lemma, which plays a key role in the proof of Johansson's theorem. In fact, Lemma 5 in [4] deals with only the case where $|G| = n_1 + \cdots + n_k$. However, the same argument works also when $|G| > n_1 + \cdots + n_k$.

Lemma 7 ([4], Lemma 5) *Let G be a graph of order at least $n_1 + \cdots + n_k$, where $n_i \geq 2$ for $1 \leq i \leq k$. If G contains a path P satisfying that*

$$|N_G(x) \cap V(P)| \geq \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_k}{2} \right\rfloor$$

for all $x \in V(G) - V(P)$, then G contains vertex-disjoint paths of order n_1, $n_2, \ldots, n_k$. □

Proof of Theorem 1. Let $d = \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \cdots + \lfloor \frac{n_k}{2} \rfloor$. Define $L = \{z \in V(G) \mid \deg_G(z) < d\}$.

Let P be a longest path in G. If P is a hamiltonian path, then since $|P| \geq n_1 + n_2 + \cdots + n_k$, we are done. If not, choose P so that $|V(P) \cap L|$ is as large as possible. Moreover, subject to the condition, we prefer the situation where many endvertices of P are in L if any.

Suppose first that P is a dominating path. If no vertex in $V(G)-V(P)$ is in L, then by Lemma 7, G contains the desired paths. So let $w \in V(G)-V(P)$ be a vertex in L. Let u and v be the endvertices of P, so that P is oriented from u to v. We claim that $N_G(u)^-$, $N_G(v)^+$ and $N_G(w)$ are pairwise disjoint subsets of $V(P)$. By the maximality of P, one can easily observe that $N_G(u)^- \cap N_G(w) = \emptyset$ and $N_G(v)^+ \cap N_G(w) = \emptyset$. Suppose that there exists a vertex $x \in N_G(u)^- \cap N_G(v)^+$. Note that one of u and v is not contained in L, since $\deg_G(u) + \deg_G(v) + \deg_G(w) \geq \sigma_3(G) \geq 3d$. Thus x is not contained in L, for otherwise we can take another longest path P' with $V(P') = V(P)$, increasing the number of endvertices contained in L, a contradiction. Then, we can easily find a longest path P'' with $V(P'') = V(P) \cup \{w\} - \{x\}$, contradicting the maximality of $|V(P) \cap L|$. This shows that $N_G(u)^- \cap N_G(v)^+ = \emptyset$. Then, $|P| \geq \deg_G(u) + \deg_G(v) + \deg_G(w) \geq \sigma_3(G) \geq 3d \geq n_1 + n_2 + \cdots + n_k$, and hence G contains the desired paths.

Suppose that P is not dominating. Let Q be a longest path in $G-V(P)$ so that $|Q| \geq 2$. By Lemmas 3 and 4, we have $|P| \geq 2d+1$ and $|P|+|Q| \geq 3d+1$. Then by Lemma 6, we may assume that $|P|+|Q| = 3d+1$, and (6a) or (6b) holds. The equality $|P|+|Q| = 3d+1$ implies that $\sigma_3(G) = 3d$ and $|P|+|Q| = \sigma_3(G)+1$. Thus by Lemma 5, (5a), (5b) or (5c) holds.

Suppose first that G satisfies (5a). Then $\sigma_3(G)$ ($= 3d$) $= a+b+c$, $|P| = b+c+1$ and $|Q| = a$. If (6a) holds, then $a+b+c = 3d = 3k$ and $a \equiv b+c+1 \equiv 2 \pmod 3$. If $b \not\equiv 2 \pmod 3$ or equivalently $c \not\equiv 2 \pmod 3$, then we can easily find k vertex-disjoint paths of order three in $G = K_1 + (K_a \cup K_b \cup K_c)$. Hence we have $a \equiv b \equiv c \equiv 2 \pmod 3$, and (1) follows. If (6b) holds, then since $|P| = 2n_1 - 1$ and $|Q| = n_1 - 1$, we have $a = b = c = n_1 - 1$, and hence (2) holds.

Next suppose that G satisfies (5b). Then $\sigma_3(G) = 3d = 3a$, $|P| = 2a+1$ and $|Q| = a$. Hence $a = d$. If (6a) holds, then $k = d = a$ and $a \equiv 2 \pmod 3$. Since $G = K_1 + tK_a$ with $t \geq 4$, G contains

$$\left\lfloor \frac{2a+1}{3} \right\rfloor + (t-2)\left\lfloor \frac{a}{3} \right\rfloor \geq \frac{2a-1}{3} + 2\,\frac{a-2}{3} = k - 1 + \frac{a-2}{3}$$

vertex-disjoint paths of order three. Thus we may assume that $a = 2$. Namely $k = 2$, and (3) is satisfied. If (6b) holds, then we have $a = d = n_1 - 1$ and hence (2) holds.

Finally, suppose that G satisfies (5c). Then from the proof of Lemma 5, we see that $\sigma_3(G) = 3d = 3s+3$, $|P| = 3s+2$ and $|Q| = 2$. Hence $d = s+1$. If (6a) holds, then (3) follows easily. If (6b) holds, then since $|P| = 2n_1 - 1$ and $|Q| = n_1 - 1$, we have $n_1 = 3$ and $s = 1$. This contradicts the condition $s \geq 2$ in (5c).

This completes the proof of Theorem 1. □

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