On the Number of Longest and Almost Longest Cycles in Cubic Graphs

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Abstract

We consider the questions: How many longest cycles must a cubic graph have, and how many may it have? For each $k \geq 2$ there are infinitely many p such that there is a cubic graph with p vertices and precisely one longest cycle of length $p-k$. On the other hand, if G is a graph with p vertices, all of which have odd degree, and its longest cycle has length $p-1$, then it has a second (but not necessarily a third) longest cycle. We presents results and conjectures on the maximum number of cycles in cubic multigraphs of girth $2, 3, 4$, respectively. For cubic cyclically 5-edge-connected graphs we have no conjecture but, we believe that the generalized Petersen graphs $P(n, k)$ are relevant. We enumerate the hamiltonian and almost hamiltonian cycles in each $P(n, 2)$. Curiously, there are many of one type if and only there are few of the other. If n is odd, then $P(2n, 2)$ is a covering graph of $P(n, 2)$. (For example, the dodecahedron graph is a covering graph of the Petersen graph). Another curiosity is that one of these has many (respectively few) hamiltonian cycles if and only if the other has few (respectively many) almost hamiltonian cycles.

1 Introduction

How few and how many longest cycles may a cubic graph with p vertices have? The analogous question for the total number of cycles were addressed in [1, 2]. A well-known consequence of Smith's theorem says a cubic hamiltonian graph cannot have fewer than 3 hamiltonian cycles. We prove that for each natural number k, where $k \geq 2$, there are infinitely many cubic graphs with exactly one longest cycle of length $p-k$. On the other hand, if the longest cycle has length $p-1$, then there must be a second one but not necessarily a third one. For this we apply a variant of Andrew Thomason's proof of Smith's theorem. Part of this was already done by Fleischner [5].

We prove that a cubic multigraph with p vertices has at most $2^{p/2}$ hamiltonian cycles, and this is best possible. We raise the question if the number of longest cycles in a cubic graph with p vertices is at most $12^{p/10}$. We have similar questions for the maximum number of hamiltonian cycles in cubic graphs of cyclic edge-connectivity $2, 3, 4$, respectively. For cubic cyclically 5-edge-connected graphs, the generalized Petersen graphs are possible candidates for those that have the maximum number of longest cycles.

Suppose n and k are two integers such that $1 \leq k \leq n-1$ and $n \geq 5$. The *generalized Petersen graph* $P(n,k)$ is defined to have vertex-set $\{u_i, v_i : i = 0, 1, \ldots, n-1\}$ and edge-set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : i = 0, 1, \ldots, n-1$ with subscripts reduced modulo $n\}$.

In [3], Bondy showed that $P(n,2)$ is *hypohamiltonian* (which means that the graph is non-hamiltonian but the resulting graph is hamiltonian if any one of its vertices is deleted) when $n \equiv 5 \pmod 6$.

In [10], Thomason showed that $P(n,2)$ has has precisely three hamiltonian cycles when $n \equiv 3 \pmod 6$.

In [8], Schwenk enumerated the number of hamiltonian cycles in the remaining generalized Petersen graph $P(n,2)$. An alternative proof using Grinberg's criterion was given in [4]. In this paper we enumerate the number of $(2n-1)$-cycles in $P(n,2)$ (see Table 1). A cycle of length k is called a *k-cycle*. Again, the proof makes use of the following remarkable theorem of Grinberg (see [7]) concerning planar hamiltonian graphs.

Theorem 1 *([7]) Suppose G is a plane graph with a hamiltonian cycle C which partitions its f_i faces of degree i into f_i' (respectively f_i'') faces of degree i in the interior (respectively exterior) of C, then*

$$\sum_{i\geq 3}(i-2)(f_i' - f_i'') = 0.$$

2 Cubic graphs with few longest cycles

By Smith's theorem (see [15]) every cubic hamiltonian graph has at least three hamiltonian cycles. Seymour (see [6]) asked if the following is true: Suppose C is a cycle in a cubic graph G. Does there exist a cycle C' distinct from C such that $V(C)$ is a subset of $V(C')$? If true this would imply that there is no cubic graph with precisely one longest cycle. Fleischner proved in [5] that the conjecture is true if C misses only one vertex of G but false in general [6] .

We shall here point out that the conjecture holds in many situations. We shall use a variant of Thomason's lollipop method [9]. That method was also used to find many hamiltonian cycles in bipartite graphs with large minimum degrees [13] and, perhaps more surprising, to prove that any longest cycle in a cubic graph has a chord [14].

Theorem 2 *Let C be a cycle in a graph G such that all vertices in C have odd degree in G and such that $G - V(C)$ is connected. Then G has a cycle C' distinct from C such that $V(C)$ is a subset of $V(C')$.*

Proof. Select a vertex x in C which has a neighbor in $G - V(C)$. Let y be a neighbor of x on C. Now define a new graph H as follows: A vertex in H is a hamiltonian path P in $G(V(C))$ which starts with the edge xy. Let z denote the other end of P. A vertex P' in H is a neighbor of P if P' can be obtained from P by adding an edge zu from z to $P - x$ and then deleting the edge succeeding u on P. Then the degree of P is even unless either z is joined to x or to a vertex in $G - V(C)$. If we delete from C the edge preceding x we get a hamiltonian path which has odd degree in H. Then H must have another vertex P' of odd degree. Either P' is contained in a hamiltonian cycle C' of $G(V(C))$, or else the end z of P' distinct from x is joined to $G - V(C)$ in which case P' can be extended to a cycle C' intersecting $G - V(C)$. In either case the cycle C' satisfies the conclusion of the theorem. □

Corollary 1 *If G is a cubic graph with p vertices and the longest cycle of G has $p - 1$ vertices, then G has at least two longest cycles.*

Corollary 1 follows also from Fleischner's paper [5]. We shall now prove that it is best possible. Consider the Petersen graph. If we delete a vertex, the resulting graph has precisely two cycles of length 9. Now let G_1 be obtained from the Petersen graph by replacing every vertex, except precisely one, by a triangle. Then G_1 has 28 vertices, it is non-hamiltonian, and it has precisely two cycles of length 27. Let e be an edge in one of these but not in the other. Let G_2 be obtained from two disjoint copies of G_1 by deleting the edge e in both copies and then adding two edges from one copy to the other. Then we get a cubic graph with 56 vertices, and it has precisely one longest cycle which has length 54. Let v be a vertex contained in the longest cycle. If we take two copies of G_2 and delete v in either copy and add three edges between the copies, then we obtain a cubic graph with 110 vertices with a unique longest cycle of length 106. If we replace a vertex contained in the longest cycle by a triangle, then we get a cubic graph with 112 vertices with a unique longest cycle of length 108, and this operation can be repeated. More generally, we get by this method for each natural number $k \geq 2$ and infinitely many natural numbers p, a cubic graph with p vertices having a unique longest cycle with $p - k$ vertices.

Corollary 2 *Every cubic, vertex-transitive, nonhamiltonian graph with p vertices containing a cycle of length $p-1$ has at least $2p$ cycles of length $p-1$.*

Proof. Let C be any cycle of length $p-1$. Let v be the vertex not in C. By Theorem 2, G has another hamiltonian cycle of $G-v$. Corollary 2 now follows by using the automorphisms of G. □

Corollary 3 *Every cubic hypohamiltonian graph with p vertices has at least $2p$ cycles of length $p-1$.*

We do not know to which extent Corollaries 2, 3 are best possible. In Corollary 3 it is not possible to replace $2p$ by $100 \times 2^{100}p$. To see this, consider the smallest planar cubic hypohamiltonian graph G in [11]. It has less that 100 vertices so it has less that 2^{100} cycles missing precisely one vertex. It also has a 4-cycle $xyzux$. Now replace this 4-cycle by a ladder, that is the cartesian product of a path with two vertices and a longer path. The resulting graph is also hypohamiltonian and has p vertices, say. For each vertex w in the ladder there are less than 2^{100} cycles missing precisely that vertex w. If w is a vertex not in the ladder, then G has less than 2^{100} cycles missing precisely that vertex w. Each such cycle can be extended to the ladder in less than p ways.

Problem 1 *Does every planar, cubic, cyclically 4-edge-connected graph with p vertices contain at least $p/2$ longest cycles?*

The cartesian product of an odd cycle with n vertices and a K_2 has precisely n hamiltonian cycles.

3 Graphs with many long cycles

We now turn to cubic graph with many longest cycles. If G is a cubic multigraph with p vertices, then its cycle space $Z(G)$ has dimension $p/2+1$. Hence the number of eulerian subgraphs is $2^{p/2+1}$. If G is nonbipartite, then the number of elements of $Z(G)$ with even cardinality equals the number of those that have odd cardinality. (To see this, let S_0 be the edge set of any fixed odd cycle in G. Let S be any element in $Z(G)$ of even cardinality. Then the symmetric difference of S and S_0 is an element of $Z(G)$ of odd cardinality.) Hence precisely half of the elements of $Z(G)$ have even cardinality, and hence the number of even cycles in G is at most $2^{p/2}$, and the number of odd cycles in G is at most $2^{p/2}$. In particular, the number of longest cycles in G is at most $2^{p/2}$. This is best possible in a sense. To see this, take a cycle of length p and replace every second edge by a double edge. Let us call this multigraph a *semi-double-cycle*.

In the bipartite case it is also true that the number of hamiltonian cycles is at most $2^{p/2}$. However we cannot use the cycles space argument above, as all elements in the cycle space now have even cardinality.

Theorem 3 *Let G be a cubic multigraph with p vertices, and let e be an edge of G which is not in a 2-edge-cut. Then G has at most $2^{p/2-1}$ hamiltonian cycles through e, unless $p = 2$.*

Proof (by induction on p). The statement is easily verified for $p = 4$, so we proceed to the induction step.

If G has a double edge we replace that edge and the two edges incident with its ends by a single edge, and we use induction. So assume that G has no double edge.

If G has a 2-edge-cut E, then we choose E such that the component H of $G-E$ not containing e is smallest possible. Then we add two edges to $G-E$ to obtain two cubic multigraphs, and we apply induction to each. When we apply induction to H with an edge added, then the added edge plays the role of e. Note that this edge is not contained in a 2-edge-cut because of the minimality of H. Also, H has more than two vertices because G has no 2-cycle. So assume that G is 3-connected.

Suppose G has a 3-edge-cut E such that $G - E$ has two components G_1, G_2 with p_1, p_2 vertices, respectively. Assume that G_2 does not contain e. We first contract G_2 to a single vertex and use induction. Then resulting graph has at most $2^{p_1/2-1/2}$ hamiltonian cycles containing e. Consider one of these, say C. Then we apply induction to G_2 with an appropriate edge (which will play the role of e) added, and with a path of length 2 replaced by a single edge. This shows that C can be extended to at most $2^{p_2/2-1/2}$ (in fact at most $2^{p_2/2-3/2}$) hamiltonian cycles in G. So assume that G is cyclically 4-edge-connected.

Now let $e = xy$, and let y, x_1, x_2 be the neighbors of x, and let x, y_1, y_2 be the neighbors of y. Now consider each of the four paths of the form $x_i x y y_j$. We delete x, y and add the edge $x_i y_j$ which we call e'. We suppress the two vertices of degree 2 so that the resulting graph G' has $p-4$ vertices. As G is cyclically 4-edge-connected, e' is not contained in a 2-edge-cut in G'. By the induction hypothesis, G' has at most $2^{p/2-3}$ hamiltonian cycles through e'. As e' can be chosen in four ways, the result follows. □

Theorem 4 *Let G be a cubic multigraph with p vertices. Then G has at most $2^{p/2}$ hamiltonian cycles. If G is not a semi-double-cycle, then G has at most $(3/4)2^{p/2}$ hamiltonian cycles.*

Proof. If G has a 2-edge-cut E, then we apply induction to each component of $G - E$ with an edge added. If the new edge is not contained in a 2-edge-cut, then we apply Theorem 3 instead. So, the result follows unless each of the two graph to which we apply induction are semi-double-cycles, and the two added edges are contained in 2-edge-cuts. Hence also G is a semi-double-cycle.

If G is 3-connected, then let v be any vertex of G. For each of the three edges incident with v, apply Theorem 3. Then we count at most 3 times $2^{p/2-1}$ hamiltonian cycles. As each hamiltonian cycle is counted twice, the result follows. □

Let H be a cubic graph with m vertices, and let e be an edge contained in some longest cycle. Let q be the number of longest cycles containing e. Consider t disjoint copies $G_1, G_2, \ldots, G_t$ of H where the indices are expressed modulo t. Now delete e from each G_i, and add an edge from G_i to G_{i+1} so that we obtain a new cubic graph G. Then G has tm vertices, and q^t longest cycles. If H is $K_{3,3}$, then we obtain in this way a cubic graph G with $p = 6t$ vertices and $4^t = 2^{p/3}$ hamiltonian cycles. If H is the Petersen graph, then we obtain in this way a cubic graph G with $p = 10t$ vertices and $12^t = 12^{p/10}$ longest cycles.

Problem 2 *Does there exist a cubic graph with p vertices and more than $12^{p/10}$ longest cycles?*

Does there exist a cubic graph with p vertices and more than $2^{p/3}$ hamiltonian cycles?

Consider a vertex v in a cubic graph G, and let v_1, v_2, v_3 be the neighbors of v. Now form a new cubic graph G' by deleting v and adding five new vertices u_1, u_2, u_3, y, z such that u_i is joined to v_i by an edge for $i = 1, 2, 3$, and y, z are joined to each of u_1, u_2, u_3. Then G' has twice as many hamiltonian cycles as G. If we start with $K_{3,3}$ and repeat this operation t times, then we obtain a graph with $6 + 4t$ vertices and $6 \cdot 2^t$ hamiltonian cycles.

Problem 3 *Does there exist a cubic, 3-connected graph with p vertices and more than $(3/\sqrt{2}) \cdot 2^{p/4}$ hamiltonian cycles?*

Consider $p/4$ disjoint 4-cycles $x_i y_i z_i u_i x_i$ where again p is divisible by 4 and the indices are expressed modulo $p/4$. For each i add the edges $y_i x_{i+1}, u_i z_{i+1}$. This graph has $(3/2) \cdot 2^{p/4}$ hamiltonian cycles.

Problem 4 *Does there exist a cubic, cyclically 4-edge-connected graph with p vertices and more than $(3/2) \cdot 2^{p/4}$ hamiltonian cycles?*

4 The number of long cycles in generalized Petersen graphs

The situation becomes more interesting when the cyclic edge-connectivity is at least 5. It is not clear which such cubic graphs have the maximum number of hamiltonian cycles. The hexagonal tilings of the torus and the Klein bottle would be interesting examples to study. Their structure is completely described [12].

Another interesting class are the generalized Petersen graphs $P(n, 2)$. Their hamiltonian cycles have been counted in [8]. We shall now count the number of cycles missing one vertex as well.

Let F_m denote the m-th Fibonacci number defined by $F_1 = F_2 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for $m > 2$.

Theorem 5 *The number of cycles of length $2n$, respectively $2n-1$, in the generalized Petersen graph $P(n, 2)$ is given in Table 1 below.*

Number of $2n$-cycles	Number of $(2n-1)$-cycles	Congruent classes
$2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1)$	$\frac{n^2}{3}$	$n \equiv 0 \pmod 6$
n	$n(F_{\frac{n+1}{2}} + 1)$	$n \equiv 1 \pmod 6$
$2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1)$	$\frac{n(n-2)}{3}$	$n \equiv 2 \pmod 6$
3	$n(F_{\frac{n+1}{2}} + \frac{n}{3})$	$n \equiv 3 \pmod 6$
$n + 2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1)$	0	$n \equiv 4 \pmod 6$
0	$n(F_{\frac{n+1}{2}} + \frac{n+1}{3})$	$n \equiv 5 \pmod 6$

Table 1. The numbers of $2n$-cycles and $(2n-1)$-cycles in $P(n, 2)$.

As Hamilton observed, the graph of the dodecahedron $P(10,2)$ is hamiltonian. In fact it has precisely 30 hamiltonian cycles. If we think of the dodecahedron as a sphere and we identify pairs of antipodal vertices, then the sphere is turned into the projective plane, and the graph of the dodecahedron is turned into the Petersen graph $P(5,2)$. By the Grinberg criterion, $P(10,2)$ has no cycle of length 19. Curiously, the Petersen graph has no hamiltonian cycle but 20 cycles of length 9. This curiosity generalizes, as $P(2n,2)$ is a covering graph of $P(n,2)$ when n is odd. (We can transforms $P(2n,2)$ into $P(n,2)$ by identifying each u_i with u_{n+i}, and identifying each v_i with v_{n+i}. $P(2n,2)$ is planar while $P(n,2)$ is projective planar.)

We also obtain the following curious facts:

$P(n,2)$ has exponentially many cycles of length $2n$, if and only if it has only polynomially many of length $2n-1$.

$P(n,2)$ has exponentially many cycles of length $2n-1$, if and only if it has only polynomially many of length $2n$.

We do not know if there are similar phenomena for other classes of cubic graphs.

As mentioned earlier, the number of hamiltonian cycles in $P(n,2)$ presented in the first column of Table 1 were counted by Bondy [3], Thomason [10] and Schwenk [8], and a unified proof was given in [4]. We shall here extend the proof to $(2n-1)$-cycles.

5 Proof of Theorem 5 when restricted to the number of $(2n-1)$-cycles

First, suppose n is odd.

Since for every pair of vertices u_i, u_j (respectively v_i, v_j), there is an automorphism of $P(n,2)$ that maps u_i to u_j (respectively v_i to v_j), we need only count the number of hamiltonian cycles in $P(n,2)-u_1$ and in $P(n,2)-v_1$ and then multiply the sum by n.

Throughout, let C denote a hamiltonian cycle in $P(n,2)-u_1$ or $P(n,2)-v_1$.

Draw $P(n,2)$ as shown in Figure 1.

We treat the case $P(n,2)-u_1$ first.

(1) Assume first that v_0v_2 is an edge in C. We claim that the number of such hamiltonian cycles in $P(n,2)-u_1$ is 1 if $n \equiv 1 \pmod 6$ and $\frac{n-2}{3}$ if $n \equiv 5 \pmod 6$, and 0 otherwise.

Suppose u_3v_3 is an edge in C. Then the paths $v_6v_4u_4u_5v_5v_7$, $v_{n-1}v_1v_3u_3$

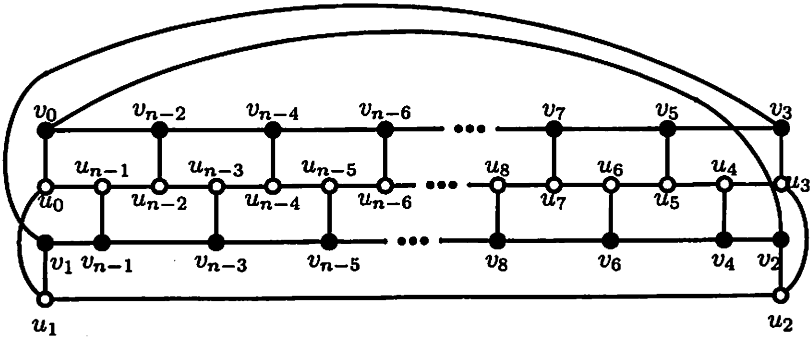

Figure 1: $P(n, 2)$ with $n \equiv 1 \pmod 2$

$u_2v_2v_0u_0u_{n-1}$ and $u_{n-2}v_{n-2}v_{n-4}$ are clearly part of C. Since the three edges v_2v_4, u_3u_4 and v_3v_5 are not in C, we may delete them from $P(n, 2)-u_1$ and obtain a plane graph whose face-degree sequence is $(5, \ldots, 5, 9, n+4)$. By Grinberg's criterion (or by a direct combinatorial argument), $P(n, 2) - u_1$ has a hamiltonian cycle only if $n \equiv 5 \pmod 6$ (since the 9-face and the $(n + 4)$-face are on different side of C). If $n \equiv 5 \pmod 6$, then the paths mentioned above extend to a unique hamiltonian cycle of $P(n, 2) - u_1$.

On the other hand, if u_3v_3 is not an edge in C, then the paths $v_6v_4u_4u_3u_2$ $v_2v_0u_0u_{n-1}$, $v_{n-1}v_1v_3v_5u_5u_6$, $v_9v_7u_7$ and $u_{n-2}v_{n-2}v_{n-4}$ are part of C.

$(O1)$ If u_6v_6 is an edge of C, then u_6u_7 and v_6v_8 are not in C. We may then draw $P(n, 2) - u_1$ as in Figure 1 but with the edge v_0v_2 intersects with v_5v_7, u_6u_7 and v_6v_8 (instead of with v_3v_5 and u_3u_4). We can then delete these three edges and obtain a plane graph whose face-degree sequence is $(5, \ldots, 5, 12, n + 1)$. Again, by Grinberg's criterion, $P(n, 2) - u_1$ has a hamiltonian cycle only if $n \equiv 5 \pmod 6$. If $n \equiv 5 \pmod 6$, then there is a unique hamiltonian cycle of $P(n, 2) - u_1$ containing these paths.

If u_6v_6 is not an edge of C, then the above four paths are replaced by $u_9u_8v_8v_6v_4u_4u_3u_2v_2v_0u_0u_{n-1}$, $v_{n-1}v_1v_3v_5u_5u_6u_7v_7v_9$, $v_{12}v_{10}u_{10}$ and $u_{n-2}v_{n-2}v_{n-4}$ respectively.

If u_9v_9 is an edge of C, then we repeat the argument as in $(O1)$, where v_5v_7, u_6u_7 and v_6v_8 are replaced by v_9v_{11}, u_9u_{10} and v_8v_{10} respectively. The face-degree sequence of the resulting graph is $(5, \ldots, 5, 15, n - 2)$. Apply Grinberg's criterion to get $n \equiv 5 \pmod 6$ in which case we have a unique hamiltonian cycle in $P(n, 2) - u_1$ containing these paths.

Continue arguing in this way, we see that, if i (≥ 2) is the smallest integer such that $u_{3i}v_{3i}$ is an edge in C, then we do the same as in $(O1)$ and then delete the edges $v_{3i-1}v_{3i+1}$, $u_{3i}u_{3i+1}$ and $v_{3i}v_{3i+2}$ to get a plane graph whose face-degree sequence is $(5, \ldots, 5, 3i+6, n+7-3i)$. By Grinberg's criterion, we have $n \equiv 5 \pmod 6$, in which case, there is only one way to extend those paths to get a unique hamiltonian cycle in $P(n,2) - u_1$.

If such i does not exist, then $n-1$ is divisible by 3, that is, $n \equiv 1 \pmod 6$, and we get a unique hamiltonian cycle.

Summarizing, we see that, if v_0v_2 is an edge in C, then $P(n,2) - u_1$ has no hamiltonian cycle unless $n \equiv 1, 5 \pmod 6$. Moreover, when $n \equiv 5 \pmod 6$, for every integer $j \in \{1, 2, \ldots, \frac{n-2}{3}\}$, there is a unique hamiltonian cycle in $P(n,2) - u_1$ avoiding the edges $v_{3j-1}v_{3j+1}$, $u_{3j}u_{3j+1}$ and $v_{3j}v_{3j+2}$. When $n \equiv 1 \pmod 6$, the hamiltonian cycle C is unique. This proves the claim in (1).

(2) Assume now that v_0v_2 is not an edge in C. We shall show that the number of hamiltonian cycles in $P(n,2) - u_1$ is $F_{\frac{n-1}{2}}$ in this case. In fact, this follows from the assertion that the number of hamiltonian cycles is equal to the number of matchings in the path $\mathcal{P}(n) = v_4v_6v_8 \ldots v_{n-1}$.

To see this, we observe that in any hamiltonian cycle C in $P(n,2) - u_1$, those edges in $\mathcal{P}(n) = v_4v_6v_8 \ldots v_{n-1}$ which are not part of C clearly form a matching on $\mathcal{P}(n)$. Observe also that the paths $v_4v_2u_2u_3$, $v_3v_1v_{n-1}$ and $u_{n-1}u_0v_0v_{n-2}$ are part of C.

On the other hand, if $i \in \{4, 6, \ldots, n-1\}$ is the smallest integer such that v_iv_{i+2} is not an edge in C, then the path $u_{i+3}u_{i+2}v_{i+2}v_{i+4}$ is part of C and the path $v_3v_1v_{n-1}$ extends to $v_{i+3}v_{i+1}u_{i+1}u_iv_iv_{i-2} \ldots v_4v_2u_2u_3 \ldots u_{i-3}u_{i-2}$ $u_{i-1}v_{i-1}v_{i-3} \ldots v_3v_1v_{n-1}$ (where the subpath $u_{i-3}u_{i-2}u_{i-1}v_{i-1}v_{i-3}$ becomes an empty path if $i = 4$).

If $j \in \{4, 6, \ldots, n-1\}$ is the second smallest integer such that v_jv_{j+2} is not an edge in C, then the configuration is repeated with i replaced by j. Therefore, if we specify a matching on the path $\mathcal{P}(n)$, there is a unique hamiltonian cycle which avoids this matching.

If there is no such integer i in $\{4, 6, \ldots, n-1\}$, then we have an empty matching on $\mathcal{P}(n)$, and there is a unique hamiltonian cycle containing all the edges in this path.

This completes the case of $P(n,2) - u_1$.

Next, we treat $P(n,2) - v_1$.

(i) Assume first that v_0v_2 is not an edge in C. We claim that the number of hamiltonian cycles in $P(n,2) - v_1$ is $F_{\frac{n-3}{2}}$ in this case.

Clearly, the paths $v_4v_2u_2u_1u_0v_0v_{n-2}$, $u_{n-2}u_{n-1}v_{n-1}v_{n-3}$ and $v_5v_3u_3u_4$ are part of C. We repeat the argument of (2) and conclude that the number of hamiltonian cycles in $P(n,2) - v_1$ is equal to the number of matchings on the path $v_5v_7\cdots v_{n-2}$ which is $F_{\frac{n-3}{2}}$.

(ii) Assume now that v_0v_2 is an edge in C. We claim that the numbers of such hamiltonian cycles in $P(n,2)-v_1$ are 0, $\frac{n}{3}$ and 1 if $n \equiv 1, 3$ and 5 (mod 6), respectively.

If u_2u_3 is not an edge of C, then we delete it and obtain a plane graph (with v_0v_2 redrawn so that it is on the unbounded face). Here the face-degree sequence is $(5,\ldots,5,\frac{n+7}{2},\frac{n+7}{2})$ and Grinberg's criterion is satisfied only if $n \equiv 3$ (mod 6) because the two $\frac{n+7}{2}$-faces are on the same side of C. This is because the paths $u_0u_1u_2v_2v_0$, $u_{n-1}v_{n-1}v_{n-3}$ and $v_6v_4u_4u_3v_3v_5u_5u_6$ are part of C. Further, there is a unique hamiltonian cycle containing these paths in the case $n \equiv 3$ (mod 6).

We are left with the case u_2u_3 is an edge of C. Here, clearly the subpaths $u_{n-1}v_{n-1}v_{n-3}$, $u_0u_1u_2u_3v_3v_5$, $v_0v_2v_4u_4u_5$ and $u_6v_6v_8$ are part of C. This immediately implies that $P(7,2)-v_1$ has no such hamiltonian cycle and that $P(5,2)-v_1$ has precisely one such hamiltonian cycle. Hence we assume that $n \geq 9$.

($O2$) If u_5v_5 is an edge of C, then v_5v_7 and u_5u_6 are not in C. We may then draw $P(n,2)-v_1$ as in Figure 1 but with the edge v_0v_2 intersects with v_5v_7, u_5u_6 and v_4v_6 (instead of with v_3v_5 and u_3u_4). We then delete these three edges and obtain a plane graph whose face-degree sequence is $(5,\ldots,5,11,n+2)$. By Grinberg's criterion, $P(n,2)-v_1$ has a hamiltonian cycle only if $n \equiv 3$ (mod 6) (because the 11-face and the $(n+2)$-face are on different side of C). If $n \equiv 3$ (mod 6), then there is a unique hamiltonian cycle of $P(n,2)-v_1$ containing these paths.

If u_5v_5 is not an edge of C, then the above four paths are replaced by $u_{n-1}v_{n-1}v_{n-3}$, $u_0u_1u_2u_3v_3v_5v_7u_7u_8$, $v_0v_2v_4u_4u_5u_6v_6v_8$ and $u_9v_9v_{11}$ respectively.

If u_8v_8 is an edge of C, then we repeat the argument in ($O2$), where v_5v_7, u_5u_6 and v_4v_6 are replaced by v_7v_9, u_8u_9 and v_8v_{10} respectively. The face-degree sequence of the resulting graph is $(5,\ldots,5,14,n-1)$. Apply Grinberg's criterion to get $n \equiv 3$ (mod 6) in which case we have a unique hamiltonian cycle in $P(n,2)-v_1$.

More generally, if j is the first integer of the form $2+3i$ (for some natural number i such that $1 \leq i \leq \frac{n-6}{3}$) such that u_jv_j is an edge in C, then we apply the argument in ($O2$) with the edges v_jv_{j+2}, u_ju_{j+1} and $v_{j-1}v_{j+1}$ deleted and obtain a plane graph with face-degree sequence $(5,\ldots,5,8+3i,n+5-3i)$. Apply Grinberg's criterion to get $n \equiv 3$ (mod 6) in which case we have a unique hamiltonian cycle in $P(n,2)-v_1$.

It remains to consider the case where u_jv_j is a not edge in C for any j of the form $2+3i$. Let j be the largest number which is of the form $2+3i$ and which is $\leq n-3$.

If $j = n-4$, then $n \equiv 3 \pmod 6$ and the four paths in the preceding argument can be extended to a unique hamiltonian cycle in $P(n,2) - v_1$ avoiding the edge $u_{n-4}v_{n-4}$.

If $j = n-3$, then $n \equiv 5 \pmod 6$ and again the four paths in the preceding argument can be extended to a unique hamiltonian cycle in $P(n,2) - v_1$ avoiding the edge $u_{n-3}v_{n-3}$.

If $j = n-5$, then $n \equiv 1 \pmod 6$ and clearly there is no hamiltonian cycle in $P(n,2) - v_1$ avoiding the edge $u_{n-5}v_{n-5}$.

This proves the claim in (ii).

We now comment on the case where n is even, that is $P(n,2)$ is planar. Draw $P(n,2)$ in the plane such that the $\frac{n}{2}$-cycle $v_1v_3 \dots v_{n-1}v_1$ is the outer face boundary.

If n is congruent to 4 modulo 6, then $P(n,2) - u_i$ and $P(n,2) - v_i$ have precisely one face whose face boundary has a length which is not congruent to 2 modulo 3, and therefore they are not hamiltonian, by the Grinberg criterion. So $P(n,2)$ has no cycle of length $2n-1$ in this case.

If n is congruent to 0 modulo 6, then $P(n,2) - u_i$ has precisely three faces whose face boundaries have length not congruent to 2 modulo 3, and these face boundaries have length 0 modulo 3. Therefore, these faces must be on the same side of the hamiltonian cycle in order to satisfy the Grinberg criterion. But this is clearly impossible. So $P(n,2) - u_1$ has no cycle of length $2n-1$ in this case.

$P(n,2) - v_i$ has precisely two faces whose face boundaries have length not congruent to 2 modulo 3. As their boundary lengths are not congruent modulo 3, these two faces must be on the same side of the hamiltonian cycle. Using this it is not difficult to see that $P(n,2) - v_i$ has precisely $\frac{n}{3}$ hamiltonian cycles. Hence $P(n,2)$ has precisely $\frac{n^2}{3}$ cycles of length $2n-1$ when n is congruent to 0 modulo 6.

If n is congruent to 2 modulo 6, then $P(n,2) - v_i$ has precisely one face whose boundary length is not congruent to 2 modulo 3. Hence it is non-hamiltonian by the Grinberg criterion. $P(n,2) - u_i$ has precisely three faces whose face boundaries have lengths not congruent to 2 modulo 3, and these face boundaries have lengths $0, 1, 1$ modulo 3. Therefore, the two faces with boundary length 1 modulo 3 must be on the same side of the hamiltonian cycle in order to satisfy the Grinberg criterion. If $i = 1$, this implies that any hamiltonian cycle must contain the paths $v_{n-1}v_1v_3$ and $u_{n-1}u_0v_0v_2u_2u_3$. Using this it is easy to see that $P(n,2) - u_i$ has $\frac{n-2}{3}$ hamiltonian cycles. Hence $P(n,2)$ has $\frac{n(n-2)}{3}$ cycles of length $2n-1$. □

Problem 5 *Does there exist a planar, cubic, cyclically* 5*-edge-connected graph with* $4n$ *vertices that contains more longest cycles than* $P(2n, 2)$*?*

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References

[1] R.E.L.ALDRED AND C. THOMASSEN, On the number of cycles in 3-connected cubic graphs, *J. Combinat. Theory B* **71** (1997) 79-84.

[2] R.E.L.ALDRED AND C. THOMASSEN, On the maximum number of cycles in a planar graph, *J. Graph Theory* **53** (2008) 255-264.

[3] J.A. BONDY, Variations on the hamiltonian theme, *Canad. Math. Bull.* **15** (1972) 57-62.

[4] G.L. CHIA AND C. THOMASSEN, Grinberg's criterion applied to some non-planar graphs, *Ars Combinat.* **100** (2011) 3-7.

[5] H. FLEISCHNER, Eulersche Linien und Kreisüberdeckungen, die vorgegebene Durchgänge in den Kanten vermeiden, *J. Combinat. Theory B* **29** (1980) 145-167.

[6] H. FLEISCHNER, Uniqueness of maximal dominating cycles in 3-regular graphs and of Hamiltonian cycles in 4-regular graphs, *J. Graph Theory* **18** (1994) 449-459.

[7] E. GRINBERG, Plane homogeneous graphs of degree three without Hamiltonian circuits, *Latvian Math. Yearbook, Izdat. Zinatne Riga* **4** (1968) 51-58 [Russian, Latvian and English summaries.]

[8] A.J. SCHWENK, Enumeration of Hamiltonian cycles in certain generalized Petersen graphs, *J. Combinat. Theory B* **47** (1989) 53-59.

[9] A.G. THOMASON, Hamiltonian cycles and uniquely edge colorable graphs, *Ann. Discrete Math.* **3** (1978) 259-268.

[10] A.G. THOMASON, Cubic graphs with three hamiltonian cycles are not always uniquely edge-colorable, *J. Graph Theory* **6** (1982) 219-221.

[11] C. THOMASSEN, Planar cubic hypohamiltonian and hypotraceable graphs, *J. Combinat. Theory B* **30** (1981) 36-44.

[12] C. THOMASSEN, Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface, *Trans. Amer. Math. Soc.* **323** (1991) 605-635.

[13] C. THOMASSEN, On the number of hamiltonian cycles in bipartite graphs, *Combinatorics, Probability and Computing* **5** (1996) 437-442.

[14] C. THOMASSEN, Chords of longest cycles in cubic graphs, *J. Combinat. Theory B* **71** (1997) 211-214.

[15] W.T. TUTTE, On Hamiltonian circuits, *J. London Math. Soc.* **21** (1946) 98-101.