Suboptimality of the Karhunen-Loève Transform for Transform Coding *

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Abstract

We examine the performance of the KLT for transform coding applications. The KLT has long been viewed as the best available block transform for transform coding. This paper treats fixed-rate and variable-rate transform codes. The fixed-rate approach uses an optimal fixed-rate scalar quantizer to describe the transform coefficients; the variable-rate approach uses a uniform scalar quantizer followed by an optimal entropy code. Earlier work shows that for the variable-rate case there exist sources on which the KLT is not unique and the optimal transform code matched to a "worst" KLT yields performance as much as 1.5 dB worse than the optimal transform code matched to a "best" KLT. In this paper, we strengthen that result to show that in both the fixed-rate and the variable-rate coding frameworks there exist sources for which the performance penalty for using a "worst" KLT can be made arbitrarily large. Further, we demonstrate in both frameworks that there exist sources for which even a best KLT gives suboptimal performance. Finally, we show that even for vector sources where the KLT yields independent coefficients, the KLT can be suboptimal for fixed-rate coding.

I Introduction

The Karhunen-Loève Transform (KLT) plays a fundamental role in a variety of disciplines, including statistical pattern matching, filtering, estimation theory, and source coding. In many of these applications, the KLT is known to be "optimal" in various senses. This paper investigates the optimality of the KLT for source coding.

The main application of the KLT in source coding is in scalar quantized transform coding. In this type of transform code, an input vector is linearly transformed into another vector of the same dimension; the components of that vector are then described to the decoder using independent scalar quantizers on the coefficients. We consider both fixed-rate and variable-rate codes. The fixed-rate code uses an optimal fixed-rate scalar quantizer; the variable-rate code uses uniform scalar quantization followed by optimal entropy coding. The decoder reconstructs the quantized transform vector and then uses a linear transformation to get an estimate of the original input vector. In both cases, the goal is to find the pair

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of linear transforms and the allocation of an average bit budget among the scalar quantizers that together minimize the end-to-end distortion. In this work, we measure distortion as mean squared error (mse). Transform codes have served as important models for gaining an understanding of both optimal quantization and low-complexity code design.

In [1], Huang and Schultheiss show that if the vector source is Gaussian and the bit budget is asymptotically large, then the KLT and its inverse are an optimal pair of transforms for fixed-rate coding. In a more recent paper, Goyal, Zhuang, and Vetterli (Proof 1 of Theorem 6 in [2]) and Telatar (Proof 2 of Theorem 6 in [2]) improve that result by showing that the KLT is optimal for Gaussian inputs without making any high resolution assumptions. Their result applies to both the fixed-rate and the variable-rate coding models.

The optimality of the KLT in transform coding of Gaussian sources is typically explained by the assertion that scalar quantization is better suited to the coding of independent random variables than to the coding of dependent random variables. Thus the optimality of the KLT for transform coding of Gaussian sources is believed to be a consequence of the fact that the KLT of a Gaussian vector yields independent transform coefficients. The application of the KLT in transform coding of non-Gaussian sources is then justified using the intuition that the KLT's coefficient decorrelation is, for general sources, the best possible approximation to the desired coefficient independence. In [3], Koschman shows that if we forgo optimal bit allocation and instead force a fixed number of the transform coefficients to be quantized at rate zero and the remaining components to be quantized with infinite accuracy, then for any stationary source the KLT minimizes the mse over all possible choices of orthogonal transforms (this is known as "zonal coding"). While this result does not address the bit allocation problem, it seems to further support the above intuition. Over the years, this intuition has blossomed into folklore, and numerous references to the "optimality" of the KLT for transform coding appear in texts and scholarly journals.

In this paper, we investigate this intuition piece by piece, demonstrating both its failures and its successes. All results apply in the high rate limit. First, we consider the question of sample decorrelation, showing that sample decorrelation is neither sufficient nor necessary for transform optimality in either fixed-rate or variable-rate transform coding. Then we show for the fixed-rate case that even for examples where decorrelation yields coefficient independence, the KLT may fail to yield the optimal performance. Finally, we show for the variable-rate case that for examples where decorrelation yields coefficient independence, the KLT guarantees the optimal performance in the high resolution limit for suitably smooth distributions.

The remainder of this paper is organized as follows. Section II introduces background material, notation, and definitions. Section III lists our main results. The outlines of proofs of these theorems appear in Sections IV–VI.

II Preliminaries

Denote the entropy of a discrete random variable Z taking on outcomes in $\{z_1, z_2, \ldots\}$ by[1] $H(Z) = -\sum_i P(Z = z_i) \log P(Z = z_i)$ and denote the differential entropy of

[1] All logarithms in this paper whose bases are omitted are base 2.

a continuous random variable Z with probability density function (pdf) f by $h(Z) = -\int f(z)\log f(z)dz$. If Z^n is a continuous random vector with components $Z_1, \ldots, Z_n$, then define the quantity $\bar{h}(Z) = (1/n)\sum_{i=1}^{n} h(Z_i)$. Denote a new norm of a pdf f as $\|f\|_{1/3} = \left(\int_{-\infty}^{\infty} f^{1/3}(x)dx\right)^3$. The usual notation $\mathcal{N}(m, \sigma^2)$ denotes the pdf of a scalar Gaussian random variable with mean m and variance σ^2.

Let source X^n be an n-dimensional random vector with real components $X_1, \ldots, X_n$. Without loss of generality, we assume that each component has mean zero, giving covariance matrix $\Phi_X = E[X^n(X^n)^t]$. Let transform T be an $n \times n$ orthogonal matrix with real elements, and let $Y^n = TX^n$ denote the transformed random vector with coefficients $Y_1, \ldots, Y_n$. We restrict attention to orthogonal transforms.

A scalar quantizer with resolution r bits is a mapping $Q : \mathbb{R} \to \mathbb{R}$ whose range space (called a *codebook*) has cardinality 2^r. The rate of a *fixed-rate* scalar quantizer Q with resolution r is $R^{(\mathrm{fr})}(Q) = r$. The rate to describe source X_i with a *variable-rate* scalar quantizer Q is $R^{(\mathrm{vr})}(Q) = H(Q(X_i))$. A *transform coder* is a system that quantizes X^n by transforming X^n by T and then applies an independent scalar quantizer Q_i to each transform coefficient Y_i, for $i \in \{1, \ldots, n\}$. Thus the per-symbol expected rate of a transform coder with transform T and quantizers $Q^n = (Q_1, \ldots, Q_n)$ is $R_T^{(\mathrm{fr})}(Q^n) = (1/n)\sum_{i=1}^{n} R^{(\mathrm{fr})}(Q_i)$ for fixed-rate transform coding and $R_T^{(\mathrm{vr})}(Q^n) = (1/n)\sum_{i=1}^{n} R^{(\mathrm{vr})}(Q_i)$ for variable-rate transform coding. The corresponding mse is $D_T(Q^n) = (1/n)E[\|X^n - T^t Q^n(TX^n)\|^2]$, where for any n-dimensional vector $x^n = (x_1, \ldots, x_n)^t$, $Q^n(x^n) = (Q_1(x_1), \ldots, Q_n(x_n))^t$ is the vector of scalar quantized components of x^n and the Euclidean distance between two arbitrary n-dimensional vectors $x^n = (x_1, \ldots, x_n)^t$ and $y^n = (y_1, \ldots, y_n)^t$ is denoted by $\|x^n - y^n\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Given an average rate budget of R bits per symbol and a fixed transform T, the operational distortion-rate function for a fixed-rate transform code based on T is defined as $D_T^{(\mathrm{fr})}(R) = \inf_{Q^n : R_T^{(\mathrm{fr})}(Q^n) \le R} D_T(Q^n)$. For variable-rate coding, we restrict attention to uniform scalar quantizers. Thus we define the operational distortion-rate function for a variable-rate transform code based on T as $D_T^{(\mathrm{vr})}(R) = \inf_{Q^n \in \mathcal{Q}_U^n : R_T^{(\mathrm{vr})}(Q^n) \le R} D_T(Q^n)$, where $Q^n \in \mathcal{Q}_U^n$ if and only if for each $1 \le i \le n$, Q_i is a *uniform* scalar quantizer. When T is the identity matrix I, we drop the subscript and use $D^{(\mathrm{fr})}(R)$ and $D^{(\mathrm{vr})}(R)$ to denote the mse corresponding to the classical fixed-rate scalar bit allocation problem and variable-rate uniform scalar bit allocation problems, respectively.

The optimality of a transform for transform coding is often considered in an asymptotic sense (e.g., [4]). Let $R_0^{(\mathrm{fr})}(T) = \sup\{R : D_T^{(\mathrm{fr})}(R) > 0\}$ and $R_0^{(\mathrm{vr})}(T) = \sup\{R : D_T^{(\mathrm{vr})}(R) > 0\}$. Then $R_0^{(\mathrm{fr})}(T)$ and $R_0^{(\mathrm{vr})}(T)$ give the minimal rate required to perfectly describe the source, with $R_0^{(\mathrm{fr})}(T) = R_0^{(\mathrm{vr})}(T) = \infty$ for continuous sources. The *fixed-rate and variable-rate coding gains* obtained by using transform T_1 instead of transform T_2 are defined as

$$G_{T_1,T_2}^{(\mathrm{fr})} = \lim_{R \to R_0^{(\mathrm{fr})}(T_1)} D_{T_2}^{(\mathrm{fr})}(R)/D_{T_1}^{(\mathrm{fr})}(R) \text{ and } G_{T_1,T_2}^{(\mathrm{vr})} = \lim_{R \to R_0^{(\mathrm{vr})}(T_1)} D_{T_2}^{(\mathrm{vr})}(R)/D_{T_1}^{(\mathrm{vr})}(R),$$

respectively. Each coding gain describes the asymptotic performance gap (in terms of sqnr) between the associated optimal transform coders. The gains measured in decibels (dBs) are $10\log_{10} G_{T_1,T_2}^{(\mathrm{fr})}$ and $10\log_{10} G_{T_1,T_2}^{(\mathrm{vr})}$.

An orthogonal transform T^* is said to be *optimal* for fixed-rate transform coding on source X^n if $G^{(\mathrm{fr})}_{T^*,T} \geq 1$ (0 dB) and for variable-rate transform coding on X^n if $G^{(\mathrm{vr})}_{T^*,T} \geq 1$ (0 dB), for all orthogonal transforms T. All of the results that follow refer to optimality in this asymptotic sense.

A *Karhunen-Loève Transform* is the linear map given by an $n \times n$ orthogonal matrix T^t such that $T^t \Phi_X T$ is a diagonal matrix. The matrix T decorrelates the random vector X^n.

III Summary of Results

We begin by showing, in Theorem 1, that decorrelation is insufficient for optimality in transform coding. We prove this theorem in Section IV by examining sources for which the KLT is not unique. While every KLT is, by definition, a decorrelating transform, there exist sources for which the KLT is not unique and not all KLTs are equally good. In particular, we show the following result.

Theorem 1 *In both fixed-rate and variable-rate transform coding, there exist sources for which the coding gain of a best KLT for transform coding over a worst KLT for transform coding can be arbitrarily large.*

In contrast, we show that the KLT is optimal for variable-rate coding.

Theorem 2 *If the KLT is unique and produces independent transform coefficients, then the KLT is optimal for variable-rate transform coding.*

We then demonstrate that even a best (or only) KLT can be suboptimal in both fixed-rate and variable-rate transform coding.

Theorem 3 *In both fixed-rate and variable-rate transform coding, there exist sources for which the coding gain of an optimal transform over a best KLT for transform coding is strictly greater than 1 (0 dB).*

Finally, we consider sources for which the KLT is unique and decorrelation yields independent transform coefficients. The intuition described in Section I suggests that the KLT should be optimal in this coding framework. In Theorem 4, we prove this intuition false for fixed-rate transform coding.

Theorem 4 *There exist sources for which the KLT is unique and produces independent transform coefficients and yet the KLT is not optimal for fixed-rate transform coding.*

Note that all of the results refer to optimality in the high resolution sense. The results for fixed-rate coding also appear in [5].

IV KLTs Are Not Uniformly Good

In this section, we give the proof of Theorem 1. That is, we show in both the fixed-rate and the variable-rate transform coding frameworks that there exist sources for which the KLT is not unique and a "best" KLT yields an infinite coding gain over a "worst" KLT. The results in both frameworks rely on the following family of examples.

Suppose that source $X^n = (X_1, \ldots, X_n)^t$ is defined as $X^n = BU^n$. Here the components $U_1, \ldots, U_n$ of vector $U^n = (U_1, \ldots, U_n)^t$ are independent and identically distributed (iid) random variables (denoted by U) with reasonably smooth pdfs and positive variances, and B is an $n \times n$ orthogonal matrix.

Let Φ_U and Φ_X denote the covariance matrices of U^n and X^n, respectively. Then for *any* orthogonal matrix B, $\Phi_U = \Phi_X = \sigma^2 I$, where I denotes the n-dimensional identity matrix. That is, any rotation of U^n creates an uncorrelated random vector X^n, and thus *any* transform matrix T is a legitimate KLT for X^n. While the KLT for X^n is not unique, practical implementations of the KLT (e.g., Householder reduction followed by the QL algorithm with implicit shifts or Jacobi's algorithm [6]) gives the identity matrix I as the KLT for X^n. Therefore using the KLT in an optimal transform coder for X^n is in practice equivalent to optimal bit allocation followed by scalar quantization on the original source X^n. We therefore calculate the coding gain of the transform B^{-1} relative to the practically achieved KLT I.[2]

If $U_1, \ldots, U_n$ are drawn iid according to a reasonably smooth pdf f_U with finite variance $\sigma^2 > 0$ and differential entropy $h(U)$, then the fixed-rate coding gain of transform B^{-1} over transform I is

$$G^{(\mathrm{fr})}_{B^{-1},I} = \lim_{R\to\infty} \frac{D^{(\mathrm{fr})}(R)}{D^{(\mathrm{fr})}_{B^{-1}}(R)} = \frac{\lim_{R\to\infty} D^{(\mathrm{fr})}(R)2^{2R}}{\lim_{R\to\infty} D^{(\mathrm{fr})}_{B^{-1}}(R)2^{2R}} = \frac{\left(\prod_{i=1}^n \|f_{X_i}\|_{1/3}\right)^{1/n}}{\|f_U\|_{1/3}}.$$

and the corresponding variable-rate coding gain is

$$G^{(\mathrm{vr})}_{B^{-1},T} = \lim_{R\to\infty} \frac{D^{(\mathrm{vr})}_T(R)}{D^{(\mathrm{vr})}_{B^{-1}}(R)} = \frac{\lim_{R\to\infty} D^{(\mathrm{vr})}_T(R)2^{2R}}{\lim_{R\to\infty} D^{(\mathrm{vr})}_{B^{-1}}(R)2^{2R}} = \frac{2^{2\bar{h}(Y^n)}}{2^{2\bar{h}(U^n)}} = 2^{2(\bar{h}(Y^n)-\bar{h}(U^n))}$$

where $Y^n = TX^n$ and T is the transform matrix. Since $U_1, \ldots, U_n$ are independent, $Y^n = TX^n = TBU^n$, and TB is nonsingular, then $\bar{h}(U^n) \leq \bar{h}(Y^n)$ by the chain rule. Thus $G^{(\mathrm{vr})}_{B^{-1},T} \geq 1$ (0 dB) with equality if and only if $Y_1, \ldots, Y_n$ are independent. Here, a transform that makes the components independent is optimal for variable-rate coding.[3] This also proves Theorem 2. Now let us go back to the iid source, where

$$G^{(\mathrm{vr})}_{B^{-1},I} = 2^{2(\bar{h}(Y^n)-\bar{h}(U^n))} = 2^{2(\bar{h}(X^n)-h(U))} \leq 2^{\log(2\pi e\sigma^2)-2h(U)}$$

with equality if and only if the marginal pdf of each X_i is $\mathcal{N}(0, \sigma^2)$. Given any choice of symmetric f_U such that the central limit theorem applies to $(1/\sqrt{n})\sum_{i=1}^n U_i$, then for sufficiently large n and carefully chosen B, the marginal density of X_i can be made arbitrarily close to the above normal density.[4]

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[2]In practice, performance problems of the KLT are often exacerbated by the KLT's sensitivity to errors in estimating the off-diagonal terms of the covariance matrix when the covariance matrix is close to I [7].

[3]While $U_1, \ldots, U_n$ are iid in this example, the result actually requires only that they be independent. In addition, this statement is true only under the high-rate assumption. It has been shown that for some sources, B^{-1} is suboptimal at certain low rates in variable-rate coding, i.e., a transform that yields independent coefficients is not always optimal for variable-rate coding when the high-rate approximation does not apply [8].

[4]This can be accomplished, for example, by letting B be $1/\sqrt{n}$ times a Hadamard matrix of order n. Hadamard matrices are known to exist at least for every n that is a power of 2. This would assure that B is orthogonal and has components all of equal magnitude, on a subsequence of $\{1, 2, \ldots, n\}$.

We next bound the coding gains for several example distributions.

Example 1 *U_i is uniform on $[-a, a]$ with $a > 0$.*

For the fixed-rate coding calculations, if $n = 2$ and

$$B = T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{1}$$

then $G^{(\text{fr})}_{B^{-1},I} \approx 2.27$ dB. If n is large and the marginal density of X_i is close to $\mathcal{N}(0, \sigma^2)$, then $G^{(\text{fr})}_{B^{-1},I} \approx 4.35$ dB. In this case, B^{-1} is not necessarily the optimal transform, so this is not necessarily the largest possible coding gain.

For variable-rate coding, B^{-1} is the optimal transform at high rate as shown earlier, and the coding gain between the best and worst KLTs for the worst-case B is $\sup_B G^{(\text{vr})}_{B^{-1},I} \approx$ 1.5 dB, which is consistent with [7]. □

Example 2 *U_i is uniform on $[-a-\delta, -a+\delta] \cup [a-\delta, a+\delta]$ with $0 < \delta < a$.*

In the fixed-rate coding calculations, if the marginal distribution of X_i is close to $\mathcal{N}(0, \sigma^2)$, then $G^{(\text{fr})}_{B^{-1},I} \approx (3\pi\sqrt{3}/8)((a^2/\delta^2) + (1/3))$. The coding gain can be made arbitrarily large by fixing a and letting $\delta \to 0$.

For variable-rate coding, $\sup_B G^{(\text{vr})}_{B^{-1},I} = (\pi e/8)((a^2/\delta^2) + (1/3))$, which can again be made arbitrarily large by fixing a and letting $\delta \to 0$. □

The problem observed above for reasonably smooth, continuous random variables becomes even more pronounced for discrete random variables. For a discrete random variable, the previous high-rate approximation does not apply since the probability mass function (pmf) is not smooth. It is still relatively easy to calculate the coding gain for certain discrete random variables, as we show in the following example.

Example 3 *U_i is discrete.*

Consider $U_1, \ldots, U_n$ drawn iid according to $p(10) = p(-10) = 1/2$. In both fixed-rate and variable-rate coding, we can quantize each U_i with distortion $D = 0$ at rate 1 bit per symbol (bps). Achieving $D = 0$ for each X_i generally requires more rate. For example, choose B so that the marginal distribution of each X_i is the binomial distribution. For $n = 64$, achieving $D = 0$ requires approximately 4 bps in variable-rate coding and $\log(64+1) \approx 6$ bps in fixed-rate coding. At rate 1 bps the mse in (either fixed-rate or variable-rate) coding of U_i is approximately 36, giving infinite coding gain for both fixed-rate and variable-rate coding ($\sup_B G^{(\text{fr})}_{B^{-1},I} = \sup_B G^{(\text{vr})}_{B^{-1},I} = \infty$). □

V A Best KLT Can Be a Suboptimal Transform

In the previous section, we showed for both fixed-rate and variable-rate transform coding that when the KLT is not unique the coding gain of a best KLT over a worst KLT can be arbitrarily large. Thus decorrelation is not sufficient for transform optimality. In this section,

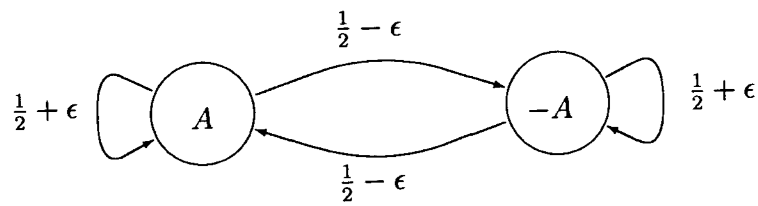

Figure 1: Stationary Markov process S_n.

we prove Theorem 3 by showing that even a best KLT can give suboptimal performance. We use the following source for both fixed-rate and variable-rate transform coding.

Let $\{U_i\}_{i=1}^{\infty}$ be an independent and identically distributed (iid) real-valued random sequence with $E[U_i] = 0$ and $E[U_i^2] = \sigma^2$ for all i, and symmetric pdf f_U. Let $\{S_i\}_{i=1}^{\infty}$ be the stationary 2-state Markov process shown in Figure 1 with $A > 0$ and $0 < |\epsilon| < 1/2$. That is, $S_i = A$ with probability 1/2, and $S_i = -A$ otherwise. The value of ϵ determines the tendency of the process to remain in its current state. Furthermore, assume that the processes U_i and S_i are independent of each other. Let $X_i = U_i + S_i$ be a scalar source and define the 2-dimensional random vector $X = (X_i, X_{i+1})^t$. Let T be the matrix defined in (1). Then the vector $Y = (Y_1, Y_2) = TX$ (note that $T = T^t$) is a KLT for vector source X, and T simply rotates X through an angle of 45° clockwise.[5]

We next show for both fixed-rate and variable-rate transform coding that (in the high-rate limit) optimal scalar quantization of the components of the correlated random vector X produces a smaller mse than optimal scalar quantization of the components of the decorrelated vector Y. (Note that Y_1 and Y_2 are uncorrelated, but they are not independent.)

Since the Markov process S_i is stationary, we assume without loss of generality that $X = (X_1, X_2)^t$. Notice that by symmetry the scalar components of X are identically distributed with pdf

$$f_{X_1}(x) = f_{X_2}(x) = \frac{1}{2} f_U(x + A) + \frac{1}{2} f_U(x - A). \quad (2)$$

We begin by setting up the coding gain calculations for both fixed-rate and variable-rate coding. We then calculate those coding gains for a variety of examples.

If $f_U(x) = 0$ for all $x \notin [-A/2, A/2]$, then the fixed-rate coding gain obtained by quantizing the correlated scalar components instead of the uncorrelated components is

$$G_{I,T}^{(\mathrm{fr})} = \frac{\|f_U * f_U\|_{1/3}}{8\|f_U\|_{1/3}} \left[(2^{1/3}\beta^{1/3} + 2\alpha^{1/3})(2^{1/3}\alpha^{1/3} + 2\beta^{1/3})\right]^{3/2} \quad (3)$$

and the corresponding variable-rate coding gain is

$$G_{I,T}^{(\mathrm{vr})} = 2^{h(Y_1)+h(Y_2)-h(X_1)-h(X_2)} = 2^{2h(f_U * f_U)+2\mathcal{H}(2\alpha)-2h(U)-2} \quad (4)$$

where $(f_U * f_U)(x) = \int_{-\infty}^{\infty} f_U(t) f_U(x-t)dt$ is the convolution of f_U with itself and $\mathcal{H}(x) = -x\log(x) - (1-x)\log(1-x)$ is the binary entropy function.

[5] There are 3 other possible KLTs of X, namely $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ which all give equal coding gains. Thus we will refer to T as "the" best KLT in this example.

This example also exploits the numerical sensitivity of computing the KLT, since only $\epsilon = 0$ yields the identity matrix, while all $\epsilon > 0$ lead to 45° rotations.

Example 4 *U_i is uniform on $[-a, a]$ with $0 < a < A/2$.*

The solid lines in Figures 2(a) and (b) show the regions of support for the two-dimensional

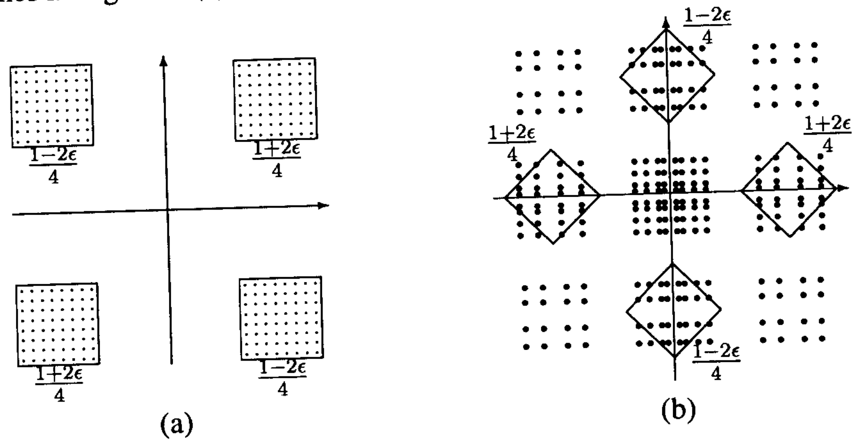

Figure 2: (a) The two-dimensional pdf of the correlated source X is uniform on each of the four squares with the heights $(1 \pm 2\epsilon)/4$ as indicated. The dots indicate reproduction locations associated with using the optimal 16-codeword scalar quantizer on each dimension; (b) The two-dimensional pdf f_Y of the uncorrelated source $Y = TX$ equals the two-dimensional pdf of X rotated 45° about the origin. The dots indicate reproduction locations associated with using the optimal 16-codeword fixed-rate scalar quantizer on each dimension.

pdfs of X and Y given a uniform distribution on U. Each pdf is uniform in its marked regions. The probability of each of those regions is marked in the figure. The dots in each figure show all possible two-dimensional reproductions when the individual components of the random vectors are quantized with the optimal fixed-rate scalar quantizers. The quantizer associated with the decorrelated random vector Y (shown in Figure 2(b)) is very inefficient and thus leads to higher mse than the quantizer associated with the correlated random vector X (shown in Figure 2(a)).

For the fixed-rate coding calculation, (3) implies that for all $\epsilon \in (0, 1/2)$, $G_{I,T}^{(\mathrm{fr})} = (27/64)\left[(2^{1/3}\beta^{1/3} + 2\alpha^{1/3})(2^{1/3}\alpha^{1/3} + 2\beta^{1/3})\right]^{3/2}$. We can show that $G_{I,T}^{(\mathrm{fr})} > 1$ (0 dB) whenever $\epsilon < 0.4998$, and $\lim_{\epsilon \to 0} G_{I,T}^{(\mathrm{fr})} \approx 5.63$ dB.

The results for variable-rate transform coding are similar. Here, $G_{I,T}^{(\mathrm{vr})} = 2^{\log e + 2\mathcal{H}(2\alpha) - 2}$, giving $G_{I,T}^{(\mathrm{vr})} > 1$ (0 dB) whenever $\epsilon < .4518$, and $\lim_{\epsilon \to 0} G_{I,T}^{(\mathrm{vr})} = e \approx 4.34$ dB. □

In the previous example, for both fixed-rate and variable-rate coding, a best KLT is 4-5 dB worse than an optimal transform. In this case, it is better to scalar quantize correlated data than uncorrelated data. The following example gives a similar outcome.

Example 5 *U_i is Gaussian with mean 0 and variance $\sigma^2 \ll A$.*

Given $f_U(x) = \mathcal{N}(0, \sigma^2)$, the marginal pdfs of X are again given by (2).

For fixed-rate coding, we can show that (3) holds in the limit as $\sigma \to 0$. Therefore $G_{I,T}^{(\mathrm{fr})} > 1$ (0 dB) whenever $\epsilon < 0.485$, for some $\sigma > 0$. For $\epsilon \approx 0$, $G_{I,T}^{(\mathrm{fr})} \approx 3.36$ dB.

For variable-rate coding, (4) holds in the limit as $\sigma \to 0$, so $\lim_{\sigma \to 0} G_{I,T}^{(\mathrm{vr})} = 2^{2\mathcal{H}(2\alpha)-1}$. In this case, $G_{I,T}^{(\mathrm{vr})} > 1$ (0 dB) whenever $\epsilon < .3899$, and $\lim_{\epsilon \to 0} \lim_{\sigma \to 0} G_{I,T}^{(\mathrm{vr})} \approx 3.01$ dB. □

Example 6 *U_i is a constant.*

Let $U_i = 0$ with probability 1. Then the pmfs of X_1 and X_2 are $p(X_1 = -A) = p(X_1 = A) = p(X_2 = -A) = p(X_2 = A) = 1/2$. At rate 1 bps, the mse obtained by using the transform matrix I is 0 for both fixed-rate and variable-rate coding. In contrast, Y_1 and Y_2 are more difficult to compress. For fixed-rate coding, a rate of $\log 3 > 1$ is required to achieve distortion 0 if $0 < \epsilon < 1/2$. For variable-rate coding, a rate of $(\mathcal{H}(Y_1) + \mathcal{H}(Y_2))/2 = 0.5 + \mathcal{H}(2\alpha)$ is needed to achieve distortion 0. Here $0.5 + \mathcal{H}(2\alpha) > 1$ if $\epsilon < .3899$. Thus, in either of these cases, the coding gain associated with using transform I rather than the KLT T is infinite at rate 1 bps. □

The next example shows that even for a source whose distribution is very close to Gaussian, the KLT may still be suboptimal.

Example 7 *The source pdf has a simply connected support and is close to normal.*

Let us consider a 2-dimensional stationary vector source $X = (X_1, X_2)^t$. Suppose the joint pdf of (X_1, X_2) is $f(x_1, x_2) = \alpha f_1(x_1, x_2) + (1 - \alpha) f_2(x_1, x_2)$, where $\alpha = .99$ and f_1 and f_2 are two 2-dimensional Gaussian pdfs. Let the means of f_1 and f_2 be $\mu_1 = (0, 0)^t$, $\mu_2 = (.5, 3)^t$ and the covariance matrices be

$$\Phi_1 = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \Phi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 289.3375 \end{bmatrix}.$$

It can be shown that $Y = TX$ is a KLT for X, where T is defined in (1), and $G_{I,T}^{(\mathrm{fr})} \approx 1.6426$ dB, $G_{I,T}^{(\mathrm{vr})} \approx .6124$ dB. Here f is still close to the Gaussian pdf, e.g., $D(f_1 \| f) \approx .0109$. □

VI KLTs with Independent Transform Coefficients Can be Suboptimal for Fixed-Rate Transform Coding

The previous examples show that even a best KLT can be suboptimal for transform coding. In those examples, the transform vector Y has coefficients that are decorrelated but not independent. In this section, we prove Theorem 4 by showing that even when the transform vector has independent coefficients, a KLT can be suboptimal for fixed-rate transform coding. Notice that this result applies only to fixed-rate coding.

We construct an example by using a two-dimensional Laplacian source $X = (X_1, X_2)^t$ with independent components. We show that the transform matrix T that rotates the source by 45° is better for fixed-rate coding on X than the identity matrix I given by the KLT.

Let b and σ be positive constants and

$$f_{X_1}(x_1) = \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|x_1|\sqrt{2}}{\sigma}\right), \quad \text{and} \quad f_{X_2}(x_2) = \frac{1}{b\sigma\sqrt{2}} \exp\left(-\frac{|x_2|\sqrt{2}}{b\sigma}\right).$$

Further, let the transform matrix T be the transform matrix defined in (1). If $b = 4/3$, then the coding gain obtained by quantizing Y instead of the independent component source X is $G_{T,I}^{(\mathrm{fr})} \approx 0.40$ dB > 0 dB.

Note that the observed problem is not limited to $b = 4/3$. For example, if $b = 1$, the KLT is not unique, and it is still not optimal to quantize the independent components. In this case, the coding gain obtained by using transform T rather than the KLT I (that is quantizing Y instead of the independent component source X) is $G_{T,I}^{(\mathrm{fr})} \approx 1.33$ dB.

VII Summary and Conclusions

In this paper, a family of sources has been demonstrated for which the KLT transform is suboptimal in a scalar quantized transform coding system (using either fixed-rate or variable-rate scalar quantizers).

We considered 3 scenarios: there are sources for which the KLT is not unique, and the worst KLT may be arbitrarily worse than the optimal transform in terms of coding gain in both fixed-rate and variable-rate transform coding; there are sources for which even a best (or only) KLT can give suboptimal performance in both fixed-rate and variable-rate transform coding. An independent work by Goyal also shows this result for variable-rate transform coding by using a different example [9]; there are sources for which the KLT that yields independent components is suboptimal in fixed-rate coding. For variable-rate coding, the transform that yields independent components is generally optimal in variable-rate coding under the high-rate assumption (note that this is not true at low rates in general).

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