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Marta Cardin and Miguel Couceiro

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An Ordinal Approach to Risk Measurement

Marta Cardin
<mcardin@unive.it>
Dept. of Applied Mathematics
University of Venice

Miguel Couceiro
<miguel.couceiro@uni.lu>
Mathematics Research Unit
University of Luxembourg

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Abstract. In this short note, we aim at a qualitative framework for modeling multivariate risk. To this extent, we consider completely distributive lattices as underlying universes, and make use of lattice functions to formalize the notion of risk measure. Several properties of risk measures are translated into this general setting, and used to provide axiomatic characterizations. Moreover, a notion of quantile of a lattice-valued random variable is proposed, which shown to retain several desirable properties of its real-valued counterpart.

Keywords: lattice, risk measure, Sugeno integral, quantile.

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Correspondence to:

Marta Cardin	Dept. of Applied Mathematics, University of Venice
	Dorsoduro 3825/e
	30123 Venezia, Italy
Phone:	[++39] (041)-234-6929
Fax:	[++39] (041)-522-1756
E-mail:	mcardin@unive.it

1 Introduction

During the last decades, researchers joined efforts to properly compare, quantify and manage risk. In this direction, risk measures constitute an important and widely studied tool. Traditionally, risk measures are thought of as mappings from a set of real-valued random variables to the real numbers. As a well-known example we have the so-called "value at risk" (VaR).

There are many different contexts in which the structure of real numbers seems to be insufficient, since it only provides a quantitative setting of risk which heavily relies on the linearly order of reals. In particular, many problems in insurance and finance involve the measurement of multivariate risk. Then modeling and measuring multivariate risks is a theoretically demanding problem and it is of major importance in practice.

In this short note we aim at a unified qualitative framework for modeling risks in such a way that qualitative evaluations are not necessarily expressed in a totally ordered universe. This goal is two-folded.

In the one hand, we take as motivating frameworks those of [4] where risk measures take values in certain partially ordered cones, and of [8] where risk measures are assumed to be vector-valued, each of which generalizing the classical real-valued setting proposed by Artzner *et al.* [1].

On the other hand, we wish to treat risks from a purely ordinal point of view, and thus abandon their numerical interpretation. Here the motivating approach is that of Chambers [5] where quantile-based risk measures are treated from an ordinal point of view and not bounded by probabilistic interpretations.

In order to unify these settings, we take completely distributive lattices as underlying universes, and consider an important class of aggregation functionals considered on these structures, namely, the class of Sugeno integrals. This setting has several appealing aspects, for it provides sufficiently rich structures well studied in the literature, which allow models and measures of risk from an ordinal point of view, and which do not depend on the usual arithmetical structure of the reals. In the next section, we survey the general background on lattice theory as well as representation and characterization results concerning Sugeno integrals on completely distributive lattices. In Section 3, we propose notions of risk measure and of quantile-based risk measure within this ordinal setting, as well as present their axiomatizations and representations. In Section 4 we briefly discuss possible directions for future work.

2 Basic notions and preliminary results

In this section we recall concepts and preliminary results relevant to studying risk measures on, not necessarily linearly ordered, distributive lattices. For further background in lattice theory we refer the reader to, e.g., Birkhoff [2], Davey and Priestley [6] or Rudeanu [12].

2.1 Basic background in lattice theory

A *lattice* is an algebraic structure $\langle L; \wedge, \vee \rangle$ where L is a nonempty set, called *universe*, and where $\wedge$ and $\vee$ are two binary operations, called *meet* and *join*, respectively, which satisfy the following axioms:

(i) (idempotency) for every $a \in L$, $a \vee a = a \wedge a = a$;

(ii) (commutativity) for every $a, b \in L$, $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;

(iii) (associativity) for every $a, b, c \in L$, $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

(iv) (absorption): for every $a, b \in L$, $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

With no danger of ambiguity, we will denote lattices by their universes. As it is well-known, every lattice L constitutes a partially ordered set endowed with the partial order $\leq$ given by: for every $x, y \in L$, write $x \leqslant y$ if $x \wedge y = x$ or, equivalently, if $x \vee y = y$. If for every $a, b \in L$, we have $a \leqslant b$ or $b \leqslant a$, then L is said to be a *chain.* A lattice L is said to be *bounded* if it has a least and a greatest element, denoted by 0 and 1, respectively..

A lattice L is said to be *distributive*, if for every $a, b, c \in L$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{or, equivalently,} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Clearly, every chain is distributive.

For an arbitrary nonempty set A and a lattice L, the set L^A of all functions from A to L constitutes a lattice under the operations $\wedge$ and $\vee$ defined pointwise, i.e.,

$$(f \wedge g)(x) = f(x) \wedge g(x) \quad \text{and} \quad (f \vee g)(x) = f(x) \vee g(x) \quad \text{for every } f, g \in L^A.$$

In particular, for any lattice L, the cartesian product L^n also constitutes a lattice by defining the lattice operations componentwise. Observe that if L is bounded (distributive), then L^A is also bounded (resp. distributive). We denote by $\mathbf{0}$ and $\mathbf{1}$ the least and the greatest elements, respectively, of L^A. Likewise, for each $c \in L$, we denote by $\mathbf{c}$ the constant c map in L^A.

Let L be a bounded lattice and A an arbitrary nonempty set. For each $X \subset A$, we denote by I_X the *characteristic function* of X in L^A, i.e.,

$$I_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Completely distributive lattices

A lattice L is said to be *complete* if for every $S \subseteq L$, its supremum $\bigwedge S := \bigwedge_{x \in S} x$ and infimum $\bigvee S := \bigvee_{x \in S} x$ exist. Clearly, every complete lattice is necessarily bounded.

A complete lattice L is said to be *completely distributive* is the following more stringent distributive law holds

$$\bigwedge_{i \in I} \Big(\bigvee_{j \in J} x_{ij} \Big) = \bigvee_{f \in J^I} \Big(\bigwedge_{i \in I} x_{i f(i)} \Big), \tag{1}$$

for every doubly indexed subset $\{x_{ij} : i \in I,\, j \in J\}$ of L. Note that every complete chain (in particular, the extended real line and each product of complete chains) is completely distributive. Moreover, complete distributivity reduces to distributivity in the case of finite lattices

Complete distributivity is a self-dual property. This was observed by Raney [11] who showed that (1) and its dual are equivalent, and thus that either is sufficient to define complete distributivity. In [15], Tunnicliffe presented a characterization of complete distributivity which relied on the notion of a "cone". We shall make use of the following alternative characterization given in [3].

Theorem 1. *A complete lattice L is completely distributive if and only if for every set A and every family $\mathcal{A}$ of nonempty subsets of A, we have*

$$P_{\mathcal{A}}(f) := \bigvee_{X\in\mathcal{A}} \bigwedge_{x\in X} f(x) = \bigwedge_{X\in\mathcal{B}} \bigvee_{x\in X} f(x) =: P^{\mathcal{B}}(f)$$

for every $f \in L^A$ where $\mathcal{B} = \{B \subseteq A : B \cap X \neq \emptyset$ for all $X \in \mathcal{A}\}$.

2.3 Lattice homomorphisms, continuity and invariance

We now recall the notion of lattice homomorphism. Let L be a lattice. A map $\gamma\colon L \to L$ is said to be a (*lattice*) *homomorphism* if it preserves $\wedge$ and $\vee$, i.e.,

$$\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y) \qquad \gamma(x \vee y) = \gamma(x) \vee \gamma(y) \quad \text{for every } x, y \in L.$$

The following notion extends that of homomorphism. A map $\gamma\colon L \to L$, where L is a complete lattice, is said to be *continuous* if it preserves arbitrary meets and and arbitrary joins, i.e., for every $S \subseteq L$,

$$\gamma(\bigwedge S) = \bigwedge \gamma(S) \quad \text{and} \quad \gamma(\bigvee S) = \bigvee \gamma(S).$$

The term continuous is justified by the following fact (see [7]): if $\gamma\colon L \to L$ is continuous, then it is continuous with respect to the Lawson topology on L.

We say that a functional $F\colon L^A \to L$ on a complete lattice L is *invariant* if, for every $f \in L^A$ and every continuous mapping $\gamma\colon L \to L$, we have

$$F(\gamma \circ f) = \gamma \circ F(f).$$

In this paper we shall also consider the following weaker property. A functional $F\colon L^A \to L$ is said to be *homogeneous* if it is invariant under continuous mappings of the form $\gamma(x) = x \wedge c$ and $\gamma(x) = x \vee c$, for every $c \in L$. Note that every homogeneous functional $F\colon L^A \to L$ is *idempotent*, i.e.,

$$F(\mathbf{c}) = c, \quad \text{for every constant map } \mathbf{c} \in L^A.$$

2.4 Sugeno integrals as lattice polynomial functionals

By a (*lattice*) *functional* on L we mean a mapping $F\colon L^A \to L$, where A is a nonempty set. The *range* of a functional $F\colon L^A \to L$ is defined by $\mathcal{R}_F = \{F(f) : f \in L^A\}$. A functional $F\colon L^A \to L$ is said to be *nondecreasing* if, for every $f, g \in L^A$ such that $f(i) \leqslant g(i)$, for every $i \in A$, we have $F(f) \leqslant F(g)$. Note that if F is nondecreasing, then $\overline{\mathcal{R}}_F = [F(\mathbf{0}), F(\mathbf{1})]$.

An *aggregation functional* on a bounded lattice L is a nondecreasing functional $F\colon L^A \to L$ such that $\overline{\mathcal{R}}_F = L$, that is, $F(\mathbf{c}) = c$ for $c \in \{0, 1\}$. For instance, each projection $F_a\colon L^A \to L$, $a \in A$, defined by $F_a(f) = f(a)$, is an aggregation functional, as well as the mappings $P_{\mathcal{A}}$ and $P^{\mathcal{A}}$ given in Theorem 1.

As mentioned, in this paper we are particularly interested in certain aggregation functionals, namely, Sugeno integrals. A convenient way to introduce the Sugeno integral is via the so-called *lattice polynomial functionals*, that is, lattice functionals which can be obtained from projections and constants by taking arbitrary meets and joins. A *Sugeno integral* on L is simply a polynomial functional $F\colon L^A \to L$ which is idempotent.

In the case when L is completely distributive, Sugeno integrals can be equivalently defined in terms of *capacities*, i.e., nondecreasing mappings $v\colon \mathcal{P}(A) \to L$, where $\mathcal{P}(A)$ denotes the set of all subsets of A. More precisely, a functional $F\colon L^A \to L$ is a Sugeno integral if and only if there exists a capacity $v\colon \mathcal{P}(A) \to L$ such that

$$F(f) = F_v(f) := \bigvee_{X \in \mathcal{P}(A)} v(X) \wedge \bigwedge_{x \in X} f(x).$$

(Sugeno integrals were introduced by Sugeno [13, 14] on linearly ordered domains. In the finitary case, Marichal [9] observed that this concept can be extended to the setting of bounded distributive lattices by defining Sugeno integrals as idempotent polynomial functions.) Note that the polynomial functionals obtained solely from projections by taking arbitrary meets and joins, coincide exactly with those Sugeno integrals associated with $\{0, 1\}$-valued capacities, i.e., nondecreasing mappings $v\colon \mathcal{P}(A) \to \{0, 1\}$ such that $v(A) \in \{0, 1\}$.

Sugeno integrals over completely distributive lattices were axiomatized in [3] in terms of nondecreasing monotonicity and homogeneity.

Theorem 2. *Let L be a completely distributive lattice, A an arbitrary nonempty set, and let $F\colon L^A \to L$ be a functional. Then F is a Sugeno integral if and only if it is nondecreasing and homogeneous.*

In order to axiomatize the subclass of those Sugeno integrals associated with $\{0, 1\}$-valued capacities, we need to strengthen the conditions of Theorem 2. As it turned out, invariance rather than homogeneity suffices.

Theorem 3. ([3]) *Let L be a completely distributive lattice, A an arbitrary nonempty set, and let $F\colon L^A \to L$ be a functional such that, for every $X \subseteq A$, $F(I_X) \in \{0, 1\}$. Then F is a Sugeno integral associated with a $\{0, 1\}$-capacity if and only if it is nondecreasing and invariant.*

In the case when L is a complete chain with at least 3 elements, Theorem 3 can be strengthened since nondecreasing monotonicity becomes redundant.

Theorem 4. ([3]) *Let $L \neq \{0,1\}$ be a complete chain, A an arbitrary nonempty set, and let $F\colon L^A \to L$ be a functional such that, for every $X \subseteq A$, $F(I_X) \in \{0,1\}$. Then F is a Sugeno integral associated with a $\{0,1\}$-capacity if and only if it is invariant.*

3 Applications to risk measurement: risk measures on completely distributive lattices

The notion of risk measure arose from the problem of quantifying risk. In the simplest setting, a risk situation is modeled as a bounded real-valued random variable. The concept of risk measure together with its axiomatic characterization was proposed in [1] for finite probability spaces and further extended to more general probabilistic settings. In particular, in [8] it is considered a more realistic situation of $\mathbb{R}^n$-valued random variables while in [4] risk measures take values in abstract cones.

In this section we aim at bringing the notions of random variable and risk measure into the more general setting of completely distributive lattices. As we will see, many of the desirable properties of risk measures can be naturally translated into the realm of completely distributive lattices, and used to provide axiomatic characterizations of risk measures similar to those found in the literature. We also propose a notion of quantile of a lattice-valued random variable, and provide an axiomatic characterization based on the results of the previous section.

3.1 Risk measures on completely distributive lattices

We suppose that a risk at a given position (e.g. time) is described by a function $f\colon \Omega \to L$ where L is a completely distributive lattice and Ω is a set of possible states. The goal is to determine a value $F(f)$ that meaningfully represents risk f. In the current ordinal setting, the natural approach is to consider *risk measures* as mappings $F\colon L^\Omega \to L$.

Several desirable properties of risk measures have been proposed in the literature (see e.g. [1, 8, 4]). Given the ordered structure of our underlying universe L, we retain the following: nondecreasing monotonicity, idempotence and homogeneity (we shall also consider the more stringent condition of invariance in Subsection 3.2). Immediately from Theorem 2, we get the following description of nondecreasing and homogeneous risk measures.

Corallary 5. *Let $F\colon L^\Omega \to L$ be a risk measure. Then F is nondecreasing and homogeneous if and only if there is a capacity $v\colon \mathcal{P}(\Omega) \to L$ such that $F = F_v$.*

3.2 Quantiles on completely distributive lattices

Quantiles of real-valued random variables have been proved to be an important tool in statistics and to have a valuable role in application fields such as economics. We consider quantile-based risk measures in a qualitative framework and we propose a notion of quantile of a lattice-valued random variable. We establish that these quantiles have several desirable features and we derive an axiomatic representation of quantiles.

Given a real-valued random variable f and a confidence level $\alpha \in (0,1)$, the $\alpha\%$ worst realizations of f, situated at the left tail of its distribution are described by the α quantile defined by

$$q_\alpha = \inf\{x \in \mathbb{R} : P(f \geq x) < \alpha\} \quad (\alpha \in (0,1)).$$

Integral representations of quantiles were obtained in [10, 5].

Proposition 6. *Let $(\Omega, 2^\Omega, P)$ is a probability space and $\mathcal{A} = \{A \subseteq \Omega : P(A) \geq \alpha\}$. Then for every bounded real-valued random variable f defined on Ω*

$$q_\alpha(f) = \bigvee_{A \in \mathcal{A}} \bigwedge_{s \in A} f(s).$$

Proof. Let f be a bounded random variable defined on Ω, and let $\mathcal{A}^\star$ be the set of subsets $A \in \mathcal{A}$ which satisfy the following condition: if $s \in A$ and $t \in \Omega$ is such that $f(t) \geq f(s)$, then $t \in A$. Thus

$$\bigvee_{A \in \mathcal{A}} \bigwedge_{s \in A} f(s) = \bigvee_{A \in \mathcal{A}^\star} \bigwedge_{s \in A} f(s).$$

Since f is bounded, for every $A \in \mathcal{A}^\star$, we have

$$f(A) := \{f(s) : s \in A\} = \{f(s) : s \in \Omega \text{ and } f(s) \geq \bigwedge_{s \in A} f(s)\}.$$

Let $x = \bigwedge_{s \in A} f(s)$, and define $A_x = \{s \in \Omega : f(s) \geq x\}$. Hence,

$$\bigvee_{A \in \mathcal{A}} \bigwedge_{s \in A} f(s) = \bigvee_{A \in \mathcal{A}^\star} \bigwedge_{s \in A} f(s) = \bigvee_{A_x \in \mathcal{A}} \bigwedge_{s \in A_x} f(s).$$

Now, if $q_\alpha(f) > x$, then $P(A_x) \geq \alpha$ and, hence, $A_x \in \mathcal{A}$. Also, if $A_x \in \mathcal{A}$, then $P(A_x) \geq \alpha$ and $q_\alpha(f) \geq x$. Hence, by the density of $\mathbb{R}$, we have $q_\alpha(f) = \bigvee_{A_x \in \mathcal{A}} x$. Moreover, for each $A_x \in \mathcal{A}$ we have $\bigwedge_{s \in A_x} f(s) = x$, and thus

$$q_\alpha(f) = \bigvee_{A_x \in \mathcal{A}} \bigwedge_{s \in A_x} f(s) = \bigvee_{A \in \mathcal{A}} \bigwedge_{s \in A} f(s),$$

which completes the proof. □

In view of Proposition 6, we propose the following definition of quantile of a lattice-valued random variable.

Let L be a completely distributive lattice and take $\alpha \in L$. For a capacity $v \colon \mathcal{P}(\Omega) \to L$, set

$$\mathcal{A}_v^\alpha := \{X \in \mathcal{P}(\Omega) : v(X) \geq \alpha\}.$$

We say that a functional $F \colon L^\Omega \to L$ is an *α-quantile* if there is a capacity $v \colon \mathcal{P}(\Omega) \to L$ such that

$$F(f) := \bigvee_{X \in \mathcal{A}_v^\alpha} \bigwedge_{x \in X} f(x).$$

Note that v can be chosen to be a $\{0, 1\}$-capacity.

In [5] quantiles are described as nondecreasing functionals which satisfy an invariance-like condition referred to as "ordinal covariance". The following corollary of Theorem 3 indicates that these desirable properties are retained by the current reformulation of quantiles of lattice-valued random variables.

Corallary 7. *Let L be a completely distributive lattice, and let $F \colon L^\Omega \to L$ be a functional such that, for every $X \subseteq \Omega$, $F(I_X) \in \{0, 1\}$. If F is nondecreasing and invariant then and only then F is an α-quantile, for some $\alpha \in L$.*

As in the case of Theorem 4, Corollary 7 can be refined when L is a complete chain.

Corallary 8. *Let $L \neq \{0, 1\}$ be a complete chain, and let $F \colon L^\Omega \to L$ be a functional such that, for every $X \subseteq \Omega$, $F(I_X) \in \{0, 1\}$. If F is invariant then and only then F is an α-quantile, for some $\alpha \in L$.*

Remark Even though our underlying universe L is implicitly assumed to be bounded, this condition is not really necessary as discussed in [3]. This boundness condition is not required in the literature, however the functionals considered are defined on spaces of measurable functions which are bounded. Thus such a requirement is also not necessary in the latter settings.

Another difference to the existing literature (e.g., the ordinal framework proposed in [5]), is that we do not assume L to be linearly ordered, and thus allowing incomparability on the values of a given domain function. Motivations to such a framework can be found in [8, 4] which consider multivariate random variables.

4 Conclusion

In this paper we have introduced a unified qualitative framework for studying risk measures, which can account for classical univariate as well as multivariate random variables. Moreover, we have illustrated how certain notions and axioms in the traditional theory of risk measures may be brought into this qualitative setting.

Looking at natural extensions to this framework, we are inevitably drawn to consider the utilitarian and multi-sorted settings. More precisely, we have considered risk measures as mappings $F \colon L^\Omega \to L$. However, it could be of interest to consider models $G \colon L^\Omega \to L'$ where risks are valued in L but their assessment is made in a possibly different lattice L', i.e., G would be decomposable into a composition $G = F \circ \varphi$ where $\varphi \colon L \to L'$ and F is a risk measure on L'. Moreover, we could further generalize and consider multivariate random variables of different sorts. Here, risks would be seen as mappings $f \colon \Omega \to \prod_{i \in I} L_i$, and their assessment attained by mappings factorizable in terms of risk measures on L composed with utility functions $\varphi_i \colon L_i \to L$.

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