Discrete Applied Mathematics 160 (2012) 1053–1063

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Directed domination in oriented graphs

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ARTICLE INFO

Article history:
Received 12 October 2010
Received in revised form 8 December 2011
Accepted 14 December 2011
Available online 29 January 2012

Keywords:
Directed domination
Oriented graph
Independence number

ABSTRACT

A directed dominating set in a directed graph D is a set S of vertices of V such that every vertex $u \in V(D) \setminus S$ has an adjacent vertex v in S with v directed to u. The directed domination number of D, denoted by $\gamma(D)$, is the minimum cardinality of a directed dominating set in D. The directed domination number of a graph G, denoted by $\Gamma_d(G)$, is the maximum directed domination number $\gamma(D)$ over all orientations D of G. The directed domination number of a complete graph was first studied by Erdös [P. Erdös, On Schütte problem, Math. Gaz. 47 (1963) 220–222], albeit in disguised form. The authors [Y. Caro, M.A. Henning, A Greedy partition lemma for directed domination, Discrete Optim. 8 (2011) 452–458] recently extended this notion to directed domination of all graphs. In this paper we continue this study of directed domination in graphs. We show that the directed domination number of a bipartite graph is precisely its independence number. Several lower and upper bounds on the directed domination number are presented.

1. Introduction

An *asymmetric digraph* or *oriented graph* D is a digraph that can be obtained from a graph G by assigning a direction to (that is, orienting) each edge of G. The resulting digraph D is called an *orientation* of G. Thus if D is an oriented graph, then for every pair u and v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D. A *directed dominating set*, abbreviated DDS, in a directed graph $D = (V, A)$ is a set S of vertices of V such that every vertex in $V \setminus S$ is dominated by some vertex of S; that is, every vertex $u \in V \setminus S$ has an adjacent vertex v in S with v directed to u. Every digraph has a DDS since the entire vertex set of the digraph is such a set. The *directed domination number* of a *directed graph* D, denoted by $\gamma(D)$, is the minimum cardinality of a DDS in D. A DDS of D of cardinality $\gamma(D)$ is called a $\gamma(D)$-set. Directed domination in digraphs is well studied (cf. [2,3,7–9,14,17,20,25,26]). We define the *lower directed domination number* of a *graph* G, denote $\gamma_d(G)$, to be the minimum directed domination number $\gamma(D)$ over all orientations D of G; that is,

$$\gamma_d(G) = \min\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\}.$$

The *upper directed domination number*, or simply the *directed domination number*, of a *graph* G, denoted by $\Gamma_d(G)$, is defined as the maximum directed domination number $\gamma(D)$ over all orientations D of G; that is,

$$\Gamma_d(G) = \max\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\}.$$

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2011.12.027

1.1. Motivation

The directed domination number of a complete graph was first studied by Erdös [13] albeit in disguised form. In 1962, Schütte [13] raised the question of given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ on $n(k)$ vertices in which for any set S of k vertices, there is a vertex u which dominates all vertices in S. Erdös [13] showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer k. The proof of the following bounds on the directed domination number of a complete graph are along identical lines to that presented by Erdös [13]. This result can also be found in [26]. Throughout this paper, log is to the base 2 while ln denotes the logarithm in the natural base e.

Theorem 1 (*Erdös [13]*)**.** *For every integer* $n \ge 2$, $\log n - 2\log(\log n) \le \Gamma_d(K_n) \le \log(n+1)$.

The authors [6] extended this notion of directed domination in a complete graph to directed domination of all graphs. In this paper, we continue this study of directed domination in graphs. In a sense, this notion of directed domination in graphs measures how "bad" an orientation of an undirected graph can be in terms of the directed domination number of the orientation. This concept of the directed domination number of a graph has a similar flavor to that of the oriented chromatic number which is very well studied in the literature.

1.2. Notation

For notation and graph theory terminology we in general follow [21,28]. Specifically, let G be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from context, we simply write $V, E, n, m, N(v)$ and $N[v]$ rather than $V, E, n, m, N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. If A and B are subsets of $V(G)$, we let $[A, B]$ denote the set of all edges between A and B in G. We denote the diameter of G by $\mathrm{diam}(G)$.

We denote the *degree* of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree among the vertices of G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. The *maximum average degree* in G, denoted by $\mathrm{mad}(G)$, is defined as the maximum of the average degrees taken over all subgraphs H of G, that is,

$$\mathrm{mad}(G) = \max_{H \subset G} \left\{ \frac{2|E(H)|}{|V(H)|} \right\}.$$

The parameter $\gamma(G)$ denotes the *domination number* of G. The parameters $\alpha(G)$ and $\alpha'(G)$ denote the (vertex) *independence number* and the *matching number*, respectively, of G, while $\chi(G)$ and $\chi'(G)$ denote the *chromatic number* and *edge chromatic number*, respectively, of G. The *covering number* of G, denoted by $\beta(G)$, is the minimum number vertices that covers all the edges of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum cardinality of a clique in G.

A vertex v in a digraph D *out-dominates*, or simply *dominates*, itself as well as all vertices u such that (v, u) is an arc of D. The *out-neighborhood* of v, denoted by $N^+(v)$, is the set of all vertices u adjacent from v in D; that is, $N^+(v) = \{u \mid (v, u) \in A(D)\}$. The *out-degree* of v is given by $d^+(v) = |N^+(v)|$, and the maximum out-degree among the vertices of D is denoted by $\Delta^+(D)$. The *in-neighborhood* of v, denoted by $N^-(v)$, is the set of all vertices u adjacent to v in D; that is, $N^-(v) = \{u \mid (u, v) \in A(D)\}$. The *in-degree* of v is given by $d^-(v) = |N^-(v)|$. The *closed in-neighborhood* of v is the set $N^-[v] = N^-(v) \cup \{v\}$. The maximum in-degree among the vertices of D is denoted by $\Delta^-(D)$.

A *hypergraph* $H = (V, E)$ is a finite set V of elements, called *vertices*, together with a finite multiset E of subsets of V, called *edges*. A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. For a digraph $D = (V, E)$, we denote by H_D the *closed in-neighborhood hypergraph*, abbreviated CINH, of D; that is, $H_D = (V, C)$ is the hypergraph with vertex set V and with edge set C consisting of the closed in-neighborhoods of vertices of V in D.

2. Observations

We show first that the lower directed domination number of a graph is precisely its domination number.

Observation 1. *For every graph* G, $\gamma_d(G) = \gamma(G)$.

Proof. Let S be a $\gamma(G)$-set and let D be an orientation obtained from G by directing all edges in $[S, V \setminus S]$ from S to $V \setminus S$ and directing all other edges arbitrarily. Then, S is a DDS of D, and so $\gamma_d(G) \le \gamma(D) \le |S| = \gamma(G)$. However if D is an orientation of a graph G such that $\gamma_d(G) = \gamma(D)$, and if S is a $\gamma(D)$-set, then S is also a dominating set of G, and so $\gamma(G) \le |S| = \gamma_d(G)$. Consequently, $\gamma_d(G) = \gamma(G)$. $\square$

In view of Observation 1, it is not interesting to ask about the lower directed domination number, $\gamma_d(G)$, of a graph G since this is precisely its domination number, $\gamma(G)$, which is very well studied. We therefore focus our attention on the (upper) directed domination number of a graph. As a consequence of Theorem 1, we establish a lower bound on the directed domination number of an arbitrary graph.

Observation 2. *For every graph G on n vertices,* $\Gamma_d(G) \geq \log n - 2\log(\log n)$.

Proof. Let D be an orientation of the edges of a complete graph K_n on the same vertex set as G such that $\Gamma_d(K_n) = \gamma(D)$. Let D_G be the orientation of G induced by arcs of D corresponding to edges of G. Then, $\Gamma_d(G) \geq \gamma(D_G) \geq \gamma(D) = \Gamma_d(K_n)$. The desired lower bound now follows from Theorem 1. □

Observation 3. *If H is an induced subgraph of a graph G, then* $\Gamma_d(G) \geq \Gamma_d(H)$.

Proof. Let $G = (V, E)$ and let $U = V(H)$. Let D_H be an orientation of H such that $\Gamma_d(H) = \gamma(D_H)$. We now extend the orientation D_H of H to an orientation D of G by directing all edges in $[U, V \setminus U]$ from U to $V \setminus U$ and directing all edges with both ends in $V \setminus U$ arbitrarily. Then, $\Gamma_d(G) \geq \gamma(D) \geq \gamma(D_H) = \Gamma_d(H)$. □

Observation 4. *If H is a spanning subgraph of a graph G, then* $\Gamma_d(G) \leq \Gamma_d(H)$.

Proof. Let D be an arbitrary orientation of G, and let D_H be the orientation of H induced by D. Since adding arcs cannot increase the directed domination number, we have that $\gamma(D) \leq \gamma(D_H)$. This is true for every orientation of G. Hence, $\Gamma_d(G) \leq \Gamma_d(H)$. □

Hakimi [19] proved that a graph G has an orientation D such that $\Delta^+(D) \leq k$ if and only if $\mathrm{mad}(G) \leq 2k$. Hence if G is a graph and k is chosen so that $2(k-1) < \mathrm{mad}(G) \leq 2k$, then G has an orientation D such that $\Delta^+(D) \leq k = \lceil \mathrm{mad}(G)/2 \rceil$. We state this as an observation.

Observation 5 (*[19]*). *Every graph G has an orientation D such that* $\Delta^+(D) \leq \lceil \mathrm{mad}(G)/2 \rceil$.

3. Bounds

In this section, we establish bounds on the directed domination number of a graph. We first present lower bounds on the directed domination number of a graph.

Theorem 2. *Let G be a graph of order n. Then the following holds.*

(a) $\Gamma_d(G) \geq \alpha(G) \geq \gamma(G)$.
(b) $\Gamma_d(G) \geq n/\chi(G)$.
(c) $\Gamma_d(G) \geq \lceil (\mathrm{diam}(G) + 1)/2 \rceil$.
(d) $\Gamma_d(G) \geq n/(\lceil \mathrm{mad}(G)/2 \rceil + 1)$.

Proof. Since every maximal independent set in a graph is a dominating set in the graph, we recall that $\gamma(G) \leq \alpha(G)$ holds for every graph G. To prove that $\alpha(G) \leq \Gamma_d(G)$, let A be a maximum independent set in G and let D be the digraph obtained from G by orienting all arcs from A to $V \setminus A$ and orienting all arcs in $G[V \setminus A]$, if any, arbitrarily. Since every DDS of D contains A, we have $\gamma(D) \geq |A|$. However the set A itself is a DDS of D, and so $\gamma(D) \leq |A|$. Consequently, $\Gamma_d(G) \geq \gamma(D) = |A| = \alpha(G)$. This establishes Part (a). Parts (b) and (c) follows readily from Part (a) and the observations that $\alpha(G) \geq n/\chi(G)$ and $\alpha(G) \geq \lceil (\mathrm{diam}(G) + 1)/2 \rceil$. By Observation 5, there is an orientation D of G such that $\Delta^+(D) \leq \lceil \mathrm{mad}(G)/2 \rceil$. Let S be a $\gamma(D)$-set. Then, $V \setminus S \subseteq \cup_{v \in S} N^+(v)$, and so $n - |S| = |V \setminus S| \leq \sum_{v \in S} d^+(v) \leq |S| \cdot \Delta^+(D)$, whence $\gamma(D) = |S| \geq n/(\Delta^+(D) + 1) \geq n/(\lceil \mathrm{mad}(G)/2 \rceil + 1)$. This establishes Part (d). □

We remark that since $\mathrm{mad}(G) \leq \Delta(G)$ for every graph G, as an immediate consequence of Theorem 2(d) we have that $\Gamma_d(G) \geq n/(\lceil \Delta(G)/2 \rceil + 1)$.

Next we consider upper bounds on the directed domination number of a graph. The following lemma will prove to be useful.

Lemma 3. *Let* $G = (V, E)$ *be a graph and let* $V_1, V_2, \ldots, V_k$ *be subsets of V, not necessarily disjoint, such that* $\cup_{i=1}^k V_i = V(G)$. *For* $i = 1, 2, \ldots, k$, *let* $G_i = G[V_i]$. *Then,*

$$\Gamma_d(G) \leq \sum_{i=1}^{k} \Gamma_d(G_i).$$

Proof. Consider an arbitrary orientation D of G. For each $i = 1, 2, \ldots, k$, let D_i be the orientation of the edges of G_i induced by D and let S_i be a $\gamma(D_i)$-set. Then, $\Gamma_d(G_i) \geq \gamma(D_i) = |S_i|$ for each i. Since the set $S = \cup_{i=1}^k S_i$ is a DDS of D, we have that $\gamma(D) \leq |S| \leq \sum_{i=1}^k |S_i| \leq \sum_{i=1}^k \Gamma_d(G_i)$. Since this is true for every orientation D of G, the desired upper bound on $\Gamma_d(G)$ follows. □

As a consequence of Lemma 3, we have the following upper bounds on the directed domination number of a graph.

Theorem 4. *Let G be a graph of order n. Then the following holds.*

(a) $\Gamma_d(G) \le n - \alpha'(G)$.
(b) *If G has a perfect matching, then* $\Gamma_d(G) \le n/2$.
(c) $\Gamma_d(G) \le n$ *with equality if and only if* $G = \overline{K}_n$.
(d) *If G has minimum degree* δ *and* $n \ge 2\delta$, *then* $\Gamma_d(G) \le n - \delta$.
(e) $\Gamma_d(G) = n - 1$ *if and only if every component of G is a* K_1*-component, except for one component which is either a star or a complete graph* K_3.

Proof. (a) Let $M = \{u_1v_1, u_2v_2, \ldots, u_tv_t\}$ be a maximum matching in G, and so $t = \alpha'(G)$. For $i = 1, 2, \ldots, t$, let $V_i = \{u_i, v_i\}$. If $n > 2t$, let $(V_{t+1}, \ldots, V_{n-2t})$ be a partition of the remaining vertices of G into $n - 2t$ subsets each consisting of a single vertex. By Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{n-t} \Gamma_d(G_i) = t + (n - 2t) = n - t = n - \alpha'(G)$. Part (b) is an immediate consequence of Part (a). Part (c) is an immediate consequence of Part (a) and the observation that $\alpha'(G) = 0$ if and only if $G = \overline{K}_n$.

(d) It is well known (see, for example, [4, pp. 87]) that if G has n vertices and minimum degree δ with $n \ge 2\delta$, then $\alpha'(G) \ge \delta$. Hence by Part (a) above, $\Gamma_d(G) \le n - \delta$.

(e) Suppose that $\Gamma_d(G) = n - 1$. Then by Part (a) above, $\alpha'(G) = 1$. However every connected graph F with $\alpha'(F) = 1$ is either a star or a complete graph K_3. Hence, either G is the vertex disjoint union of a star and isolated vertices or of a complete graph K_3 and isolated vertices. □

4. Relation to other parameters

We establish next that the directed domination number of a bipartite graph is precisely its independence number. For this purpose, recall that König [24] and Egerváry [12] showed that if G is a bipartite graph, then $\alpha'(G) = \beta(G)$. Hence by Gallai's Theorem [15], if G is a bipartite graph of order n, then $\alpha(G) + \alpha'(G) = n$.

Theorem 5. *If G is a bipartite graph, then* $\Gamma_d(G) = \alpha(G)$.

Proof. Since G is a bipartite graph, we have that $n - \alpha'(G) = \alpha(G)$. Thus by Theorems 2(a) and 4(b), we have that $\alpha(G) \le \Gamma_d(G) \le n - \alpha'(G) = \alpha(G)$. Consequently, we must have equality throughout this inequality chain. In particular, $\Gamma_d(G) = \alpha(G)$. □

The upper domination number $\Gamma(G)$ of a graph G is the maximum cardinality of a minimal dominating set in G. Cheston and Fricke [10] and Jacobson and Peters [23] proved a similar result to Theorem 5 for the upper domination number of a graph. Hence for the class of bipartite graphs G, we have that $\Gamma_d(G) = \Gamma(G)$. A natural question is whether there is any further relationship between the upper domination number Γ and the (upper) directed domination number Γ_d.

However in general the difference $\Gamma_d(G) - \Gamma(G)$ can be made arbitrary large, as can the difference $\Gamma(G) - \Gamma_d(G)$. For example, if $G = K_n$, then $\Gamma(G) = 1$ but by Theorem 1, we have $\Gamma_d(G) \ge \log n - 2\log(\log n)$. Hence for n sufficiently large, $\Gamma_d(G) - \Gamma(G)$ can be made arbitrary large. On the other hand, let G be obtained from two complete graphs K_n by adding a perfect matching M between the two cliques. Then, $\Gamma(G) = n$. Taking $H = 2K_n$ to be the spanning subgraph of G obtained from G by deleting the edges of M, we have by Theorem 1 and Observation 4 that $\Gamma_d(G) \le \Gamma_d(H) \le 2\log(n + 1)$. Hence for n sufficiently large, $\Gamma(G) - \Gamma_d(G)$ can be made arbitrary large.

The following result establishes an upper bound on the directed domination of a graph in terms of its independence number and chromatic number.

Theorem 6. *For every graph G, we have* $\Gamma_d(G) \le \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.

Proof. Let G have order n. If $\chi(G) = 1$, then G is the empty graph, $\overline{K}_n$ and so $\Gamma_d(G) = n = \alpha(G)$, while if $\chi(G) = 2$, then G is a bipartite graph, and so by Theorem 5, $\Gamma_d(G) = \alpha(G)$. In both cases, $\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$, and so $\Gamma_d(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$. Hence we may assume that $\chi(G) \ge 3$. If $\chi(G) = 2k$ for some integer $k \ge 2$, then let $V_1, V_2, \ldots, V_{2k}$ denote the color classes of G. For $i = 1, 2, \ldots, k$, let G_i be the subgraph $G[V_{2i-1} \cup V_{2i}]$ of G induced by V_{2i-1} and V_{2i} and note that G_i is a bipartite graph. By Theorem 5, $\Gamma_d(G_i) = \alpha(G_i) \le \alpha(G)$ for all $1, 2, \ldots, k$. Hence by Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{k} \Gamma_d(G_i) \le k\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$, as desired. If $\chi(G) = 2k+1$ for some integer $k \ge 1$, then let $V_1, V_2, \ldots, V_{2k+1}$ denote the color classes of G. For $i = 1, 2, \ldots, k$, let H_i be the subgraph of G induced by V_{2i-1} and V_{2i} and note that H_i is a bipartite graph. Further let $H_{k+1} = G[V_{2k+1}]$, and so H_{k+1} is an empty graph on $|V_{2k+1}| \le \alpha(G)$ vertices. By Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{k+1} \Gamma_d(H_i) \le (k+1)\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$. □

As shown in the proof of Theorem 6, the upper bound of Theorem 6 is always attained if $\chi(G) \le 2$. We remark that if $\chi(G) = 3$ or $\chi(G) = 4$, then the upper bound of Theorem 6 is achievable by taking, for example, $G = rK_t$ where $t \in \{3, 4\}$ and r is some positive integer. In this case, $\chi(G) = t$ and $\Gamma_d(G) = 2r = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.

Theorem 7. *If G is a graph of order n, then* $\Gamma_d(G) \le n - \lfloor \chi(G)/2 \rfloor$.

Proof. If $\chi(G) = 1$, then the bound is immediate since $\Gamma_d(G) \le n$ by Theorem 4(c). Hence we may assume that $\chi(G) = k \ge 2$. Let $V_1, V_2, \ldots, V_k$ denote the color classes of G. By the minimality of the coloring, there is an edge between every two color classes. In particular for $i = 1, 2, \ldots, \lfloor k/2 \rfloor$, there is an edge between V_{2i-1} and V_{2i}, and so $\alpha'(G) \ge \lfloor k/2 \rfloor$. Hence by Theorem 4(a), $\Gamma_d(G) \le n - \alpha'(G) \le n - \lfloor k/2 \rfloor$. □

We remark that the bound of Theorem 7 is achievable for graphs with small chromatic number as may be seen by considering the graph $G = \overline{K}_{n-k} \cup K_k$ where $1 \le k \le 4$ and $n > k$. We show next that the directed domination of a graph is at most the average of its order and independence number. For this purpose, we recall the Gallai–Milgram Theorem [16] for oriented graphs which states that in every oriented graph $G = (V, E)$, there is a partition of V into at most $\alpha(G)$ vertex disjoint directed paths.

Theorem 8. *If G is a graph of order n, then $\Gamma_d(G) \le (n + \alpha(G))/2$.*

Proof. Let D be an orientation of G. By the Gallai–Milgram Theorem for oriented graphs, there is a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ of $V(D)$ into t vertex disjoint directed paths where $t \le \alpha(G)$. For $i = 1, 2, \ldots, t$, let $|P_i| = p_i$, and so $\sum_{i=1}^{t} p_i = n$. By Lemma 3, $\Gamma_d(G) \le \sum_{i=1}^{t} \Gamma_d(P_i) = \sum_{i=1}^{t} \lceil p_i/2 \rceil \le \sum_{i=1}^{t} (p_i + 1)/2 = (\sum_{i=1}^{t} p_i/2) + t/2 = (n + \alpha(G))/2$. □

That the bound of Theorem 8 is best possible, may be seen by considering, for example, the graph $G = rK_3 \cup sK_1$ of order $n = 3r + s$ with $\alpha(G) = r + s$ and $\Gamma_d(G) = 2r + s = (n + \alpha(G))/2$.

The following result establishes an upper bound on the directed domination of a graph in terms of the chromatic number of its complement.

Theorem 9. *If G is a graph of order n, then $\Gamma_d(G) \le \chi(\overline{G}) \cdot \log\left(\left\lceil \frac{n}{\chi(\overline{G})} \right\rceil + 1\right)$.*

Proof. Let $t = \chi(\overline{G})$ and consider a $\chi(\overline{G})$-coloring of the complement $\overline{G}$ of G into t color classes $Q_1, Q_2, \ldots, Q_t$, where $|Q_i| = q_i$ for $i = 1, 2, \ldots, t$. For each $i = 1, 2, \ldots, t$, the subgraph $G[Q_i]$ of G induced by Q_i is a clique. We now consider an arbitrary orientation D of G, and we let $D_i = D[Q_i]$ denote the orientation of the edges of the clique $G[Q_i]$ induced by D. Then,

$$\gamma(D) \le \sum_{i=1}^{t} \gamma(D_i) \le \sum_{i=1}^{t} \Gamma_d(Q_i) = \sum_{i=1}^{t} \Gamma_d(K_{q_i}).$$

This is true for every orientation D of G, and so, by Theorem 1, we have that $\Gamma_d(G) \le \sum_{i=1}^{t} \log(q_i + 1)$, where $\sum_{i=1}^{t} q_i = n$. By convexity the right hand side attains its maximum when all summands are as equal as possible; that is, some of the summands are $\lfloor n/t \rfloor$ and some are $\lceil n/t \rceil$. Hence, $\Gamma_d(G) \le t \log(\lceil n/t \rceil + 1)$. □

As a consequence of Theorem 9, we have the following result on the directed domination number of a dense graph with large minimum degree. Recall that an equitable coloring is a coloring in which the numbers of vertices in any two color classes differ by at most one. The well-known Hajnal–Szemerédi Theorem [18] states that every graph with maximum degree Δ has an equitable coloring with $\Delta + 1$ colors.

Theorem 10. *If G is a graph on n vertices with minimum degree $\delta(G) \ge (k-1)n/k$ where k divides n, then $\Gamma_d(G) \le n \log(k+1)/k$.*

Proof. Since $k \mid n$, we note that $n = kt$ for some integer t, implying that $\delta(G) \ge (k-1)t$ and $\Delta(\overline{G}) = n - \delta(G) - 1 \le t - 1$. By the Hajnal–Szemerédi Theorem [18], the graph $\overline{G}$ is t-colorable with all color classes of size $\lfloor n/t \rfloor = \lceil n/t \rceil = k$. Hence, G contains t vertex disjoint copies of K_k. Further, $\chi(\overline{G}) \le t$. Thus applying Theorem 9, we have that $\Gamma_d(G) \le t \log(k+1) = n \log(k+1)/k$. □

5. Special families of graphs

In this section, we consider the (upper) directed domination number of special families of graph. As remarked earlier, the directed domination number of a complete graph K_n is determined by Erdös [13] in Theorem 1, while the directed domination number of a bipartite graph is precisely its independence number (see Theorem 5).

5.1. Regular graphs

For each given $\delta \ge 1$, applying Theorem 2(a) to the graph $G = K_{\delta,n-\delta}$ yields $\Gamma_d(G) \ge n - \delta$. Hence without regularity, we observe that for each fixed $\delta \ge 1$, there exists a graph G of order n and minimum degree δ satisfying $\Gamma_d(G) \ge n - \delta$. With regularity, the directed domination number of a graph may be much smaller. For a given r, let $n = k(r+1)$ for some integer k and let G consist of the disjoint union of k copies of K_{r+1}. Let $G_1, G_2, \ldots, G_k$ denote the components of G. Each component of G is r-regular, and by Theorem 1, $\Gamma_d(G) = \sum_{i=1}^{k} \Gamma_d(G_i) = \sum_{i=1}^{k} \Gamma_d(K_{r+1}) \le k \log(r+2) = n \log(r+2)/(r+1)$. Hence there exist r-regular graphs of order n with $\Gamma_d(G) \le n \log(r+2)/(r+1)$. In view of these observations it is of interest to investigate the directed domination number of regular graphs.

In 1964, Vizing proved his important edge-coloring result which states that every graph G satisfies $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$. As a consequence of Vizing's Theorem, we have the following upper bound on the directed domination number of a regular graph.

Theorem 11. *For $r \geq 2$, if G is an r-regular graph of order n, then*

$$\Gamma_d(G) \leq n(r+2)/2(r+1).$$

Proof. By Vizing's Theorem, $\chi'(G) \leq r+1$. Consider an edge coloring of G using $\chi'(G)$-colors. The edges in each color class form a matching in G, and so the matching number of G is at least the size of a largest color class in G. Hence if G has size m, we have $\alpha'(G) \geq m/\chi'(G) \geq m/(r+1) = nr/2(r+1)$. Hence by Theorem 4(a), $\Gamma_d(G) \leq n - \alpha'(G) \leq n - nr/2(r+1) = n(r+2)/2(r+1)$. □

As a special case of Theorem 11, we have that $\Gamma_d(G) \leq 2n/3$ if G is a 2-regular graph. We next characterize when equality is achieved in this bound.

Proposition 1. *Let G be a 2-regular graph on $n \geq 3$ vertices. Then the following holds.*

(a) *If G is connected, then $\Gamma_d(G) = \lceil n/2 \rceil$.*
(b) *$\Gamma_d(G) \leq 2n/3$ with equality if and only if G consists of disjoint copies of K_3.*

Proof. (a) Suppose that G is a cycle C_n. If n is even, G has a perfect matching, and so, by Theorem 4(c), $\Gamma_d(G) \leq n/2$. If n is odd, then $\alpha'(G) = (n-1)/2$. By Theorem 4(b), $\Gamma_d(G) \leq n - \alpha'(G) = n - (n-1)/2 = (n+1)/2$. In both cases, $\Gamma_d(G) \leq \lceil n/2 \rceil$. To show that $\Gamma_d(G) \geq \lceil n/2 \rceil$, we note that if D is a directed cycle C_n, then every vertex out-dominates itself and exactly one other vertex, and so $\Gamma_d(G) \geq \gamma(D) = \lceil n/2 \rceil$. This proves part (a).

(b) To prove part (b), let $G_1, G_2, \ldots, G_k$ be the components of G, where $k \geq 1$. For $i = 1, 2, \ldots, k$, let G_i have order n_i. Since each component is a cycle, $n \geq 3k$. Applying the result of part (a) to each component of G, we have

$$\Gamma_d(G) = \sum_{i=1}^{k} \Gamma_d(G_i) \leq \sum_{i=1}^{k} \left(\frac{n_i + 1}{2} \right) = \frac{n+k}{2} \leq \frac{2n}{3},$$

with equality if and only if $n = 3k$, i.e., if and only if $G_i = C_3$ for each $i = 1, 2, \ldots, k$. □

We remark that the upper bound of Theorem 11 can be improved using tight lower bounds on the size of a maximum matching in a regular graph established in [22]. Applying Theorem 4(a) to these matching results in [22], we have the following result. We remark that the $(n+1)/2$ bound in the statement of Theorem 12 is only included as it is necessary when n is very small or $r = 2$.

Theorem 12. *For $r \geq 2$, if G is a connected r-regular graph of order n, then*

$$\Gamma_d(G) \leq \begin{cases} \max\left\{ \left(\dfrac{r^2+2r}{r^2+r+2} \right) \times \dfrac{n}{2}, \dfrac{n+1}{2} \right\} & \text{if } r \text{ is even} \\ \dfrac{(r^3+r^2-6r+2)\,n + 2r - 2}{2(r^3-3r)} & \text{if } r \text{ is odd}. \end{cases}$$

We close this section with the following observation. Graphs G satisfying $\chi'(G) = \Delta(G)$ are called *class* 1 and those with $\chi'(G) = \Delta(G) + 1$ are *class* 2.

Observation 6. *Let G be an r-regular graph of order n. Then the following holds.*

(a) *If G is of class 1, then $\Gamma_d(G) \leq n/2$.*
(b) *If $r \geq n/2$, then $\Gamma_d(G) \leq \lceil n/2 \rceil$.*

Proof. (a) Consider a r-edge coloring of G. The edges in each color class form a perfect matching in G, and so, by Theorem 4(b), $\Gamma_d(G) \leq n/2$.

(b) If $n = 2$, then the result is immediate. Hence we may assume that $n \geq 3$. By Dirac's theorem, G is Hamiltonian, and so $\alpha'(G) \geq \lfloor n/2 \rfloor$. By Theorem 4(b), $\Gamma_d(G) \leq n - \alpha'(G) \leq n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$. □

5.2. Outerplanar graphs

Let $\mathcal{OP}_n$ denote the family of all maximal outerplanar graphs of order n. We define $\mathrm{Mop}(n) = \max\{\Gamma_d(G)\}$ where the maximum is taken over all graphs $G \in \mathcal{OP}_n$.

Theorem 13. $\mathrm{Mop}(n) = \lceil n/2 \rceil$.

Proof. Let $G \in \mathcal{OP}_n$. Since every maximal outerplanar graph is Hamiltonian, we observe by Observation 4 and Proposition 1(a), that $\Gamma_d(G) \le \Gamma_d(C_n) = \lceil n/2 \rceil$. Since this is true for an arbitrary graph G in $\mathcal{OP}_n$, we have $\mathrm{Mop}(n) \le \lceil n/2 \rceil$. Hence it suffices for us to prove that $\mathrm{Mop}(n) \ge \lceil n/2 \rceil$. If $n = 3$, then by Observation 4, $\Gamma_d(G) \ge \Gamma_d(C_n) = \lceil n/2 \rceil$, as desired. Hence we may assume that $n \ge 4$, for otherwise the desired result follows.

For $n \ge 4$ even, we take a directed cycle $\overrightarrow{C_n}$ on $n \ge 4$ vertices and a selected vertex v on the cycle, and we add arcs from every vertex u, where u is neither the in-neighbor nor the out-neighbor of v on $\overrightarrow{C_n}$, to the vertex v. The resulting orientation D of the underlying maximal outerplanar graph has $\gamma_d(D) = n/2$. Hence for $n \ge 4$ even, we have $\mathrm{Mop}(n) = n/2$.

It remains for us to show that for $n \ge 5$ odd, $\mathrm{Mop}(n) = (n+1)/2$. For $n \ge 5$ odd, we take a directed cycle $\overrightarrow{C_n} : v_1 v_2 \dots v_n v_1$ on n vertices. We now add the arcs from v_i to v_1 for all odd i, where $3 \le i \le n-2$, and we add the arcs from v_1 to v_i for all even i, where $4 \le i \le n-1$. Let G denote the resulting underlying maximal outerplanar graph and let D denote the resulting orientation of D. We now consider an arbitrary DDS S in D.

Suppose first that $v_1 \in S$. In order to dominate the $(n-1)/2$ vertices v_{2i+1}, where $1 \le i \le (n-1)/2$, in D we must have that $|S \cap \{v_{2i}, v_{2i+1}\}| \ge 1$ for all $i = 1, 2, \dots, (n-1)/2$. Hence in this case when $v_1 \in S$, we have $|S| \ge (n+1)/2$.

Suppose next that $v_1 \notin S$. Then, $v_2 \in S$. In order to dominate the $(n-3)/2$ vertices v_{2i}, where $2 \le i \le (n-1)/2$, in D we must have that $|S \cap \{v_{2i}, v_{2i-1}\}| \ge 1$ for all $i = 2, \dots, (n-1)/2$. In order to dominate v_1, there is a vertex $v_j \in S$ for some odd j, where $3 \le j \le n$. Let j be the largest such odd subscript for which $v_j \in S$. If $j = n$, then $v_n \in S$ and $|S| \ge (n+1)/2$, as desired. Hence we may assume that $j < n$. In order to dominate the vertex v_i for i odd with $j < i \le n$, we must have $v_{i-1} \in S$. In particular, we have that $v_{j+1} \in S$ to dominate v_{j+2}, implying that $|S \cap \{v_j, v_{j+1}\}| = 2$ while for i odd where $i \ne j$ and $3 \le i \le n-2$, we have $|S \cap \{v_i, v_{i+1}\}| \ge 1$, implying that $|S| \ge (n+1)/2$.

In both cases, $|S| \ge (n+1)/2$. Since S is an arbitrary DDS in D, we have $\gamma(D) \ge (n+1)/2$. Hence, $\Gamma_d(G) \ge (n+1)/2$, implying that $\mathrm{Mop}(n) = (n+1)/2$. □

5.3. *Perfect graphs*

Recall that a *perfect graph* is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. Characterization of perfect graphs was a longstanding open problem. The first breakthrough was due to Lovász in 1972 who proved the Perfect Graph Theorem.

Perfect Graph Theorem. *A graph is perfect if and only if its complement is perfect.*

Let $\alpha \ge 1$ be an integer and let $\mathcal{G}_\alpha$ be the class of all graphs G with $\alpha \ge \alpha(G)$. We are now in a position to present an upper bound on the directed domination number of a perfect graph in terms of its independence number.

Theorem 14. *If $G \in \mathcal{G}_\alpha$ is a perfect graph of order $n \ge \alpha$, then*

$$\Gamma_d(G) \le \alpha \log\left(\lceil n/\alpha \rceil + 1\right).$$

Proof. By the Perfect Graph Theorem, the complement $\overline{G}$ of G is perfect. Hence, $\chi(\overline{G}) = \omega(\overline{G}) = \alpha(G)$. The desired result now follows from Theorem 9. □

6. Interplay between transversals and directed domination

In this section, we present upper bounds on the directed domination number of a graph by demonstrating an interplay between the directed domination number of a graph and the transversal number of a hypergraph. We shall need the following upper bounds on the transversal number of a uniform hypergraph established by Alon [1] and Chvátal and McDiarmid [11]. Applying probabilistic arguments, Alon [1] showed the following result.

Theorem 15 (*Alon [1]*)**.** *For $k \ge 2$, if H is a k-uniform hypergraph with n vertices and m edges, then $\tau(H) \le (m+n)(\ln k)/k$.*

Theorem 16 (*Chvátal, McDiarmid [11]*)**.** *For $k \ge 2$, if H is a k-uniform hypergraphs with n vertices and m edges, then $\tau(H) \le (n + \lfloor \frac{k}{2} \rfloor m)/\lfloor \frac{3k}{2} \rfloor$. bound is sharp.*

We proceed further with two lemmas. For this purpose, we shall need the Szekeres–Wilf Theorem, where we recall that a graph G is *k-degenerate* if every induced subgraph of G has minimum degree at most k.

Theorem 17 (*Szekeres–Wilf [27]*)**.** *If G is a k-degenerate graph, then $\chi(G) \le k+1$.*

Lemma 18. *If G is a graph and D is an orientation of G such that $\Delta^-(D) \le k$ for some fixed integer $k \ge 0$, then $\chi(G) \le 2k+1$.*

Proof. It suffices to show that G is $2k$-degenerate, since then the desired result follows from the Szekeres–Wilf theorem. Assume, to the contrary, that G is not $2k$-degenerate. Then there is a subset S of $V(G)$ such that the subgraph $G_S = G[S]$ induced by S has minimum degree at least $2k+1$ and hence contains at least $(2k+1)|S|/2$ edges. Let $D_S = D[S]$ be the orientation of D induced by S. Since $\Delta^-(D) \le k$, we have that $\Delta^-(D_S) \le k$ and

$$k|S| \ge \sum_{v \in V(D_S)} d^-(v) = |E(G_S)| \ge (2k+1)|S|/2 > k|S|,$$

a contradiction. □

Lemma 19. *Let D be an orientation of a graph G. If G contains n_k vertices with in-degree at most k in D for some fixed integer $k \ge 0$, then $n_k \le (2k+1)\alpha(G)$.*

Proof. Let V_k denote the set of all vertices of G with in-degree at most k in D, and so $n_k = |V_k|$. Let $G_k = G[V_k]$ and let $D_k = D[V_k]$. Then, D_k is an orientation of G_k such that $\Delta^-(D_k) \le k$, and so by Lemma 18, $\chi(G_k) \le 2k+1$. Since every color class of G_k is an independent set, and therefore has cardinality at most $\alpha(G)$, we have that $n_k = |V_k| \le \chi(G_k)\alpha(G) \le (2k+1)\alpha(G)$. □

Let $f(n,k)$, $g(n,k)$, and $h(n,k)$ be the functions of n and k defined as follows.

$$f(n,k) = 2n\ln(k+2)/(k+2) + (2k+1)\alpha(G)$$
$$g(n,k) = n(k+2)/3k + 2(2k+1)\alpha(G)/3$$
$$h(n,k) = n(k+1)/(3k-1) + 2k(2k+1)\alpha(G)/(3k-1).$$

Theorem 20. *If G is a graph on n vertices, then*

$$\Gamma_d(G) \le \begin{cases} \min\limits_{k\ge 0}\{f(n,k), g(n,k)\} & \text{if } k \text{ is even} \\ \min\limits_{k\ge 1}\{f(n,k), h(n,k)\} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let D be an arbitrary orientation of the graph G and let $k \ge 0$ be an arbitrary integer. Let V_k denote the set of all vertices of G with in-degree at most k in D and let $n_k = |V_k|$. Let $V_{>k} = V(G) \setminus V_k$, and so all vertices in $V_{>k}$ have in-degree at least $k+1$ in D. Let $H_{>k}$ be the hypergraph obtained from the CINH H_D of D by deleting the n_k edges corresponding to closed in-neighborhoods of vertices in V_k. Each edge in $H_{>k}$ has size at least $k+2$.

We now define the hypergraph H as follows. For each edge e_v in $H_{>k}$ corresponding to the closed in-neighborhood of a vertex v in $V_{>k}$, let e'_v consist of v and exactly $k+1$ vertices from $N^-(v)$. Thus, $e'_v \subseteq e_v$ and e'_v has size $k+2$. Let H be the hypergraph obtained from $H_{>k}$ by shrinking all edges e_v of $H_{>k}$ to the edges e'_v. Then, H is a $(k+2)$-uniform hypergraph with n vertices and $n - n_k$ edges.

Every transversal T in H contains a vertex from the closed in-neighborhood of each vertex from the set $V_{>k}$ in D, and therefore $T \cup V_k$ is a DDS in D. In particular, taking T to be a minimum transversal in H, we have that $\gamma(D) \le \tau(H) + n_k$. By Lemma 19, $n_k \le (2k+1)\alpha(G)$. Applying Theorem 15 to the hypergraph H, we have that

$$\tau(H) \le (n + n - n_k)\ln(k+2)/(k+2) \le 2n\ln(k+2)/(k+2),$$

and so $\gamma(D) \le \tau(H) + n_k \le 2n\ln(k+2)/(k+2) + \alpha(G)(2k+1) = f(n,k)$. Applying Theorem 16 to the hypergraph H for k even, we have that

$$\tau(H) \le (2n + k(n-n_k))/3k = n(k+2)/3k - n_k/3,$$

and so $\gamma(D) \le \tau(H) + n_k \le n(k+2)/3k + 2n_k/3 \le n(k+2)/3k + 2(2k+1)\alpha(G)/3 = g(n,k)$. Thus for k even, we have that $\Gamma_d(G) \le \min\{f(n,k), g(n,k)\}$. Applying Theorem 16 to the hypergraph H for k odd, we have that

$$\tau(H) \le (2n + (k-1)(n-n_k))/(3k-1) = n(k+1)/(3k-1) - (k-1)n_k/(3k-1),$$

and so $\gamma(D) \le \tau(H) + n_k \le n(k+1)/(3k-1) + 2kn_k/(3k-1) \le n(k+1)/(3k-1) + 2k(2k+1)\alpha(G)/(3k-1) = h(n,k)$. Thus for k odd, we have that $\Gamma_d(G) \le \min\{f(n,k), h(n,k)\}$. □

Let $f_n(\alpha)$, $g_n(\alpha)$, and $h_n(\alpha)$ be the functions of n and α defined as follows.

$$f_n(\alpha) \doteq \sqrt{2n\alpha}\left(\ln(\sqrt{2n/\alpha}) + 2\right) - 2\alpha$$

$$g_n(\alpha) \doteq \frac{1}{3}\left(n + 2\alpha + 4\sqrt{2n\alpha}\right)$$

$$h_n(\alpha) \doteq \frac{1}{3}\left(n + \frac{14}{3}\alpha + \frac{\sqrt{2\alpha}\,(27n + 20\alpha)}{3\sqrt{5\alpha + 6n}}\right).$$

As a consequence of Theorem 20, we have the following upper bound on the directed domination of a graph.

Theorem 21. *If G is a graph on n vertices with independence number α, then*

$$\Gamma_d(G) \le \min\{f_n(\alpha), g_n(\alpha), h_n(\alpha)\}.$$

Proof. By Theorem 20, we need to optimize the functions $f(n,k)$, $g(n,k)$ and $h(n,k)$ over k to obtain an upper bound on $\Gamma_d(G)$. To simplify the notation, let $\alpha = \alpha(G)$. Optimizing the function $g(n,k)$ over k (treating n as fixed), we get $g(n,k) \le g_n(\alpha)$, while optimizing the function $h(n,k)$ over k (treating n as fixed), we get $h(n,k) \le h_n(\alpha)$. Optimization of the function $f(n,k)$ is complicated. Hence to simplify the computations, we choose a value k^* for k and show that $f(n,k^*) \le f_n(\alpha)$.

Suppose $\alpha \ge n/2$. Then, $\alpha = cn$ with $1 \ge c \ge 1/2$. Substituting this into $f_n(\alpha)$ we get $f_n(\alpha) = n\sqrt{2c}(\ln(2/c)+2)-2cn = n\left(\sqrt{2c}(\ln(2/c)+2)-2c\right) \ge n$, and so the inequality $\Gamma_d(G) \le f_n(\alpha)$ holds trivially. Hence we may assume that $\alpha \le n/2$. We now take $k = \sqrt{2n/\alpha} - 2 \ge 0$. Substituting into $f(n,k) = 2n\ln(k+2)/(k+2) + (2k+1)\alpha$, we get

$$\begin{aligned} f(n,k) &= 2n\ln(\sqrt{2n/\alpha}\,)/\sqrt{2n/\alpha} + (2\sqrt{2n/\alpha} - 3)\alpha \\ &= \sqrt{2n\alpha}\ln(\sqrt{2n/\alpha}\,) + 2\alpha\sqrt{2n/\alpha} - 3\alpha \\ &= \sqrt{2n\alpha}\left(\ln(\sqrt{2n/\alpha}\,) + 2\right) - 3\alpha \\ &< f_n(\alpha), \end{aligned}$$

as desired. □

If every edge of a hypergraph H has size at least r, we define an *r-transversal* of H to be a transversal T such that $|T \cap e| \ge r$ for every edge e in H. The *r-transversal number* $\tau_r(H)$ of H is the minimum size of an r-transversal in H. In particular, we note that $\tau_1(H) = \tau(H)$. For integers $k \ge r$ where $k \ge 2$ and $r \ge 1$, we first establish general upper bounds on the r-transversal number of a k-uniform hypergraph. Our next result generalizes that of Theorem 15 due to Alon [1], as well as generalizes results due to Caro [5].

Theorem 22. *For integers $k \ge r$ where $k \ge 2$ and $r \ge 1$, let H be a k-uniform hypergraph with n vertices and m edges. Then, $\tau_r(H) \le n\ln k/k + rm(2\ln k)^r/k$.*

Proof. Since every minimal transversal in H contains no isolated vertex, we may assume that $\delta(H) \ge 1$. When $k = 2$ and $r = 1$, the result follows from Theorem 15. When $k = r = 2$, we have that $\tau_r(H) = n$ and the desired result follows. Hence we may assume that $k \ge 3$ and $r \ge 1$. Pick every vertex of $V(H)$ randomly with probability p to be determined later but such that $(1-p) > 1/2$. Let X be the set of randomly picked vertices and let E_X be the set of edges of $E(H)$ whose intersection with X is at most $r-1$. For every fixed edge $e \in E(H)$, the probability that e is in E_X is exactly

$$\Pr(e \in E_X) = \sum_{i=0}^{r-1}\binom{k}{i}p^i(1-p)^{k-i} = (1-p)^k\sum_{i=0}^{r-1}\binom{k}{i}\left(\frac{p}{1-p}\right)^i. \quad (1)$$

We now choose $p = \ln k/k$. With this choice of p, we have that $(1-p) > 1/2$. Hence, $1/(1-p)^i < 2^i$ for all $i \ge 1$. Since $1 - x \le e^{-x}$ for all $x \in R$, we note that $(1-p)^k \le e^{-pk} = e^{-\ln k} = 1/k$. Substituting $p = \ln k/k$ into Eq. (1) we therefore get

$$\Pr(e \in E_X) \le \frac{1}{k}\sum_{i=0}^{r-1}\frac{k^i}{i!}\cdot\frac{p^i}{(1-p)^i} \le \frac{1}{k}\sum_{i=0}^{r-1}\frac{(2kp)^i}{i!} \le \frac{1}{k}\sum_{i=0}^{r-1}(2\ln k)^i \le \frac{1}{k}(2\ln k)^r,$$

since $1 + q + q^2 + \cdots + q^{r-1} = (q^r - 1)/(q-1) \le q^r$ for $q > 2$ and $r \ge 1$ (recall that $k \ge 3$, and so $2\ln k > 2$). For each edge $e \in E_X$, we add $r - |e \cap X|$ (which is at most r) vertices from $e \setminus X$ to a set Y. Then, $T = X \cup Y$ is a r-transversal in H and $|Y| \le r|E_X|$. By the linearity of expectation, $E(T) = E(X) + E(Y) \le E(X) + rE(E_X) = n\ln k/k + rm(2\ln k)^r/k$. □

For $r \ge 1$, an *r-directed dominating set* in a directed graph $D = (V, A)$ is a set S of vertices of V such that every vertex in $V \setminus S$ is dominated by at least r vertices of S; that is, every vertex $u \in V \setminus S$ is adjacent from r vertices v in S with v directed to u. The *directed r-domination number* of a graph G, denoted by $\Gamma_d(G, r)$, is defined as the maximum r-directed domination number $\gamma_r(D)$ over all orientations D of G. Using r-transversals in hypergraphs, we obtain the following bound on the directed r-domination number of a graph.

Theorem 23. *For $r \ge 1$ an integer, if G is a graph on n vertices, then*

$$\Gamma_d(G, r) \le \min_{k \ge r}\left\{(2k-1)\alpha(G) + n\ln(k+1)/(k+1) + rn(2\ln(k+1))^r/(k+1)\right\}.$$

Proof. Let D be an arbitrary orientation of the graph G and let $k \ge r$ be an arbitrary integer. Let $V_{<k}$ denote the set of all vertices of G with in-degree at most $k-1$ in D and let $n_{<k} = |V_{<k}|$. Let $G_{<k}$ be the subgraph of G induced by the set $V_{<k}$ and let $D_{<k}$ be the orientation of $G_{<k}$ induced by D. Then, $\Delta^-(D_{<k}) \le k-1$, and so, by Lemma 18, $\chi(G_{<k}) \le 2k-1$, implying that $n_{<k} \le (2k-1)\alpha(G)$.

Let $V_k = V(G) \setminus V_{<k}$, and so all vertices in V_k have in-degree at least k in D. Let H_k be the hypergraph obtained from the CINH H_D of D by deleting the $n_{<k}$ edges corresponding to closed in-neighborhoods of vertices in $V_{<k}$. Each edge in H_k has size at least $k + 1$. We now define the hypergraph H as follows. For each edge e_v in H_k corresponding to the closed in-neighborhood of a vertex v in V_k, let e'_v consist of v and exactly k vertices from $N^-(v)$. Thus, $e'_v \subseteq e_v$ and e'_v has size $k+1$. Let H be the hypergraph obtained from H_k by shrinking all edges e_v of H_k to the edges e'_v. Then, H is a $(k+1)$-uniform hypergraph with n vertices and $n - n_{<k}$ edges.

Every r-transversal T in H contains at least r vertices from the closed in-neighborhood of each vertex from the set V_k in D, and therefore $T \cup V_{<k}$ is a r-directed dominating set in D. In particular, taking T to be a minimum r-transversal in H, we have that $\gamma_r(D) \le \tau_r(H) + n_{<k}$. By Lemma 19, $n_{<k} \le (2k-1)\alpha(G)$. Noting that $k+1 \ge r+1 \ge 2$, we can apply Theorem 22 to the hypergraph H yielding $\tau_r(H) \le n\ln(k+1)/(k+1) + r(n-n_{<k})(2\ln(k+1))^r/(k+1)$, and so $\gamma_r(D) \le \tau_r(H) + n_{<k} \le (2k-1)\alpha(G) + n\ln(k+1)/(k+1) + rn(2\ln(k+1))^r/(k+1)$. Since this is true for every integer $k \ge r$, the desired upper bound on $\Gamma_d(G, r)$ follows. □

7. Open questions

We close with a list of open questions and conjectures that we have yet to settle. Let $\mathcal{R}_n$ denote the family of all r-regular graphs of order n. We define $m(n, r) = \min\{\Gamma_d(G)\}$ and $M(n, r) = \max\{\Gamma_d(G)\}$, where the minimum and maximum are taken over all graphs $G \in \mathcal{R}_n$. Then, $m(n, 1) = M(n, 1) = n/2$. By Proposition 1, $m(n, 2) = n/2$ while $M(n, 2) = 2n/3$. We remark that by Theorem 11, for $r \ge 2$, we know that

$$\frac{n}{2} \le M(n, r) \le \left(\frac{r+2}{r+1}\right) \cdot \frac{n}{2} \qquad (2)$$

(and this upper bound on $M(n, r)$ can be improved slightly by Theorem 12).

Conjecture 1. *For $r \ge 3$, $M(n, r) = n/2$.*

By Theorem 2(a), we know that if $G \in \mathcal{R}_n$, then $\Gamma_d(G) \ge \alpha(G) \ge n/(r+1)$, and so $n/(r+1) \le m(n, r)$. Moreover taking $n/(r+1)$ copies of K_{r+1}, we have by Theorem 1 that $m(n, r) \le n\log(r+2)/(r+1)$. We pose the following question.

Question 1. For $r \ge 3$, does there exists a constant c such that $m(n, r) \le cn/(r+1)$?

Let $\mathcal{OP}_n$ denote the family of all maximal outerplanar graphs of order n and define $\mathrm{mop}(n) = \min\{\Gamma_d(G)\}$, where the minimum is taken over all graphs $G \in \mathcal{OP}_n$. Since outerplanar graphs are 3-colorable, we note by Theorem 2(b) that for every graph $G \in \mathcal{OP}_n$, $\Gamma_d(G) \ge n/3$, implying that $\mathrm{mop}(n) \ge n/3$. By Theorem 13, we know that $\mathrm{mop}(n) \le \lceil n/2 \rceil$. Thus, $n/3 \le \mathrm{mop}(n) \le \lceil n/2 \rceil$.

Problem 1. Find good lower and upper bounds on $\mathrm{mop}(n)$.

Let $\mathcal{P}_n$ denote the family of all maximum planar graphs of order n. We define $\mathrm{mp}(n) = \min\{\Gamma_d(G)\}$ and $\mathrm{Mp}(n) = \max\{\Gamma_d(G)\}$, where the minimum and maximum are taken over all graphs $G \in \mathcal{P}_n$.

Problem 2. Find good lower and upper bounds on $\mathrm{mp}(n)$ and $\mathrm{Mp}(n)$.

Acknowledgments

The second author's research was supported in part by the South African National Research Foundation and the University of Johannesburg.

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