A Review of the Decomposition Method in Applied Mathematics

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The decomposition method can be an effective procedure for analytical solution of a wide class of dynamical systems without linearization or weak nonlinearity assumptions, closure approximations, perturbation theory, or restrictive assumptions on stochasticitiy. © 1988 Academic Press, Inc.

An advantage of the decomposition method is that it can provide analytical approximation to a rather wide class of nonlinear (and stochastic) equations without linearization, perturbation, closure approximations, or discretization methods which can result in massive numerical computation. The usually desired closed-form analytical solutions of a nonlinear problem necessitate making some simplifying and restrictive assumptions in order to make it solvable. The result, however elegant, may not be physically realistic. In other words, the solution of the simpler mathematical problem may not be a good approximation to the solution of the original problem. "Weak" nonlinearity and "small" perturbations are common assumptions. However, nature is nonlinear and stochastic in general. The deterministic case may be regarded as a limiting case in which randomness can be ignored. Similarly, linearity can be regarded as a limiting case, as can cases where perturbation theory is adequate. The decomposition solution is also an approximation, but one which does not change the problem. Therefore it is often physically more realistic. While the solution obtained by decomposition is generally an infinite series, an n-term approximation, ϕ_n, usually serves as a practical solution. As shown in the closing examples of this paper, an accurate solution is often obtained with very small values of n. The decomposition method has some features in common with other methods, but it is distinctly different on closer examination, and it offers several significant advantages. A general description of the method follows.

Begin with an equation $Fu(t) = g(t)$, where F represents a general non-linear ordinary differential operator involving both linear and nonlinear terms. The linear term is decomposed into $L + R$, where L is easily invertible and R is the remainder of the linear operator. For convenience, L may be taken as the highest order derivative which avoids difficult integrations which result when complicated Green's functions are involved. Thus the equation may be written

$$Lu + Ru + Nu = g, \tag{1}$$

where Nu represents the nonlinear terms. Solving for Lu,

$$Lu = g - Ru - Nu. \tag{2}$$

Because L is invertible, an equivalent expression is

$$L^{-1}Lu = L^{-1}g + L^{-1}Ru - L^{-1}Nu. \tag{3}$$

If this corresponds to an initial-value problem, the integral operator L^{-1} may be regarded as definite integrals from t_0 to t. If L is a second-order operator, L^{-1} is a twofold integration operator and $L^{-1}Lu = u - u(t_0) - (t - t_0) u'(t_0)$. For boundary value problems (and, if desired, for initial-value problems as well), indefinite integrations are used and the constants are evaluated from the given conditions. Solving (3) for u yields

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{4}$$

The nonlinear term Nu will be equated to $\sum_{n=0}^{\infty} A_n$, where the A_n are special polynomials to be discussed, and u will be decomposed into $\sum_{n=0}^{\infty} u_n$, with u_0 identified as $A + Bt + L^{-1}g$:

$$\sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n.$$

Consequently, we can write

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}A_0 \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1 \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n. \end{aligned} \tag{5}$$

The polynomials A_n are generated for each nonlinearity so that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends on u_0, u_1, u_2, etc. [1]. All of the u_n components are calculable, and $u = \sum_{n=0}^{\infty} u_n$. If the series converges, the n-term partial sum $\phi_n = \sum_{i=0}^{n-1} u_i$ will be the approximate

solution since $\text{limit}_{n \to \infty} \phi_n = \sum_{i=0}^{\infty} u_i = u$ by definition. It is important to emphasize that the A_n can be calculated for complicated nonlinearities of the form $f(u, u', \ldots)$ or $f(g(u))$. As an example, the A_n for e^{-x^2} are $A_0 = e^{-x_0^2}$, $A_1 = -2x_0 x_1 e^{-x_0^2}$, $A_2 = \{2x_0^2 x_1^2 - x_1^2 - 2x_0 x_2\} e^{-x_0^2}$, etc. For a simple example such as $x = k + e^{-x^2}$, the error in a four-term approximation ϕ_4 is already less than 0.001 %. In [1] it is shown that a similar result holds for $Ny = (\alpha x^2 + \beta)^{1/2}$ with less than a 0.5 % error in only four terms. Also, since we are not linearizing or assuming "weak nonlinearity," the solutions tend to be much more physically correct than those obtained by other methods of approximation based on simplifying assumptions. Furthermore, numerical values can be computed if desired, and it is then easy to see convergence as we calculate terms [1]. Since the solutions are analytic (and verifiable by substitution), physical insight into functional relationships follows.

Two formulations have been developed for the A_n polynomials: one set designated as A_n and the other by $\hat{A}_n$; either may be used as indicated above. Consider an equation for which $u(x)$ is the solution, containing a nonlinear term $Nu \equiv f(u) = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \hat{A}_n$. These $\hat{A}_n$ polynomials are defined by

$$\hat{A}_0 = f(u_0)$$

$$\hat{A}_1 = u_1 (d/du_0) f(u_0) + (u_1^2/2!)(d^2/du_0^2) f(u_0) + (u_1^3/3!)(d^3/du_0^3) f(u_0) + \cdots$$

$$= (e^{u_1 d/du_0} - 1) f(u_0)$$

$$\hat{A}_2 = u_2 (d/du_0) f(u_0) + (u_2^2/2!)(d^2/du_0^2) f(u_0) + (u_2^3/3!)(d^3/du_0^3) f(u_0) + \cdots$$

$$+ u_1 u_2 (d^2/du_0^2) f(u_0) + (1/2)(u_1^2 u_2 + u_2^2 u_1)(d^3/du_0^3) f(u_0) + \cdots$$

$$= (e^{u_2 d/du_0} - 1) f(u_0) + (e^{u_1 d/du_0} - 1)(e^{u_2 d/du_0} - 1) f(u_0)$$

$$\vdots \tag{6}$$

Thus

$$\sum_{i=0}^{n} \hat{A}_i = f(u_0) + (u_1 + u_2 + \cdots + u_n) \, df/du_0 + \cdots$$

$$\simeq f(u_0) + (u - u_0) \, df/du_0 + \cdots;$$

i.e., the partial sum consists of the essential terms of a Taylor expansion about the function $u_0(t)$ rather than about a point, since $u_1 + u_2 + \cdots + u_n \simeq u_1 + u_2 + \cdots$. Thus, addition of the first $n + 1$ terms of the $\hat{A}_0, \ldots, \hat{A}_n$ approaches $\sum_{n=0}^{\infty} ((u - u_0)^n/n!)(d^n f/du_0^n) = e^{ud/du_0} f(u_0)$. Later terms are divided by $n!$ and are relatively insignificant. With the product terms,

products of $n!$ occur in the denominator which also can be ignored after some n. This formulation has been termed the accelerated form of the A_n polynomials [13]; we have found the equally correct but somewhat slower converging set A_n to be more convenient. They are defined by [1]

$$A_0 = f(u_0)$$
$$A_1 = u_1(d/du_0)\, f(u_0)$$
$$A_2 = u_2 d/du_0\, f(u_0) + (u_1^2/2!)(d^2/du_0^2)\, f(u_0) \qquad (7)$$
$$A_3 = u_3(d/du_0)\, f(u_0) + u_1 u_2(d^2/du_0^2)\, f(u_0) + (u_1^3/3!)\, d^3/du_0^3\, f(u_0)$$
$$\vdots$$

Alternative definitions and formulas have been discussed in [1] and elsewhere. One form for A_n is

$$A_n = (1/n!) \sum_{v=1}^{n} c(v, n)\, d^v f/du^v, \qquad (8)$$

where the second index in the coefficient is the order of the derivative and the first index progresses from 1 to n along with the order of the derivative. In the linear case $f(u) = u$, and the A_n reduce to u_n. Otherwise $A_n = A_n(u_0, u_1, ..., u_n)$ as seen in [1]. For $f(u) = u^2$, for example, $A_0 = u_0^2$, $A_1 = 2u_0 u_1$, $A_2 = u_1^2 + 2u_0 u_2$, $A_3 = 2u_1 u_2 + 2u_0 u_3$, ... It is to be noted that in this scheme, the sume of the subscripts in each term of the A_n are equal to n. The accelerated form $\hat{A}_n$ can be given as

$$\hat{A}_n = \mathscr{A}_n\, f(u_0),$$

where

$$\mathscr{A}_0 = 1$$
$$\mathscr{A}_n = \xi_n \sum_{v=0}^{n-1} \mathscr{A}_v \qquad n \geqslant 1$$
$$\xi_n = e^{u_n d/du_0} - 1.$$

Thus

$$\mathscr{A}_0 = 1$$
$$\mathscr{A}_1 = \xi_1 \mathscr{A}_0 = \xi_1$$
$$\mathscr{A}_2 = \xi_2(\mathscr{A}_0 + \mathscr{A}_1) = \xi_2 + \xi_2 \xi_1$$
$$\vdots$$

We observe that $\hat{A}_0 = A_0 = f(u_0)$. As an example, for $f(u) = e^{-u}$, one can write

$$\hat{A}_0 = f(u_0) = e^{-u_0}$$

$$\hat{A}_1 = \xi_1 e^{-u_0} = (e^{u_1 d/du_0} - 1) e^{-u_0}$$

$$\hat{A}_2 = [\xi_2(1 + \xi_1)] e^{-u_0}$$

$$\vdots$$

For a nonlinear equation in u, one may express any given function $f(u)$ in the A_n or $\hat{A}_n$ by $f(u) = \sum_{n=0}^{\infty} A_n$ or $\sum_{n=0}^{\infty} \hat{A}_n$. Thus, the simple equation

$$du/dt - u^2 = 0 \qquad u(0) = 1$$

is solved by writing

$$u = \sum_{n=0}^{\infty} u_n = u(0) + L^{-1} \sum_{n=0}^{\infty} \hat{A}_n$$

$$u_0 = u(0) = 1$$

$$u_1 = L^{-1}[1] = t$$

$$u_2 = L^{-1}\hat{A}_1 = L^{-1}(t^2 + 2t) = t^3/3 + t^2$$

$$\vdots$$

The exact answer is $u = 1/(1 - t)$; hence $u(1/2) = 2$ and the approximations $\phi_n = \sum_{i=0}^{n-1} u_i$ are given by

$$\phi_1 = 1.00$$

$$\phi_2 = 1.50$$

$$\phi_3 = 1.79$$

$$\phi_4 = 1.93$$

$$\vdots$$

The series based on the A_n converges a little more slowly, but it is evident that

$$u_0 = 1$$

$$u_1 = t$$

$$u_2 = t^2$$

$$u_3 = t^3$$

$$\vdots$$

$$u = \sum_{n=0}^{\infty} t^n = 1/(1 - t).$$

This is less obvious with the first series although one might notice that the first terms of u_2, u_3 add to give t^3, etc.

If we consider the equation $y' + y - y^2 = 0$, it can be written in the form

$$Ly = -y + \sum_{n=0}^{\infty} A_n$$

$$y = y(0) - L^{-1} \sum_{n=0}^{\infty} y_n + L^{-1} \sum_{n=0}^{\infty} A_n$$

$$y_0 = y(0)$$

$$y_1 = -L^{-1} y_0 + L^{-1} A_0 = -L^{-1} y_0 + L^{-1} y_0^2$$

$$y_2 = -L^{-1} y_1 + L^{-1} A_1 = -L^{-1} y_1 + L^{-1}(2 y_0 y_1)$$

$$\vdots$$

If $y(0) = 2$ we have $y = 2 + 2t + 3t^2 + 13t^3/3 + 75t^4/12 + \cdots$ for $t < \ln 2$. Note we can verify solutions at any n-term approximation ϕ_n. Thus for the equation $y' + y^2 = t^2 + 1$ with $y(0) = 0$, we get the two-term approximation

$$\phi_2 = y_0 + y_1 = (t + t^3/3) - (t^3/3 + 2t^5/15 + t^7/63).$$

If we substitute onto the original equation only y_0 is used in the nonlinear term, not $y_0 + y_1$ as in the y' term. In approximating y', $\sum_{n=0}^{\infty} A_n$ is limited to the A_0 term since y_1 depends only on A_0.

Let us now discuss some simple examples to clarify use of the method:

EXAMPLE.   If we consider the anharmonic oscillator described by

$$d^2\theta/dt^2 + k^2 \sin \theta = 0$$

with $k^2 = g/l$ and large amplitude motion and assuming $\theta(0) = \gamma$ and $\theta'(0) = 0$ we write

$$L\theta + N\theta = 0.$$

We obtain

$$\theta = \theta(0) - L^{-1} N\theta = \theta(0) - L^{-1} \sum_{n=0}^{\infty} A_n,$$

where $N\theta = k^2 \sin \theta$. Since for $N\theta = \sin \theta$ we have

$$A_0 = \sin \theta_0$$

$$A_1 = \theta_1 \cos \theta_0$$

$$A_2 = -(\theta_1^2/2) \sin \theta_0 + \theta_2 \cos \theta_0$$

$$A_3 = -(\theta_1^3/6) \cos \theta_0 - \theta_1 \theta_2 \sin \theta_0 + \theta_3 \cos \theta_0,$$

etc., we get

$$\theta_0 = \gamma$$

$$\theta_1 = -L^{-1} k^2 A_0$$

$$\theta_2 = -L^{-1} k^2 A_1$$

$$\vdots$$

Since L^{-1} represents a twofold definite integration from 0 to t,

$$\theta_1 = -(k^2 t^2/2!) \sin \gamma$$

$$\theta_2 = (k^4 t^4)/4!) \sin \gamma \cos \gamma$$

$$\theta_3 = -(k^6 t^6/6!)[\sin \gamma \cos^2 \gamma - 3 \sin^3 \gamma]$$

$$\vdots$$

and $\phi_n = \sum_{i=0}^{n-1} \theta_i$ is our approximation.

EXAMPLE. Mathieu's equation $\ddot{y} + f(t)y = 0$ with $f(t) = \alpha + \beta \cos \omega t$. α and β are constants. Let $L = d^2/dt^2$. Then we have

$$Ly + (\alpha + \beta \cos \omega t)y = 0$$

$$Ly = -(\alpha + \beta \cos \omega t)y$$

$$y = y(0) + ty'(0) - (\alpha + \beta \cos \omega t) \sum_{n=0}^{\infty} y_n$$

$$y_0 = y(0) + ty'(0)$$

$$y_{n+1} = -(\alpha + \beta \cos \omega t)y_n \qquad (n \geqslant 0).$$

EXAMPLE. Airy's equation $y'' - ty = 0$; $y(0) = 1$, $y'(0) = 1$. The equation is written in the form

$$Ly - Ry = 0$$

with $L = d^2/dt^2$, $R = t$, $L^{-1} = \int_0^t (\int_0^t [\cdot] \, dt)$. Then operating with L^{-1} we obtain

$$y(t) = y(0) + ty'(0) + L^{-1} Ry$$

$$y_0 = 1 + t$$

$$y_1 = L^{-1} Ry_0 = L^{-1} t(1 + t) = \frac{t^3}{2 \cdot 3} + \frac{t^4}{3 \cdot 4} = \frac{1 \cdot t^3}{3!} + \frac{2 \cdot t^4}{4!}$$

$$y_2 = L^{-1} Ry_1 = \frac{t^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{t^7}{3 \cdot 4 \cdot 6 \cdot 7} = \frac{1 \cdot 4 \cdot 6 \cdot t^6}{6!} + \frac{2 \cdot 5 \cdot t^7}{7!}$$

$$\vdots$$

$$y_n = \frac{1 \cdot 4 \cdot 7 \cdots (3n - 2) t^{3n}}{(3n)!} + \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1) t^{3n + 1}}{(3n + 1)!}$$

$$y(t) = \sum_{n=0}^{\infty} y_n \qquad \text{is Airy's function.}$$

EXAMPLE. Consider the equation $d^2u/dx^2 - kx^p u = g$ with $u(1) = u(-1) = 0$. Using the decomposition method [1], write $L = d^2/dx^2$ and $Lu = g + kx^p u$. Operating with L^{-1}, we have $L^{-1} Lu = L^{-1} g + L^{-1} kx^p u$. Then

$$u = c_1 + c_2 x + gx^2/2 + L^{-1} kx^p u.$$

Let $u = \sum_{n=0}^{\infty} u_n$ with $u_0 = c_1 + c_2 x + gx^2/2$. Then $u_{m+1} = L^{-1} kx^p u_m$ with $m \geqslant 0$. Thus $u = \sum_{m=0}^{\infty} (L^{-1} kx^p)^m u_0$ or $u = \sum_{m=0}^{\infty} (L^{-1} kx^p)^m c_1 + \sum_{m=0}^{\infty} (L^{-1} kx^p)^m c_2 x + \sum_{m=0}^{\infty} (L^{-1} kx^p)^m gx^2/2$, and finally $u = c_1 \Phi_1(x) + c_2 \Phi_2(x) + \Gamma(x)$ where

$$\Phi_1(x) = \sum_{m=0}^{\infty} k^m x^{mp + 2m}/(mp + 2m - 1)(mp + 2m)$$

$$\Phi_2(x) = \sum_{m=0}^{\infty} k^m x^{mp + 2m + 1}/(mp + 2m)(mp + 2m - 1)$$

$$\Gamma(x) = \sum_{m=0}^{\infty} (1/2) gk^m x^{mp + 2m + 2}/(mp + 2m + 1)(mp + 2m + 2).$$

Since $u(1) = u(-1) = 0$ we have $c_1 \Phi_1(1) + c_2 \Phi_2(1) + \Gamma(1) = 0$ and $c_1 \Phi_1(-1) + c_2 \Phi_2(-1) + \Gamma(-1) = 0$ or

$$\begin{pmatrix} \phi_1(1) & \phi_2(1) \\ \phi_1(-1) & \phi_2(-1) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\Gamma(1) \\ -\Gamma(-1) \end{pmatrix}$$

or $\phi C = \Gamma$ or $C = \phi^{-1}\Gamma$ with

$$\phi^{-1} = \frac{\begin{pmatrix} \phi_2(-1) & -\phi_2(1) \\ -\phi_1(-1) & \phi_1(1) \end{pmatrix}}{\phi_1(1)\,\phi_2(-1) - \phi_2(1)\,\phi_1(-1)}.$$

Thus

$$c_1 = [\phi_2(1)\Gamma(-1) - \phi_2(-1\ \Gamma(1)]/[\phi_1(1)\,\phi_2(-1) - \phi_2(1)\,\phi_1(-1)]$$

$$c_2 = [\phi_1(-1)\Gamma(1) - \phi_1(1)\ \Gamma(-1)]/[\phi_1(1)\,\phi_2(-1) - \phi_2(1)\,\phi_1(-1)]$$

and the complete solution has been determined. The same procedure solves nonlinear equations if the A_n polynomials are used for nonlinear terms.

Suppose in the above example, we let $k = 40$, $p = 1$, $g = 2$. Thus we consider the equation $d^2u/dx^2 - 40xu = 2$ with $u(-1) = u(1) = 0$. This is the one-dimensional case of the elliptic equation $\Delta^2 u = f(x, y, z) + k(x, y, z)u$ arising in problems of physics and engineering. Here $L = d^2/dx^2$ and we have $Lu = 2 + 40xu$. This is a relatively stiff case because of the large coefficient of u, and the nonzero forcing function yields an additional Airy-like function. Operating with L^{-1} yields $u = A + Bx + L^{-1}(2) + L^{-1}(40xu)$. Let $u_0 = A + Bx + L^{-1}(2) = A + Bx + x^2$ and let $u = \sum_{n=0}^{n} u_n$ with the components to be determined so that the sum is u. We identify $u_{n+1} = L^{-1}(40xu_n)$. Then all components can be determined, e.g., $u_1 = (20/3)Ax^3 + (10/3)Bx^4 + 2x^5$ and $u_2 = (80/9)Ax^6 + (200/63)Bx^7 + (10/7)x^8$. An n-term approximate $\phi_n = \sum_{i=0}^{n-1} u_i$ with $n = 12$ for $x = 0.2$ is given by -0.135649, for $x = 0.4$ is given by -0.113969, for $x = 0.6$ is given by -0.083321, for $x = 0.8$ is given by -0.050944, and for $x = 1.0$ is of course zero. These easily obtained results are correct to seven digits. We see that a better solution is obtained and much more easily than by variational methods. The solution is found just as easily for nonlinear versions without linearization.

EXAMPLE. *The Duffing and Van der Pol oscillators*: The Duffing oscillator and the Van der Pol oscillator can be considered without linearization or "smallness" assumptions. The treatment can also include randomness in coefficients or inputs without customary restrictions to special processes or perturbation theory. When we consider, for example, the equation for a simple pendulum, we usually approximate $\sin x$ by x to obtain the harmonic oscillator equation. Suppose we go a step further and write $\sin x = x - x^3/3!$, i.e., use the first two terms of the series for $\sin x$ assuming small x. We then get the equation $x'' + \omega^2 x + \varepsilon x^3 = 0$, which is the Duffing equation with ε as a "small" parameter. This is a perturbation

method, and one seeks a solution in the form $x(t) = x_0(t) + \varepsilon x_1(t) + \cdots$. We will consider such equations without smallness assumptions.

In dealing with stochastic oscillators, we depart again from usual procedures which require some sort of approximation in order to determine the second-order response statistics. A common procedure in this connection is *statistical linearization*. This procedure simply replaces the original nonlinear equation with a so-called "equivalent" linear system. Thus, if we write an oscillator equation in the form

$$x'' + \alpha x' + \omega_0 x + \beta f(x) = F(t),$$

where $x(t)$ is a displacement, α is a damping constant, ω_0 is a linear frequency, $\beta f(x)$ is a nonlinear restoring force, and $F(t)$ is a stationary process, the process of statistical linearization substitutes

$$x'' = \alpha x' + \gamma^2 x = F(t),$$

where γ^2 is determined in such a way that the mean square error due to the replacement is minimized, and the mean displacement is the same for both systems. It is customary to assume $F(t)$ is Gaussian and δ-correlated with zero-mean, or, $\langle F(t) \rangle = 0$ and $\langle F(t)F(t') \rangle = D\delta(t - t')$. This latter assumption is, of course, made for mathematical, not physical, reasons and is physically unrealistic. We propose none of these restrictions and will solve the actual nonlinear equation.

The *Duffing oscillator* is described by the equation

$$y'' + \alpha y' + \beta y + \gamma y^3 = x(t)$$

in our standard $\mathscr{F}y = x(t)$ form.

The *Van der Pol equation* is generally given as

$$y'' + \varepsilon y^2 y' - \varepsilon y' + y = x(t)$$

or by

$$y'' + \varepsilon y'(y^2 - 1) + y = x(t)$$

which we rewrite again as

$$y'' + \alpha y' + \beta y + \gamma(d/dt)y^3 = x(t)$$

since $y^2 y' = (d/dt)(y^3/3)$. Thus $\alpha = -\varepsilon$, $\beta = 1$, $\gamma = \varepsilon/3$ relates the last equation to the two previously given forms. We now have our standard form $\mathscr{F}y = \mathscr{L}y + \mathscr{N}y = x$, or $Fy = Ly + Ny = x$ if no stochasticity is involved. We will consider the equations to be deterministic here.

The linear operator in both equations is given by $d^2/dt^2 + \alpha d/dt + \beta$. The nonlinear term Ny is a simple cubic nonlinearity γy^3 in the case of the Duffing oscillator, and $\gamma(d/dt) y^3$ in the case of the Vab der Pol oscillator. These terms will, of course, be expanded in our A_n polynomials generated for the specific nonlinearity.

The treatment of the linear operator offers some alternatives. We can use the entire linear operator as L which enhances speed of convergence, but the inverse and consequent integrations become more difficult. We can also use part of the above operator which could be $L = d^2/dt^2$, $L = d^2/dt^2 + \alpha d/dt$, or $L = d^2/dt^2 + \beta$. We prefer in most cases to use $L = d^2/dt^2$, i.e., the highest order differential operator. We expect this to give the slowest convergence but much easier integrations and less actual computation time. The remainder of the linear operator will be called R, the "remainder" operator. If $L = d^2/dt^2$, $R = \alpha d/dt + \beta$. (When we consider stochasticity, we will use a script latter $\mathscr{R}$ for a random part of the operator and may have $L + R + \mathscr{R}$.)

The choice made here (that $L = d^2/dt^2$) yields the simplest Green's function for computation. In this case, L^{-1} is the twofold definite integral from 0 to t. Generally, this choice of the highest ordered derivative for L is the most desirable because the integrations are the simplest. If we invert the entire linear operator, convergence is expected to be much faster. It is interesting to examine a compromise here which can be used to advantage on occasion.

If we choose $L = d^2/dt^2 + \beta$, $R = \alpha d/dt$, we gain something in convergence rate over the previous case and expect to lose something is easy computability. The interesting aspect that suggests the compromise is that we see we will get sine and cosine functions for solutions of the homogeneous equation. For the Duffing equation we now have

$$Ly = x - Ry - \gamma y^3$$

$$y = c_1 \phi_1(t) + c_2 \phi_2(t) + L^{-1}x - L^{-1}Ry - \gamma L^{-1}y^3,$$

where ϕ_1, ϕ_2 satisfy $L\phi = 0$ or $d^2\phi/dt^2 + \beta\phi = 0$. Consequently,

$$\phi_1(t) = \cos\sqrt{\beta}\,t$$
$$\phi_2(t) = (1/\sqrt{\beta})\sin\sqrt{\beta}\,t.$$

Now

$$\sum_{n=0}^{\infty} y_n = c_1\phi_1(t) + c_2\phi_2(t) + L^{-1}x - L^{-1}Ry - \gamma L^{-1}\sum_{n=0}^{\infty} A_n$$

or

$$\sum_{n=0}^{\infty} y_n = c_1 \cos \sqrt{\beta} t + (c_2/\sqrt{\beta}) \sin \sqrt{\beta} t + L^{-1} x$$

$$- \alpha L^{-1}(d/dt) \sum_{n=0}^{\infty} y_n - \gamma L^{-1} \sum_{n=0}^{\infty} A_n,$$

where the A_n are the appropriate polynomials for $Ny = y^3$. These are given by

$A_0 = y_0^3$

$A_1 = 3 y_0^2 y_1$

$A_2 = 3 y_0 y_1^2 + 3 y_0^2 y_2$

$A_3 = y_1^3 + 6 y_0 y_1 y_2 + 3 y_0^2 y_3$

$A_4 = 3 y_1^2 y_2 + 3 y_0 y_2^2 + 6 y_0 y_1 y_3 + 3 y_0^2 y_4$

$A_5 = 3 y_1 y_2^2 + 3 y_1^2 y_3 + 6 y_0 y_2 y_3 + 6 y_0 y_1 y_4 + 3 y_0^2 y_5$

$A_6 = y_2^3 + 6 y_1 y_2 y_3 + 3 y_1^2 y_4 + 3 y_0 y_3^2 + 6 y_0 y_2 y_4 + 6 y_0 y_1 y_5 + 3 y_0^2 y_6$

$\vdots$

Since $L = d^2/dt^2 + \beta$ now, L^{-1} is no longer the simple twofold integral, and we must determine Green's function for this L. The G will satisfy the equation $LG(t, \tau) = \delta(t - \tau)$ or

$$d^2 G(t, \tau)/dt^2 + \beta G(t, \tau) = \delta(t - \tau).$$

G, of course, is determinable in a number of ways. We will again use the decomposition method itself and write

$$d^2 G/dt^2 = \delta(t - \tau) - \beta G;$$

so we again have a simple second-order operator to invert. Hence

$$G(t, \tau) = G(0, \tau) + t G_t(0, \tau) + L^{-1}\delta(t - \tau) - \beta L^{-1} \sum_{n=0}^{\infty} G_n.$$

Thus

$$\sum_{n=0}^{\infty} G_n = G(0, \tau) + L G_t(0, \tau) + L^{-1}\delta(t - \tau) - \beta L^{-1} \sum_{n=0}^{\infty} G_n$$

$$= G_0 - \beta L^{-1} \sum_{n=0}^{\infty} G_n,$$

where

$$G_0 = G(0, \tau) + G_t(0, \tau) + tH(t - \tau)$$

and

$$G_1 = -\beta L^{-1} G_0 = -\beta\{G(0, \tau) t^2/2! + G_t(0, \tau)t^3/3! + (t^3/3!) H(t - \tau)\}$$

$$= -\beta G(0, \tau)[t^2/2! + t^3/3!] - \beta(t^3/3!) H(t - \tau),$$

etc., for G_2, G_3, ... An appropriate n-term approximation can now be used in the L^{-1} integrations.

The example $m\ddot{y} + \omega^2 y + \alpha y^3 = 0$ occurs in the theory of nonlinear vibrating mechanical systems and in some nonlinear electrical systems which is the Duffing case above. Suppose we have the specified conditions $y = a$ at $t = 0$ and $y' = 0$ at $t = 0$. Write $L = d^2/dt^2$ and

$$Ly = -(\omega^2/m)y - (\alpha/m)y^3$$

$$y = y(0) - L^{-1}(\omega^2/m) \sum_{n=0}^{\infty} y_n - (\alpha/m) \sum_{n=0}^{\infty} A_n,$$

where

$$y_0 = a$$

$$y_1 = -(\omega^2/m) L^{-1} y_0 - (\alpha/m) A_0$$

$$= -a\omega^2 t^2/2m - \alpha a^3/m$$

$$\vdots$$

so that $y = a[1 - \omega^2 t^2/2m - \alpha a^2/m \cdots]$. For the Duffing equation with $L = d^2/dt^2$ we have $y = \sum_{n=0}^{\infty} y_n$ where

$$y_0 = y(0) + ty'(0) + L^{-1}x(t)$$

$$y_1 = -L^{-1}\alpha(d/dt) y_0 - L^{-1}\beta y_0 - L^{-1}\gamma A_0$$

$$y_2 = -L^{-1}\alpha(d/dt) y_1 - L^{-1}\beta y_1 - L^{-1}\gamma A_1$$

$$y_3 = -L^{-1}\alpha(d/dt) y_2 - L^{-1}\beta y_2 - L^{-1}\gamma A_2,$$

etc. For the Van der Pol equation we have

$$y_0 = y(0) + ty'(0) + L^{-1}x(t)$$

$$y_1 = -L^{-1}\alpha(d/dt) y_0 - L^{-1}\beta y_0 - L^{-1}\gamma(d/dt)A_0$$

$$y_2 = -L^{-1}\alpha(d/dt) y_1 - L^{-1}\beta y_1 - L^{-1}\gamma(d/dt)A_1,$$

etc.

Stochastic case.   We could have stochastic fluctuations in α, β, or γ in addition, of course, to stochastic $x(t)$ or initial conditions. Thus, in general we could write

$$\alpha = \langle \alpha \rangle + \varepsilon$$

$$\beta = \langle \beta \rangle + \eta$$

$$\gamma = \langle \gamma \rangle + \sigma,$$

where ε, η, σ are zero-mean random processes. The solution process can now be obtained from

$$Ly = x - \alpha(d/dt)\, y - \beta y - \gamma \sum_{n=0}^{\infty} A_n - \varepsilon(d/dt)\, y - ny - \sigma \sum_{n=0}^{\infty} A_n,$$

where the A_n summation represents y^3 in the Duffing case and $(d/dt)y^3$ in the Van der Pol case. Thus,

$$y_0 = y(0) + ty'(0) + L^{-1}x$$

$$y_1 = -\alpha(d/dt)\, y_0 - \beta y_0 - \gamma A_0 - \varepsilon(d/dt)\, y_0 - \eta y_0 - \sigma A_0$$

$$y_2 = -\alpha(d/dt)\, y_1 - \beta y_1 - \gamma A_1 - \varepsilon(d/dt)\, y_1 - \eta y_1 - \sigma A_1$$

$$\vdots$$

Then $y(t) = \sum_{n=0}^{\infty} y(t)$ yields a stochastic series from which statistics can now be obtained without problems of statistical separability of quantities such as $\langle \mathscr{R}y \rangle$ where $R = \varepsilon(d/dt) - \eta$ which normally require closure approximations and truncations [1].

Systems of differential and partial differential equations are also solvable by decomposition. In the differential equation system we use a decomposition for each independent variable u, v, ... and determine the set u_0, v_0, ..., then find the set u_1, v_1, ... in terms of the first set, etc. For a partial differential equation involving several linear operators we solve for each linear operator term in turn. Thus $\nabla^2 u$ becomes $L_x u + L_y u + L_z u$. Solving for each of these terms yields three equations. Inverting the operators, we get three equations in u. Adding and dividing by three we get a final equation for u which is solved exactly like a differential equation. For a system of partial differential equations both the above ideas are incorporated. Consider some examples, noting that we are illustrating procedures so results can be checked by other methods.

EXAMPLE.

$$u_t = x^2 - (1/4)(u_x)^2 \qquad u(x, 0) = 0.$$

Writing $L_t = \partial/\partial t$, we have

$$L_t u = x^2 - (1/4)(u_x)^2.$$

The inverse $L_t^{-1} = \int_0^t [\cdot]\, dt$, hence

$$L_t^{-1} L_t u = L_t^{-1} x^2 - (1/4)\, L_t^{-1}(u_x)^2.$$

Since the left side is $u - u(0) = u$ we have

$$\sum_{n=0}^{\infty} u_n = u_0 - (1/4)\, L_t^{-1} \sum_{n=0}^{\infty} A_n,$$

where we let $u = \sum_{n=0}^{\infty} u_n$ identify $u_0 = L_t^{-1} x^2 = x^2 t$, and replace the non-linearity $(u_x)^2$ by the A_n polynomials. The A_n polynomials for u^2 are $A_0 = u_0^2$, $A_1 = 2u_0 u_1$, $A_2 = u_1^2 + 2u_0 u_2$, ... See, e.g., [1]. Consequently

$$u_1 = -(1/4)\, L_t^{-1}(u_{0_x})^2 = -(1/4)\, L_t^{-1}(4x^2 t^2) = -x^2 t^3/3$$

$$u_2 = -(1/4)\, L_t^{-1}(2u_{0_x} u_{1_x}) = (2/15)\, x^2 t^5$$

$$u_3 = -(1/4)\, L_t^{-1}[u_{1_x}^2 + 2u_{0_x} u_{2_x}]$$

$$\vdots$$

so that

$$u = x^2(t - t^3/3 + 2t^5/15 - \cdots)$$

$$u = x^2 \tanh t \qquad |t| < \pi/2$$

which is easily verified not only for $u = x^2 \tanh t$ but for the series $\sum_{i=0}^{n-1} u_i$ for any n.

EXAMPLE.   Consider the system

$$u_t = uu_x + vu_y$$

$$v_t = uv_x + vv_y$$

$$u(x, y, 0) = x^2$$

$$v(x, y, 0) = y.$$

Writing L_t for $\partial/\partial t$, L_x for $\partial/\partial x$, L_y for $\partial/\partial y$,

$$L_t u = uL_x u + vL_y u$$

$$L_t v = uL_x x + vL_y v.$$

The inverse for L_t^{-1} is a definite integration from 0 to t. (The quantity $uL_x u = uu_x$ and $vL_u u = vu_y$.) Thus

$$u = u(x, y, 0) + L_t^{-1} uL_x u + L_t^{-1} vL_y u$$
$$v = v(x, y, 0) + L_t^{-1} uL_x v + L_t^{-1} vL_y v.$$

With $u = \sum_{n=0}^{\infty} u_n$, $v = \sum_{n=0}^{\infty} v_n$, $u_0 = u(x, y, 0) = x^2$, $v_0 = v(x, y, 0) = y$ we have

$$u_1 = L_t^{-1}(x^2) L_x(x^2) + L_t^{-1}(y) L_y(x^2) = L_t^{-1}(2x^3) = 2x^3 t$$

$$v_1 = L_t^{-1}(x^2) L_x(y) + L_t^{-1} yL_y y = yt$$

$$u_2 = L_t^{-1}[(x^2) L_x(2x^3 t) + (2x^3 t) L_x(x^2)] + L_t^{-1}[yL_y(2x^3 t) + (yt) L_y(x^2)]$$
$$= 3x^4 t^2 + 2x^4 t^2 = 5x^4 t^2$$

$$v_2 = L_t^{-1}[x^2 L_x(yt) + (2x^3 t) L_x(y)] + L_t^{-1}[yL_y(yt) + (yt) L_y(y)]$$
$$= yt^2/2 + yt^2/2 = yt^2$$
$$\vdots$$

so that

$$u = x^2(1 + 2tx + 5t^2 x^2 + \cdots)$$
$$v = y(1 + t + t^2 + \cdots) = y/(1 - t).$$

If more than one linear operator term exists, we myst solve for each, invert, and add the equations for each of the independent variables. The following example illustrates the procedure:

$$u_t + u_x = v^2 - u^2$$
$$v_t - v_x = u^2 - v^2.$$

This can be considered an evolution equation for a vector $\bar{u} = (u, v)$ defined on the domain $Q = \{(x, t): x \geqslant 0, \ t \geqslant 0\}$. We consider the initial value problem for which $\bar{u}(x, t)$ for $t > 0$ is bound. Writing $L_t = \partial/\partial t$ and $L_x = \partial/\partial x$ we have

$$L_t u + L_x u = v^2 - u^2 \tag{9}$$

$$L_t v - L_x v = u^2 - v^2. \tag{10}$$

Solve (9) for $L_t u$ and for $L_x u$ and (10) for $L_t v$ and $L_x v$. Thus

$$L_t u = -L_y u + v^2 - u^2$$
$$L_x u = -L_t u + v^2 - u^2$$
$$L_t v = L_x v + u^2 - v^2$$
$$L_x v = L_t v + u^2 - v^2. \tag{11}$$

With the inversions

$$u = u(x, 0) - L_t^{-1} L_x u + L_t^{-1} v^2 - L_t^{-1} u^2$$
$$u = u(0, t) - L_x^{-1} L_t u + L_x^{-1} v^2 - L_x^{-1} u^2$$
$$v = v(x, 0) + L_t^{-1} L_x v + L_t^{-1} u^2 - L_t^{-1} v^2$$
$$v = v(0, t) + L_x^{-1} L_t v + L_x^{-1} u^2 - L_x^{-1} v^2. \tag{12}$$

Add the equations for u and divide by two and add the equations for v and divide by two. Then

$$u = (1/2)\{u(x, 0) + u(0, t)\} - (1/2)\{L_t^{-1} L_x + L_x^{-1} L_t\} u$$
$$+ (1/2)\{L_t^{-1} + L_x^{-1}\} v^2 - (1/2)\{L_t^{-1} + L_x^{-1}\} u^2$$
$$v = (1/2)\{v(x, 0) + v(0, t)\} + (1/2)\{L_t^{-1} L_x + L_x^{-1} L_t\} v$$
$$+ (1/2)\{L_t^{-1} + L_x^{-1}\} u^2 - (1/2)\{L_t^{-1} + L_x^{-1}\} v^2.$$

Let $u_0 = (1/2)\{u(x, 0) + u(0, t)\}$ and $v_0 = ((1/2))\{x, 0) + v(0, t)\}$. Let $u = \sum_{n=0}^{\infty} u_n$ and $v = \sum_{n=0}^{\infty} v_n$. Replace u^2 by $\sum_{n=0}^{\infty} A_n(u^2)$ and v^2 by $\sum_{n=0}^{\infty} A_n(v^2)$ (the notation $A_n(u^2)$ means A_n for u^2). We have now

$$u = u_0 - (1/2)\{L_t^{-1} L_x + L_x^{-1} L_t\} \sum_{n=0}^{\infty} u_n$$

$$+ (1/2)\{L_t^{-1} + L_x^{-1}\} \sum_{n=0}^{\infty} A_n(v^2)$$

$$- (1/2)\{L_t^{-1} + L_x^{-1}\} \sum_{n=0}^{\infty} A_n(u^2)$$

$$v = v_0 + (1/2)\{L_t^{-1} L_x + L_x^{-1} L_t\} \sum_{n=0}^{\infty} v_n$$

$$+ (1/2)\{L_t^{-1} + L_x^{-1}\} \sum_{n=0}^{\infty} A_n(u^2)$$

$$- (1/2)\{L_t^{-1} + L_x^{-1}\} \sum_{n=0}^{\infty} A_n(v^2).$$

Now with u_0, v_0 specified, all other components are determined from

$$u_{n+1} = -(1/2)\{L_t^{-1}L_x + L_x^{-1}L_t\}u_n + (1/2)\{L_t^{-1} + L_x^{-1}\} A_n(v^2)$$
$$- (1/2)\{L_t^{-1} + L_x^{-1}\} A_n(u^2)$$
$$v_{n+1} = (1/2)\{L_t^{-1}L_x + L_x^{-1}L_t\}v_n + (1/2)\{L_t^{-1} + L_x^{-1}\} A_n(u^2)$$
$$- (1/2)\{L_t^{-1} + L_x^{-1}\} A_n(v^2)$$

for $n \geqslant 0$. Given u_0, v_0 and the A_n for u^2 and v^2 or

$$A_0(u^2) = u_0^2 \qquad\qquad A_0(v^2) = v_0^2$$
$$A_1(u^2) = 2u_0 u_1 \qquad\qquad A_1(v^2) = 2v_0 v_1$$
$$A_2(u^2) = u_1^2 + 2u_0 u_2 \qquad A_2(v^2) = v_1^2 + 2v_0 v_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$

u_1 and v_1 can be determined in terms of u_0, v_0. Similarly u_{n+1}, v_{n+1} are determined in terms of the preceding components for $n \geqslant 0$ so that $u = \sum_{n=0}^{\infty} u_n$ and $v = \sum_{n=0}^{\infty} v_n$ are determined.

The method has now been applied in wide classes of nonlinear equations by a number of researchers as seen in the references, however, general statements about errors and convergence rates cannot be made at this time although when numerical results are obtained, one sees rapid stabilization to any desired accuracy. To show accuracy of the method, computer checks have now been made on a wide range of examples of elliptic, parabolic, and hyperbolic equations.

EXAMPLE.   Consider the solution $\phi(x, t)$ of the nonlinear partial differential equation previously considered by Oguztöreli, Shuhubi, and Leung using numerical methods

$$\partial\phi/\partial t + a\phi^m \, \partial\phi/\partial x = b \, \partial^n\phi/\partial x^n \qquad (t > 0, x > 0),$$

where $m = 1, 2$ and $n = 2, 3, 4, \ldots$ subject to the initial condition

$$\phi(x, 0) = f(x) \qquad (x \geqslant 0)$$

and boundary conditions

$$\phi(0, t) = g(t)$$
$$\lim_{x \to \infty} \phi(x, t) = 0 \qquad (t \geqslant 0),$$

where a, b are real constants and $f(x)$ and $g(t)$ are given sufficiently smooth functions of x and t, respectively, satisfying the condition $f(0) = g(0)$.

Our objective is an analytic solution which is obtained in a rapidly convergent series form without linearization using the decomposition method [2–5]. Questions of existence and uniqueness are left to discussion elsewhere. Write the given equation in the form

$$L_t\phi + aN\phi = bL_x\phi,$$

where $L_t = \partial/\partial t$, $L_x = \partial^n/\partial x^n$, $N\phi = \phi^m\phi_x$, and $m = 1, 2$ and $n = 2, 3, 4, \ldots$. Solving for the linear terms as discussed in [1, 13, 14], we get two equations

$$L_x\phi = b^{-1}L_t\phi + ab^{-1}N\phi \tag{13}$$

$$L_t\phi = bL_x\phi - aN\phi. \tag{14}$$

The inverse operator L_t^{-1} can conveniently be taken as the definite integral from 0 to t. The inverse L_x^{-1} is taken as an indefinite (see [1, 2]) n-fold integration. (Thus $L_x^{-1}L_x\phi = \phi + x\alpha(t) + \beta(t)$ if L_x is second order, and is changed suitably if L_x is of different order; $L_t^{-1}L_t\phi = \phi - \phi(x, 0)$ for first-order L_t as shown. If L_t were second order, we would have $\phi - \phi(x, 0) - t\phi'(x, 0)$ for initial conditions specified.) To consider the possibilities simultaneously, we write $L_x^{-1}L_x\phi = \phi - \Phi$, where $\Phi = x\alpha(t) + \beta(t)$ in the second-order case, for example, and can be considered known from the specified boundary conditions. (Thus for $\Phi = x\alpha(t) + \beta(t)$, using the boundary conditions, we see that $\beta = g(t)$, $\alpha = 0$.) Thus, applying the inverse operators L_x^{-1} to Eq. (13) and L_t^{-1} to Eq. (14) and solving for ϕ,

$$\phi = g(t) + b^{-1}L_x^{-1}L_t\phi + ab^{-1}L_x^{-1}N\phi$$

$$\phi = f(x) + bL_t^{-1}L_x\phi - aL_t^{-1}N\phi.$$

Adding and dividing by two, and defining

$$\phi_0 = (1/2)\{f(x) + g(t)\}$$

we have

$$\phi = \phi_0 + (1/2)\{bL_t^{-1}L_x + b^{-1}L_x^{-1}L_t\}\phi + (1/2)\{ab^{-1}L_x^{-1} - aL_t^{-1}\}N\phi.$$

Let $\phi = \sum_{n=0}^{\infty} \phi_n$ and $N\phi = \sum_{n=0}^{\infty} A_n$ where the A_n are Adomian's A_n polynomials generated for $\phi^m\phi_x$. The generating schemes are discussed in [1, 12, 13]. For this case, they can be written

$$A_0 = \phi_0^m(\partial/\partial x)\phi_0$$

$$A_1 = \phi_0^m(\partial/\partial x)\phi_1 + \phi_1^m(\partial/\partial x)\phi_0$$

$$A_2 = \phi_0^m(\partial/\partial x)\phi_2 + \phi_1^m(\partial/\partial x)\phi_1 + \phi_2^m(\partial/\partial x)\phi_0.$$

Thus we have

$$\phi = \phi_0 + (1/2)\{bL_t^{-1}L_x + b^{-1}L_x^{-1}L_t\} \sum_{n=0}^{\infty} \phi_n$$

$$+ (1/2)\{ab^{-1}L_x^{-1} - aL_t^{-1}\} \sum_{n=0}^{\infty} A_n$$

which leads to

$$\phi_1 = (1/2)\{bL_t^{-1}L_x + b^{-1}L_x^{-1}L_t\}\phi_0 + (1/2)\{ab^{-1}L_x^{-1} - aL_t^{-1}\} A_0$$
$$\phi_2 = (1/2)\{bL_t^{-1}L_x + b^{-1}L_x^{-1}L_t\}\phi_1 + (1/2)\{ab^{-1}L_x^{-1} - aL_t^{-1}\} A_1$$
$$\vdots$$

Thus for $n \geq \phi$

$$\phi_{n+1} = (1/2)\{bL_t^{-1}L_x + b^{-1}L_x^{-1}L_t\}\phi_n + (1/2)\{ab^{-1}L_x^{-1} - aL_t^{-1}\} A_n$$

allows us to determine all components. The sum $\sum_{n=0}^{\infty} u_n$ is the complete solution and the n-term approximation $\sum_{i=0}^{n-1} u_i$ is the practical solution converging for reasonable n as discussed elsewhere [1]. Thus the necessary components are easily calculated once the initial/boundary conditions have been specified.

A straightforward extension to three space dimensions is easily carried out. Let us consider the case with $m = 1$, $n = 2$, and for convenience, $a = b = 1$, and write

$$\phi_t + a\phi\phi_x = b\nabla^2\phi$$

or

$$[L_t - L_x - L_y - L_z]\phi = \phi\phi_x,$$

where $L_t = \partial/\partial t$, $L_x = \partial^2/\partial x^2$, $L_y = \partial^2/\partial y^2$, $L_z = \partial^2/\partial z^2$. Solve for each linear operator term in turn. Then

$$L_t\phi = L_x\phi + L_y\phi + L_z\phi + \phi\phi_x$$
$$L_x\phi = L_t\phi - L_y\phi - L_z\phi - \phi\phi_x$$
$$L_y\phi = L_t\phi - L_z\phi - L_x\phi - \phi\phi_x$$
$$L_z\phi = L_t\phi - L_x\phi - L_y\phi - \phi\phi_x.$$

Operating on each equation by the appropriate inverse, we have

$$\phi = \alpha_1 + L_t^{-1}(L_x + L_y + L_z)\phi + L_t^{-1}\phi\phi_x$$

$$\phi = (\alpha_2 + \alpha_3 x) + L_x^{-1}L_t\phi - L_t^{-1}L_y\phi - L_x^{-1}L_z\phi - L_x^{-1}\phi\phi_x$$

$$\phi = (\alpha_4 + \alpha_5 y) + L_y^{-1}L_t\phi - L_y^{-1}L_z\phi - L_y^{-1}L_x\phi - L_y^{-1}\phi\phi_x$$

$$\phi = (\alpha_6 + \alpha_7 z) + L_z^{-1}L_t\phi - L_z^{-1}L_x\phi - L_z^{-1}L_y\phi - L_z^{-1}\phi\phi_x.$$

Adding an dividing by four

$$\begin{aligned}
\phi = {} & (1/4)\{\alpha_1 + (\alpha_2 + \alpha_3 x) + (\alpha_4 + \alpha_5 y) + (\alpha_6 + \alpha_7 z)\} \\
& + (1/4)\{L_t^{-1}(L_x + L_y + L_z) + L_x^{-1}(L_t - L_y - L_z) \\
& + L_y^{-1}(L_t - L_z - L_x) + L_z^{-1}(L_t - L_x - L_y)\}\phi \\
& + (1/4)\{L_t^{-1} - L_x^{-1} - L_y^{-1} - L_z^{-1}\}\phi\phi_x,
\end{aligned}$$

where the α's have been individually evaluated and are therefore known in terms of initial/boundary conditions.
  Define

$$\phi_0 = (1/4)\{\alpha_1 + (\alpha_2 + \alpha_3 x) + (\alpha_4 + \alpha_5 y) + (\alpha_6 + \alpha_7 z)\}.$$

Let $\phi = \sum_{n=0}^{\infty} \phi_n$ and let the nonlinear term $\phi\phi_x$ be expressed in the A_n polynomials; thus $\phi\phi_x = \sum_{n=0}^{\infty} A_n$ where

$$A_0 = \phi_0(d/dx)\phi_0$$

$$A_1 = \phi_0(d/dx)\phi_1 + \phi_1(d/dx)\phi_0$$

$$A_2 = \phi_0(d/dx)\phi_2 + \phi_1(d/dx)\phi_1 + \phi_2(d/dx)\phi_0$$

$$\vdots$$

Now we also define

$$\begin{aligned}
K = {} & (1/4)\{L_t^{-1}(L_x + L_y + L_z) + L_x^{-1}(L_t - L_y - L_z) \\
& + L_y^{-1}(L_t - L_z - L_x) + L_z^{-1}(L_t - L_x - L_y)\} \\
G = {} & (1/4)(L_t^{-1} - L_x^{-1} - L_y^{-1} - L_z^{-1})
\end{aligned}$$

so that for $n \geq 0$

$$\phi_{n+1} = K\phi_n + GA_n$$

solves the equation for all components of ϕ desired to some approximation. Other coefficients a, b offer no problem, and other forms of nonlinear terms

mean only an appropriate change of the A_n polynomials. Thus, many of the equations of physics appear to be solvable anylytically without linearization or perturbation or discretization.

EXAMPLE. Consider the equation $d^2u/dx^2 - kxu = g$. The solution by decomposition is

$$u = c_1 \left[1 + \sum_{m=1}^{\infty} k^m x^{3m} \Big/ \prod_{\mu=1}^{m} (3\mu - 1)(3\mu) \right]$$

$$+ c_2 x \left[1 + \sum_{m=1}^{\infty} k^m x^{3m} \Big/ \prod_{\mu=1}^{m} (3\mu)(3\mu + 1) \right]$$

$$+ gx^2 \sum_{m=0}^{\infty} k^m x^{3m} \Big/ \prod_{\mu=0}^{m} (3\mu + 1)(3\mu + 2),$$

where c_1, c_2 are evaluated from given conditions. We arrive at this result as follows:

$$Lu = g + kxu$$

$$L^{-1}Lu = L^{-1}g + L^{-1}kxu$$

$$L^{-1}Lu = u - c_1 - c_2 x$$

$$u = c_1 + c_2 x + gx^2/2 + kL^{-1}xu.$$

Let $u = \sum_{n=0}^{\infty} u_n$ and define $u_0 = c_1 + c_2 x + gx^2/2$. Then

$$u = u_0 + L^{-1}kx \sum_{n=0}^{\infty} u_n$$

$$u_{n+1} = L^{-1}kxu_n \qquad n \geq 0$$

$$u_1 = L^{-1}kxu_0$$

$$u_1 = c_1 kx^3/2 \cdot 3 + c_2 kx^4/3 \cdot 4 + (g/2) kx^5/4 \cdot 5$$

$$u_2 = L^{-1}kxu_1$$

$$= L^{-1} [c_1 k^2 x^4/2 \cdot 3 + c_2 k^2 x^5/3 \cdot 4 + (g/2) kx^6/4 \cdot 5]$$

$$u_2 = c_1 k^2 x^6/2 \cdot 3 \cdot 5 \cdot 6 + c_2 k^2 x^7/3 \cdot 4 \cdot 6 \cdot 7 + (g/2) k^2 x^8/4 \cdot 5 \cdot 7 \cdot 8.$$

Continuing in the same manner

$$u_3 = c_1 k^3 x^9/2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 + c_2 k^3 x^{10}/3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10$$

$$+ (g/2) k^3 x^{11}/4 \cdot 5 \cdot 7 \cdot 8 \cdot 10 \cdot 11$$

$$\vdots$$

Writing $u = u_0 + \sum_{m=1}^{\infty} u_m$,

$$u = c_1 + c_2 x + (g/2)x^2$$

$$+ \sum_{m=1}^{\infty} \left\{ c_1 k^m x^{3m} \middle/ \prod_{\mu=1}^{m} (3\mu - 1)(3\mu) + c_2 k^m x^{3m+1} \middle/ \prod_{\mu=1}^{m} (3\mu)(3\mu + 1) \right.$$

$$\left. + (g/2) k^m x^{3m+2} \middle/ \prod_{\mu=1}^{m} (3\mu + 1)(3\mu + 2) \right\}.$$

If we include the u_0 term in the summation, we get the result given in (13).

Using our usual [1] notation for the n-term approximation $\phi_n = \sum_{i=0}^{n-1} u_i$, we can write it in the form

$$\phi_m = c_1 \xi_m + c_2 \eta_m + (g/2)\zeta_m,$$

where

$$\xi_m = \sum_{\mu=0}^{m-1} \left\{ k^\mu x^{3\mu} \middle/ \prod_{v=0}^{\mu} (3v - 1)(3v) \right\}$$

$$\eta_m = \sum_{\mu=0}^{m-1} \left\{ k^\mu x^{3\mu+1} \middle/ \prod_{v=0}^{\mu} (3v)(3v + 1) \right\}$$

$$\zeta_m = \sum_{\mu=0}^{m-1} \left\{ k^\mu x^{3\mu+2} \middle/ \prod_{v=0}^{\mu} (3v + 1)(3v + 2) \right\}.$$

Next we will evaluate the constants from the (approximate) boundary conditions. $u(1) = 0$ implies that

$$\phi_m(1) = c_1 \xi_m(1) + c_2 \eta_m(1) + (g/2)\zeta_m(1) = 0$$

and $u(-1) = 0$ implies that

$$\phi_m(-1) = c_1 \xi_m(-1) + c_2 \eta_m(-1) + (g/2)\zeta_m(-1) = 0$$

which gives us the matrix equation

$$\begin{pmatrix} \xi_m(1) & \eta_m(1) \\ \xi_m(-1) & \eta_m(-1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -(g/2)\zeta_m(1) \\ -(g/2)\zeta_m(-1) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^{-1} \begin{pmatrix} -(g/2)\zeta_m(1) \\ -(g/2)\zeta_m(-1) \end{pmatrix},$$

where

$$
A^{-1} = \left[
\begin{array}{c|c}
\dfrac{\eta_m(-1)}{\xi_m(1)\,\eta_m(-1) - \eta_m(1)\,\xi_m(-1)} & \dfrac{-\eta_m(1)}{\xi_m(1)\,\eta_m(-1) - \eta_m(1)\,\xi_m(-1)} \\[4mm]
\dfrac{-\xi_m(-1)}{\xi_m(1)\,\eta_m(-1) - \eta_m(1)\,\xi_m(-1)} & \dfrac{\xi_m(1)}{\xi_m(1)\,\eta_m(-1) - \eta_m(1)\,\xi_m(-1)}
\end{array}
\right].
$$

Thus

$$
c_1 = \frac{\eta_m(1)(g/2)\,\zeta_m(-1) - \eta_m(-1)(g/2)\,\zeta_m(1)}{\xi_m(1)\,\eta_m(-1) - \eta_m(1)\,\xi_m(-1)}
$$

$$
c_2 = \frac{\xi_m(-1)(g/2)\,\zeta_m(1) - \xi_m(1)(g/2)\,\zeta_m(-1)}{\xi_m(1)\,\eta_m(-1) - \eta_m(1)\,\xi_m(-1)}
$$

so the explicit solution satisfying the given conditions is obtained.

It makes no real difference whether we have an initial value or a boundary value problem in the one-dimensional case. Consider a simple example: $d^2u/dx^2 + \alpha u = g$ now letting α, g be constants purely for convenience in evaluation. The same procedure gives us $u = c_1 + c_2 x + L^{-1}g - L^{-1}\alpha u$ or

$$
u_0 = c_1 + c_2 x + L^{-1}g = c_1 + c_2 x + gx^2/2
$$

$$
u_{n+1} = -L^{-1}\alpha u_n
$$

for $n \geqslant 0$;

$$
u = c_1 + c_2 x + gx^2/2 - c_1\alpha x^2/2 - c_2\alpha x^3/3! - g\alpha x^4/4! + \cdots
$$

or

$$
u = c_1 \sum_{n=0}^{\infty} (-1)^n \alpha^n x^{2n}/(2n)! + c_2 \sum_{n=0}^{\infty} (-1^n \alpha^n x^{2n+1}/(2n+1)!
$$

$$
+ g \sum_{n=0}^{\infty} (-1)^n a^n x^{2n+2}/(2n+2)!
$$

whose analytic sum is

$$
u = c_1 \cos\sqrt{a}x + (c_2/\sqrt{\alpha})\sin\sqrt{\alpha}x + (g/\alpha)[1 - \cos\sqrt{\alpha}x].
$$

If this is an initial-value problem, we trivially identify $c_1 = u(0)$ and $c_2 = u'(0)$. If it is a boundary-value problem, we will have, in general, conditions such as

$$
u'(x_1) + \beta_1 u(x_1) = \gamma_1
$$

$$
u'(x_2) + \beta_2 u(x_2) = \gamma_2.
$$

Since normally we will not have the sum of the series but only the n-term approximate $\phi_n = \sum_{i=0}^{n-1} u_i$ we will use ϕ_n to satisfy the conditions. Although formal estimates are not available, except as discussed in [14], convergence is seen to be very rapid and the results are seen to be accurate for very low n as we can observe in the following case.

A final example [14] using strongly coupled boundary conditions is the following (and it is to be emphasized that it works equally well if the conditions or the equations are nonlinear or stochastic):

$$d^2u/dt^2 + v = 0$$

$$d^2v/dt^2 + u = 0$$

on the interval $[\pi/4, \pi/2]$ supposing we are given the boundary conditions:

$$B_1(u, v) = \frac{d}{dt} u(\pi/4) + \tfrac{1}{2}u(\pi/4) + 2v(\pi/4) = 3$$

$$B_2(u, v) = \frac{d}{dt} u(\pi/2) + \tfrac{1}{3}u(\pi/2) + 4v(\pi/2) = 5$$

$$B_3(u, v) = \frac{d}{dt} v(\pi/4) + \tfrac{1}{4}v(\pi/4) + 8u(\pi/4) = 7$$

$$B_4(u, v) = \frac{d}{dt} v(\pi/2) + \tfrac{1}{5}v(\pi/2) + 16u(\pi/2) = 11.$$

If $L = d^2/dt^2$, we have $Lu = -v$ and $Lv = -u$. Then

$$u = c_1 + c_2 t - L^{-1} v$$

$$v = k_1 + k_2 t - L^{-1} u$$

so that

$$u_0 = c_1 + c_2 t$$

$$v_0 = k_1 + k_2 t$$

$$u_1 = -L^{-1}[k_1 + k_2 t] = -k_1 t^2/2 - k_2 t^3/3!$$

$$v_1 = -L^{-1}[c_1 + c_2 t] = -c_1 t^2/2 - c_2 t^3/3!.$$

Continuing in this way we can write the n-term approximations $\phi_n(t)$ and $\theta_n(t)$. Using only three terms, i.e., ϕ_3 and θ_3 substituted under the given conditions, we find $c_1 = 0.20300$, $c_2 = 1.00769$, $k_1 = 0.49390$, and $k_2 = 0.98480$. Verification by substitution yields three decimal place accuracy.

Often accurate solutions can be obtained with very few terms. Consider, for example,

$$y' = y - 2y^2 \qquad y(0) = 0.$$

The decomposition solution is

$$y = 1 - t + 3t^2/2 - 13t^3/6 + \cdots.$$

Taking only a two-term approximation $\phi_2 = y_0 + y_1$ we have $\phi_2 = 0.99$. The exact solution is $y = 1/(2 - e^t) = 0.99$ to two places.

The solutions are verifiable by substitution, as seen in the following example:

The equation $y' + y^2 = x(t)$ where $x(t) = t^2 + 1$ and $y(0) = 0$ gives us

$$\phi_2 = (t^3/3 + t) - (t^7/63 + 2t^5/15 + t^3/3)$$

again taking only two terms. Using $y \simeq \phi_2$,

$$y' \simeq t^2 + 1 - t^6/9 - 2t^4/3 - t^2.$$

Thus, $y' + y^2 = t^2 + 1$, verifying our solution. Note that in substituting for y^2, we use only y_0, not $y_0 + y_1$. Approximating the derivative to y_1 requires using only A_0 since y_1 depends only on A_0. Thus

$$\sum_{i=0}^{n} y_i' + \sum_{i=0}^{n-1} A_i = x.$$

And as $n \to \infty$, we have the original equation.

Let us consider the behavior of the decomposition components in the previous equation with $Ny = y^2$. We have

$$y_1 = -L^{-1} y_0^2 = -\int_0^t y_0^2 \, d\tau.$$

Suppose the initial condition is zero, for convenience,

$$y_1 = -\int_0^t |L^{-1} x L^{-1} x| \, dx = M^2 t^3/3!,$$

where M is the bound for $x(t)$ in interval $[0, T]$

$$y_2 = -L^{-1}(2y_0 y_1) = -2L^{-1} L^{-1} x L^{-1} L^{-1} x L^{-1} x$$
$$|y_2| = M^3 t^5/5!,$$

etc.; i.e., the $|y_{n+1}| \to 0$. If we write

$$\phi_n = y_0 - L^{-1} \sum_{i=0}^{n-1} A_i$$

$$\phi_{n+1} = y_0 - L^{-1} \sum_{i=0}^{n} A_i = y_0 - L^{-1} \sum_{i=0}^{n-1} A_i - L^{-1} A_n$$

we have

$$|\phi_{n+1} - \phi_n| = -|L^{-1} A_n| = -|y_{n+1}| \to 0.$$

Convergence has now been demonstrated in in a number of ways. Numerical results in many problems show a high degree of accuracy for rather low n, and sometimes the series can be summed.

In [1], error estimates are provided to show the effect of calculating more terms. However, when numerical results are desired, one sees the approach to a stable solution for the desired number of decimal places.

It is easy to see the effect of linearization in a specific problem by linearizing Ny. For example, if $Ny = e^y$, we can use $e^y \simeq 1 + y$, calculate the result, then substitute $Ny = \sum_{n=0}^{\infty} A_n$ where the A_n are determined for e^y. Nonlinear equations arise is every area of application, and the correct solution of dynamical systems modeled by nonlinear ordinary differential equations, systems of differential equations, partial differential equations, and systems of partial differential equations is vital to progress in many fields. In order to make these equations tractable, it is quite common to linearize equations or assume "weak" nonlinearity, etc., because adequate methods simply have not been otherwise available. It is known, of course, that the linearized solution can deviate considerably from the actual solution of the nonlinear problem and that linearization procedures require proof that the solution is valid. For example, writing $x'' = a \sin x$ in the form $x'' = ax$ requires a priori proof that x is sufficiently small. The decomposition method has substantially improved our ability to solve a wide class of nonlinear and/or stochastic equations. For example, it is now possible to obtain very accurate and verifiable solutions of nonlinear, or even nonlinear stochastic equations for all of the above types even if nonlinear, stochastic, or coupled boundary conditions are involved. Usage of linearization has become rather standardized; however, solution of the actual nonlinear form is clearly preferable to a linear approximation. Since the linearized problem is a different problem, the usage of linearization requires justification that it is adequate in a particular problem to change it in this way. The practice of approximating a nonlinear function with a linearized version arose from the need to make equations tractable by simple analysis, since numerical solutions from computers have drawbacks

and methods of analytical solution of nonlinear equations are generally inadequate. The belief that linearization and perturbation are essential procedures, the reluctance to give up the convenient analytical tool of superposition, and the faith that faster computers will solve everything are factors in preserving the status quo.

Exact linearization is possible for a nonlinear equation—as in the case of Burger's equation—so that a convenient check can be made of the decomposition solution. As an example, consider the nonlinear equation for $\phi(x, t)$

$$\phi_t + \phi_x + a\phi + \phi^2 = 0$$

with specified conditions. The transformation $\phi = 1/\psi(x, t)$ leads to the linear equation

$$\psi_t + \psi_x - \alpha\psi = 1$$

and conditions specified on ψ; however, the nonlinear equation can be solved directly as follows. Using decomposition we write

$$L_t\phi + L_x\phi + a\phi + \phi^2 = 0.$$

Let $N\phi = a\phi^2 = \sum_{n=0}^{\infty} A_n$ where

$$A_0 = \phi_0^2$$
$$A_1 = 2\phi_0\phi_1$$
$$A_2 = \phi_1^2 + 2\phi_0\phi_2$$
$$A_3 = 2\phi_1\phi_2 + 2\phi_0\phi_3.$$

Solving for $L_t\phi$ and for $L_x\phi$

$$L_t\phi = -L_x\phi - a\phi - \phi^2$$
$$L_x\phi = -L_t\phi - a\phi - \phi^2.$$

Thus

$$\phi = \phi(x, 0) - L_t^{-1}L_x\phi - aL_t^{-1}\phi - L_t^{-1}\phi^2$$
$$\phi = \phi(0, t) - L_x^{-1}L_t\phi - aL_x^{-1}\phi - L_x^{-1}\phi^2.$$

Adding

$$\phi = (1/2)\{\phi(x, 0) + \phi(0, t) - (L_t^{-1}L_x + L_x^{-1}L_t)\phi$$
$$- a(L_t^{-1} + L_x^{-1})\phi + (L_t^{-1} + L_x^{-1})\phi^2\}.$$

We define

$$\phi_0 = (1/2)\{\phi(x, 0) + \phi(0, t)\}$$

and substitute $\phi = \sum_{n=0}^{\infty} \phi_n$ and $\phi^2 = \sum_{n=0}^{\infty} A_n$ to obtain

$$\phi_{n+1} = -(1/2)\{L_t^{-1}L_x + L_x^{-1}L_t\}\phi_n$$
$$- (1/2) a\{L_t^{-1} + L_x^{-1}\} \phi_n - (1/2)\{L_t^{-1} + L_x^{-1}\} A_n$$

for $n \geqslant 0$ so that all components are determined.

In the analysis of systems it is common to suppose that everything is linear or sufficiently close to linear so that linearized analyses will be adequate. Thus in a mechanical system assumed to be linear, displacements and accelerations are proportional to forces. If the system is electrical, currents are proportional to voltages, etc. Thus we write output $y(t)$ as proportional to input $x(t)$ or $y = kx$ where k is a constant independent of t or x. Now suppose that the system deviates only slightly from linearity by adding a small term which is nonlinear; e.g., $y = kx + \varepsilon x^2$. If we now consider the input $x = A \cos \omega_1 t + B \cos \omega_2 t$, we get not only the linear term

$$k[A \cos \omega_1 t + B \cos \omega_2 t]$$

but also

$$k\varepsilon[A^2 \cos^2 \omega_1 t + B^2 \cos^2 \omega_2 t + 2AB \cos \omega_1 t \cos \omega_2 t].$$

The first two of these produce constant terms and second harmonic terms as before. Also sum and difference frequencies arise from the cross product term

$$2AB \cos \omega_1 t \cos \omega_2 t = AB[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t].$$

Thus, a nonlinear system produces new effects not present in linear systems: these effects are proportional to ε (and to products of amplitudes A^2, B^2, or AB). Clearly then, if ε is not small, such effects become important.

Solutions are generally carried out only under the assumption that ε is small so that perturbation theory will be applicable, i.e., when we consider a "slightly nonlinear" or a "weakly nonlinear" system.

Since, in general, solutions of nonlinear equations are made by linearizing the equations, it is natural to ask what the effect of linearization is on the actual solutions. Let us consider the equation

$$Ly + Ry + Ny = x(t),$$

where L is the invertible linear operator, R is the remaining linear operator, and Ny is the nonlinear term. We have

$$Ly = x - Ry - Ny$$

$$y = \phi + L^{-1}x - L^{-1}Ry - L^{-1}Ny,$$

where $L\phi = 0$. We assume the solution decomposition $y = \sum_{n=0}^{\infty} y_n$ with $y_0 = \phi + L^{-1}x$ and also the decomposition of the nonlinear term Ny into $\sum_{n=0}^{\infty} A_n$ where the A_n are generated for the specific Ny. Then the components after y_0 are determinable in terms of y_0 as

$$y_1 = -L^{-1}Ry_0 - L^{-1}A_0(y_0)$$

$$y_2 = -L^{-1}Ry_1 - L^{-1}A_1(y_0, y_1)$$

$$y_3 = -L^{-1}Ry_2 - L^{-1}A_2(y_0, y_1, y_2)$$

$$\vdots$$

$$y_n = -L^{-1}Ry_{n-1} - L^{-1}A_{n-1}(y_0, ..., y_{n-1})$$

or, equivalently,

$$y_1 = (-L^{-1}R)y_0 - L^{-1}A_0$$

$$y_2 = (-L^{-1}R)^2y_0 - (-L^{-1}R)L^{-1}A_0 - L^{-1}A_1$$

$$y_3 = (-L^{-1}R)^3y_0 - (-L^{-1}R)^2L^{-1}A_0 - (-L^{-1}R)L^{-1}A_1 - L^{-1}A_2$$

$$\vdots$$

$$y_n = (-L^{-1}R)^n y_0 - \sum_{v=0}^{n-1} (-L^{-1}R)^{n-1-v} L^{-1}A_v$$

for $n \geq 1$. The solution is $y = \sum_{n=0}^{\infty} y_n$ or

$$y = \sum_{n=0}^{\infty} (-L^{-1}R)^n y_0 - \sum_{n=1}^{\infty} \sum_{v=0}^{n-1} (-L^{-1}R)^{n-1-v} - L^{-1}A_v.$$

It has been shown by Adomian that $A_n(y_0, y_1, ..., y_n)$ reduces to y_n if $Ny = f(y) = y$. Then the solution is

$$y = \sum_{n=0}^{\infty} (-L^{-1}R)^n y_0 - \sum_{n=1}^{\infty} \sum_{v=0}^{n-1} (-L^{-1}R)^{n-1-v}L^{-1}y_v,$$

i.e., the solution corresponds now to the equation $Ly + Ry + y = x$ which yields

$$y = y_0 - L^{-1}Ry - L^{-1}y$$

with

$$y_1 = -L^{-1} R y_0 - L^{-1} y_0$$

$$y_2 = -L^{-1} R y_1 - L^{-1} y_0$$

$$= (-L^{-1} R)^2 y_0 - (-L^{-1} R) L^{-1} y_0 - L^{-1} y_0$$

$$y_3 = -L^{-1} R y_2 - L^{-1} y_1$$

$$= (-L^{-1} R)^3 y_0 - (-L^{-1} R)^2 L^{-1} y_0 - (-L^{-1} R) L^{-1} y_0 - L^{-1} y_1$$

$$\vdots$$

The result on the solution of replacing $Ny = f(y)$ by y is seen by plotting the nonlinear and the linearized results. Similarly replacing $f(y)$ by another perhaps more sophisticated linearization is seen by simply calculating the A_n for the linearized function replacing $f(y)$. For a general linear stochastic system it has been shown that the decomposition solution reduces to the results of perturbation theory in cases where perturbation theory is applicable; however, the solution is not restricted to "small" fluctuations as in the perturbation result. This is also true of nonlinear stochastic or nonlinear deterministic systems; the methods proposed involve no "small fluctuation" or "small nonlinearity" assumptions.

EXAMPLE. *Exponential nonlinearity*: Let us consider a simple nonlinear ordinary differential equation with an exponential nonlinearity

$$dy/dx + e^y = 0 \qquad y(0) = 1.$$

In our usual standard form (1983) this is written

$$Ly + Ny = x$$

with $L = d/dx$ and $Ny = e^y$. We solve for Ly, i.e., $Ly = -Ny$, then write $L^{-1} Ly = -L^{-1} Ny$ with L^{-1} defined as the integration over x. Thus

$$y = y(0) - L^{-1} Ny;$$

the nonlinear term $Ny = e^y$ is replaced by $\sum_{n=0}^{\infty} A_n$ where the A_n can be written as $A_n(e^y)$ to emphasize that they are generated for this specific function. Thus,

$$y = y(0) - L^{-1} \sum_{n=0}^{\infty} A_n(e^y).$$

Now the decomposition of the solution y into $\sum_{n=0}^{\infty} y_n$ leads to the term-by-term identification

$$y_0 = y(0) = 1$$
$$y_1 = -L^{-1} A_0$$
$$y_2 = -L^{-1} A_1$$
$$\vdots$$
$$y_{n+1} = -L^{-1} A_n$$

The $A_n(e^y)$ are given by

$$A_0 = e^{y_0}$$
$$A_1 = y_1 e^{y_0}$$
$$A_2 = (y_1^2/2 + y_2) e^{y_0}$$
$$A_3 = (y_1^3/6 + y_1 y_2 + y_3) e^{y_0}$$
$$\vdots$$

Thus

$$y_0 = 1$$
$$y_1 = -ex$$
$$y_2 = e^2 x^2/2$$
$$y_3 = -e^3 x^3/3!$$
$$\vdots$$
$$y = 1 + \sum_{n=1}^{\infty} (-1)^n e^n x^n/n!$$

which can also be written $y = 1 - \ln[1 + ex]$ for $x < 1$.

Suppose we replace e^y by $1 + y$, dropping all terms of the series for e^y except the constant and the linear term. The differential equation becomes

$$dy/dx = -(1 + y).$$

Then

$$y = y(0) - L^{-1}(1 + y)$$
$$= 1 - L^{-1}[1] - L^{-1} \sum_{n=0}^{\infty} y_n$$

so that

$$y_0 = 1 - x$$
$$y_1 = -L^{-1}(1 - x)$$
$$\vdots$$
$$y_n = -L^{-1}y_{n-1} \qquad (n \geqslant 1).$$

Since y is the sum of the components, we have

$$y = \sum_{n=0}^{\infty} \{(-1)^n x^n/n! + (-1)^{n+1}x^{n+1}/(n+1)!\}$$
$$y = e^{-x} + (e^{-x} - 1) = 2e^{-x} - 1.$$

We will identify this linearized solution as y and compare with the solution y of the nonlinear equation. The results are given in the following table.

x	y	y_l
0	1.0000	1.0000
0.1	0.7595	0.8097
0.2	0.5658	0.6375
0.3	0.4036	0.4816
0.4	0.2641	0.3406
0.5	0.1417	0.2131

EXAMPLE. *Anharmonic oscillator*: Consider the anharmonic oscillator $d^2\theta/dt^2 + k^2 \sin \theta = 0$ for $\theta(0) = \gamma = $ constant and $\theta'(0) = 0$. Using the decomposition method, the solution is found to be

$$\theta(t) = \gamma - [(kt)^2/2!] \sin \gamma + [(kt)^4/4!] \sin \gamma \cos \gamma$$
$$- [(kt)^6/6!](\sin \gamma \cos^2 \gamma - 3 \sin^3 \gamma) + \cdots$$

which becomes

$$\theta(t) = \gamma[1 - (k^2 t^2/2!) + (kt)^4/4! - \cdots]$$

in the linearized case, i.e., for "small amplitude" motion which offers interesting comparison when the smallness assumption is inappropriate.

EXAMPLE. *Hyperbolic sine nonlinearity*: Consider the equation

$$du/dt - k \sinh u/\alpha,$$

where $u(0) = c > 0$ for $t > 0$.

If we assume that we can approximate $\sinh u/\alpha \simeq u/\alpha$, we have $du/dt - ku/\alpha = 0$. Now solving by decomposition with $u = \sum_{n=0}^{\infty} vu_n$, $L = d/dt$, and L^{-1} as the definite integral from 0 to t

$$Lu - ku/\alpha = 0$$

$$L^{-1}Lu = L^{-1}(k/\alpha)u$$

$$u = u(0) + L^{-1}(k/\alpha)u = \sum_{n=0}^{\infty} u_n$$

$$u_0 = u(0) = c$$

$$u_1 = (k/\alpha)ct = ckt/\alpha$$

$$u_2 = c(kt/\alpha)^2/2!$$

$$u_3 = c(kt/\alpha)^3/3!$$

$$\vdots$$

$$u_m = c(kt/\alpha)^m/m!$$

$$\vdots$$

i.e., $u = ce^{kt/\alpha}$. To solve the original equation with $\sinh u/\alpha$

$$u = \sum_{n=0}^{\infty} u_n = c + kL^{-1} \sum_{n=0}^{\infty} A_n,$$

where the A_n are generated for $Nu = \sinh u/\alpha$. These are given by

$$A_0 = \sinh(u_0/\alpha)$$

$$A_1 = (u_1/\alpha) \cosh(u_0/\alpha)$$

$$A_2 = (u_2/\alpha) \cosh(u_0/\alpha) + (1/2!)(u_1^2/\alpha^2) \sinh(u_0/\alpha)$$

$$A_3 = (u_3/\alpha) \cosh(u_0/\alpha) + (u_1/\alpha)(u_2/\alpha) \sinh(u_0/\alpha)$$
$$+ (1/3!)(u_1^3/\alpha^3) \cosh(u_0/\alpha)$$

$$A_4 = (u_4/\alpha) \cosh(u_0/\alpha) + [(1/2!)(u_2^2/\alpha^2)$$
$$+ (u_1/\alpha)(u_2/\alpha)] \sinh(u_0/\alpha)$$
$$+ (1/2!)(u_1^2/\alpha^2)(u_2/\alpha) \cosh(u_0/\alpha)$$
$$+ (1/4!)(u_1^4\alpha^4) \sinh(u_0/\alpha)$$

$$A_5 = (u_5/\alpha)\cosh(u_0/\alpha) + [(u_2/\alpha)(u_3/\alpha)$$
$$+ (u_1/\alpha)(u_4/\alpha)]\sinh(u_0/\alpha) + [(u_1/\alpha)(1/2!)(u_2^2/\alpha^2)$$
$$+ (1/2!)(u_1^2/\alpha^2)(u_3/\alpha)]\cosh(u_0/\alpha)$$
$$+ (1/3!)(u_1^3/\alpha^3)(u_2/\alpha)\sinh(u_0/\alpha)$$
$$+ (1/5!)(u_1^5/\alpha^5)\cosh(u_0/\alpha)$$
$$A_6 = (u_6\alpha)\cosh(u_0/\alpha) + [(1/2!)(u_3^2/\alpha^2) + (u_2/\alpha)(u_4/\alpha)$$
$$+ (u_1/\alpha)(u_5/\alpha)]\sinh(u_0/\alpha)$$
$$+ [(1/3!)(u_2^3/\alpha^3) + (u_1/\alpha)(u_2/\alpha)(u_3/\alpha)$$
$$+ (1/2!)(u_1^2/\alpha^2)(u_4/\alpha)]\cosh(u_0/\alpha)$$
$$+ [(1/2!)(u_1^2/\alpha^2)(1/2!)(u_2^2/\alpha^2)$$
$$+ (1/3!)(u_1^3/\alpha^3)(u_3/\alpha)]\sinh(u_0/\alpha)$$
$$+ (1/4!)(u_1^4/\alpha^4)(u_2/\alpha)\cosh(u_0/\alpha)$$
$$+ (1/6!)(u_1^6/\alpha^6)\sinh(u_0/\alpha)$$
$$\vdots$$

Now

$$u_0 = c$$
$$u_1 = kL^{-1}A_0 = kL^{-1}[\sinh u_0/\alpha] = kt\sinh c/\alpha$$
$$u_2 = kL^{-1}A_1 = kL^{-1}[(u_1/\alpha)\cosh(u_0/\alpha)]$$
$$= kL^{-1}[(kt/\alpha)\sinh(c/\alpha)\cosh(c/\alpha)]$$
$$= (k^2t^2/2!\,\alpha)\sinh(c/\alpha)\cosh(c/\alpha)$$
$$u_3 = kL^{-1}A_2 = kL^{-1}[(u_2/\alpha)\cosh(u_0/\alpha) + (1/2)(u_1^2/\alpha^2)\sinh(u_0/\alpha)]$$
$$= kL^{-1}[(k^2t^2/2)(1/\alpha)^2\sinh(c/\alpha)\cosh^2(c/\alpha)$$
$$+ (1/2)(k^2t^2)(1/\alpha)^2\sinh^3(c/\alpha)]$$
$$= (k^3t^3/3!)(1/\alpha)^2\sinh(c/\alpha)[\sinh^2(c/\alpha) + \cosh^2(c/\alpha)]$$
$$u_4 = (k^4t^4/4!)(1/\alpha)^3[\sinh(c/\alpha)\cosh(c/\alpha)][5\sinh^2(c/\alpha) + \cosh^2(c/\alpha)]$$
$$\vdots$$

Thus, the correct solution is the sum of the u_n above while the linearized solution is

$$u = c[1 + kt/\alpha + (kt/\alpha)^2/2! + (kt/\alpha)^3/3! + \cdots].$$

If we assume $\alpha = c = k = 1$, we have the results $u = e^t$ in the linear case which can be compared with the nonlinear solution.

t	Linear solution	Nonlinear solution
0	1	1
0.1	1.105170918	1.127402502
0.2	1.221402758	1.278624737
0.3	1.349858808	1.462380247
0.4	1.491824698	1.693614375
0.5	1.648721271	2.001468676
0.6	1.8221188	2.456234584
0.7	2.013752707	3.325545159
0.75	2.118000017	4.512775469
0.76	2.13827622	5.121285511
0.77	2.159766254	6.939848656
0.7719	2.163873711	10.9022661
0.771936	2.163951611	14.69149181
0.77193683	2.163953407	20.34929139
0.7719368329	2.163953414	28.3241683
0.7719368330	2.163953414	∞

The error in the linear solution at t is given by:

t	% Error
0	0
0.1	1.97
0.2	4.48
0.3	7.69
0.4	11.92
0.5	17.62
0.6	25.82
0.7	39.45
0.75	53.09
0.76	58.25
0.77	68.88
0.7719	80.15
0.771936	85.27
0.77193683	89.37
0.7719368329	92.36
0.7719368330	100.

A comparison with numerical integration. The decomposition method is extremely accurate and in most cases ϕ_n, the *n*-term approximation, is accurate for quite low values of n. To emphasize this point we consider only a two-term approximation in the following example with the added comment that additional terms are extremely easy to obtain.

Consider the equation $dy/dt = t + y^{-1}$. We now have $L = d/dt$, $Ny = -y^{-1}$, $x(t) = t$. Assume $y(0) = k$ is an integer. Then

$$L^{-1}Ly = L^{-1}x - L^{-1}Ny$$

$$y - y(0) = L^{-1}t + L^{-1} \sum_{n=0}^{\infty} A_n,$$

where the $A_n = A_n(y^{-1})$.

$$y = y(0) + L^{-1} \sum_{n=0}^{\infty} A_n$$

$$y_0 = k + t^2/2$$

and since

$$A_0 = y_0^{-1}, \ A_1 = -y_0^{-2}y_1, \ A_2 = -y_0^{-2}y_2 + y_0^{-3}y_1^2, \ ...,$$

$$y_1 = \int_0^t y_0^{-1} \, dt = \int_0^t (k + t^2/2)^{-1} \, dt$$

$$y_1 = (2/k)^{1/2} \tan^{-1} [t/(2k)^{1/2}]$$

$$\vdots$$

Let us consider a *two-term* approximation $\phi_2 = y_0 + y_1$. (The complete solution, of course, is $\sum_{n=0}^{\infty} y_n$.) Then

$$\phi_2 = k + t^2/2 + (2/k)^{1/2} \tan^{-1} [t/(2k)^{1/2}].$$

Table I compares this approximation with results of a numerical integration using $k = 4$. With only a little more effort we could go to a higher $\phi_n(\phi_5$, for example) for a better approximation, which is unnecessary since the percentage error is already extremely small. The worst case is less than 0.4%. However, if we go to ϕ_3 we find the worst case has an error less than 0.02%—this for only a three-term approximation! Thus, we have very rapid convergence.

Computational time. In numerical solutions of physical problems, it is common to make computations at discrete space or time intervals. Computer methods are based on changing continuous problems to discrete

TABLE I

t	Decomposition method ϕ_1	Numerical integration y	$\Delta = \phi_1 - y$	% Error
0	4.0	4.0	0	
0.5	4.25	4.25	0	0
1.0	4.74	4.73	0.01	0.21
1.5	5.47	5.46	0.01	0.18
2.0	6.44	6.42	0.02	0.31
2.5	7.64	7.61	0.03	0.39
3.0	9.08	9.05	0.03	0.33
4.0	12.68	12.64	0.04	0.32
5.0	17.25	17.21	0.04	0.23
10.0	54.92	54.88	0.04	0.07
20.0	205.01	204.97	0.04	0.02

problems. Thus, in solving a differential equation, one must solve the equation at each point of time. Since these points must be close together to approximate the total solution, massive computations are needed and the resulting numerical printouts yield little insight into dependences. Clearly we have *linearized approximations* in each integral. It is also clear that laborious computation is involved which becomes more accurate as the mesh gets finer but at the expense of increasing the computation (and the need for gigaflop computers).

If the time points for solution are close for a reasonably accurate solution, the number of computations becomes enormous! For example, solving even the simple linear equation

$$d^2y/dt^2 + dy/dt + y = 0$$

with $y(0) = 1$ and $y'(0) = 0$ requires a *thousand* solutions of the above equation or 8000 actual computations to get the value of y every 0.01 sec for 10 sec. For one minute this means 6000 solutions or 48,000 actual computations. Note that if one considers an equation in two independent variables x, t, we require a Δx and Δt, the computations will go up several orders of magnitude. One can easily visualize a problem lending to a billion coupled difference equations to solve. For a simple scalar elliptic equation, we have one unknown at each mesh point. For more complex problems, there can be many unknowns at each mesh point and the resulting systems of difference equations (instead of being linear as in the previous case) may be nonlinear, time-dependent, and very large (inclusion of stochastic coefficients, etc., is still another matter). To solve massive systems, iterative procedures are used to solve simpler systems, then substitution to get

"residuals" and repetitions of the process to produce corrections and until the error is (or is felt to be) within tolerable limits. To get accuracy the mesh must become very fine and computations required finally exceed any conceivable computer capability for complicated equations in x, y, z, t.

Solution by the decomposition method, on the other hand, is *continuous*, analytical, and requires no discretization; it corresponds to the results obtained by an infinite number of computations by discretization—*no linearization is involved, and the solution is accurate*. If variable coefficients, several independent variables, and nonlinearities are involved, the decomposition method is clearly preferable. In the case of stochastic equations, the decomposition method is particularly appropriate; it requires no perturbative or truncation methods or a priori assumptions of special behavior. Computer results, on the other hand, are not correct when stochastic processes are discretized.

Supercomputers are developing rapidly because of urgent need in meterology, fluid dynamics, fusion research, intelligent missile guidance, and weapons design. In fluid dynamics, for example, they are considered essential for the solution of the equations which are relevant to turbulence, internal waves in the ocean, and future development of hypersonic flight vehicles and engines. Supercomputers are also essential for VLSI devices, seismology, reservoir modeling, bioengineering, and studies of the national economy.

To solve, for example, the relevant problem of hypersonic aircraft for eventual single-stage flight to a space station on contemplated next-generation computers, a three-dimensional mesh is generated which discretizes the system of nonlinear partial differential equations into a million, a hundred million, or perhaps a billion coupled difference equations in as many unknowns. One begins to see then the tremendous data-handling problem, the necessity for improved algorithms, and the need for still greater computational speed. We may also have many unknowns at each point, and, as we have pointed out, the system nonlinearities and random fluctuations need to be taken into consideration. Since usually solutions are iterative—first solving an approximation to the original system of differential equations and then improving the solution by repeated substitution of each new solution—parallel processing is complicated by the difficulty of partitioning the work so each processor can work independently. This is being pursued by many ingenious ideas necessitated by the brute force method of discretization.

In all such problems we need to be able to solve coupled systems of nonlinear (and generally stochastic as well) partial differential equations with complex boundary conditions and possible delayed effects. These systems are *linearized* and *discretized* (and the stochastic aspects either ignored or improperly dealt with) so the various numerical approximation methods

can be used. This requires faster and faster supercomputers to do these computations in a reasonable time. Fifth generation computers will operate at speeds up to 1000 megaflops (a gigaflop) or 10^9 operations per second.

Unfortunately the further developments in supercomputers can quite possibly still give wrong answers because even a single one-dimensional nonlinear differential equation without stochasticity in coefficients, inputs, and boundary conditions—let alone vector partial differential equations in space and time with nonlinear and/or stochastic parameters—are not solved exactly. Real systems are nonlinear and stochastic. When you throw out these "complications," you have a different problem! When you linearize and use perturbative methods, you solve a mathematized problem, not the physical problem. The model equations, even before the linearization, discretization, etc., are already wrong because the stochastic behavior is generally not incorporated or is incorporated incorrectly as an afterthought. In a real ocean, velocity, density, and pressure are stochastic, not constants. Present treatment of the fluid flow equations solves a simplistic model, not real behavior. Turbulence, for example, is a strongly non-linear, strongly stochastic phenomenon and cannot be understood by linearized perturbative treatments. The theories of physics are perturbative theories and the theories of mathematics are for linear operators (other than some ad hoc methods for special nonlinear equations). What is needed is a way of solving one or more nonlinear stochastic operator equations whether algebraic, differential, delay-differential, partial-differential, or systems of such equations. To solve the national economy or control it, one needs to solve systems involving the real fluctuations and nonlinearities which are present. A supercomputer is, after all, a fast adding machine, and its computational accuracy is dependent on the sophistication of the mathematical methods prgrammed into it. Typical calculations consider millions of discrete time intervals made small enough so trajectories between them can be taken as low-order polynomials, e.g., quadratics. If stochasticity is involved, then Monte Carlo methods are used, which insert randomness but not the properly correlated randomness which is present in the physical problem.

When one studies airflow about aircraft surfaces, computations are made at tens of millions of points, and it is felt that by increasing the volume of computation to the limit in an ultimate extrapolation, supercomputers will yield complete accuracy. Not only does this ignore stochaticity, it ignores the sensitivity of nonlinear stochastic systems to very slight changes in the model—in fact, to changes essentially undeterminable by measurement.

Discussion of decomposition method. Solution of nonlinear equations arising in the modeling of a physical system generally begins with some form of linearization, assumptions of weak nonlinearity, and "smallness." Yet physical systems are nonlinear, and real systems generally involve random

fluctuations. The general case is a nonlinear stochastic system, and "linear" and "deterministic" are *special cases*. We wish ultimately to solve physical problems realistically, not simply easy artificial versions of those problems in order to make the mathematics "tractable." It is true that we have gone a long way with the earlier methods, and in many problems they are completely adequate; in others, they are a good first approximation to gain some insight. Yet, it is easy to find cases in which they are not adequate; clearly not all systems are linear and deterministic. Further, all effects and behaviors in complex systems are not instantaneous, and a serious attempt to explain behavior of a complex dynamical system, and eventually to control it, must sometimes consider delays or retarded effects. To solve such problems, we must be able to solve equations or systems of equations which may involve differential or partial differential operators, be linear or nonlinear, deterministic or stochastic, involve delays, etc.

As ambitious as it may appear, it is fortuitous that the decomposition method appears to be capable of such solutions in a fairly wide class of problems. The superficial resemblance of this method to some other methods can be misleading; the proof is in the fact that it solves problems not solvable by other methods, or only solvable with much more difficulty or computation. The method is an "approximation" method, not a "closed form" solution. The usual significance of these terms is that in the one case, we have an exact answer and in the other an approximate one. Clearly, however, a method of solution which changes the problem to a *different, easier mathematical problem* and then solves it exactly is not to be preferred to one in which the *actual* nonlinear and/or stochastic model is treated with an "approximate" method which provides *accurate*, rapidly *convergent*, and *computable* series of terms.

Summary. The decomposition method can be an effective method for solution of a wide class of problems providing generally a rapidly convergent series solution. It has some distinct advantages over usual approximation methods in that it is computationally convenient, provides analytic, verifiable solutions not requiring perturbation, linearization, or discretization and resulting massive computation. Since it solves nonlinear problems rather than linearizing them, the resulting solutions are more physically realistic. The given references provide further insights [1–58].

Of course difficulties remain which present interesting areas for study. In particular, it would be desirable to determine easier ways of generating the A_n polynomials and to study their properties. Convergence and error studies for various classes of equations is a further need and it is clear that many theorems will follow.

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