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Vague soft hemirings

Yunqiang Yin [a,b,*], Young Bae Jun [c], Jianming Zhan [d]

[a] *Key Laboratory of Radioactive Geology and Exploration Technology, Fundamental Science for National Defense, East China Institute of Technology, Fuzhou, Jiangxi 344000, China*

[b] *School of Mathematics and Information Sciences, East China Institute of Technology, Fuzhou, Jiangxi 344000, China*

[c] *Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Republic of Korea*

[d] *Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province 445000, China*

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ABSTRACT

Xu et al. introduced the concept of vague soft sets, which is an extension to the soft set and the vague set. In this paper, we apply the concept of vague soft sets to hemiring theory. The notion of $(\in, \in \vee q)$-vague (soft) left h-ideals of a hemiring is introduced and some related properties are investigated.

1. Introduction

Uncertain or imprecise data are inherent and pervasive in many important applications in the areas such as economics, engineering, environment, social science, medical science and business management. Uncertain data in those applications could be caused by data randomness, information incompleteness, limitations of measuring instruments, delayed data updates, etc. Due to the importance of these applications and the rapidly increasing amount of uncertain data collected and accumulated, research on effective and efficient techniques that are dedicated to modeling uncertain data and tackling uncertainties has attracted much interest in recent years and has yet remained challenging at large. There have been a great amount of research and applications in the literature concerning some special tools like probability theory, (intuitionistic) fuzzy set theory [1], rough set theory [2], vague set theory [3] and interval mathematics [4,5]. However, all of these have their advantages as well as inherent limitations in dealing with uncertainties. One major problem shared by those theories is their incompatibility with the parameterization tools. To overcome these difficulties, Molodtsov [6] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. This theory has been proven useful in many different fields such as decision making [7–11], data analysis [12,13], forecasting [14] and so on.

Up to the present, research on soft sets has been very active and many important results have been achieved in the theoretical aspect. Maji et al. [15] defined and studied several operations on soft sets. Ali et al. [16] further presented and investigated some new algebraic operations for soft sets. Maji et al. [17] and Majumdar and Samanta [18] extended (classical) soft sets to fuzzy soft sets, respectively. Irfan Ali and Shabir [19] investigated De Morgan's law in fuzzy soft sets. Maji et al. [20]

* Corresponding author at: Key Laboratory of Radioactive Geology and Exploration Technology, Fundamental Science for National Defense, East China Institute of Technology, Fuzhou, Jiangxi 344000, China.

E-mail addresses: yunqiangyin@gmail.com, yinyunqiang@126.com (Y. Yin), skywine@gmail.com (Y.B. Jun), zhanjianming@hotmail.com (J. Zhan).

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extended (classical) soft sets to intuitionistic fuzzy soft sets. Xu et al. [21] introduced the notion of vague soft set which is an extension to the soft set and vague set, and discussed the basic properties of vague soft sets. Aktaş and Çağman [22] compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups, and derived some related properties. Feng et al. [23] investigated soft semirings by using the soft set theory. Feng et al. [24,25] investigated the connections among fuzzy sets, rough sets and soft sets. They introduced the concepts of soft rough sets and soft rough fuzzy sets and studied some related properties. Jun [26] introduced and investigated the notion of soft BCK/BCI-algebras. Jun and Park [27] and Jun et al. [28] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras, respectively. Aygünoğlu and Aygün [29] introduced and studied (normal) fuzzy soft groups. Zhan and Jun [30] studied soft BL-algebras based on fuzzy sets.

The purpose of this paper is to deal with the algebraic structure of hemirings by applying vague (soft) set theory. The concept of $(\in, \in \vee q)$-vague (soft) left h-ideals of a hemiring is introduced, their characterization and algebraic properties are investigated. The notion of φ-compatible $(\in, \in \vee q)$-vague (soft) left h-ideals of a hemiring, and the homomorphism properties of $(\in, \in \vee q)$-vague (soft) left h-ideals are discussed.

The rest of this paper is organized as follows. Section 2 summarizes some basic concepts which will be used throughout the paper. Sections 3 and 4 introduce the concept of $(\in, \in \vee q)$-vague (soft) left h-ideals of a hemiring and investigate some related properties. Section 5 introduces the notion of φ-compatible $(\in, \in \vee q)$-vague (soft) left h-ideals of a hemiring and discuss the homomorphism properties of $(\in, \in \vee q)$-vague (soft) left h-ideals. Some conclusions are given in the last section.

2. Preliminaries

2.1. Hemirings

A *semiring* is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a \cdot (b + c) = a \cdot b + b \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

are satisfied for all $a, b, c \in S$.

By *zero* of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring $(S, +, \cdot)$ with zero is called a *hemiring* if $(S, +)$ is commutative. For the sake of simplicity, we shall omit the symbol "$\cdot$", writing ab for $a \cdot b$ $(a, b \in S)$.

A subset A in a hemiring S is called a *left* ideal of S if A is closed under addition and $SA \subseteq A$. A left ideal A of S is called a *left h-ideal* if $x, z \in S$, $a, b \in A$, and $x + a + z = b + z$ implies $x \in A$.

2.2. Vague sets

Let X be a non-empty set. A fuzzy subset μ in X is defined as a mapping from X to $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $\mathscr{F}(X)$ the set of all fuzzy subsets in X. For any $\mu \in \mathscr{F}(X)$, the complement of μ, denoted by μ^C, is defined as $\mu^C(x) = 1 - \mu(x)$ for all $x \in X$. For any non-empty subset P in S, the characteristic function of S is denoted by χ_P. A fuzzy subset μ in X of the form

$$\mu(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is said to be a *fuzzy point with support x and value r* and is denoted by x_r, where $r \in (0, 1]$. For a fuzzy point x_r and a fuzzy subset μ in X, we say that

(1) $x_r \in \mu$ if $\mu(x) \geq r$.
(2) $x_r \, q \, \mu$ if $\mu(x) + r > 1$.
(3) $x_r \in \vee q \, \mu$ if $x_r \in \mu$ or $x_r \, q \, \mu$.
(4) $x_r \in \wedge q \, \mu$ if $x_r \in \mu$ and $x_r \, q \, \mu$.

A *vague set* A in X is characterized by a truth-membership function t_A and a false-membership function f_A,

$$t_A : X \to [0, 1], \qquad f_A : X \to [0, 1],$$

where for any $x \in X$, $t_A(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, $f_A(x)$ is a lower bound on the negation of x derived from the evidence against x, and $t_A(x) + f_A(x) \leq 1$. The grade of membership of x in the vague set is bounded to a subinterval $[t_A(x), 1 - f_A(x)]$ of $[0, 1]$. The vague value $[t_A(x), 1 - f_A(x)]$ indicates that the exact grade of membership $\mu_A(x)$ of x may be unknown, but it is bounded by $t_A(x) \leq \mu_A(x) \leq 1 - f_A(x)$, where $t_A(x) + f_A(x) \leq 1$. Denote $A = \{(x, [t_A(x), 1 - f_A(x)])|x \in X\}$. The set of all vague sets in X is denoted by $\mathscr{V}(X)$.

Lemma 2.1. *Let $A = \{(x, [t_A(x), 1 - f_A(x)])|x\}$ and $B = \{(x, [t_B(x), 1 - f_B(x)])|x\}$ be vague sets in X and let $r, t \in [0, 1]$. Define*

(1) $\Box A = \{(x, [t_A(x), 1 - t_A^c(x)])|x \in X\}$,
(2) $\Diamond A = \{(x, [f_A^c(x), 1 - f_A(x)])|x \in X\}$,

(3) $P_{r,t}(A) = \{(x, [\max\{r, t_A(x)\}, 1 - \min\{t, f_A(x)\}])|x \in X\}$ *for* $r + t \leq 1$,
(4) $Q_{r,t}(A) = \{(x, [\min\{r, t_A(x)\}, 1 - \max\{t, f_A(x)\}])|x \in X\}$ *for* $r + t \leq 1$.

Then $\Box A$, $\Diamond A$, $P_{r,t}(A)$ *and* $Q_{r,t}(A)$ *are vagues sets in* X.

Proof. The proof is straightforward by the definition of vague sets. $\Box$

Definition 2.2 ([3]). Let A and B be two vague sets in X. If $[t_A(x), 1 - f_A(x)] = [t_B(x), 1 - f_B(x)]$ for all $x \in X$, then the vague sets A and B are called *equal*, denoted as $A = B$.

Definition 2.3 ([3]). Let A and B be two vague sets in X. If $t_A(x) \leq t_B(x)$, $1 - f_A(x) \leq 1 - f_B(x)$ for all $x \in X$, then the vague set A is said to be *included by* B, denoted as $A \subseteq B$.

Definition 2.4 ([3]). The *union* of two vague sets A and B is a vague set C, written as $C = A \cup B$, whose truth-membership and false-membership functions are related to those of A and B defined by

$$t_C(x) = \max\{t_A(x), t_B(x)\}$$

and

$$1 - f_C(x) = \max\{1 - f_A(x), 1 - f_B(x)\} = 1 - \min\{f_A(x), f_B(x)\}$$

for all $x \in X$, that is, $t_C = t_A \cup t_B$ and $f_C = f_A \cap f_B$.

Definition 2.5 ([3]). The *intersection* of two vague sets A and B is a vague set C, written as $C = A \cap B$, whose truth-membership and false-membership functions are related to those of A and B defined by

$$t_C(x) = \min\{t_A(x), t_B(x)\}$$

and

$$f_C(x) = \min\{(1 - f_A(x)), (1 - f_B(x))\} = 1 - \max\{f_A(x), f_B(x)\}$$

for all $x \in X$, that is, $t_C = t_A \cap t_B$ and $f_C = f_A \cup f_B$.

The notions of union and intersection of two vague sets can be extended to a family of vague sets in X. Let $\{A_i\}_{i \in I}$ be a family of vague sets in X, where I is an index set. Define $\cap_{i \in I} A_i$ and $\cup_{i \in I} A_i$ as follows:

$$t_{\cap_{i \in I} A_i}(x) = \inf_{i \in I} t_{A_i}(x), \qquad 1 - f_{\cap_{i \in I} A_i}(x) = \inf_{i \in I}(1 - f_{A_i}(x)) = 1 - \sup_{i \in I} f_{A_i}(x)$$

and

$$t_{\cup_{i \in I} A_i}(x) = \sup_{i \in I} t_{A_i}(x), \qquad 1 - f_{\cup_{i \in I} A_i}(x) = \sup_{i \in I}(1 - f_{A_i}(x)) = 1 - \inf_{i \in I} f_{A_i}(x).$$

If I is a finite set, say $I = \{1, 2, \ldots, n\}$, we sometimes denote

$$t_{\cap_{i \in I} A_i}(x) = \min\{t_{A_1}(x), \ldots, t_{A_n}(x)\}, \qquad 1 - f_{\cap_{i \in I} A_i}(x) = 1 - \max\{f_{A_1}(x), \ldots, f_{A_n}(x)\}$$

and

$$t_{\cup_{i \in I} A_i}(x) = \max\{t_{A_1}(x), \ldots, t_{A_n}(x)\}, \qquad 1 - f_{\cup_{i \in I} A_i}(x) = 1 - \min\{f_{A_1}(x), \ldots, f_{A_n}(x)\}.$$

For the sake of simplicity, we use $A = [t_A, 1 - f_A]$ to denote the vague set $A = \{(x, [t_A(x), 1 - f_A(x)])|x \in X\}$. For any $r, s \in [0, 1]$, denote $A^{(r,s)} = \{x \in X|t_A(x) > r$ and $f_A(x) < s\}$, which is called the (r, s)-*strong level set* of A.

It follows easily that for all $A, B \in \mathscr{V}(X)$,

(1) $A \subseteq B$ and $r, s \in [0, 1]$ imply $A^{(r,s)} \subseteq B^{(r,s)}$.
(2) $r_1 \leq r_2$ and $s_2 \leq s_1$ for $r_1, r_2, s_1 s_2 \in [0, 1]$ imply $A^{(r_2,s_2)} \subseteq A^{(r_1,s_1)}$.
(3) $A = B$ if and only if $A^{(r_1,s_1)} = A^{(r_2,s_2)}$ for all $r_1, r_2, s_1 s_2 \in [0, 1]$.

Let us now introduce two new ordering relations on $\mathscr{F}(X)$, denoted as "$\subseteq \vee q$" and "$\subseteq \wedge q$", as follows: $\forall \mu, \nu \in \mathscr{F}(X)$

(1) By $\mu \subseteq \vee q\, \nu$ we mean that $x_r \in \mu$ implies $x_r \in \vee q\, \nu$ for all $x \in X$ and $r \in (0, 1]$.
(2) By $\mu \subseteq \wedge q \nu$ we mean that $x_r \in \wedge q\, \mu$ implies $x_r \in \nu$ for all $x \in X$ and $r \in (0, 1]$.

In what follows, unless otherwise stated, $\overline{\alpha}$ means α does not hold, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q, \subseteq, \subseteq \vee q, \subseteq \wedge q\}$.

Lemma 2.6. *Let* $\mu, \nu \in \mathscr{F}(X)$. *Then:*

(1) $\mu \subseteq \vee q\, \nu$ *if and only if* $\nu(x) \geq \min\{\mu(x), 0.5\}$ *for all* $x \in X$.
(2) $\mu \subseteq \wedge q \nu$ *if and only if* $\mu(x) \leq \max\{\nu(x), 0.5\}$ *for all* $x \in X$.

Proof. We prove (2). (1) can be similarly proved. Assume that $\mu \subseteq \wedge q\nu$. Let $x \in X$. If $\mu(x) > \max\{\nu(x), 0.5\}$, then there exists r such that $\mu(x) > r > \max\{\nu(x), 0.5\}$. Hence $x_r \in \mu$ and $x_r q\, \mu$, that is, $x_r \in \wedge q\, \mu$, but $x_r \overline{\in} \nu$, a contradiction. Therefore, $\mu(x) \leq \max\{\nu(x), 0.5\}$.

Conversely, assume that $\mu(x) \leq \max\{\nu(x), 0.5\}$ for all $x \in S$. If $\mu \overline{\subseteq \wedge q} \nu$, then there exists $x_r \in \wedge q\, \mu$ but $x_r \overline{\in}\, \nu$, and so $\mu(x) \geq r$, $\mu(x) + r > 1$ and $\nu(x) < r$, which imply $\mu(x) > 0.5$. Hence $\mu(x) > \max\{\nu(x), 0.5\}$, a contradiction. Therefore, $\mu \subseteq \wedge q\nu$. □

Lemma 2.7. *Let $\mu, \nu, \omega \in \mathscr{F}(X)$.*

(1) *If $\mu \subseteq \vee q\, \nu$ and $\nu \subseteq \vee q\, \omega$, then $\mu \subseteq \vee q\, \omega$.*
(2) *If $\mu \subseteq \wedge q\nu$ and $\nu \subseteq \wedge q\omega$, then $\mu \subseteq \wedge q\omega$.*

Proof. It is straightforward by Lemma 2.6. □

Based on the ordering relations "$\subseteq \vee q$" and "$\subseteq \wedge q$" on $\mathscr{F}(X)$, we introduce a new ordering relation "$\Subset$" on $\mathscr{V}(X)$ as follows.

For any two vague sets A and B, by $A \Subset B$ we mean that $t_A \subseteq \vee q\, t_B$ and $f_B \subseteq \wedge q f_A$.

2.3. Vague soft sets

Let U be an initial universe, E a set of parameters, $\mathscr{P}(U)$ the power set of U, and $X \subseteq E$. Molodtsov [6] defined the concept a soft set, which is a mapping from a set of parameters into the power set of a universe set. However, the notion of soft set, as given in its definition, cannot be used to represent the vagueness of the associated parameters. To overcome this, Xu et al. [21] further introduced the concept of a vague soft set based on soft set theory and vague set theory as follows.

Definition 2.8 (*[21]*)**.** A pair $(\tilde{F}, X)$ is called a *vague soft set* over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : X \mapsto \mathscr{V}(U)$.

In other words, a vague soft set over U is a parameterized family of vague sets of the universe U. For $\varepsilon \in X$, $\tilde{F}(\varepsilon)$ is regarded as the set of ε-approximate elements of the vague soft set $(\tilde{F}, X)$.

For a vague soft set, Xu et al. [21] have introduced the following definitions concerning its operations.

Definition 2.9 (*[21]*)**.** If $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ are two vague soft sets over a universe U. "$(\tilde{F}, X)$ AND $(\tilde{G}, Y)$", denoted by $(\tilde{F}, X)\tilde{\wedge}(\tilde{G}, Y)$, is defined by $(\tilde{F}, X)\tilde{\wedge}(\tilde{G}, Y) = (\tilde{H}, X \times Y)$, where

$$t_{\tilde{H}(\alpha,\beta)}(x) = \min\{t_{\tilde{F}(\alpha)}(x), t_{\tilde{G}(\beta)}(x)\}$$

and

$$1 - f_{\tilde{H}(\alpha,\beta)}(x) = \min\{1 - f_{\tilde{F}(\alpha)}(x), 1 - f_{\tilde{G}(\beta)}(x)\} = 1 - \max\{f_{\tilde{F}(\alpha)}(x), f_{\tilde{G}(\beta)}(x)\}$$

for all $(\alpha, \beta) \in (X, Y)$ and $x \in U$.

Definition 2.10 (*[21]*)**.** If $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ are two vague soft sets over a universe U. "$(\tilde{F}, X)$ OR $(\tilde{G}, Y)$", denoted by $(\tilde{F}, X)\tilde{\vee}(\tilde{G}, Y)$, is defined by $(\tilde{F}, X)\tilde{\vee}(\tilde{G}, Y) = (\tilde{O}, X \times Y)$, where

$$t_{\tilde{O}(\alpha,\beta)}(x) = \max\{t_{\tilde{F}(\alpha)}(x), t_{\tilde{G}(\beta)}(x)\}$$

and

$$1 - f_{\tilde{O}(\alpha,\beta)}(x) = \max\{1 - f_{\tilde{F}(\alpha)}(x), 1 - f_{\tilde{G}(\beta)}(x)\} = 1 - \min\{f_{\tilde{F}(\alpha)}(x), f_{\tilde{G}(\beta)}(x)\}$$

for all $(\alpha, \beta) \in (X, Y)$ and $x \in U$.

Definition 2.11 (*[21]*)**.** The *union* of two vague soft sets $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ over a universe U is a vague soft set denoted by $(\tilde{H}, Z)$, where $Z = X \cup Y$ and

$$t_{\tilde{H}(\alpha)}(x) = \begin{cases} t_{\tilde{F}(\alpha)}(x) & \text{if } \alpha \in X - Y, \\ t_{\tilde{G}(\alpha)}(x) & \text{if } \alpha \in Y - X, \\ \max\{t_{\tilde{F}(\alpha)}(x), t_{\tilde{G}(\alpha)}(x)\} & \text{if } \alpha \in X \cap Y, \end{cases}$$

and

$$1 - f_{\tilde{H}(\alpha)}(x) = \begin{cases} 1 - f_{\tilde{F}(\alpha)}(x) & \text{if } \alpha \in X - Y, \\ 1 - f_{\tilde{G}(\alpha)}(x) & \text{if } \alpha \in Y - X, \\ 1 - \min\{f_{\tilde{F}(\alpha)}(x), f_{\tilde{G}(\alpha)}(x)\} & \text{if } \alpha \in X \cap Y, \end{cases}$$

for all $\alpha \in Z$ and $x \in U$. This is denoted by $(\tilde{H}, Z) = (\tilde{F}, X)\tilde{\cup}(\tilde{G}, Y)$.

Definition 2.12 (*[21]*). The *intersection* of two vague soft sets $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ over a universe U is a vague soft set denoted by $(\tilde{H}, Z)$, where $Z = X \cup Y$ and

$$t_{\tilde{H}(\alpha)}(x) = \begin{cases} t_{\tilde{F}(\alpha)}(x) & \text{if } \alpha \in X - Y, \\ t_{\tilde{G}(\alpha)}(x) & \text{if } \alpha \in Y - X, \\ \min\{t_{\tilde{F}(\alpha)}(x), t_{\tilde{G}(\alpha)}(x)\} & \text{if } \alpha \in X \cap Y, \end{cases}$$

and

$$1 - f_{\tilde{H}(\alpha)}(x) = \begin{cases} 1 - f_{\tilde{F}(\alpha)}(x) & \text{if } \alpha \in X - Y, \\ 1 - f_{\tilde{G}(\alpha)}(x) & \text{if } \alpha \in Y - X, \\ 1 - \max\{f_{\tilde{F}(\alpha)}(x), f_{\tilde{G}(\alpha)}(x)\} & \text{if } \alpha \in X \cap Y, \end{cases}$$

for all $\alpha \in Z$ and $x \in U$. This is denoted by $(\tilde{H}, Z) = (\tilde{F}, X)\tilde{\cap}(\tilde{G}, Y)$.

In contrast with the above definition of vague soft set intersections, we may sometimes adopt a different definition of intersection given as follows.

Definition 2.13. Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two vague soft sets over a universe U such that $X \cap Y \neq \emptyset$. The *bi-intersection* of $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ is defined to be the soft set $(\tilde{H}, Z)$, where $C = A \cap B$ and $\tilde{H}(\alpha) = \tilde{F}(\alpha) \cap \tilde{G}(\alpha)$ for all $\alpha \in Z$. This is denoted by $(\tilde{H}, Z) = (\tilde{F}, X)\tilde{\sqcap}(\tilde{G}, Y)$.

The notions of AND, OR and bi-intersection operations of vague soft sets can be extended to a family of vague soft sets over U. Let $\{(\tilde{F}_i, X_i)\}_{i \in I}$ be a non-empty family of vague soft sets over a common universe U, where I is an index set. The *AND-vague soft set* $\tilde{\wedge}_{i\in I}(\tilde{F}_i, X_i)$ of these soft sets is defined to be the soft set $(\tilde{G}, Y)$ such that $Y = \prod_{i\in I} X_i$ and $\tilde{G}(\alpha) = \cap_{i\in I} \tilde{F}_i(\alpha)$ for all $\alpha = (\alpha_i)_{i\in I} \in Y$. Similarly, the *OR-vague soft set* $\tilde{\vee}_{i\in I}(\tilde{F}_i, X_i)$ of these soft sets is defined to be the soft set $(\tilde{G}, Y)$ such that $Y = \prod_{i\in I} X_i$ and $\tilde{G}(\alpha) = \cup_{i\in I} \tilde{F}_i(\alpha)$ for all $\alpha = (\alpha_i)_{i\in I} \in Y$. The *bi-intersection* vague soft set $\sqcap_{i\in I}(\tilde{F}_i, X_i)$ of these soft sets is defined to be the soft set $(\tilde{G}, Y)$ such that $Y = \cap_{i\in I} X_i$ and $\tilde{G}(\alpha) = \cap_{i\in I} \tilde{F}_i(\alpha)$ for all $\alpha \in Y$.

Next, we introduce some new relations based on the ordering relation “ $\Subset$” on $\mathscr{V}(U)$.

Definition 2.14. For two vague soft sets $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ over a universe U, we say that $(\tilde{F}, X)$ is a *vague soft subset* in $(\tilde{F}, Y)$, if

$$X \subseteq Y \quad \text{and} \quad \tilde{F}(\varepsilon) \Subset \tilde{G}(\varepsilon)$$

for all $\varepsilon \in X$. This relationship is denoted by $(\tilde{F}, X)\tilde{\subseteq}(\tilde{G}, Y)$. Similarly, $(\tilde{F}, X)$ is said to be a *vague soft superset* in $(\tilde{G}, Y)$, if $(\tilde{G}, Y)$ is a vague soft subset in $(\tilde{F}, X)$. We denote it by $(\tilde{F}, X)\tilde{\supseteq}(\tilde{G}, Y)$.

Definition 2.15. Two vague soft sets $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ over a universe U are said to *vague soft equal* if $(\tilde{F}, X)\tilde{\subseteq}(\tilde{G}, Y)$ and $(\tilde{G}, Y)\tilde{\subseteq}(\tilde{F}, X)$.

3. $(\in, \in \vee q)$-vague left h-ideals of a hemiring

In this section, we introduce the concept of $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring and investigate some of their basic properties. Let us first provide the following definitions.

Definition 3.1. Let A and B be two vague sets in a hemiring S. Define the *h-sum* of A and B by $A +_h B = [t_A \widetilde{+_h} t_B, 1 - f_A \widehat{+_h} f_B]$, where

$$(t_A \widetilde{+_h} t_B)(x) = \sup_{x+a_1+b_1+z=a_2+b_2+z} \min\{t_A(a_1), t_A(a_2), t_B(b_1), t_B(b_2)\}$$

and

$$(f_A \widehat{+_h} f_B)(x) = \inf_{x+a_1+b_1+z=a_2+b_2+z} \max\{f_A(a_1), f_A(a_2), f_B(b_1), f_B(b_2)\}$$

for all $x \in S$.

Definition 3.2. Let A and B be two vague sets in a hemiring S. Define the *h-intrinsic product* of A and B by $A \odot_h B = [t_A \widetilde{\odot_h} t_B, 1 - f_A \widehat{\odot_h} f_B]$, where

$$(t_A \widetilde{\odot_h} t_B)(x) = \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \min\{t_A(a_i), t_A(a'_j), t_B(b_i), t_B(b'_j)\}$$

and

$$(f_A\widehat{\odot_h}f_B)(x)=\inf_{x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z}\max\{f_A(a_i),f_A(a'_j),f_B(b_i),f_B(b'_j)\}$$

for all $i=1,\ldots,m;j=1,\ldots,n$, and $(t_A\widetilde{\odot_h}t_B)(x)=0$ and $(f_A\widehat{\odot_h}f_B)(x)=1$ if x cannot be expressed as $x+\sum_{i=1}^{m}a_ib_i+z=\sum_{j=1}^{n}a'_jb'_j+z$ for all $a_i,b_i,a'_j,b'_j,z\in S$.

By the direct calculation we obtain immediately the following results.

Proposition 3.3. *Let A and B be two vague sets in a hemiring S. Then both $A+_hB$ and $A\odot_hB$ are vague sets in S.*

Lemma 3.4. *Let S be a hemiring and $A_1,A_2,B_1,B_2\in\mathscr{V}(S)$ such that $A_1\Subset A_2$ and $B_1\Subset B_2$. Then*

(1) $A_1+_hB_1\Subset A_2+_hB_2$.
(2) $A_1\odot_hB_1\Subset A_2\odot_hB_2$.
(3) $A_1\cap B_1\Subset A_2\cap B_2$.

Definition 3.5. A vague set A in a hemiring S is called an $(\in,\in\vee q)$-*vague left h-ideal* if it satisfies:

(V1a) $A+_hA\Subset A$;
(V2a) $\chi_S\odot_hA\Subset A$;
(V3a) $a_r,b_s\in t_A\to x_{\min\{r,s\}}\in\vee q\,t_A$ and $x_{\max\{r,s\}}\in\wedge q\,f_A\to a_r\in f_A$ or $b_s\in f_A$ for all $a,b,x,z\in S$ and $r,s\in(0,1]$ such that $x+a+z=b+z$.

Note that if A is an $(\in,\in\vee q)$-vague left h-ideal of S, then $t_A(0)\ge\min\{t_A(x),0.5\}$ and $f_A(0)\le\max\{f_A(x),0.5\}$. In fact, for any $x\in S$, since $0+x+0=x+0$, if $t_A(0)<r=\min\{t_A(x),0.5\}$, then $x_r\in t_A$ and $t_A(0)+r<r+r\le1$, that is, $0_r\overline{\in\vee q}\,t_A$, which contradicts to (V3a). Hence $t_A(0)\ge\min\{t_A(x),0.5\}$. Similarly, $f_A(0)\le\max\{f_A(x),0.5\}$.

Example 3.6. Let S be as in Example 2.4 in [31]. Define a vague set A in S by

$$t_A(x)=\begin{cases}0.6 & \text{if } x\in\begin{pmatrix}0&0\\0&0\end{pmatrix},\\ 0.4 & \text{if } \begin{pmatrix}a&0\\b&c\end{pmatrix}\ (a,b\in\mathbb{P},c\in\mathbb{N}),\\ 0 & \text{otherwise},\end{cases}$$

and

$$1-f_A(x)=\begin{cases}0.7 & \text{if } x\in\begin{pmatrix}0&0\\0&0\end{pmatrix},\\ 0.4 & \text{if } \begin{pmatrix}a&0\\b&c\end{pmatrix}\ (a,b\in\mathbb{P},c\in\mathbb{N}),\\ 0 & \text{otherwise}.\end{cases}$$

Then A is an $(\in,\in\vee q)$-vague left h-ideal of S.

In what follows, we provide some characterizations of $(\in,\in\vee q)$-vague left h-ideals of a hemiring S.

Theorem 3.7. *A vague set A in a hemiring S is an $(\in,\in\vee q)$-vague left h-ideal of S if and only if it satisfies the following conditions: $\forall a,b,x,y,z\in S$*

(V1b) $t_A(x+y)\ge\min\{t_A(x),t_A(y),0.5\}$ *and* $f_A(x+y)\le\max\{f_A(x),f_A(y),0.5\}$;
(V2b) $t_A(xy)\ge\min\{t_A(y),0.5\}$ *and* $f_A(xy)\le\max\{f_A(y),0.5\}$;
(V3b) $t_A(x)\ge\min\{t_A(a),t_A(b),0.5\}$ *and* $f_A(x)\le\max\{f_A(a),f_A(b),0.5\}$ *if* $x+a+z=b+z$.

Proof. Assume that A is an $(\in,\in\vee q)$-vague left h-ideal of S. Now, if there exist $x,y\in S$ such that $f_A(x+y)>\max\{f_A(x),f_A(y),0.5\}$. Choose r such that $f_A(x+y)>r>\max\{f_A(x),f_A(y),0.5\}$. Then $f_A(x+y)>r$ and $f_A(x+y)+r>r+r>1$, and so $(x+y)_r\in\wedge q\,f_A$. On the other hand, we have $f_A(0)\le\max\{f_A(x),0.5\}$ and so

$$\begin{aligned}(f_A\widehat{+_h}f_A)(x+y)&=\inf_{x+y+a_1+b_1+z=a_2+b_2+z}\max\{f_A(a_1),f_A(a_2),f_A(b_1),f_A(b_2)\}\\&\le\max\{f_A(0),f_A(x),f_A(y)\}\le\max\{f_A(x),f_A(y),0.5\}<r.\end{aligned}$$

Hence $(x+y)_r\notin f_A\widehat{+_h}f_A$, which contradicts $f_A\subseteq\wedge q\,f_A\widehat{+_h}f_A$. Therefore $f_A(x+y)\le\max\{f_A(x),f_A(y),0.5\}$. Similarly, $t_A(x+y)\ge\min\{t_A(x),t_A(y),0.5\}$. Hence, condition (V1b) is satisfied. Condition (V2b) can be similarly proved. For condition (V3b), if there exist $a,b,x,z\in S$ with $x+a+z=b+z$ and $s\in(0,1]$ such that $f_A(x)>s>\max\{f_A(a),f_A(b),0.5\}$, then $f_A(x)>s,f_A(x)+s\ge2s>1$ and $s>\max\{f_A(a),f_A(b)\}$, and so $x_s\in\wedge q\,f_A$, but $a_s\notin f_A$ and $b_s\notin f_A$, a contradiction. Hence $f_A(x)\le\max\{f_A(a),f_A(b),0.5\}$. Similarly, $t_A(x)\ge\min\{t_A(a),t_A(b),0.5\}$. Hence, condition (V3b) is valid.

Conversely, assume that the given conditions hold. For any fuzzy points x_r in S, if $x_r \in \wedge q f_A$, then $f_A(x) \geq r$ and $f_A(x) + r > 1$, and so $f_A(x) > 0.5$. Let $a_1, a_2, b_1, b_2, x, z \in S$ be such that $x + a_1 + b_1 + z = a_2 + b_2 + z$, then by conditions (V1b) and (V3b), we have

$$\begin{aligned} 0.5 < f_A(x) &\leq \max\{f_A(a_1 + b_1), f_A(a_2 + b_2), 0.5\} \\ &\leq \max\{\max\{f_A(a_1), f_A(b_1), 0.5\}, \max\{f_A(a_2), f_A(b_2), 0.5\}, 0.5\} \\ &= \max\{f_A(a_1), f_A(b_1), f_A(a_2), f_A(b_2), 0.5\}, \end{aligned}$$

which implies $f_A(x) \leq \max\{f_A(a_1), f_A(b_1), f_A(a_2), f_A(b_2)\}$. Hence we have

$$\begin{aligned} (f_A \widehat{+_h} f_A)(x) &= \inf_{x+a_1+b_1+z=a_2+b_2+z} \max\{f_A(a_1), f_A(b_1), f_A(a_2), f_A(b_2)\} \\ &\geq \inf_{x+a_1+b_1+z=a_2+b_2+z} f_A(x) = f_A(x) \geq r. \end{aligned}$$

Hence $x_r \in f_A \widehat{+_h} f_A$ and so $f_A \subseteq \wedge q f_A \widehat{+_h} f_A$. Similarly, $t_A \widetilde{+_h} t_A \subseteq \vee q\ t_A$, and so $A +_h A \Subset A$, that is, condition (V1a) is satisfied. Condition (V2a) can be similarly proved. For condition (V3a), if there exist $a, b, x, z \in S$ and $r, s \in (0, 1]$ with $x + a + z = b + z$ and $x_{\max\{r,s\}} \in \wedge q f_A$ such that $a_r \notin f_A$ and $b_s \notin f_A$, then $f_A(x) \geq \max\{r, s\} > \max\{f_A(a), f_A(b)\}$ and $2f_A(x) \geq f_A(x) + \max\{r, s\} > 1$, which imply $f_A(x) > 0.5$ and so $f_A(x) > \max\{f_A(a), f_A(b), 0.5\}$, a contradiction. Thus $x_{\max\{r,s\}} \in \wedge q f_A$ implies $a_r \in f_A$ or $b_s \in f_A$. Similarly, we may show that $a_r, b_s \in t_A$ implies $x_{\min\{r,s\}} \in \vee q\ t_A$ for $r, s \in (0, 1]$ and $x, z, a, b \in S$ such that $x + a + z = b + z$. Hence, condition (V3a) is valid. Therefore, A is an $(\in, \in \vee q)$-vague left h-ideal of S. □

Theorem 3.8. *A vague set A in a hemiring S is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if it satisfies condition* (V3a) *and $\forall x, y \in S$ and $r, s \in [0, 1]$*

(V1c) $x_r, y_s \in t_A \to (x + y)_{\min\{r,s\}} \in \vee q t_A$ *and* $(x + y)_{\max\{r,s\}} \in \wedge q f_A \to x_r \in f_A$ *or* $y_s \in f_A$;
(V2c) $y_r \in t_A \to (xy)_r \in \vee q t_A$ *and* $(xy)_r \in \wedge q f_A \to x_r \in f_A$.

Proof. The proof is similar to that of Theorem 3.7. □

For any $r \in [0, 1]$ and fuzzy subset μ in S, denote $\mu_{\hat{r}} = \{x \in S | \mu(x) > r\}$, $\langle \mu \rangle_r = \{x \in S | x_r\, q\, \mu\}$, $[\mu]_r = \{x \in S | x_r \in \vee q \mu\}$, $\widehat{\mu}_r = \{x \in S | \mu(x) < r\}$ and $\widehat{[\mu]}_r = \{x \in S | x_r \overline{\in} \vee \overline{q} \mu\}$. Clearly, $A^{(r,s)} = t_{A\hat{r}} \cap \widehat{f}_{As}$ for all $r, s \in [0, 1]$.

The next theorem presents the relationships between $(\in, \in \vee q)$-vague left h-ideals and crisp left h-ideals of a hemiring S.

Theorem 3.9. *Let S be a hemiring and A a vague set in S. Then:*

(1) *A is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if non-empty subsets $t_{A\hat{r}}$ and $\widehat{f}_{As}$ are left h-ideals of S for all $r \in [0, 0.5)$ and $s \in (0.5, 1]$.*
(2) *A is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if non-empty subsets $t_{A\hat{r}}$ and $\widehat{[f_A]}_s$ are left h-ideals of S for all $r \in [0, 0.5)$ and $s \in (0, 1]$.*
(3) *A is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if non-empty subsets $\langle t_A \rangle_r$ and $\widehat{f}_{As}$ are left h-ideals of S for all $r, s \in (0.5, 1]$.*
(4) *A is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if non-empty subsets $\langle t_A \rangle_r$ and $\widehat{[f_A]}_s$ are left h-ideals of S for all $r \in (0.5, 1]$ and $s \in (0, 1]$.*
(5) *A is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if non-empty subsets $[t_A]_r$ and $\widehat{f}_{As}$ are left h-ideals of S for all $r \in (0, 1]$ and $s \in (0.5, 1]$.*
(6) *A is an $(\in, \in \vee q)$-vague left h-ideal of S if and only if non-empty subsets $[t_A]_r$ and $\widehat{[f_A]}_s$ are left h-ideals of S for all $r, s \in (0, 1]$.*

Proof. We only show (1) and (2). The other properties can be similarly proved.

(1) Let A be an $(\in, \in \vee q)$-vague left h-ideal of S and assume that $t_{A\hat{r}} \neq \emptyset$ and $\widehat{f}_{As} \neq \emptyset$ for some $r \in [0, 0.5)$ and $s \in (0.5, 1]$. We only show that $\widehat{f}_{As}$ is a left h-ideal of S. The case for $t_{A\hat{r}}$ can be similarly proved. Let $x, y \in \widehat{f}_{As}$. Then $f_A(x) < s$ and $f_A(y) < s$. Since A is an $(\in, \in \vee q)$-vague left h-ideal of S, we have $f_A(x+y) \leq \max\{f_A(x), f_A(y), 0.5\} < s$. Hence $x+y \in \widehat{f}_{As}$. In a similar way, we may show that $xy, yx \in \widehat{f}_{As}$ for all $x \in \widehat{f}_{As}$ and $y \in S$ and that $x + a + z = b + z$ for $x, z \in S$ and $a, b \in \widehat{f}_{As}$ implies $x \in \widehat{f}_{As}$. Therefore, $\widehat{f}_{As}$ is a left h-ideal of S.

Conversely, assume that the given conditions hold. We only show that f_A satisfies conditions (V1b)–(V3b). The case for t_A can be similarly proved. Now, let $x, y \in S$. Choose $s = \max\{f_A(x), f_A(y), 0.5\} + \varepsilon$, where $\varepsilon > 0$. Then $s \in (0.5, 1]$ and $x, y \in \widehat{f}_{As}$. Since $\widehat{f}_{As}$ is a left h-ideal of S, we have $x + y \in \widehat{f}_{As}$, and so $f_A(x + y) < s = \max\{f_A(x), f_A(y), 0.5\} + \varepsilon$. Hence $f_A(x + y) \leq \max\{f_A(x), f_A(y), 0.5\}$ since ε is arbitrary. Similarly, we may show that $f_A(xy) \leq \max\{f_A(y), 0.5\}$ for all $x, y \in S$ and that $f_A(x) \leq \max\{f_A(a), f_A(b), 0.5\}$ for all $x, z, a, b \in S$ such that $x + a + z = b + z$. It follows from Theorem 3.7 that A is an $(\in, \in \vee q)$-vague left h-ideal of S.

(2) Let A be an $(\in, \in \vee q)$-vague left h-ideal of S and assume that $t_{A\hat{r}} \neq \emptyset$ and $\widehat{[f_A]}_s \neq \emptyset$ for some $r \in [0, 0.5)$ and $s \in (0, 1]$. By (1), is suffices to show that $\widehat{[f_A]}_s$ is a left h-ideal of S, let $x, y \in \widehat{[f_A]}_s$. Then $x_s \overline{\in} \vee \overline{q} f_A$ and $y_s \overline{\in} \vee \overline{q} f_A$, that is, $f_A(x) < s$ or $f_A(x) + s \leq 1$, and $f_A(y) < s$ or $f_A(y) + s \leq 1$. We consider the following cases.

Case 1. $s \in (0, 0.5]$. Then $1 - s \geq 0.5 \geq s$.

(1) If $f_A(x) < s$ and $f_A(y) < s$, then $f_A(x + y) \leq \max\{f_A(x), f_A(y), 0.5\} = 0.5 \leq 1 - s$, that is, $(x + y)_s \, \overline{q} \, f_A$.
(2) If $f_A(x) + s \leq 1$ or $f_A(y) + s \leq 1$, then $f_A(x + y) \leq \max\{f_A(x), f_A(y), 0.5\} \leq 1 - s$, that is, $(x + y)_s \, \overline{q} \, f_A$.

Case 2. $s \in (0.5, 1]$. Then $s > 0.5 > 1 - s$.

(1) If $f_A(x) < s$ or $f_A(y) < s$, then $f_A(x + y) \leq \max\{f_A(x), f_A(y), 0.5\} < s$, that is, $(x + y)_s \overline{\in} f_A$.
(2) If $f_A(x)+s \leq 1$ and $f_A(y)+s \leq 1$, then $f_A(x+y) \leq \max\{f_A(x), f_A(y), 0.5\} \leq \max\{1-s, 1-s, 0.5\} < s$, that is, $(x+y)_s \overline{\in} f_A$.

Thus, in any case, $(x + y)_s \overline{\in} \vee \overline{q} f_A$, that is, $x + y \in \widehat{[f_A]}_s$. Similarly, we can show that $xy, yx \in \widehat{[f_A]}_s$ for all $x \in \widehat{[f_A]}_s$ and $y \in S$ and that $x + a + z = b + z$ for $x, z \in S$ and $a, b \in \widehat{[f_A]}_s$ imply $x \in \widehat{[f_A]}_s$. Therefore, $\widehat{[f_A]}_s$ is a left h-ideal of S.

Conversely, assume that the given conditions hold. By (1), conditions (V1b)–(V3b) hold for t_A. Now, if there exist $x, y \in S$ and $s \in (0, 1]$ such that $f_A(x + y) > s > \max\{f_A(x), f_A(y), 0.5\}$. Then $x_s, y_s \overline{\in} f_A$ but $(x + y)_s \in \wedge q f_A$, that is, $x, y \in \widehat{[f_A]}_s$ but $x, y \notin \widehat{[f_A]}_s$, a contradiction. Hence $f_A(x + y) \leq \max\{f_A(x), f_A(y), 0.5\}$. Similarly, we may show that $f_A(xy) \leq \max\{f_A(y), 0.5\}$ for all $x, y \in S$ and that $f_A(x) \leq \max\{f_A(a), f_A(b), 0.5\}$ for all $x, z, a, b \in S$ such that $x + a + z = b + z$. It follows from Theorem 3.7 that A is an $(\in, \in \vee q)$-vague left h-ideal of S. □

Next let us consider the $(\in, \in \vee q)$-vague left h-ideals of S induced by an $(\in, \in \vee q)$-vague left h-ideal of S.

Proposition 3.10. *If A is an $(\in, \in \vee q)$-vague left h-ideal of a hemiring S, then so are (1) $\Box A$, (2) $\Diamond A$, (3) $P_{r,t}(A)$, (4) $Q_{r,t}(A)$, where $r, t \in [0, 1]$ and $r + t \leq 1$.*

Proof. It is straightforward. □

In view of Proposition 3.10, it is easy to verify that the following theorem is valid.

Theorem 3.11. *A vague fuzzy set A in S is an $(\in, \in \vee q)$-vague left h-ideal of a hemiring S if and only if $\Box A$ and $\Diamond A$ are $(\in, \in \vee q)$-vague left h-ideals of S.*

Corollary 3.12. *$A = [\chi_S, 1 - \chi_S^c]$ is an $(\in, \in \vee q)$-vague left h-ideal of a hemiring S.*

Theorem 3.13. *A non-empty set P in a hemiring S is a left h-ideal of S if and only if $A = [\chi_P, 1 - \chi_P^c]$ is an $(\in, \in \vee q)$-vague left h-ideal of S.*

Proof. It is straightforward. □

Theorem 3.14. *Let A and B be $(\in, \in \vee q)$-vague left h-ideals of a hemiring S. Then so are $A \cap B = [t_A \cap t_B, 1 - (f_A \cup f_B)]$ and $A +_h B = [t_A \widetilde{+}_h t_B, 1 - (f_A \widehat{+}_h f_B)]$.*

Proof. We show that $A +_h B$ is an $(\in, \in \vee q)$-vague left h-ideal of a hemiring S. The proof for $A \cap B$ is straightforward.

(1) For any $x, y \in S$, we have

$$
\begin{aligned}
(t_A \widetilde{+}_h t_B)(x + y) &= \sup_{x+y+a_1+b_1+z=a_2+b_2+z} \min\{t_A(a_1), t_A(a_2), t_B(b_1), t_B(b_2)\} \\
&\geq \min\left\{ \sup_{x+c_1+d_1+z_1=c_2+d_2+z_1} \min\{t_A(c_1), t_A(c_2), t_B(d_1), t_B(d_2)\}, \right. \\
&\qquad \left. \sup_{y+e_1+f_1+z_2=e_2+f_2+z_2} \min\{t_A(e_1), t_A(e_2), t_B(f_1), t_B(f_2)\}, 0.5 \right\} \\
&= \min\{(t_A \widetilde{+}_h t_B)(x), (t_A \widetilde{+}_h t_B)(y), 0.5\}.
\end{aligned}
$$

Similarly, $(f_A \widehat{+}_h f_B)(x + y) \leq \max\{(f_A \widehat{+}_h f_B)(x), (f_A \widehat{+}_h f_B)(y), 0.5\}$.

(2) For any $x, y \in S$, we have

$$
\begin{aligned}
\min\{(t_A \widetilde{+}_h t_B)(y), 0.5\} &= \min\left\{ \sup_{y+a_1+b_1+z=a_2+b_2+z} \min\{t_A(a_1), t_A(a_2), t_B(b_1), t_B(b_2)\}, 0.5 \right\} \\
&\leq \sup_{xy+xa_1+xb_1+xz=xa_2+xb_2+xz} \min\{t_A(xa_1), t_A(xa_2), t_B(xb_1), t_B(xb_2)\} \\
&\leq \sup_{xy+c_1+d_1+z_1=c_2+d_2+z_1} \min\{t_A(c_1), t_A(c_2), t_B(d_1), t_B(d_2)\} \\
&= (t_A \widetilde{+}_h t_B)(xy).
\end{aligned}
$$

Similarly, $(f_A \widehat{+}_h f_B)(xy) \leq \max\{(f_A \widehat{+}_h f_B)(y), 0.5\}$.

(3) Let a, b, x and z_1 be any elements of S such that $x+a+z_1=b+z_1$, and let $c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, z_2, z_3 \in S$ be such that

$$a+c_1+d_1+z_2=c_2+d_2+z_2 \quad \text{and} \quad b+e_1+f_1+z_3=e_2+f_2+z_3.$$

Then we have $x+c_2+d_2+e_1+f_1+z_4=c_1+d_1+e_2+f_2+z_4$, where $z_4=z_1+z_2+z_3$, and so

$$\begin{aligned}(t_A\widetilde{+}_h t_B)(x) &= \sup_{x+a_1+b_1+z=a_2+b_2+z} \min\{t_A(a_1), t_A(a_2), t_B(b_1), t_B(b_2)\}\\ &\geq \min\{t_A(c_2+e_1), t_A(c_1+e_2), t_B(d_2+f_1), t_B(d_1+f_2)\}\\ &\geq \min\{\min\{t_A(c_2), t_A(e_1), 0.5\}, \min\{t_A(c_1), t_A(e_2), 0.5\},\\ &\quad \min\{t_B(d_2), t_B(f_1), 0.5\}, \min\{t_B(d_1), t_B(f_2), 0.5\}\}\\ &= \min\{\min\{t_A(c_1), t_A(c_2), t_B(d_1), t_B(d_2)\}, \min\{t_A(e_1), t_A(e_2), t_B(f_1), t_B(f_2)\}, 0.5\},\end{aligned}$$

this gives

$$\begin{aligned}(t_A\widetilde{+}_h t_B)(x) &\geq \min\left\{\sup_{a+c_1+d_1+z_2=c_2+d_2+z_2} \min\{t_A(c_1), t_A(c_2), t_B(d_1), t_B(d_2)\},\right.\\ &\quad \left.\sup_{b+e_1+f_1+z_3=e_2+f_2+z_3} \min\{t_A(e_1), t_A(e_2), t_B(f_1), t_B(f_2)\}, 0.5\right\}\\ &= \min\{(t_A\widetilde{+}_h t_B)(a), (t_A\widetilde{+}_h t_B)(b), 0.5\}.\end{aligned}$$

Similarly, $(f_A\widehat{+}_h f_B)(x) \leq \max\{(f_A\widehat{+}_h f_B)(a), (f_A\widehat{+}_h f_B)(b), 0.5\}$.

Summing up the above statements, $A+_h B$ is an $(\in, \in \vee q)$-vague left h-ideal of a hemiring S. $\square$

In what follows, we denote by $GVLI(S)$ the set of all $(\in, \in \vee q)$-vague left h-ideals of a hemiring S with the same tip, that is, $t_A(0)=t_B(0)$ and $f_A(0)=f_B(0)$ for all $A, B \in GVLI(S)$. Theorem 3.14 gives that $(GVLI(S), +_h)$ is a semigroup.

Theorem 3.15. *Let S be a hemiring. Then $(GVLIT(S), +_h, \cap)$ is a complete lattice with minimal element $[\chi_S^c, 1-\chi_S^c]$ and maximal element $[\chi_S, 1-\chi_S]$ under the ordering relation "$\Subset$".*

Proof. Let $A, B \in GVLIT(S)$. It follows from Theorem 3.14 that $A \cap B \in GVLIT(S)$ and $A+_h B \in GVLIT(S)$. It is clear that $A \cap B$ is the greatest lower bound of A and B. We now show that $A+_h B$ is the least upper bound of A and B. Since $t_A(0)=t_B(0)$, for any $x \in S$, we have

$$\begin{aligned}(t_A\widetilde{+}_h t_B)(x) &= \sup_{x+a_1+b_1+z=a_2+b_2+z} \min\{t_A(a_1), t_A(a_2), t_B(b_1), t_B(b_2)\}\\ &\geq \min\{t_A(0), t_A(x), t_B(0), t_B(0)\} = \min\{t_A(0), t_A(x)\}\\ &\geq \min\{t_A(x), 0.5\},\end{aligned}$$

hence $t_A \subseteq \vee q\ t_A\widetilde{+}_h t_B$. Similarly, $f_A\widehat{+}_h f_B \subseteq \wedge q f_A$. Thus $A \Subset A+_h B$. In a similar way, we have $B \Subset A+_h B$. Now, let $C \in GVLIT(S)$ be such that $A, B \Subset C$. Then, we have $A+_h B \Subset C+_h C \Subset C$. Hence $A \vee B = A+_h B$. There is no difficulty in replacing the $\{A, B\}$ with an arbitrary family of $GVLIT(S)$ and so $(GVLIT(S), +_h, \cap)$ is a complete lattice under the relation "$\Subset$". It is easy to see that $[\chi_S^c, 1-\chi_S^c]$ and $[\chi_S, 1-\chi_S]$ are the minimal element and maximal element in $(GVLIT(S), +_h, \cap)$, respectively. $\square$

Theorem 3.16. *Given any chain of left h-ideals $S_0 \subset S_1 \subset \cdots \subset S_n = S$ of a hemiring S, there exists an $(\in, \in \vee q)$-vague left h-ideal of S whose non-empty strong level sets are precisely the members of the chain with $A^{(0.5,0.5)} = S_0$.*

Proof. Let $\{r_i | r_i \in (0, 0.5), i=1, 2, \ldots, n\}$ and $\{t_i | t_i \in (0, 0.5), i=1, 2, \ldots, n\}$ be such that $r_1 > r_2 > \cdots > r_n$, $t_1 < t_2 < \cdots < t_n$ and $r_i + t_i \leq 1$ for all $i=1, \ldots, n$. Let t_A and f_A be fuzzy subsets in S such that

$$t_A(x) = \begin{cases} r_0 > 0.5 & \text{if } x \in S_0,\\ r_1 & \text{if } x \in S_1 - S_0,\\ \cdots & \\ r_n & \text{if } x \in S_n - S_{n-1},\end{cases} \qquad f_A(x) = \begin{cases} t_0 < 0.5 & \text{if } x \in S_0,\\ t_1 & \text{if } x \in S_1 - S_0,\\ \cdots & \\ t_n & \text{if } x \in S_n - S_{n-1},\end{cases}$$

for all $x \in S$. Then it is easy to see that $A = [t_A, 1-f_A]$ is an vague set in S, and

$$t_{Ar} = \begin{cases} S_0 & \text{if } r \in [r_1, r_0),\\ S_1 & \text{if } r \in [r_2, r_1),\\ \cdots & \\ S_n & \text{if } r \in [0, r_n),\end{cases} \qquad \widehat{f_{At}} = \begin{cases} S_0 & \text{if } t \in (t_0, t_1],\\ S_1 & \text{if } t \in (t_1, t_2],\\ \cdots & \\ S_n & \text{if } t \in (t_n, 1].\end{cases}$$

Hence, for any $r \in [0, r_0)$ and $t \in (t_0, 1]$, there exist $i, j \in \{1, 2, \cdots, n\}$ such that $r \in [r_i, r_{i-1})$ and $t \in (t_j, t_{j+1}]$. Without loss of generality, we may assume that $i \le j$, then $t_{A\widehat{r}} = S_{i-1}, \widehat{f}_{At} = S_j$ and $A^{(r,t)} = t_{A\widehat{r}} \cap \widehat{f}_{At} = S_{i-1} \cap S_j = S_{i-1}$. Thus, our assumption and Theorem 3.9 give that A is an $(\in, \in \vee q)$-vague left h-ideal of S whose non-empty strong level sets are precisely the members of the chain. Clearly, $A^{(0.5,0.5)} = S_0$. □

4. $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring

In this section, we consider $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring. Let us first introduce the following definition.

Definition 4.1. Let $(\tilde{F}, X)$ be a vague soft set over a hemiring S. Then $(\tilde{F}, X)$ is called an $(\in, \in \vee q)$-*vague soft left h-ideal* over S if $\tilde{F}(\alpha)$ is an $(\in, \in \vee q)$-vague left h-ideal of S for all $\alpha \in X$.

Example 4.2. The set $\mathbb{N}_0$ of all non-negative integers with usual addition and multiplication is a hemiring. Define a vague soft set $(\tilde{F}, X)$ over $\mathbb{N}_0$, where $X = \mathbb{N}_0$, by

$$t_{\tilde{F}(\alpha)}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1}{\alpha} & \text{if } x \in (\alpha) - \{0\}, \\ 0 & \text{otherwise}, \end{cases} \qquad 1 - f_{\tilde{F}(\alpha)}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1}{\alpha} & \text{if } x \in (\alpha) - \{0\}, \\ 0 & \text{otherwise}, \end{cases}$$

for all $\alpha, x \in \mathbb{N}_0$. Then $(\tilde{F}, X)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over $\mathbb{N}_0$.

Proposition 4.3. *Let $(\tilde{F}, X)$ be a vague soft set over a hemiring S and let $Y \subset X$. If $(\tilde{F}, X)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S, then so is $(\tilde{F}, Y)$ whenever it is non-null.*

Proof. It is straightforward by Definition 4.1. □

Theorem 4.4. *Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S. Then $(\tilde{F}, X)\tilde{\wedge}(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. By Definition 2.9, we can write $(\tilde{F}, X)\tilde{\wedge}(\tilde{G}, Y) = (\tilde{H}, Z)$, where $Z = X \times Y$ and $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta)$ for all $(\alpha, \beta) \in Z$. Now for any $(\alpha, \beta) \in Z$, since $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ are $(\in, \in \vee q)$-vague soft left h-ideals over S, we have both $\tilde{F}(\alpha)$ and $\tilde{G}(\beta)$ are $(\in, \in \vee q)$-vague left h-ideals of S. Thus it follows from Theorem 3.14 that $\tilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta)$ is an $(\in, \in \vee q)$-vague left h-ideal of S. Therefore, $(\tilde{F}, X)\tilde{\wedge}(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S. □

Theorem 4.5. *Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S. If for any $\alpha \in X$ and $\beta \in Y$, either $\tilde{F}(\alpha) \subseteq \tilde{G}(\beta)$ or $\tilde{G}(\beta) \subseteq \tilde{F}(\alpha)$, then $(\tilde{F}, X)\tilde{\vee}(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. By Definition 2.10, we can write $(\tilde{F}, X)\tilde{\vee}(\tilde{G}, Y) = (\tilde{O}, Z)$, where $Z = X \times Y$ and $\tilde{O}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta)$ for all $(\alpha, \beta) \in Z$. Now for any $(\alpha, \beta) \in Z$, by the assumption, either $\tilde{F}(\alpha) \subseteq \tilde{G}(\beta)$ or $\tilde{G}(\beta) \subseteq \tilde{F}(\alpha)$, say $\tilde{F}(\alpha) \subseteq \tilde{G}(\beta)$, that is, $t_{\tilde{F}(\alpha)} \subseteq t_{\tilde{G}(\beta)}$ and $f_{\tilde{G}(\beta)} \subseteq f_{\tilde{F}(\alpha)}$. Now, for any $x, y \in S$, we have

$$\begin{aligned} t_{\tilde{O}(\alpha,\beta)}(x+y) &= (t_{\tilde{F}(\alpha)} \cup t_{\tilde{G}(\beta)})(x+y) = \max\{t_{\tilde{F}(\alpha)}(x+y), t_{\tilde{G}(\beta)}(x+y)\} \\ &= t_{\tilde{G}(\beta)}(x+y) \ge \min\{t_{\tilde{G}(\beta)}(x), t_{\tilde{G}(\beta)}(y), 0.5\} \\ &= \min\{(t_{\tilde{F}(\alpha)} \cup t_{\tilde{G}(\beta)})(x), (t_{\tilde{F}(\alpha)} \cup t_{\tilde{G}(\beta)})(y), 0.5\} \\ &= \min\{t_{\tilde{O}(\alpha,\beta)}(x), t_{\tilde{O}(\alpha,\beta)}(y), 0.5\} \end{aligned}$$

and

$$\begin{aligned} f_{\tilde{O}(\alpha,\beta)}(x+y) &= (f_{\tilde{F}(\alpha)} \cap f_{\tilde{G}(\beta)})(x+y) = \min\{f_{\tilde{F}(\alpha)}(x+y), f_{\tilde{G}(\beta)}(x+y)\} \\ &= f_{\tilde{G}(\beta)}(x+y) \le \max\{f_{\tilde{G}(\beta)}(x), f_{\tilde{G}(\beta)}(y), 0.5\} \\ &= \max\{(f_{\tilde{F}(\alpha)} \cap f_{\tilde{G}(\beta)})(x), (f_{\tilde{F}(\alpha)} \cap f_{\tilde{G}(\beta)})(y), 0.5\} \\ &= \max\{f_{\tilde{O}(\alpha,\beta)}(x), f_{\tilde{O}(\alpha,\beta)}(y), 0.5\}. \end{aligned}$$

Similarly, we may show that $t_{\tilde{O}(\alpha,\beta)}(xy) \ge \min\{t_{\tilde{O}(\alpha,\beta)}(y), 0.5\}$ and $f_{\tilde{O}(\alpha,\beta)}(xy) \le \max\{f_{\tilde{O}(\alpha,\beta)}(y), 0.5\}$ for all $x, y \in S$, and that $t_{\tilde{O}(\alpha,\beta)}(x) \ge \min\{t_{\tilde{O}(\alpha,\beta)}(a), t_{\tilde{O}(\alpha,\beta)}(b), 0.5\}$ and $f_{\tilde{O}(\alpha,\beta)}(x) \le \max\{f_{\tilde{O}(\alpha,\beta)}(a), f_{\tilde{O}(\alpha,\beta)}(b), 0.5\}$ for all $x, z, a, b \in S$ such that $x + a + z = b + z$.

Therefore, $(\tilde{F}, X)\tilde{\vee}(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S. □

If there exist $\alpha \in X$ and $\beta \in Y$ such that $\tilde{F}(\alpha)\overline{\subseteq}\tilde{G}(\beta)$ and $\tilde{G}(\beta)\overline{\subseteq}\tilde{F}(\alpha)$ in Theorem 4.5, then Theorem 4.5 is not true in general as seen in the following example.

Example 4.6. Let $\mathbb{N}_0/_{(6)} = \{[0], [1], \ldots, [5]\}$ be the hemiring of non-negative integers module 6 with usual addition and multiplication. Let $X = \mathbb{N}_0/_{(6)}$ and $Y = \{[2]\}$. Define vague soft sets $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ over $\mathbb{N}_0/_{(6)}$ by

$$t_{\tilde{F}([\alpha])}(x) = \begin{cases} 1 & \text{if } x = [0], \\ \dfrac{1}{\alpha} & \text{if } x \in ([\alpha]) - \{[0]\}, \\ 0 & \text{otherwise}, \end{cases} \qquad 1 - f_{\tilde{F}([\alpha])}(x) = \begin{cases} 1 & \text{if } x = [0], \\ \dfrac{1}{\alpha} & \text{if } x \in ([\alpha]) - \{[0]\}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$t_{\tilde{G}[(2)]}(x) = \begin{cases} \dfrac{1}{3} & \text{if } x \in [(3)], \\ 0 & \text{otherwise}, \end{cases} \qquad 1 - f_{\tilde{G}[(2)]}(x) = \begin{cases} \dfrac{1}{3} & \text{if } x \in [(3)], \\ 0 & \text{otherwise}, \end{cases}$$

for all $\alpha \in \mathbb{N}_0$ and $x \in \mathbb{N}_0/_{(6)}$. Then both $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ are $(\in, \in \vee q)$-vague soft left h-ideals over $\mathbb{N}_0/_{(6)}$, and $\tilde{F}[(2)] \overline{\subseteq} \tilde{G}[(2)]$ and $\tilde{G}[(2)] \overline{\subseteq} \tilde{F}[(2)]$. However, $(\tilde{O}, Z) = (\tilde{F}, X) \tilde{\vee} (\tilde{G}, Y)$ is not an $(\in, \in \vee q)$-vague soft left h-ideal over $\mathbb{N}_0/_{(6)}$ since

$$t_{\tilde{O}([2]\times[2])}([2] + [3]) = \max\{t_{\tilde{F}[(2)]}([2] + [3]), t_{\tilde{G}[(2)]}([2] + [3])\} = 0,$$

while

$$\begin{aligned} \min\{t_{\tilde{O}([2]\times[2])}[(2)], t_{\tilde{O}([2]\times[2])}[(3)], 0.5\} &= \min\{\max\{t_{\tilde{F}[(2)]}[(2)], t_{\tilde{G}[(2)]}[(2)]\}, \max\{t_{\tilde{F}[(2)]}[(3)], t_{\tilde{G}[(2)]}[(3)]\}, 0.5\} \\ &= \min\left\{\max\left\{\frac{1}{2}, 0\right\}, \max\left\{0, \frac{1}{3}\right\}, 0.5\right\} = \frac{1}{3}, \end{aligned}$$

that is, $t_{\tilde{O}([2]\times[2])}([2] + [3]) < \min\{t_{\tilde{O}([2]\times[2])}[(2)], t_{\tilde{O}([2]\times[2])}[(3)], 0.5\}$.

Theorem 4.7. *Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S. If $X \cap Y \neq \emptyset$, then $(\tilde{F}, X) \tilde{\sqcap} (\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. By Definition 2.13, we can write $(\tilde{F}, X) \tilde{\sqcap} (\tilde{G}, Y) = (\tilde{H}, Z)$, where $Z = X \cap Y$ and $\tilde{H}(\alpha) = \tilde{F}(\alpha) \cap \tilde{G}(\alpha)$ for all $\alpha \in Z$. Now for any $\alpha \in Z$, since $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ are $(\in, \in \vee q)$-vague soft left h-ideals over S, we have both $\tilde{F}(\alpha)$ and $\tilde{G}(\alpha)$ are $(\in, \in \vee q)$-vague left h-ideals of S. Thus it follows from Theorem 3.14 that $\tilde{H}(\alpha) = \tilde{F}(\alpha) \cap \tilde{G}(\alpha)$ is an $(\in, \in \vee q)$-vague left h-ideal of S. Therefore, $(\tilde{F}, X) \tilde{\sqcap} (\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S. □

As a generalization of Theorems 4.4–4.5 and 4.7, we have the following result.

Theorem 4.8. *Let $(\tilde{F}_i, X_i)_{i \in I}$ be a family of $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S, where I is an index set. Then*

(1) *$\tilde{\wedge}_{i \in I} (\tilde{F}_i, X_i)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*
(2) *If either $\tilde{F}_i(\alpha_i) \subseteq \tilde{F}_j(\alpha_j)$ or $\tilde{F}_j(\alpha_j) \subseteq \tilde{F}_i(\alpha_i)$ for all $(\alpha_i)_{i \in I} \in \prod_{i \in I} X_i$ and $i, j \in I$, then $\tilde{\vee}_{i \in I} (\tilde{F}_i, X_i)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*
(3) *If $\cap_{i \in I} X_i \neq \emptyset$, then $\tilde{\sqcap}_{i \in I} (\tilde{F}_i, X_i)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. The proof is similar to that of Theorems 4.4–4.5 and 4.7. □

Theorem 4.9. *Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S. Then $(\tilde{F}, X) \tilde{\cap} (\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. By Definition 2.12, we can write $(\tilde{F}, X) \tilde{\cap} (\tilde{G}, Y) = (\tilde{H}, Z)$, where $Z = X \cup Y$ and

$$\tilde{H}(\alpha) = \begin{cases} \tilde{F}(\alpha) & \text{if } \alpha \in X - Y, \\ \tilde{G}(\alpha) & \text{if } \alpha \in Y - X, \\ \tilde{F}(\alpha) \cap \tilde{G}(\alpha) & \text{if } \alpha \in X \cap Y. \end{cases}$$

for all $\alpha \in Z$.

Now for any $\alpha \in Z$, we consider the following cases.

Case 1. $\alpha \in X - Y$. Then $\tilde{H}(\alpha) = \tilde{F}(\alpha)$ is an $(\in, \in \vee q)$-vague left h-ideal of S since $(\tilde{F}, X)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.

Case 2. $\alpha \in Y - X$. Then $\tilde{H}(\alpha) = \tilde{G}(\alpha)$ is an $(\in, \in \vee q)$-vague left h-ideal of S since $(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.

Case 3. $\alpha \in X \cap Y$. Then $\tilde{H}(\alpha) = \tilde{F}(\alpha) \cap \tilde{G}(\alpha)$ is an $(\in, \in \vee q)$-vague left h-ideal of S by the assumption and Theorem 4.13.

Thus, in any case, $\tilde{H}(\alpha)$ is an $(\in, \in \vee q)$-vague left h-ideal of S. Therefore, $(\tilde{F}, X) \tilde{\cap} (\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S. □

Theorem 4.10. *Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S. If X and Y are disjoint, then $(\tilde{F}, X)\tilde{\cup}(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. It is straightforward by the proof of Theorem 4.9. $\square$

Note that Example 4.6 also shows that Theorem 4.10 is not true in general if X and Y are not disjoint.
Next let us introduce the h-sum of two vague soft sets over a hemiring S.

Definition 4.11. Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two vague soft sets over a hemiring S. The *h-sum* of $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ is defined to be the vague soft set $(\tilde{H}, Z)$, where $Z = X \cup Y$ and

$$t_{\tilde{H}(\alpha)}(x) = \begin{cases} t_{\tilde{F}(\alpha)}(x) & \text{if } \alpha \in X - Y, \\ t_{\tilde{G}(\alpha)}(x) & \text{if } \alpha \in Y - X, \\ (t_{\tilde{F}(\alpha)} \widetilde{+}_h t_{\tilde{G}(\alpha)})(x) & \text{if } \alpha \in X \cap Y, \end{cases}$$

and

$$1 - f_{\tilde{H}(\alpha)}(x) = \begin{cases} 1 - f_{\tilde{F}(\alpha)}(x) & \text{if } \alpha \in X - Y, \\ 1 - f_{\tilde{G}(\alpha)}(x) & \text{if } \alpha \in Y - X, \\ 1 - (f_{\tilde{F}(\alpha)} \widehat{+}_h f_{\tilde{G}(\alpha)})(x) & \text{if } \alpha \in X \cap Y, \end{cases}$$

for all $\alpha \in Z$ and $x \in S$, that is,

$$\tilde{H}(\alpha) = \begin{cases} \tilde{F}(\alpha) & \text{if } \alpha \in X - Y, \\ \tilde{G}(\alpha) & \text{if } \alpha \in Y - X, \\ \tilde{F}(\alpha) +_h \tilde{G}(\alpha) & \text{if } \alpha \in X \cap Y. \end{cases}$$

for all $\alpha \in Z$. This is denoted by $(\tilde{H}, Z) = (\tilde{F}, X) +_h (\tilde{G}, Y)$.

By Theorem 3.14, we can obtain the following result.

Theorem 4.12. *Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two $(\in, \in \vee q)$-vague soft left h-ideals over a hemiring S. Then $(\tilde{F}, X) +_h (\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. The proof is similar to that of Theorem 4.9. $\square$

Denote by $GVSLIT(S)$ the set of all $(\in, \in \vee q)$-vague soft left h-ideals with the same tip t, that is, $t_{\tilde{F}(\alpha)}(0) = t_{\tilde{G}(\beta)}(0)$ and $f_{\tilde{F}(\alpha)}(0) = f_{\tilde{G}(\beta)}(0)$ for all $(\tilde{F}, X), (\tilde{G}, Y) \in GVSLIT(S)$ and $\alpha \in X$, $\beta \in Y$. Then we have the following result.

Theorem 4.13. *Let S be a hemiring. Then $(GVSLIT(S), +_h, \tilde{\sqcap})$ is lattice under the ordering relation "$\tilde{\subseteq}$".*

Proof. The proof is similar to that Theorem 3.15. $\square$

5. The homomorphism properties of $(\in, \in \vee q)$-vague (soft) left h-ideals of a hemiring

Definition 5.1. Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ a homomorphism of hemirings. A left h-ideal L of S is called *φ-compatible* if, for all $x, z, \in S$ and $a, b \in A$, $\varphi(x + a + z) = \varphi(b + z)$ implies $x \in A$. An $(\in, \in \vee q)$-vague left h-ideal A of S is called *φ-compatible* if, for all $x, z, a, b \in S$, $\varphi(x + a + z) = \varphi(b + z)$ implies

$$t_A(x) \geq \min\{t_A(a), t_A(b), 0.5\} \quad \text{and} \quad f_A(x) \leq \max\{f_A(a), f_A(b), 0.5\}.$$

The next theorem presents the relationships between φ-compatible $(\in, \in \vee q)$-vague left h-ideals and φ-compatible left h-ideals of a hemiring.

Example 5.2. Denote by $\mathbb{N}_0$ and $\mathbb{N}_0/_{(10)}$ the hemiring of non-negative integers and the hemiring of non-negative integers module 10, respectively. Define a mapping $\varphi : \mathbb{N}_0 \to \mathbb{N}_0/_{(10)}$ by $\varphi(x) = [x]$. Then φ is an epimorphism of $\mathbb{N}_0$ onto $\mathbb{N}_0/_{(10)}$. Now define a vague set A in $\mathbb{N}_0$ by

$$t_A(x) = \begin{cases} 0.6 & \text{if } x = [(10)], \\ 0.2 & \text{otherwise}, \end{cases} \qquad 1 - f_A(x) = \begin{cases} 0.6 & \text{if } x = [(10)], \\ 0.2 & \text{otherwise}, \end{cases}$$

for all $x \in \mathbb{N}_0$. Then A is an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of $\mathbb{N}_0$.

Theorem 5.3. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings and A an $(\in, \in \vee q)$-vague left h-ideal of S. Then*

(1) *A is φ-compatible if and only if non-empty subsets $t_{A\hat{r}}$ and $\widehat{f_{A_s}}$ are φ-compatible for all $r \in [0, 0.5)$ and $s \in (0.5, 1]$.*
(2) *A is φ-compatible if and only if non-empty subsets $t_{A\hat{r}}$ and $\widehat{[f_A]}_s$ are φ-compatible for all $r \in [0, 0.5)$ and $s \in (0, 1]$.*
(3) *A is φ-compatible if and only if non-empty subsets $\langle t_A \rangle_r$ and $\widehat{f_{A_s}}$ are φ-compatible for all $r, s \in (0.5, 1]$.*

(4) *A is φ-compatible if and only if non-empty subsets $\langle t_A\rangle_r$ and $[\widehat{f_A}]_s$ are φ-compatible for all $r \in (0.5, 1]$, $s \in (0, 1]$.*
(5) *A is φ-compatible if and only if non-empty subsets $[t_A]_r$ and $\widehat{f_{A_s}}$ are φ-compatible for all $r \in (0, 1]$ and $s \in (0.5, 1]$.*
(6) *A is φ-compatible if and only if non-empty subsets $[t_A]_r$ and $[\widehat{f_A}]_s$ are φ-compatible for all $r, s \in (0, 1]$.*

Proof. The proof is similar to that of Theorem 3.9. □

Proposition 5.4. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ a mapping from S into S'. Let A and B be vague sets in S and S', respectively. Then the image $\varphi(A) = [\varphi(t_A), 1 - \varphi(f_A)]$ of A is a vague set in S' defined by*

$$\varphi(t_A)(x') = \begin{cases} \sup\limits_{x\in\varphi^{-1}(x')} t_A(x) & \text{if } \varphi^{-1}(x') \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$1 - \varphi(f_A)(x') = \begin{cases} 1 - \inf\limits_{x\in\varphi^{-1}(x')} f_A(x) & \text{if } \varphi^{-1}(x') \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x' \in S'$. And the inverse image $\varphi^{-1}(B) = [\varphi^{-1}(t_B), 1 - \varphi^{-1}(f_B)]$ of B is a vague set in S defined by $\varphi^{-1}(t_B)(x) = t_B(\varphi(x))$, and $1 - \varphi^{-1}(f_B)(x) = 1 - f_B(\varphi(x))$, for all $x \in S$.

Proof. It is straightforward. □

Theorem 5.5. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ an epimorphism of hemirings. If A is an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of S, then $\varphi(A)$ is an $(\in, \in \vee q)$-vague left h-ideal of S'.*

Proof. Let A be an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of S.

(1) Let $x', y' \in S'$. Then

$$\begin{aligned}\varphi(t_A)(x' +' y') &= \sup_{\varphi(a)=x'+'y'} t_A(a) \geq \sup_{\varphi(x)=x',\varphi(y)=y'} t_A(x+y) \geq \sup_{\varphi(x)=x',\varphi(y)=y'} \min\{t_A(x), t_A(y), 0.5\} \\ &= \min\left\{\sup_{\varphi(x)=x'} t_A(x), \sup_{\varphi(y)=y'} t_A(y), 0.5\right\} = \min\{\varphi(t_A)(x'), \varphi(t_A)(y'), 0.5\}.\end{aligned}$$

Similarly, $\varphi(f_A)(x' +' y') \leq \max\{\varphi(f_A)(x'), \varphi(f_A)(y'), 0.5\}$.

(2) Let $x', y' \in S'$. Analogous to (1), we have $\varphi(t_A)(x'y') \geq \min\{\varphi(t_A)(y'), 0.5\}$ and $\varphi(f_A)(x'y') \leq \max\{\varphi(f_A)(y'), 0.5\}$.

(3) Let $x, a, b, z \in S$ and $x', a', b', z' \in S'$ be such that $x' +' a' +' z' = b' +' z'$, $\varphi(x) = x'$, $\varphi(a) = a'$, $\varphi(b) = b'$ and $\varphi(z) = z'$. Then $\varphi(x + a + z) = \varphi(b + z)$. Since A is φ-compatible, we have $\varphi(t_A)(x) \geq \min\{\varphi(t_A)(a), \varphi(t_A)(b), 0.5\}$ and $\varphi(f_A)(x) \leq \max\{\varphi(f_A)(a), \varphi(f_A)(b), 0.5\}$. Thus we have $\varphi(t_A)(x') = \sup_{\varphi(x)=x'} t_A(x) \geq \min\{t_A(a), t_A(b), 0.5\}$, and so

$$\begin{aligned}\varphi(t_A)(x') &\geq \sup_{\varphi(a)=a',\varphi(b)=b'} \min\{t_A(a), t_A(b), 0.5\} = \min\left\{\sup_{\varphi(a)=a'} t_A(a), \sup_{t_A(b)=b'} t_A(b), 0.5\right\} \\ &= \min\{\varphi(t_A)(a'), \varphi(t_A)(b'), 0.5\}.\end{aligned}$$

Similarly, $\varphi(f_A)(x') \leq \max\{\varphi(f_A)(a'), \varphi(f_A)(b'), 0.5\}$.

Summing up the above arguments, $\varphi(A)$ is an $(\in, \in \vee q)$-vague left h-ideal of S'. □

Theorem 5.6. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ a homomorphism of hemirings. If A' is an $(\in, \in \vee q)$-vague left h-ideal of S, then $\varphi^{-1}(A')$ is an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of S.*

Proof. The proof is similar to that Theorem 5.5. □

Combining Theorems 5.5 and 5.6, we have the following result.

Theorem 5.7. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ an epimorphism of hemirings. Then the mapping $\psi : A \to \varphi(A)$ defines a one-to-one correspondence between the set of all φ-compatible $(\in, \in \vee q)$-vague left h-ideals of S and the set of all $(\in, \in \vee q)$-vague left h-ideals of S'.*

Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a mapping from S into S'. Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be vague soft sets over S and S', respectively. Then we can define a vague soft set $(\varphi(\tilde{F}), X)$ over S', where $\varphi(\tilde{F}) : X \to \mathscr{V}(S')$ is given by $\varphi(\tilde{F})(\alpha) = \varphi(\tilde{F}(\alpha))$ for all $\alpha \in X$, and a vague soft set $(\varphi^{-1}(\tilde{G}), Y)$ over S, where $\varphi^{-1}(\tilde{G}) : Y \to \mathscr{V}(S)$ is given by $\varphi^{-1}(\tilde{G})(\beta) = \varphi^{-1}(\tilde{G}(\beta))$ for all $\beta \in Y$.

Definition 5.8. Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings and $(\tilde{F}, X)$ a vague soft set over S. Then $(\tilde{F}, X)$ is called an *φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal* of S if $\tilde{F}(\alpha)$ is an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of S for all $\alpha \in X$.

The next theorem presents the relationships between φ-compatible $(\in, \in \vee q)$-vague soft left h-ideals and φ-compatible left h-ideals of a hemiring.

Theorem 5.9. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings and $(\tilde{F}, X)$ an $(\in, \in \vee q)$-vague soft left h-ideal over S. Then*

1. *$(\tilde{F}, X)$ is φ-compatible if and only if non-empty subsets $t_{\tilde{F}(\alpha)\hat{r}}$ and $\widehat{f_{\tilde{F}(\alpha)_s}}$ are φ-compatible for all $r \in [0, 0.5)$, $s \in (0.5, 1]$ and $\alpha \in X$.*
2. *$(\tilde{F}, X)$ is φ-compatible if and only if non-empty subsets $t_{\tilde{F}(\alpha)\hat{r}}$ and $\widehat{[f_{\tilde{F}(\alpha)}]}_s$ are φ-compatible for all $r \in [0, 0.5)$, $s \in (0, 1]$ and $\alpha \in X$.*
3. *$(\tilde{F}, X)$ is φ-compatible if and only if non-empty subsets $\langle t_{\tilde{F}(\alpha)}\rangle_r$ and $\widehat{f_{\tilde{F}(\alpha)_s}}$ are φ-compatible for all $r, s \in (0.5, 1]$ and $\alpha \in X$.*
4. *$(\tilde{F}, X)$ is φ-compatible if and only if non-empty subsets $\langle t_{\tilde{F}(\alpha)}\rangle_r$ and $\widehat{[f_{\tilde{F}(\alpha)}]}_s$ are φ-compatible for all $r \in (0.5, 1]$, $s \in (0, 1]$ and $\alpha \in X$.*
5. *$(\tilde{F}, X)$ is φ-compatible if and only if non-empty subsets $[t_{\tilde{F}(\alpha)}]_r$ and $\widehat{f_{\tilde{F}(\alpha)_s}}$ are φ-compatible for all $r \in (0, 1]$, $s \in (0.5, 1]$ and $\alpha \in X$.*
6. *$(\tilde{F}, X)$ is φ-compatible if and only if non-empty subsets $[t_{\tilde{F}(\alpha)}]_r$ and $[\widehat{f_{\tilde{F}(\alpha)}}]_s$ are φ-compatible for all $r, s \in (0, 1]$ and $\alpha \in X$.*

Proof. It is straightforward by Theorem 5.3. □

Theorem 5.10. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ an epimorphism of hemirings. If $(\tilde{F}, X)$ is an φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal over S, then $(\varphi(\tilde{F}), X)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S'.*

Proof. For any $\alpha \in X$, since $(\tilde{F}, X)$ is an φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal over S, $\tilde{F}(\alpha)$ is an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of S. Hence $\varphi(\tilde{F})(\alpha) = \varphi(\tilde{F}(\alpha))$ is an $(\in, \in \vee q)$-vague left h-ideal of S' by Theorem 5.5, and so $(\varphi(\tilde{F}), X)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S'. □

Theorem 5.11. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ a homomorphism of hemirings. If $(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S, then $(\varphi^{-1}(\tilde{G}), Y)$ is an φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. The proof is similar to that Theorem 5.10. □

Combining Theorems 5.10 and 5.11, we have the following result.

Theorem 5.12. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ an epimorphism of hemirings. Then the mapping $\psi : (\tilde{F}, X) \to (\varphi(\tilde{F}), X)$ defines a one-to-one correspondence between the set of all φ-compatible $(\in, \in \vee q)$-vague soft left h-ideals over S and the set of all $(\in, \in \vee q)$-vague soft left h-ideals over S'.*

Definition 5.13. Let $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ be two vague soft sets over S and S', respectively. Let $\varphi : S \to S'$ and $\psi : X \to Y$ be two mappings. Then the pair (φ, ψ) is called a *vague soft hemiring homomorphism* if it satisfies the following conditions:

(1) φ is an epimorphism of hemirings.
(2) ψ is a surjective mapping.
(3) $\varphi(\tilde{F}(\alpha)) = \tilde{G}(\psi(\alpha))$ for all $\alpha \in X$.

If there exists a vague soft hemiring homomorphism between $(\tilde{F}, X)$ and $(\tilde{G}, Y)$, we say that $(\tilde{F}, X)$ is *vague soft homomorphic to* $(\tilde{G}, Y)$, which is denoted by $(\tilde{F}, X) \sim (\tilde{G}, Y)$. Moreover, if φ is an isomorphism of hemirings and ψ is a bijective mapping, then (φ, ψ) is called a *vague soft hemiring isomorphism*. In this case, we say that $(\tilde{F}, X)$ is *vague soft isomorphic to* $(\tilde{G}, Y)$, which is denoted by $(\tilde{F}, X) \backsimeq (\tilde{G}, Y)$.

Example 5.14. Denote by $\mathbb{N}_0$ and $\mathbb{N}_0 /_{(4)}$ the hemiring of non-negative integers and the hemiring of non-negative integers module 4, respectively. Let $X = \mathbb{N}_0$ and $Y = \mathbb{N}_0 /_{(4)}$. Define vague soft sets $(\tilde{F}, X)$ over $\mathbb{N}_0$ and $(\tilde{G}, Y)$ over $\mathbb{N}_0 /_{(4)}$ by

$$t_{\tilde{F}(\alpha)}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1}{\alpha} & \text{if } x \in (\alpha) - \{0\}, \\ 0 & \text{otherwise}, \end{cases} \qquad 1 - f_{\tilde{F}(\alpha)}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1}{\alpha} & \text{if } x \in (\alpha) - \{0\}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$t_{\tilde{G}(\beta)}(y) = \begin{cases} \dfrac{1}{4} & \text{if } y \in (\beta), \\ 0 & \text{otherwise}, \end{cases} \qquad 1 - f_{\tilde{G}(\alpha)}(y) = \begin{cases} \dfrac{1}{4} & \text{if } y \in (\beta), \\ 0 & \text{otherwise}, \end{cases}$$

respectively, for all $\alpha, x \in \mathbb{N}_0$, $\beta, y \in \mathbb{N}_0 /_{(4)}$. Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0 /_{(4)}$ be the natural mapping defined by $\varphi(x) = [x]$ for all $x \in \mathbb{N}_0$. Evidently, φ is an epimorphism of hemirings. Now, it is easy to check that (φ, φ) is a vague soft homomorphic from $(\tilde{F}, X)$ to $(\tilde{G}, Y)$.

Theorem 5.15. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ vague soft sets over S and S', respectively, and (φ, ψ) a vague soft homomorphic from $(\tilde{F}, X)$ to $(\tilde{G}, Y)$. If $(\tilde{F}, X)$ is an φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal over S, then $(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S'.*

Proof. Since (φ, ψ) is a vague soft homomorphic from $(\tilde{F}, X)$ to $(\tilde{G}, Y)$, by Definition 5.13, ψ is a surjective mapping from X onto Y and φ is an epimorphism of from S onto S'. Now, for any $\beta \in Y$, since ψ is surjective, there exists $\alpha \in X$ such that $\psi(\alpha) = \beta$. From Theorem 5.5, since φ is an epimorphism, $\tilde{G}(\beta) = \tilde{G}(\psi(\alpha)) = \varphi(\tilde{F}(\alpha))$ is an $(\in, \in \vee q)$-vague left h-ideal of S', and so $(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S'. □

Theorem 5.16. *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $(\tilde{F}, X)$ and $(\tilde{G}, Y)$ vague soft sets over S and S', respectively, and (φ, ψ) a vague soft homomorphic from $(\tilde{F}, X)$ to $(\tilde{G}, Y)$. If $(\tilde{G}, Y)$ is an $(\in, \in \vee q)$-vague soft left h-ideal over S and φ is an isomorphism of hemirings, then $(\tilde{F}, X)$ is an φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal over S.*

Proof. Let $(\tilde{G}, Y)$ be an $(\in, \in \vee q)$-vague soft left h-ideal over S. Then, for any $\alpha \in X$, $\tilde{G}(\psi(\alpha))$ is an $(\in, \in \vee q)$-vague left h-ideal of S'. Now, since φ is an isomorphism, it follows from Theorem 5.6 that $\tilde{F}(\alpha) = \varphi^{-1}(\varphi(\tilde{F}(\alpha))) = \varphi^{-1}(\tilde{G}(\psi(\alpha)))$ is an φ-compatible $(\in, \in \vee q)$-vague left h-ideal of S. Therefore, $(\tilde{F}, X)$ is an φ-compatible $(\in, \in \vee q)$-vague soft left h-ideal over S. □

6. Conclusions

In this paper, our aim is to promote research and the development of vague (soft) technology by studying the vague (soft) hemirings. The goal is to explain new methodological development in hemirings which will also be of growing importance in the future. The obtained results can be applied the other algebraic structures. Our future work on this topic will focus on studying the relationships among hemirings, BL-algebras and IS-algebras.

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References

[1] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.
[2] Z. Pawlak, Rough sets, Int. J. Inform. Comput. Sci. 11 (1982) 341–356.
[3] W.L. Gau, D.J. Buehrer, Vague sets, IEEE Trans. Syst. Man Cybern. 23 (2) (1993) 610–614.
[4] K. Atanassov, Operators over interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 64 (1994) 159–174.
[5] M.B. Gorzalzany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems 21 (1987) 1–17.
[6] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (4–5) (1999) 19–31.
[7] F. Feng, Y.B. Jun, X. Liu, L. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (1) (2010) 10–20.
[8] N. Çağman, S. Enginoğlu, Soft matrix theory and its decision making, Comput. Math. Appl. 59 (10) (2010) 3308–3314.
[9] N. Çağman, S. Enginoğlu, Soft set theory and uni-int decision making, European J. Oper. Res. 207 (2) (2010) 848–855.
[10] A.R. Roy, P.K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412–418.
[11] P.K. Maji, A.R. Roy, R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[12] D. Chen, E.C.C. Tsang, D.S. Yeung, X. Wang, The parameterization reduction of soft sets and its applications, Comput. Math. Appl. 49 (2005) 757–763.
[13] Y. Zou, Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowl. Based Syst. 21 (8) (2008) 941–945.
[14] Z. Xiao, K. Gong, Y. Zou, A combined forecasting approach based on fuzzy soft sets, J. Comput. Appl. Math. 228 (1) (2009) 326–333.
[15] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[16] M. Irfan Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[17] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy soft sets, J. Fuzzy. Math. 9 (3) (2001) 589–602.
[18] P. Majumdar, S.K. Samanta, Generalised fuzzy soft sets, Comput. Math. Appl. 59 (4) (2010) 1425–1432.
[19] M. Irfan Ali, M. Shabir, Comments on De Morgan's law in fuzzy soft sets, J. Fuzzy Math. 18 (3) (2010) 679–686.
[20] P.K. Maji, R. Biswas, A.R. Roy, Intuitionistic fuzzy soft sets, J. Fuzzy. Math. 9 (3) (2001) 677–692.
[21] W. Xu, J. Ma, S.Y. Wang, G. Hao, Vague soft sets and their properties, Comput. Math. Appl. 59 (2010) 787–794.
[22] H. Aktaş, N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[23] F. Feng, Y.B. Jun, X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[24] F. Feng, C.X. Li, B. Davvaz, M.I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Comput. 14 (2010) 899–911.
[25] F. Feng, Y.M. Li, V. Leoreanu-Fotea, Y.B. Jun, Soft sets and soft rough sets, Inform. Sci. 181 (2011) 1125–1137.
[26] Y.B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408–1413.
[27] Y.B. Jun, C.H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466–2475.
[28] Y.B. Jun, K.J. Lee, C.H. Park, Soft set theory applied to ideals in d-algebras, Comput. Math. Appl. 57 (2009) 367–378.
[29] A. Aygünoğlu, H. Aygün, Introduction to fuzzy soft groups, Comput. Math. Appl. 58 (2009) 1279–1286.
[30] J. Zhan, Y.B. Jun, Soft *BL*-algebras based on fuzzy sets, Comput. Math. Appl. 59 (2010) 2037–2046.
[31] Y. Yin, H. Li, The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings, Inform. Sci. 178 (2008) 3451–3464.