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HIGHER SET THEORY AND MATHEMATICAL PRACTICE *

Harvey M.FRIEDMAN
Stanford University

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Introduction

When we examine the classical set-theoretic foundations of mathematics, we see that the only sets that play a role are sets of restricted type; at the risk of understatement, only sets of rank $< \omega + \omega$. Further examination reveals four fundamental principles about sets used: the existence of an infinite set; the existence of the power set of any set; every property determines a subset of any set; and the axiom of choice. The theory based on these four principles is known as Zermelo set theory together with the axiom of choice. and is written **Z** in this paper. Then **Z** adequately formalizes mathematical practice (excluding modern set theory) in an elegant and straightforward way.

In modern set theory, however, the object of study is the notion (or notions) of set of transfinite rank. Whether or not there is a single meaningful notion of set of transfinite type, rather than, instead only a multitute of notions of set obtained by prescribing a definite "number" of iterations of the power set operation, remains a controversial issue. In any case, what is completely clear is that no notion of: set of arbitrary transfinite type, or even notions of set obtained by some definite iteration (beyond $\omega + \omega$) of the power set operation, is relevant, as of now, to mathematical practice, or even understood by mathematicians. We refer to this characteristic aspect of modern set theory, the consideration of sets of transfinite rank, or of sets obtained by more than finite-

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ly many iterations of the power set operation applied to the hereditarily finite sets, as higher set theory.

What is the significance of this sociology for us? It suggests to us consideration of the following conjecture:

*) every sentence of mathematical discourse (excluding, of course, higher set theory) which can be decided using fundamental principles about sets of *transfinite* rank (like: **Z** consists of fundamental principles about sets of rank $< \omega + \omega$), can already be decided in mathematical practice.

It is beyond the scope of this paper to thoroughly discuss whether certain formal systems do or do not codify fundamental principles about sets of transfinite rank, but certain cases are clear cut. (It is, of course, the case that no one today knows how to provide a theoretical description of what is a fundamental principle and what is not; a general theory of notions and principles is nowhere in sight). That **Z** codifies fundamental principles about sets of transfinite rank is clear, even though it was intended to codify only fundamental principles about sets of rank $< \omega + \omega$. That the theory $\mathbf{Z}(\Omega) = \mathbf{Z}$ together with "there is a rank function defined on every countable well-ordering" does, is fairly clear cut. That, say, Zermelo-Fraenkel set theory together with the existence of a measurable cardinal, or, say, Zermelo-Fraenkel together with the existence of nonconstructible sets of natural numbers *does not* is also fairly clear cut. There is nothing in the phrase "set of transfinite rank" which even remotely suggests that all sets are constructible or that all cardinals are nonmeasurable.

With these rough guidelines in mind, the reader can appreciate the following important open question, which has turned out to be connected with attempts at settling *):

**) are there fundamental principles about sets of transfinite rank which refute or prove the axiom of constructibility?

No answer to **) is in sight.

Perhaps some more rough guidelines may be useful in helping the reader appreciate *). Clearly Con(**Z**) can be proved in $\mathbf{Z}(\Omega)$ but not in

Z itself. Does this constitute a refutation of *)? No, because Con(**Z**) is really about (formal systems of) set theory of rank $< \omega + \omega$, and to *understand what* a set of rank $< \omega + \omega$ is, one has to go beyond use of sets of rank $< \omega + \omega$, and so, go beyond (our model of) mathematical practice. Thus Con(**Z**) is considered outisde of mathematical discourse.

The main obstacle in obtaining a genuine negative solution to *) is that the only sentences of mathematical discouse which are known to be independent of **Z** at the same time which have proofs in higher set theory (even using, say, the existence of a measurable cardinal) are also known to imply, within **Z**, the existence of nonconstructible sets; so, if one wishes to solve *) using such sentences, then one will also have to solve **).

Our approach avoids this nonconstructible trouble by producing a sentence of mathematical discouse about Borel sets which is Π^1_3 (hence provably relativizes to constructible sets) and giving a proof of independence of this Π^1_3 sentence from **Z** and conjecturing that this Π^1_3 sentence is provable within $\mathbf{Z}(\Omega)$. That the Π^1_3 sentence is provable within $\mathbf{Z}(\Omega)$ seems like a reasonable conjecture because of

1) examination of the proofs of independence given here;
2) the Π^1_3 sentence is known to be provable using the existence of Ramsey cardinals (D.Martin [4]);
3) this proof of Martin uses partition properties of cardinals directly, and the cardinal of $V(\Omega)$ is the first cardinal satisfying certain important weaker partition properties.

The Π^1_3 sentence under investigation here is Borel determinateness, written here as $(\forall\alpha)(D(\alpha))$, (see Definitions 1.4 and 1.5). Our independence result from **Z** is given in the Corollary to Theorem 1.6. Actually, the independence proofs work equally well for the following consequence of Borel determinateness, which *reads* like (but by our independence proof *is* not) a standard Theorem in the classical theory of the Borel hierarchy: to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a continuous function F which either uniformizes Y or uniformizes $[(f,g)\colon (g,f) \notin Y]$; see Section 4 for elaboration.

The paper is organized as follows. In Section 1 we proceed directly to the many independence result which is Theorem 1.6 (and Corollary), making use of detailed information about the model, $L^{\omega+\omega}$, (see Definition 1.16) of **Z** used in the independence proof. Section 2 is entirely devoted to an outline of a proof of this detailed information. Thus Section 1 comprises the body of the independence proof, and Section 2 comprises the routine detailed machinery needed. Section 3 considers various refinements, including the independence from 2nd-order arithmetic of determinateness for $G_{\delta\sigma\ \delta\sigma}$ sets; this is to be compared with M.Davis [2], which gives a mathematical practice type proof of determinateness for $G_{\delta\sigma}$ sets (easily formalizable in 2nd-order arithmetic). Neither our independence methods nor the methods of [2] (or any other mathematical practice methods) seem to apply to $G_{\delta\sigma\delta}$.

Apparently, determinateness was first introduced by Gale and Stewart in [3]. Determinateness in various forms (for analytic sets, projective sets, ordinal definable sets, all sets, to mention some divisions) have been under intensive investigation in recent years. For a recent survey, see A.Mathias [5].

Section 1

The purpose of this Section is to prove Theorem 1.6 and its Corollary.

We let ω be $[0, 1, 2, \ldots]$, 2^ω be the set of all functions from ω into $[0, 1]$, and Ω be the first uncountable ordinal.

The Borel subsets of 2^ω are the least σ-algebra containing all open and closed subsets of 2^ω. It is well known that the Borel subsets of 2^ω are just those subsets which lie in some B_α, $\alpha < \Omega$, as defined below. But first we define the open subsets of 2^ω.

Definition 1.1. We say $Y \subset 2^\omega$ is *open* if and only if $(\forall x)(x \in Y \to \to (\exists n \in \omega)(\forall y \in 2^\omega)((\forall m \leq n)(y(m) = x(m)) \to y \in Y))$. We say $Y \subset 2^\omega$ is *closed* if and only if $2^\omega - Y$ is open.

Definition 1.2. Define $B_1 = [Y \subset 2^\omega : Y$ is open or Y is closed$]$, $B_{\alpha+1} = [Y \subset 2^\omega : Y$ is the intersection of some countable (or finite) subset of B_α or Y is the union of some countable subset of $B_\alpha]$, $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$, where $\alpha, \lambda < \Omega$, λ is a limit ordinal.

We can associate in informal terms, to each $Y \subset 2^\omega$, a discrete two-person game of infinite duration. The players are designated I, II. The players alternately produce (or play) either 0 or 1, starting with I. If the resulting element of 2^ω is in Y then I is considered the winner; if not, then II is. The question arises as to whether there is a perfect strategy for winning available to one of the two players.

We now wish to give the well known formal analysis of the above.

Definition 1.3. A 0, 1-*sequence* is a function s whose domain is an initial segment (possibly empty) of ω and whose range is a subset of $[0, 1]$. We write $\ln(s)$ to be such that $\mathrm{Dom}(s) = [i : i < \ln(s)]$. If s, t are 0, 1-sequences then we say t extends s if and only if $\ln(s) \leq \ln(t)$ and $(\forall i < \ln(s))(s(i) = t(i))$. If s is a 0, 1-sequence and $f \in 2^\omega$ then f extends s means that $(\forall i)(i < \ln(s) \to s(i) = f(i))$.

Definition 1.4. Let $Y \subset 2^\omega$. We write $S(Y, \mathrm{I}, f)$ if and only if

1) f is a function from the 0, 1-sequences into $[0, 1]$,
2) $(\forall g \in 2^\omega)(\lambda n(g((n-1)/2)$ if n is odd; $f(g \restriction [i : i < n/2])$ if n is even$) \in Y)$.

We write $S(Y, \mathrm{II}, g)$ if and only if
1) g is a function from the 0, 1-sequences into $[0, 1]$
2) $(\forall f \in 2^\omega)(\lambda n(f(n/2)$ if n is even; $g(f \restriction [i : i < (n+1)/2])$ if n is odd$) \in 2^\omega - Y)$.

We write $D(Y)$ if and only if $(\exists f)(S(Y, \mathrm{I}, f) \vee S(Y, \mathrm{II}, f))$.

Thus $S(Y, \mathrm{I}, f)$ expresses that f is a winning strategy for I in the game associated with Y; $S(Y, \mathrm{II}, f)$ for II. And $D(Y)$ expresses that either I or II has a winning strategy.

In this paper we are only concerned with $D(Y)$ for Borel Y.

Definition 1.5. Let $1 < \alpha < \Omega$. Then $D(\alpha)$ means $(\forall Y \in B_\alpha)(D(Y))$.

We use some notions from ordinary recursion theory.

Definition 1.6. For $f \in 2^\omega$ we write φ_e^f for the eth partial function of one argument on ω that is partial recursive in f, according to some customary enumeration. We write $g \leq_T f$ for $(\exists e)(g = \varphi_e^f)$. We write $g =_T f$ for $g \leq_T f$ & $f \leq_T g$, and we write $f <_T g$ for $f \neq_T g$ & $f \leq_T g$.

Thus $g \leq_T f$ is read "g is partial recursive in f". The T stands for Turing.

Definition 1.7. We write $J(f)$ for the Turing jump of $f \in 2^\omega$. Define $J^{n+1}(f) = J(J^n(f))$, $0 < n$. Define $J^\omega(f) = \lambda m((J^a(f))(b)$ if $0 < a$, $0 \leq b$ and $m = 2^a 3^b$; 0 otherwise$)$.

Definition 1.8. A *Turing set* is a $Y \subset 2^\omega$ such that $(\forall f)(\forall g)((f \in Y$ & $f =_T g) \to g \in Y)$. A Turing cone is a $Y \subset 2^\omega$ such that $(\exists f \in 2^\omega)(\forall g)(g \in Y \equiv f \leq_T g)$.

Unless we specify otherwise, wherever we quantify over functions we are quantifying only over 2^ω.

We now present a theorem of D.Martin modified and specialized to suit our purposes.

Theorem 1.1. *Suppose* $(\forall \alpha)(D(\alpha))$. *Then for all* α, *every Turing set* $Y \in B_\alpha$ *either contains or is disjoint from a Turing cone.*

Proof Take X as $[f: \lambda n(f(2n)) \in Y \;\&\; \lambda n(f(2n+1)) \leq_T \lambda n(f(2n))]$. If $S(X, \mathrm{I}, g)$, then $[\alpha \in 2^\omega : h \leq_T \alpha] \subset Y$. If $S(X, \mathrm{II}, g)$, then $[\alpha \in 2^\omega : h \leq_T \alpha] \cap Y = \phi$.

Definition 1.9. LST is the *language of set theory*; i.e. the predicate calculus with equality (=) and membership ($\in$).

Definition 1.10. **Z** is *Zermelo set theory*, a theory in LST, whose non-logical axioms are

1) $(\exists y)(\forall z)(z \in y \equiv z \subset x)$

2) $(\exists z)(\forall w)(w \in z \equiv (w = x \vee w = y))$

3) $x = y \equiv (\forall z)(z \in x \equiv z \in y)$

4) $(\exists x)(\forall y)(y \in x \equiv (y \in a \;\&\; F))$, where F is a formula in LST which does not mention x free

5) $(\exists y)(\forall z)(z \in y \equiv (\exists w)(z \in w \;\&\; w \in x))$

6) $(\exists x)(\phi \in x \;\&\; (\forall y)(y \in x \to (\exists z)(z \in x \;\&\; (\forall w)(w \in z \equiv (w \in y \vee w = y)))))$. Here $z \subset y$ is an abbreviation for $(\forall x)(x \in z \to x \in y)$, and $\phi \in x$ is an abbreviation for $(\exists y)(\forall z)(z \notin y \;\&\; y \in x)$

7) $x \neq \phi \to (\exists y)(y \in x \;\&\; (\forall y)(z \in x \to z \notin y))$

8) the Axiom of Choice.

We now describe the model of **Z** we will use in this Section, and which we analyze in Section 2.

Definition 1.11. If x is a set then ϵ_x is the *binary relation* on x given by $\epsilon_x(a, b) \equiv (a \in x \;\&\; b \in x \;\&\; a \in b)$.

Definition 1.12. Define $V(0) = \phi$, $V(\alpha + 1) = \mathrm{P}(V(\alpha))$, $V(\lambda) = \bigcup_{\alpha<\lambda} V(\alpha)$, where $\mathrm{P}(x)$ is $[y : y \subset x]$ and λ is a limit ordinal.

Definition 1.13. A structure is a system (A, R), where A is a nonempty set, R is a binary relation on A. An assignment in (A, R) is a function $f: \omega \to A$ with finite range. We write $\mathrm{Sat}((A, R), F, f)$ to express that the formula F of LST holds in the structure (A, R) when ϵ is interpreted as R, $=$ as equality, and each free variable v_i in F is interpreted as $f(i)$. If F has no free variables then we may write $\mathrm{Sat}((A, R), F)$.

Definition 1.14. For structures (A, R), (B, S) we write $\mathrm{Inj}(f, (A, R), (B, S))$ to express that $f: A \to B$, f 1–1, and $(\forall x, y \in A)(R(x, y) \equiv S(f(x), f(y)))$. We write $\mathrm{Iso}(f, (A, R), (B, S))$ if the above holds and f is onto. We write $(A, R) \approx (B, S)$ for $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S)))$.

Definition 1.15. For structures (A, R) we take $\mathrm{FODO}((A, R)) = [x \subset A$: for some formula F and assignment f we have $x = [y: \mathrm{Sat}((A, R), F, f_y^0)]]$, where $f_y^0(i) = f(i)$ if $i \neq 0$; y if $i = 0$.

FODO stands for "first order definable over".
Often we abbreviate (x, ϵ_x) by (x, ϵ).

Definition 1.16. Define $L(0) = V(\omega)$, $L(\alpha + 1) = \mathrm{FODO}((L(\alpha), \epsilon_{L(\alpha)}))$, $L(\lambda) = \bigcup_{\alpha<\lambda} L(\alpha)$, where λ is a limit ordinal. Define $L^{\omega+\omega}(0) = \phi$, $L^{\omega+\omega}(\alpha + 1) = \mathrm{FODO}((L^{\omega+\omega}(\alpha), \epsilon)) \cap V(\omega + \omega)$, $L^{\omega+\omega}(\lambda) = \bigcup_{\alpha<\lambda} L^{\omega+\omega}(\alpha)$, where λ is a limit ordinal. Define $L^{\omega+\omega} = [x: (\exists\alpha)(x \in L^{\omega+\omega}(\alpha))]$.

Thus our L is the usual constructible hierarchy.

Lemma 1.2.1. *Each $L^{\omega+\omega}(\alpha)$ is transitive. In addition, $L^{\omega+\omega}$ is transitive.*

Lemma 1.2.2. *For all transitive sets x and all $f: \omega \to x$ with finite range we have* $\mathrm{Sat}((x, \epsilon), v_0 \subset v_1, f) \equiv f(0) \subset f(1)$.

Lemma 1.2.3. *$V(\omega + \omega)$ is closed under subset and power set and union.*

Theorem 1.2. *$L^{\omega+\omega}$ satisfies* **Z**.

Proof. There is an α such that $L^{\omega+\omega}(\alpha) = L^{\omega+\omega}(\alpha + 1)$. Choose α least with this property. 3), 5), and 6) follow from the lemmas; to check 1), 2), and 4), note first that $L^{\omega+\omega}(\alpha) = L^{\omega+\omega}$. For 1), note that $[x \in L^{\omega+\omega}(\alpha): x \subset y] \in L^{\omega+\omega}(\alpha + 1)$ for any $y \in L^{\omega+\omega}(\alpha)$. For 2), note that $[x \in L^{\omega+\omega}(\alpha): x = y \vee x = y] \in L^{\omega+\omega}(\alpha + 1)$ for any y, $z \in L^{\omega+\omega}(\alpha)$. For 4), note that $[y \in a: \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, f_y^0)] \in L^{\omega+\omega}(\alpha + 1)$ for all $a \in L^{\omega+\omega}(\alpha)$, all assignments f, all formulae F in LST.

Definition 1.17. We assume a fixed primitive recursive total one-one onto Gödel numbering of the formulae in LST. We let 'φ' be the Gödel number of φ. Let (A, R) be a structure. We write $\mathrm{Def}((A, R), n, x)$ if and only if n is the Godel number of the formula $F(v_0)$ with only the free variables shown and x is the unique element of A with $\mathrm{Sat}((A, R), F(v_0), \lambda n(x))$, and furthermore n is the least integer with this property that x is the unique element of A with $\mathrm{Sat}((A, R), F(v_0), \lambda n(x))$.

Definition 1.18. Let (A, R) be a structure. Then we let $\mathrm{Th}((A, R))$ be $[n: n$ is the Gödel number of the sentence F and $\mathrm{Sat}((A, R), F)]$.

Definition 1.19. If $x \subset \omega$ then we write $\mathrm{Ch}(x)$ for $\lambda n(1$ if $n \in x$; 0 if $n \notin x)$.

We need to draw on one fact about the construction of $L^{\omega+\omega}$; Section 2 is devoted to a detailed outline of a proof of the following.

Theorem 2. *There are formulae $\varphi_1(v_0, v_1)$, $\varphi_2(v_0, v_1)$, and $\varphi_3(v_0, v_1)$ in* LST *with only the free variables shown such that for each $x \subset \omega$, $x \in L^{\omega+\omega}$, there is a limit ordinal λ such that*

1) $x \in L^{\omega+\omega}(\lambda)$

2) $(\forall y \in L^{\omega+\omega}(\lambda))(\exists n)(\mathrm{Def}((L^{\omega+\omega}(\alpha), \epsilon)n, y))$

3) $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$

4) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_1(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) < (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

5) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_2(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) = (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

6) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_3(v_0, v_1), f)$ *if and only if* $f(1) = (\mu n \in \omega)(f(0) \in V(\omega + n))$.

We make the following Definition 1.21 modelled after Theorem 2, using the φ_1, φ_2, and φ_3 of the statement of that Theorem.

Definition 1.20. We fix a structure (A^0, R^0) such that $A^0 = [i : i$ is odd$]$, R^0 is a recursive relation, and (A^0, R^0) is isomorphic to $(V(\omega), \epsilon)$. By $\bar{n}$ we mean that element of A^0 which is satisfied, in (A^0, R^0), to be n.

Definition 1.21. A *towered structure* is a structure (A, R) such that

1) $A \subset \omega$ and the relation $x \sim y \equiv \mathrm{Sat}((A, R), \varphi_2(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ is an equivalence relation on A

2) the relation $x < y \equiv \mathrm{Sat}((A, R), \varphi_1(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ has that $(\forall x, y \in A)((x < y \;\&\; \sim y < x) \vee (y < x \;\&\; \sim x < y) \vee (x \sim y \;\&\; \sim x < y \;\&\; \sim y < x))$ and $(\forall x, y, z \in A)(((x \sim z \;\&\; x < y) \rightarrow z < y) \;\&\; ((x \sim z \;\&\; y < x) \rightarrow y < z))$, and $<$ has no maximal element

3) $A^0 = [i : i \in A \;\&\; (\forall j)(\sim j < i)]$, $R^0 = R \upharpoonright A^0$

4) we have $(\forall x \in A)(\exists ! y)(\mathrm{Sat}((A, R), \varphi_3(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$, and so we let F be given by $(\forall x \in A)(\mathrm{Sat}((A, R), \varphi_3(v_0, v_1), \lambda n(x \text{ if } n = 0; F(x) \text{ if } n \neq 0))$. Then we want $(\forall x \in A)(\exists n)(F(x) = \bar{n})$, and $(\forall x \in A^0)(F(x) = \bar{0})$

5) $(\forall x \in A - A^0)(F(x) = \bar{n}$ where n is the least integer greater than every i such that $(\exists y)(R(y, x) \;\&\; F(y) = \bar{i}))$

6) suppose $x \in A$. Then $\mathrm{FODO}(([i: i < x], R \upharpoonright [i: i < x])) = [z \subset [i: i < x] : (\exists j)((j < x \vee j \sim x) \;\&\; z = [k: R(k, j)])]$

7) $(\forall x, y \in A)(R(x, y) \rightarrow x < y)$

8) (A, R) satisfies the axiom of extensionality

9) $(\forall i \in A - A^0)(\exists j)(\mathrm{Def}((A, R), j, 2j) \;\&\; i = 2j)$

10) $[i: i \in \mathrm{Th}((A, R))] \in \mathrm{FODO}(\mathrm{FODO}((A, R)), \epsilon)$

11) for all nonempty $x \subset A$ with $\mathrm{Ch}(x) \leq_T J(J^\omega(\mathrm{Ch}(\mathrm{Th}((A, R)))))$ there exists a $y \in x$ such that for all $z \in x$ we have $\sim z < y$.

We presume knowledge of the effective Borel hierarchy. In particular, we will make use of the notion of: being in $B_{\omega+\omega}$ with recursive code.

Lemma 1.3.1. $[f \in 2^\omega : f$ codes $\mathrm{Th}((A, R))$ for some towered structure $(A, R)]$ *is in* $B_{\omega+\omega}$ *with recursive code. In other words,* $\mathcal{S} = [f \in 2^\omega : f = \mathrm{Ch}(\mathrm{Th}((A, R)))$ for some towered structure $(A, R)]$ *is in* $B_{\omega+\omega}$ *with recursive code.*

Proof. A more detailed proof of a more delicate version of this is given as Lemma 3.2.2; we will only mention some basic points for this present version. To "test" whether $f \in \mathcal{S}$ first construct the relational structure (A, R) given by $A^0 \subset A$, $R^0 = R \upharpoonright A^0$, $A - A^0 = [2i: i$ is the Gödel number of some formula $F(v_0)$ such that '$(\exists ! v_0)(F(v_0))$' $\in [k: f(k) = 1]$ and $(\forall j < i)$ (if j is the Gödel number of some formula $G(v_0)$ then '$(\exists ! v_0)(G(v_0)) \;\&\; (\exists v_0)(G(v_0) \;\&\; F(v_0))$' $\in [k: f(k) = 0]]$, $R(2i, 2j)$, for $2i, 2j \in A$, holds if and only if for the corresponding F, G we have '$(\exists v_0)(F(v_0) \;\&\; (\exists v_1)(G(v_1) \;\&\; v_0 \in v_1))$' $\in [k: f(k) = 1]$, $R(2i, 2j + 1)$ is always false, $R(2i + 1, 2j)$ holds if and only if '$(\exists v_0)(P(v_0) \;\&\; (\exists v_1)(G(v_1) \;\&\; v_0 \in v_1))$' $\in [k: f(k) = 1]$, where P is the *canonical* definition of $2i + 1$ in (A^0, R^0). Then check whether clauses 1) – 11) hold for this (A, R). It is clear that if there is any (A, R) with $\mathrm{Th}((A, R)) = [k: f(k) = 1]$ it must be this (A, R) above.

Lemma 1.3.2. *If* $Y \subset 2^\omega$ *is in* $B_{\omega+\omega}$ *with recursive code then* $Y \cap L^{\omega+\omega}$ *must be in* $L^{\omega+\omega}$ *and* $L^{\omega+\omega}$ *must satisfy that* $Y \cap L^{\omega+\omega}$ *is in* $B_{\omega+\omega}$ *with recursive code.*

Proof. This is a well known absoluteness property of the effective Borel hierarchy.

Theorem 1.3. *$\mathcal{S} \cap L^{\omega+\omega} \in L^{\omega+\omega}$ and is satisfied in $L^{\omega+\omega}$ to be an element of $B_{\omega+\omega}$ with recursive code.*

Theorem 1.4. *For all $f \in 2^\omega \cap L^{\omega+\omega}$ there is a $g \in \mathcal{S} \cap L^{\omega+\omega}$ such that $f \leq_T g$.*

Proof. Take this f. Let $x = [k : f(k) = 1]$. Choose λ according to Theorem 2. We must choose the appropriate towered structure $(A, R) \approx (L^{\omega+\omega}(\alpha), \epsilon)$. We will define a g such that $\mathrm{Iso}(g, (L^{\omega+\omega}(\lambda), \epsilon), (A, R))$. Take $g \upharpoonright V(\omega)$ to be the isomorphism from $(V(\omega), \epsilon)$ onto (A^0, R^0). For $y \in L^{\omega+\omega}(\lambda) - V(\omega)$ take $g(y)$ to be $2n$ where $\mathrm{Def}((L^{\omega+\omega}(\lambda), \epsilon), n, y)$. Take R to be the relation on $\mathrm{Rng}(g)$ induced by g. Conditions 1) – 10) in the definition of towered structure are easily verified. Condition 11) also is satisfied since $<$ will be a well-founded relation.

Definition 1.22. Let $f, g \in 2^\omega$. The join of f, g, written (f, g), is $\lambda n(f(n/2)$ if n is even; $g((n-1)/2)$ if n is odd).

Lemma 1.5.1. *Suppose (A, R), (B, S) are towered structures such that* $\mathrm{Ch}(\mathrm{Th}((A, R))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((B, S))))$ *and* $\mathrm{Ch}(\mathrm{Th}((B, S))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((A, R))))$. *Then either $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S)))$ or $(\exists f)(\mathrm{Inj}(f, (A, R), (B, S))$ and $(\exists x \in B)(\mathrm{Rng}(f) = [y \in B : y < x]$, where $<$ is as in (B, S) as in* Definition 1.21)), *or $(\exists f)(\mathrm{Inj}(f, (B, S), (A, R))$ and $(\exists x \in A)(\mathrm{Rng}(f) = [y \in A : y < x]$, where $<$ is as in (A, R) as in* Definition 1.21)).

Proof. Let $T_1 = \mathrm{Th}((A, R))$, $T_2 = \mathrm{Th}((B, S))$. Let $\sim_1, <_1, F_1$ be as in Definition 1.21 for (A, R); $\sim_2, <_2, F_2$ be as in Definition 1.21 for (B, S).

Define the predicate $P(n, i, j)$ by recursion on n. $P(0, i, j) \equiv i \in A^0$ & $i = j$. $P(n+1, i, j) \equiv F_1(i) = F_2(j) = \overline{n+1}$ & $(\forall a)(R(a, i) \rightarrow (\exists b)(\exists k)(S(b, j)$ & $P(k, a, b)$ & $F_1(a) = F_2(b) = \bar{k}))$ & $(\forall a)(S(a, j) \rightarrow (\exists b)(\exists k)(R(b, i)$ & $P(k, b, a)$ & $F_2(a) = F_1(b) = \bar{k}))$. It is easily seen that, uniformly, for each k, the relation $P(k, a, b)$ is recursive in

$J^k((\mathrm{Ch}(T_1), \mathrm{Ch}(T_2)))$. Hence, uniformly, for each k, the relation $P(k, a, b)$ is recursive in both $J^{k+1}(\mathrm{Ch}(T_1))$ and $J^{k+1}(\mathrm{Ch}(T_2))$.

We now wish to prove by induction on n that for each i there is at most one j such that $P(n, i, j)$. The case $n = 0$ is trivial. Suppose true for all $k \leq n$ and let $P(n + 1, i, j)$, $P(n + 1, i, a)$. Let $S(x, j)$. Then $F_2(x) = \bar{k}$ for some $k \leq n$. Then for some $x_0 \in A$ we have $P(k, x_0, x)$ and $R(x_0, i)$. Hence by $P(n + 1, i, a)$ we must have for some $y \in B$, $P(k, x_0, y)$ and $S(y, a)$. But since $k \leq n$ we must have $x = y$. So $S(x, a)$. Hence $(\forall x)(S(x, j) \to S(x, a))$. Symmetrically, $(\forall x)(S(x, a) \to S(x, j))$. So $a = j$, and we are done.

Symmetrically, for each j there is at most one i such that $P(n, i, j)$.

Clearly $(P(n, i, j) \;\&\; R(a, i)) \to (\exists b)(\exists k)(P(k, a, b) \;\&\; S(b, j))$; the only nontrivial case is when $j \in A^0$, in which case $a \in A^0$ by clause 7) of Definition 1.21. Also $(P(n, i, j) \;\&\; S(a, j)) \to (\exists b)(\exists k)(P(k, b, a) \;\&$ $R(b, i))$.

Thus roughly speaking, P defines a partial isomorphism between (A, R) and (B, S).

Consider $K = [\, i \in A : (\forall j)(j \sim_1 i \to (\exists n)(\exists m)(\exists a)(\exists b)(P(n, i, a)$ $\&\; P(m, j, b) \;\&\; a \sim_2 b \;\&\; (\forall c)(c \sim_2 b \to (\exists d)(\exists r)(d \sim_1 i \;\&\; P(r, d, c)))$ $\&\; (\forall c)(c <_2 b \to (\exists d)(\exists r)(d <_1 i \;\&\; P(r, d, c)))))\,]$. Then clearly $\mathrm{Ch}(A - K) \leq_T J(J^\omega(\mathrm{Ch}(T_1)))$. We now break into cases.

Case 1. $A - K = \phi$, $(\forall j \in B)(\exists n)(\exists i)(P(n, i, j))$. Then obviously $(A, R) \approx (B, S)$, given by P.

Case 2. $A - K = \phi$, $(\exists j \in B)(\forall n)(\forall i)(\sim P(n, i, j))$. Note that then $\mathrm{Ch}([\, j \in B : (\forall n)(\forall i)(\sim P(n, i, j))\,]) \leq_T J(J^\omega(\mathrm{Ch}(T_2)))$ and is non-empty. Choose $x \in B$ with $(\forall n)(\forall i)(\sim P(n, i, j))$ & $(\forall y < x)(\exists n)(\exists i)(P(n, i, j))$. Then since $K = A$ we must have that $(\forall j)[(\exists n)(\exists i)(P(n, i, j)) \to j <_2 x]$. Hence set $f(i)$ to be the unique j such that $(\exists n)(P(n, i, j))$. Then $\mathrm{Inj}(f, (A, R), (B, S))$ & $\mathrm{Rng}(f) =$ $[\, j : j < x\,]$.

Case 3. $A - K \neq \phi$, and $(\exists x)(x \in A - K \;\&\; (\forall y)(y <_1 x \to y \in K)\;\&$ $x \notin A^0)$. Fix this x. Note $\mathrm{Ch}([\, j \in B : (\forall n)(\forall i)(i <_1 x \to$ $\sim P(n, i, j))\,]) \leq_T J^\omega(\mathrm{Ch}(T_2))$. If $(\forall j \in B)(\exists n)(\exists i)(i <_1 x \;\&$ $P(n, i, j))$ then take $f(j)$ to be the unique i such that $(\exists n)(P(n, i, j))$. Then $\mathrm{Inj}(f, (B, S), (A, R))$ & $\mathrm{Rng}(f) = [\, y : y <_1 x\,]$. If

$(\exists j \in B)(\forall n)(\forall i)(i <_1 x \to \sim P(n, i\ j))$, then choose $y \in B$ such that $(\forall n)(\forall i)(i <_1 x \to \sim P(n, i, y))$ and $(\forall j <_2 y)(\exists n)(\exists i)(i <_1 x \;\&\; P(n, i, j))$. Now note that $([i: i <_1 x], R \upharpoonright [i: i <_1 x]) \approx ([j: j <_2 y], S \upharpoonright [j: j <_2 y])$ and let f be the isomorphism given by $f(i)$ = the unique j such that $(\exists n)(P(n, i, j))$. We obtain a contradiction by showing that $x \in K$. It suffices to show that $(\forall a)(a \sim_1 x \to (\exists n)(\exists b)(P(n, a, b) \;\&\; b \sim_2 y)) \;\&\; (\forall a)(a \sim_2 y \to (\exists n)(\exists b)(P(n, b, a) \;\&\; b \sim_1 x))$. By symmetry it suffices to obtain the first conjunct. Let $a \sim_1 x$. Then $[i: R(i, a)] \in \mathrm{FODO}([i: i <_1 x], R \upharpoonright [i: i <_1 x])$. In particular let G be a formula and g an assignment such that $[i: R(i, a)] = [i: \mathrm{Sat}(([i: i <_1 x], \underline{R \upharpoonright} [i: i <_1 x]), G, g_i^0]$. Now there must be a k such that $F_1(a) = \overline{k+1}$. Choose the unique $a^* \in B$ such that $[j: S(j, x^*)] = [j: \mathrm{Sat}(([j: j <_2 y], S \upharpoonright [j: j <_2 y]), G, (f \circ g)_j^0)]$. Then since f is an isomorphism we must have $a^* \notin [j: j <_2 y]$ since $a \notin [i: i <_1 x]$. But $a^* \in \mathrm{FODO}([j: j <_2 y], S \upharpoonright [j: j <_2 y])$, and so we have $a^* \sim_2 y$. Also since f is an isomorphism, we have that $\mathrm{Rng}(f \upharpoonright [i: R(i, a)]) = [j: S(j, a^*)]$, and hence by the way f is defined, we have $P(k+1, a, a^*)$.

Case 4. $A - K \neq \phi$, and $(A - K) \cap A^0 \neq \phi$. But this is obviously impossible since $A^0 \subset K$.

Lemma 1.5.2. *Let* (A, R), (B, S) *be towered structures,* $\mathrm{Inj}(f, (A, R), (B, S))$, $x \in B$, $\mathrm{Rng}(f) = [i: i <_2 x]$, *where* $<_2$ *refers to* (B, S). *Then* $J(\mathrm{Ch}(\mathrm{Th}((A, R)))) <_T \mathrm{Ch}(\mathrm{Th}((B, S)))$.

Proof. We use the notation of the proof of Lemma 1.5.1. Fix f, x. Note that $<_2$ has no maximum element. Let x_1 = any $<_2$-least element of $[i: x <_2 i]$. Let x_2 = any $<_2$-least element of $[i: x <_2 i]$. Then $[\bar{i}: i \in \mathrm{Th}((A, R))] \in \mathrm{FODO}(\mathrm{FODO}((A, R)), \epsilon)$ as in 10) of Definition 1.21. Hence there is a $y \sim_2 x_2$ with $S(z, y) \equiv z$ is some $\bar{i}$ with $i \in \mathrm{Th}((A, R))$. Next it is easy to find a formula $P(v_0, v_1)$ such that $\mathrm{Sat}((B, S), P(v_0, v_1), f_y^1) \equiv f(0)$ is some $\bar{j}$ with $J^2(\mathrm{Ch}(\mathrm{Th}(A, R)))(j) = 1$. Hence clearly $J^2(\mathrm{Ch}(\mathrm{Th}((A, R)))) \leq_T \mathrm{Ch}(\mathrm{Th}(B, S))$, since $(\exists n)\,\mathrm{Def}((B, S), n, y)$. Since $J(\mathrm{Ch}(\mathrm{Th}((A, R)))) <_T J^2(\mathrm{Ch}(\mathrm{Th}(A, R))))$, we must have $J(\mathrm{Ch}(\mathrm{Th}((A, R)))) <_T \mathrm{Ch}(\mathrm{Th}((B, S)))$.

Lemma 1.5.3. *Suppose (A,R), (B,S) are towered structures such that* $\mathrm{Ch}(\mathrm{Th}((A,R))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((B,S))))$ *and* $\mathrm{Ch}(\mathrm{Th}((B,S))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((A,R))))$. *Then* $(A\ R) = (B,S)$.

Proof. Assume hypotheses. Then either $(\exists f)(\mathrm{Iso}(f,(A,R),(B,S)))$ or $(\exists f)(\mathrm{Inj}(f,(A,R),(B,S))$ and $(\exists x \in B)(\mathrm{Rng}(f) = [y \in B : y <_2 x]))$, or vice versa. The latter two cases contradict our hypothesis by Lemma 1.5.2. Hence $\mathrm{Iso}(f,(A,R),(B,S))$ for some f. Hence $\mathrm{Th}((A,R)) = \mathrm{Th}((B,S))$, and so obviously for all i, $\mathrm{Def}((A,R)\ i,x) \equiv \mathrm{Def}((B,S),i,f(x))$. Hence by clause 9) of Definition 1.21, f must be the identity. Hence $(A,R) = (B,S)$, and we are done.

Theorem 1.5. *For all* $f \in 2^\omega \cap L^{\omega+\omega}$ *there is a* g *such that* $f \leq_T g$ *and* $(\forall \alpha \in 2^\omega)(g =_T \alpha \rightarrow \alpha \in (2^\omega - \mathcal{S}) \cap L^{\omega+\omega})$.

Proof. Fix $f \in 2^\omega \cap L^{\omega+\omega}$. By Theorem 1.4, choose $h \in \mathcal{S} \cap L^{\omega+\omega}$ with $f \leq_T h$, and let $[\,i : h(i) = 1\,] = \mathrm{Th}((A,R))$, where (A,R) is a towered structure. Then $J(h) \in L^{\omega+\omega}$ and so $(\forall \alpha)(\alpha =_T J(h) \rightarrow \alpha \in L^{\omega+\omega})$. Clearly $f \leq_T J(h)$. Now $J(h) \leq_T J(h)$ and $h \leq_T J(J(h))$, and so by Lemma 1.5.3 there must not be a towered (B,S) with $J(h) =_T \mathrm{Th}((B,S))$. In other words, $(\forall \alpha)(g =_T \alpha \rightarrow \alpha \in 2^\omega - \mathcal{S})$.

Theorem 1.6. $L^{\omega+\omega}$ *satisfies that there exists an element of* $B_{\omega+\omega}$ *with recursive code which is a Turing set but does not contain nor is disjoint from a Turing cone. In particular,* $L^{\omega+\omega}$ *satisfies* $\sim D(\omega+\omega)$ *by* Theorem 1.1.

Proof. Take the Turing set X to be $[f \in 2^\omega : (\exists g \in \mathcal{S})(f =_T g)]$. Then using Theorem 1.3 it is easily seen that $X \cap L^{\omega+\omega} \in L^{\omega+\omega}$ and is satisfied to be an element of $B_{\omega+\omega}$ with recursive code and to be a Turing set. From Theorem 1.4 one has that X is satisfied to intersect every Turing cone, because of the absoluteness of Turing reducibility. By Theorem 1.5, X is satisfied to not contain any Turing cone.

Corollary. *By* Theorem 1.2, $D(\omega+\omega)$ *is not provable in* $\mathbf{Z}$.

Section 2

We have defined **Z** in Definition 1.10, and $L^{\omega+\omega}(\alpha)$, $L^{\omega+\omega}$ in Definition 1.6, and have remarked that each $L^{\omega+\omega}(\alpha)$ is transitive and that $\mathrm{Sat}((L^{\omega+\omega}, \epsilon), F, f)$ for all $F \in \mathbf{Z}$ and assignments f (see Definition 1.13). Furthermore, we have the special structure (A^0, R^0) of Definition 1.20.

The purpose of this Section is to give a detailed outline of a proof of the fact about the $L^{\omega+\omega}(\alpha)$ needed in Section 1; namely, Theorem 2.

Definition 2.1. We let $\langle x, y\rangle = [x, [x, y]]$. We write $\mathrm{Fcn}(x)$ for $(\forall y \in x)(\exists a)(\exists b)(y = \langle a, b\rangle)$ & $(\forall a)(\forall b)(\forall c)((\langle a, b\rangle \in x$ & $\langle a, c\rangle \in x) \to b = c)$. We write $\mathrm{Dom}(x)$ for $[a: (\exists b)(\langle a, b\rangle \in x)]$, $\mathrm{Rng}(x)$ for $[a: (\mathrm{E}b)(\langle b, a\rangle \in x)]$. We let $(\ ) = \phi$, $(x) = [\langle 0, x\rangle]$, $(x_0, \ldots, x_k) = [\langle i, x_i\rangle: 0 \leq i \leq k]$. We write $\mathrm{ln}((x_0, \ldots, x_{k-1})) = k$, $(x_0, \ldots, x_{k-1})(i) = x_i$, $i < k$. We take $\mathrm{Seq}(x) = [y: \mathrm{Fcn}(y)$ & $(\exists k \in \omega)(k \neq \phi$ & $\mathrm{Dom}(y) = k)$ & $\mathrm{Rng}(y) \subset x]$. We take $a_0 * a_1 * \ldots * a_k$, for $a_i \in \mathrm{Seq}(x)$, to be the result of concatenation.

Definition 2.2. We assume a one-one Gödel numbering from formulae onto ω. A formula is a formula using $\forall$, $\exists$, &, v, ~, $\in$, =, $v_0, v_1, \ldots$. For formulae F we let 'F' be the Gödel number of F. For $n \in \omega$ we let $|n|$ be that formula with Gödel number n.

Definition 2.3. We write $\mathrm{LO}(x)$, (x is a linear ordering) for $x = \langle A, R\rangle$ and $A \neq \phi$ and $R \subset [\langle a, b\rangle: a \in A$ & $b \in A]$ and $A \cap V(\omega) = \phi$ and (A, R) constitutes a linear ordering on all of A. We write $A = \mathrm{Field}(x)$, $R = \mathrm{Rel}_1(x)$.

Definition 2.4. If $\mathrm{LO}(x)$ we take $O(x, y) \equiv y \in A$ & $(\forall z)(\langle z, y\rangle \notin \mathrm{Rel}_1(x))$, $\mathrm{Suc}(x, y, z) \equiv \langle z, y\rangle \in R_1$ & ~ $(\exists a)(\langle z, a\rangle \in R_1$ & $\langle a, y\rangle \in R_1)$, $\mathrm{Lim}(x, y) \equiv y \in A$ & $(\forall z)(\langle z, y\rangle \in R_1 \to (\exists a)(\langle z, a\rangle \in R_1$ & $\langle a, y\rangle \in R_1))$.

Definition 2.5. We write $\mathrm{CS}(x)$, (x is a coded structure), for $x = \langle A, R\rangle$ and $A \neq \phi$ and $R \subset [\langle a, b\rangle : a \in A$ & $b \in A]$. We write $A = \mathrm{Field}(x)$, and whenever we write $\mathrm{CS}(x)$, we write Rel_2 for R.

Definition 2.6. We write SLO(x), (x is a structured linear ordering), for $x = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$ and LO$(\langle A, R_1\rangle)$ and CS$(\langle A, R_2\rangle)$, and $F\colon A \to \omega$. We write Field$(x) = A$, Rel$_1(x) = R_1$, Rel$_2(x) = R_2$, Fn$(x) = F$.

Definition 2.7. We write Sati(x, n, y) for CS(x) & $n \in \omega$ & $y \in$ Seq(x) & $y = (a_0, ..., a_k)$, $0 \leq k$, & $x = \langle A, R\rangle$ & Sat$(\langle A, R\rangle, |n|, f)$, where $f(i) = y(i)$ for $i <$ ln(y); $y(\ln(y) - 1)$ for $i \geq \ln(y)$.

Definition 2.8. Let K be the least class satisfying

1) $A^0 \subset K$. See Definition 1.20

2) whenever $a_0, ..., a_k \in K$, $0 \leq k$, $n \in \omega$, $x \notin V(\omega)$ we have $(x, n, a_0, ..., a_k) \in K$. Let F_0 be the function on K given by $F_0(n) = n$ if $n \in A^0$; $F_0((x, n, a_0, ..., a_k)) = ([1], x, n) * F_0(a_0) * ... * F_0(a_k) * * ([2])$.

Lemma 2.1. *F_0 is a one-one function on K.*

Proof. We prove by induction on ln(s) that if $F_0(y_1) = s$, $F_0(y_2) = s$, then $y_1 = y_2$. Assume $F_0(y_1) = s$, $F_0(y_2) = s$. Then if s is not a sequence then $s = n$ for some $n \in A^0$, in which case $y_1 = y_2 = n$. So s is a sequence. Clearly s must be of the form $([1], x, n) * F_0(a_0) * ... * F_0(a_k) * ([2])$. Now we must show that the $a_0, ..., a_k, x, n$ above are unique. Let $s = ([1], y, m) * F_0(b_0) * ... * F_0(b_r) * ([2])$. Obviously $x = y$, $n = m$. If $F_0(a_0) \in A^0$ then obviously $F_0(b_0) \in A^0$ and $F_0(a_0) = F_0(b_0)$. If $F_0(a_0) \notin A^0$ then $F_0(a_0)$ starts with $[1]$ and ends with $[2]$, and no $[1]$ or $[2]$ occurs in between. Therefore $F_0(a_0) = F_0(b_0)$, and so on. So we obtain that $k = r$ and each $F_0(a_i) = F_0(b_i)$. Since each $F(a_i)$ has shorter length than s, we are done by induction hypothesis.

Definition 2.9. We write $<(x, a, b)$ for LO(x) & $a, b \in$ Seq(Field(x)) & a comes before b in the lexicographic ordering on Seq(Field(x)) induced by x.

Definition 2.10. We write Defn(x, a, k) for SLO(x) and $a = (n, b_0, ..., b_m)$, $0 \leq m$, and each $b_i \in$ Field(x) and $Y = \{b\colon$ Sati(\langle Field(x),

$\mathrm{Rel}_2(x)\rangle, n, (b, b_0, ..., b_m))]$ satisfies the following conditions:

a) the range of $\mathrm{Fn}(x) \upharpoonright Y$ contains $k-1$ as an element and is a subset of k and $k \in \omega - [0]$.

b) $Y \neq [b: \langle b, c\rangle \in \mathrm{Rel}_2(x)]$ for all $c \in \mathrm{Field}(x)$,

c) $Y \neq [b: \mathrm{Sati}(\langle \mathrm{Field}(x), \mathrm{Rel}_2(x)\rangle, r, (b, b_0, ..., b_m))]$ for all $r < n$,

d) $Y \neq [b: \mathrm{Sati}(\langle \mathrm{Field}(x), \mathrm{Rel}_2(x)\rangle, n, (b, c_0, ..., c_r))]$ whenever $<(x, (c_0, ..., c_r), (b_0, ..., b_m))$.

Definition 2.11. We write $\mathrm{CHY}(x,f)$ for

1) $\mathrm{LO}(x)$

2) $\mathrm{Fcn}(f)$ & $\mathrm{Dom}(f) = \mathrm{Field}(x)$ & $(\forall y)(y \in \mathrm{Field}(x) \to (\mathrm{SLO}(f(y))$ & $\mathrm{Field}(f(y)) \subset A^0 \cup \mathrm{Seq}(V(\omega) \cup x)))$

3) $0(x,y) \to f(y) = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$, where $A = A^0$, $R_2 = R^0$, $R_1 = \epsilon \upharpoonright A^0$, $F(a) = 0$ for all $a \in A$

4) $\mathrm{Suc}(x,a,b) \to f(a) = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$, where $A = \mathrm{Field}(f(b)) \cup [([1], b, n) * b_0 * ... * b_m * ([2]): \mathrm{Defn}(f(b), (n, b_0, ..., b_m), k)$ for some $k]$, $R_1 = \mathrm{Rel}_1(f(b)) \cup [\langle a, s\rangle: a \in \mathrm{Field}(f(b))$ & $s \in A\text{-}\mathrm{Field}(f(b))] \cup [\langle a, s\rangle: a, s \in A\text{-}\mathrm{Field}(f(b))$ & $a = ([1], b, n) * b_0 * ... * b_m * ([2])$ & $s = ([1], b, m) * c_0 * ... * c_r([2])$ & $(n < m \vee < ((\mathrm{Field}(f(b)), \mathrm{Rel}_1(f(b))), (b_0, ..., b_m), (c_0, ..., c_r)))]$, $R_2 = \mathrm{Rel}_2(f(b)) \cup [\langle a, s\rangle: a \in \mathrm{Field}(f(b))$ & $s \in A\text{-}\mathrm{Field}(f(b))$ & $s = ([1], b, n) * b_0 * ... * b_m * ([2])$ & $\mathrm{Sati}(\langle \mathrm{Field}(f(b)), \mathrm{Rel}_2(f(b))\rangle, n, (a, b_0, ..., b_m))]$, $F(a) = \mathrm{Fn}(f(b))(a)$ if $a \in \mathrm{Field}(f(b))$; if $a \in A\text{-}\mathrm{Field}(f(b))$, $a = ([1], b, n) * b_0 * ... * b_m * ([2])$, then $F(a) = k$ where $\mathrm{Defn}(f(b), (n, b_0, ..., b_m), k)$

5) $\mathrm{Lim}(x,a) \to f(a) = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$, where F, A, R_1, R_2 are the unions, over those b with $\langle b, a\rangle \in \mathrm{Rel}_1(x)$, of $\mathrm{Fn}(f(b))$, $\mathrm{Field}(f(b))$, $\mathrm{Rel}_1(f(b))$, $\mathrm{Rel}_2(f(b))$, respectively. $\mathrm{CHY}(x,f)$ reads "f is a coded hierarchy on x".

Definition 2.11. A limit ordinal λ is an ordinal > 0 with no immediate predecessor. Whenever we write λ we mean a limit ordinal.

Lemma 2.2. *There is a formula* $P_1(v_0, v_1, v_2)$ *and a sentence* Q_1 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_1)$, *and for all transitive sets* A *such that* $\mathrm{Sat}((A, \epsilon), Q_1)$ *we have* : $\mathrm{Sat}((A, \epsilon), P_1, f) \equiv \mathrm{Sati}(f(0), f(1), f(2))$, *for all assignments* f *in* A, *and* $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists x)(\forall y)(y \in x \equiv (y = \langle v_1, v_2\rangle \;\&\; P_1(v_0, v_1, v_2))))$.

Lemma 2.3. *There is a formula* $P_2(v_0, v_1, v_2)$ *and a sentence* Q_2 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_2)$, *and for all transitive sets* A *such that* $\mathrm{Sat}((A, \epsilon), Q_2)$ *we have* $\mathrm{Sat}((A, \epsilon), P_2, f) \equiv\, < (f(0), f(1), f(2))$, *for all assignments* f *in* A, *and have also* $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists x)(\forall y)(y \in x \equiv (y = \langle v_1, v_2\rangle \;\&\; P_2(v_0, v_1, v_2))))$.

Lemma 2.4. *There is a formula* $P_3(v_0, v_1, v_2)$ *and a sentence* Q_3 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_3)$, *and for all transitive sets* A *with* $\mathrm{Sat}((A, \epsilon), Q_3)$ *we have*: $\mathrm{Sat}((A, \epsilon), P_3, f) \equiv \mathrm{Defn}(f(0), f(1), f(2))$, *for all assignments* f *in* A, *and* $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists x)(\forall y)(y \in x \equiv (y = \langle v_1, v_2\rangle \;\&\; P_2(v_0, v_1, v_2))))$.

Lemma 2.5. *There is a formula* $P_4(v_0, v_1)$ *and a sentence* Q_4 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_4)$ *and for all transitive sets* A *with* $\mathrm{Sat}((A, \epsilon), Q_4)$ *we have* $\mathrm{Sat}((A, \epsilon), P_4, f) \equiv \mathrm{CHY}(f(0), f(1))$, *for all assignments* f *in* A.

Definition 2.12. We write $\mathrm{WO}(x)$ for $\mathrm{LO}(x) \;\&\; (\forall y \subset \mathrm{Field}(x))(y \neq \phi \to (\exists a \in y)(\forall b \in y)(\langle b, a\rangle \notin \mathrm{Rel}_1(x)))$. We write $(A, R) \approx (B, S)$ for $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S))$. If $\mathrm{LO}(x)$ and $a \in \mathrm{Field}(x)$, then we write x_a for $[b : \langle b, a\rangle \in \mathrm{Rel}_1(x)]$.

Lemma 2.6. *For all* $x \in V(\omega + \omega)$ *with* $\mathrm{WO}(x)$ *there is a unique* f *such that* $\mathrm{CHY}(x, f) \;\&\; f \in V(\omega + \omega)$. *Furthermore,*

1) *for all* $a \in \mathrm{Field}(x)$ *we have that* $(\exists !\, g_a)(\mathrm{Iso}(g_a, (\mathrm{Field}(f(a)), \mathrm{Rel}_2(f(a))), (L^{\omega+\omega}(\beta), \epsilon)))$, *where* $(x_a, \mathrm{Rel}_1 \upharpoonright x_a) \approx (\beta, \epsilon)$

2) *for all* $a \in \mathrm{Field}(x)$ *and for all* $b \in \mathrm{Field}(f(a))$ *we have that* $\mathrm{Fn}(f(a))(b) = \mu n(g_a(b) \in V(\omega + n))$

3) *for all* $a \in \mathrm{Field}(x)$ *we have* $\mathrm{WO}(\langle \mathrm{Field}(f(a)), \mathrm{Rel}_1(f(a))\rangle)$.

Lemma 2.7. *Let* LO(x), (Field(x), Rel$_1(x)$) $\approx (\alpha, \epsilon)$, $x \in L^{\omega+\omega}(\beta)$. *Then* $(\exists f)$(CHY(x,f) & $f \in L^{\omega+\omega}(\beta + \alpha + \omega)$). *Furthermore for each* $a \in$ Field(x) *and* k *there is a* $g_a^k \in L^{\omega+\omega}(\beta + \alpha + \omega)$ *such that* Iso(g_a^k, (Field$(f(a))$ $\cap$ $[b$: Fn$(f(a))(b) \leq k]$, Rel$_2(f(a))$ $\restriction$ Field$(f(a))$ $\cap$ $[b$: Fn$(f(a))(b) \leq k]$), $(L^{\omega+\omega}(\gamma) \cap V(\omega + k), \epsilon)$), *and* $L^{\omega+\omega}(\gamma) \cap V(\omega + k) \in L^{\omega+\omega}(\beta + \alpha + \omega)$, where $(\gamma, \epsilon) \approx (x_a$, Rel$_1(x) \restriction x_a)$.

Proof. Fix β. Then argue by induction on α. The basis case is trivial. Argue the limit case through use of Lemma 2.6, which gives unicity below the limit and which assures that the types needed are bounded below by $V(\omega + n_0)$, and by Lemma 2.5, which gives a first-order description below the limit. Argue the successor case by Lemma 2.4.

The g_a^k are developed by induction on k.

Definition 2.13. We say $L^{\omega+\omega}(\alpha)$ is pure just in case $\omega < \alpha$ and for all $\beta < \alpha$ there is an $x \in L^{\omega+\omega}(\alpha)$ with LO(x) and $(\beta, \epsilon) \approx$ (Field(x), Rel$_1(x)$), and for all $\beta < \alpha$ we have $L^{\omega+\omega}(\beta) \neq L^{\omega+\omega}(\beta + 1)$.

Lemma 2.8. *Let* $L^{\omega+\omega}(\alpha)$ *be pure,* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, WO$(v_0)$, $\lambda k(x))$. *Then either* WO(x) *or for all* $\beta < \alpha$ *there is an* $a \in$ Field(x) *with* $(\beta, \epsilon) \approx ([b : \langle b, a\rangle \in$ Rel$_1(x)]$, Rel$_1(x) \restriction [b : \langle b, a\rangle \in$ Rel$_1(x)]$).

Proof. Let $x \in L^{\omega+\omega}(\alpha)$, Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, WO$(v_0)$, $\lambda k(x))$, and assume $\beta < \alpha$, $\sim$ WO(x), and β is the order type of the maximal well-ordered initial segment of (Field(x), Rel$_1(x)$). We wish to obtain a contradiction. By purity, let $y \in L(\alpha)$ have LO(y) & $(\beta, \epsilon) \approx$ (Field(y), Rel$_1(y)$), and choose $\gamma < \alpha$ with $x, y \in L^{\omega+\omega}(\gamma)$. Then a straightforward inductive argument will reveal the existence of an isomorphism from the ordering defined by y onto the maximal well-ordered initial segment of the ordering defined by x, which lies in $L^{\omega+\omega}(\gamma + \beta + \omega)$. But then Sat$((L^{\omega+\omega}(\gamma + \beta + \omega), \epsilon)$, $\sim$ WO(v_0), $\lambda k(x))$, and hence Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, $\sim$ WO(v_0), $\lambda k(x))$, which is a contradiction.

Lemma 2.9. *Let* $L^{\omega+\omega}(\alpha)$ *be pure,* $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$, $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, *and* Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, WO$(v_0)$, $\lambda n(x))$. *Then*

$(\exists f \in L^{\omega+\omega}(\alpha))(\mathrm{CHY}(x,f))$ *if and only if* $(\exists \beta < \alpha)((\mathrm{Field}(x), \mathrm{Rel}_1(x)) \approx (\beta, \epsilon))$.

Proof. Suppose $\sim \mathrm{WO}(x)$. Then by Lemma 2.8 the maximal well-ordered initial segment of x must be at least α. Note that we can define $g_a^k \in L^{\omega+\omega}(\alpha)$ as in Lemma 2.7, for each $a \in \mathrm{Field}(x)$, even though $\sim \mathrm{WO}(x)$. In fact, let $x \in L^{\omega+\omega}(\beta)$. Then the g_a^k are in $L^{\omega+\omega}(\beta+\omega)$. Consider $S = [a \in \mathrm{Field}(x) \colon (\exists k)(\exists b \in \mathrm{Rng}(g_a^k))(\forall c)(\langle c, a\rangle \in \mathrm{Rel}_1(x) \to (\forall p)(b \notin \mathrm{Rng}(g_c^p)))]$. Then clearly S contains the initial segment of x of type α. Now, S is in $L^{\omega+\omega}(\beta+\omega+\omega)$. If α is the type of the maximal well-ordered initial segment of x then, since $\mathrm{WO}(x)$ holds in $L^{\omega+\omega}(\alpha)$, we must have $(\exists a \in S)$ (a is beyond the maximal well-ordered initial segment of x). If there is a well-ordered initial segment of x of type $\alpha + 1$ then since $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$, we must again have $(\exists a \in S)$ (a is bevond the maximal well-ordered initial segment of x). Fixing this a, form $g_a^k \in L^{\omega+\omega}(\beta+\omega)$. Then by definition of S, we will have a $y \in L^{\omega+\omega}(\beta+\omega)$ which does not lie in $L^{\omega+\omega}(\alpha)$, which is a contradiction. The converse is by Lemma 2.7.

Lemma 2.10. *There is a sentence* Q_5 *such that*

1) *for all pure* $L^{\omega+\omega}(\alpha)$ *with* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$ *and* $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), Q_5)$

2) *if* A *is transitive and* $\mathrm{Sat}((A, \epsilon), Q_5)$ *and for all assignments* f *in* A, $\mathrm{Sat}((A, \epsilon), (\exists v_1)(P_4(v_0, v_1)), f) \to \mathrm{WO}(f(0))$, *then* $(\exists \beta)(A = L^{\omega+\omega}(\beta)\ \&\ (\forall \gamma)(\gamma < \beta \to \gamma + \gamma < \beta))$.

Lemma 2.11. *There is a formula* $P_5(v_0, v_1)$ *such that for all pure* $L^{\omega+\omega}(\alpha)$ *with* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$ *and* $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$ *we have* $\mathrm{WO}(\langle A, R\rangle)$, *where* $A = L^{\omega+\omega}(\alpha)$ *and* $R = [\langle a, b\rangle \colon \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), P_5, \lambda n(a \text{ if } n = 0;\ b \text{ if } n \neq 0))]$.

Proof. We will just define the R. Take $R = [\langle g_y^k(a), g_y^p(b)\rangle \colon (\exists x)(\exists y)(\exists f)(\mathrm{WO}(x)\ \&\ f \in L^{\omega+\omega}(\alpha)\ \&\ \mathrm{CHY}(x,f)\ \&\ y \in \mathrm{Field}(x)\ \&\ a, b \in \mathrm{Field}(f(y))\ \&\ \langle a, b\rangle \in \mathrm{Rel}_1(f(y))\ \&\ \mathrm{Fn}(f(y)(a) = k\ \&\ \mathrm{Fn}(f(y))(b) = p)]$. Of course, g_y^k, g_y^p depend on x, f as in Lemma 2.7.

Lemma 2.12. *Let $L^{\omega+\omega}(\alpha)$ be pure, $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$, $x \in L^{\omega+\omega}(\alpha + 1)$, where $x = [a : \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, \lambda n(a))]$. Then there is a transitive set $A \subset L^{\omega+\omega}(\alpha)$ such that*

1) $\mathrm{Sat}((A, \epsilon), Q_4 \,\&\, Q_5)$

2) $\mathrm{TC}(x) \subset A \,\&\, x \in A$ *and* $(\forall a \in x)(\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, \lambda n(a)) \equiv \mathrm{Sat}((A, \epsilon), F, \lambda n(a)))$

3) $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists v_1)(P_4(v_0, v_1)) \to \mathrm{WO}(v_0)))$

4) *for all $y \in A$ we have* $[\mathrm{Sat}((A, \epsilon), \mathrm{WO}(v_0), \lambda n(y)) \equiv \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), \mathrm{WO}(v_0), \lambda n(y))]$ & $[\mathrm{Sat}((A, \epsilon), (\exists f)(P_4(y, f)), \lambda n(y)) \equiv \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), (\exists f)(P_4(y, f)), \lambda n(y))]$

5) *there is a partial function G which is from the cartesian product of ω with* $\mathrm{TC}(x)$ *onto A and a formula $P_6(v_0, v_1, v_2, v_3)$ such that $G(a, b) = c$ if and only if* $\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), P_6(v_0, v_1, v_2, v_3), \lambda n(a \text{ if } n = 0; b \text{ if } n = 1; c \text{ if } n = 2; x \text{ if } n > 2))$.

Proof. Using Lemma 2.11, employ a standard closure of $\mathrm{TC}(x) \cup [x]$ under the Skolem functions for the finite number of formulae needed. This can be described in $L^{\omega+\omega}(\alpha)$ because of the bound in complexity of the formulae. Then perform the isomorphy onto the transitive set A. This isomorphism can also be described in $L^{\omega+\omega}(\alpha)$, and will result in a subset of $L^{\omega+\omega}(\alpha)$. This isomorphism will carry well-orderings into well-orderings.

Lemma 2.13. *Let $L^{\omega+\omega}(\alpha)$ be pure, $(\forall \beta < \alpha)(\beta + \beta < \alpha)$. Furthermore, suppose $L^{\omega+\omega}(\alpha + 1) - L^{\omega+\omega}(\alpha) \neq \phi$. Then there is a partial function G, and P_6 such that* 5) *in* Lemma 2.12 *holds and $A = L^{\omega+\omega}(\alpha)$.*

Proof. Choose A as in Lemma 2.12, using any $x \in L^{\omega+\omega}(\alpha + 1) - L^{\omega+\omega}(\alpha)$ of the form $[a : \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, \lambda n(a))]$. Such an x can be found by Lemma 2.11. It suffices to prove that $A = L^{\omega+\omega}(\alpha)$. Note that by Lemma 2.10 we have $A = L^{\omega+\omega}(\beta)$ for some β. Note by 2) of Lemma 2.12 that $x \in L^{\omega+\omega}(\beta + 1)$. Hence $\alpha = \beta$.

Lemma 2.14. *Let $L^{\omega+\omega}(\alpha)$ be pure, $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$. Then $L^{\omega+\omega}(\alpha + 1)$ is pure.*

Proof. We use the G, P_6 of Lemma 2.13, for some $x \in L^{\omega+\omega}(\alpha+1) - L^{\omega+\omega}(\alpha)$, and P_5 of Lemma 2.11. It suffices to produce a linear ordering $y \in L^{\omega+\omega}(\alpha+1)$ with $(\alpha, \epsilon) \approx (\mathrm{Field}(y), \mathrm{Rel}_1(y))$. Take $y = \langle A, R\rangle$, where $A = \mathrm{Dom}(G)$, $R = [\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle : \langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in A$ & $\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), P_5(v_0, v_1), \lambda n(G(x_1, y_1)$ if $n = 0$; $G(x_2, y_2)$ if $n > 0))]$. If this $\langle A, R\rangle$ is longer than (α, ϵ) then take the appropriate initial segment; this $\langle A, R\rangle$ must be a well-ordering.

Lemma 2.15. *If $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$ and $\omega < \alpha$ then $L^{\omega+\omega}(\alpha+1)$ and $L^{\omega+\omega}(\alpha)$ are pure.*

Proof. Straightforward from Lemma 2.14 by transfinite induction.

Lemma 2.16. *Suppose $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$. Then $L^{\omega+\omega}(\alpha \times \omega) \neq L^{\omega+\omega}((\alpha \times \omega)+1)$.*

Proof. Suppose $L^{\omega+\omega}(\alpha \times \omega) = L^{\omega+\omega}((\alpha \times \omega)+1)$. By Lemma 2.15, there is a well-ordering in $L^{\omega+\omega}(\alpha+1)$ of type α. Hence there is a well-ordering $y \in L^{\omega+\omega}(\alpha \times \omega)$ of type $(\alpha \times \omega)+1$. Since $(L^{\omega+\omega}(\alpha+\omega), \epsilon)$ satisfies Z, there must be an $f \in L^{\omega+\omega}(\alpha \times \omega)$ with $\mathrm{CHY}(y, f)$. Hence $\mathrm{TC}(f) \in L^{\omega+\omega}(\alpha \times \omega)$ since $(L^{\omega+\omega}(\alpha \times \omega), \epsilon)$ satisfies Z. In addition $(L^{\omega+\omega}(\alpha \times \omega), \epsilon)$ must satisfy that every set has smaller cardinality than $\mathrm{TC}(f)$. But $(L^{\omega+\omega}(\alpha \times \omega), \epsilon)$ satisfies the power set axiom and Cantor's Theorem, and so we have a contradiction.

Lemma 2.17. *Let $y \subset \omega$, $y \in L^{\omega+\omega}$. Then there is a λ such that $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$ and $y \in L^{\omega+\omega}(\lambda)$ and a formula $P_7(v_0, v_1, v_2)$ such that* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), (\forall v_1)(\exists! v_0)(v_0 \in \omega$ & $P_7(v_0, v_1, v_2)), \lambda n(z))$, *for some $z \in L^{\omega+\omega}(\lambda)$.*

Proof. Choose α least such that $y \in L^{\omega+\omega}(\alpha)$, $\omega < \alpha$. Then $\alpha = \beta + 1$. Set $\lambda = \beta \times \omega$. Note that by Lemma 2.16, $L^{\omega+\omega}(\lambda)$ satisfies the hypotheses of Lemma 2.12, using y for x. Using Lemma 2.10, the resulting A must be $L^{\omega+\omega}(\lambda)$. Using the P_6 of Lemma 2.12 one easily constructs the desired P_7 since $\mathrm{TC}(y) = \omega$, or y is finite.

Lemma 2.18. *Let $y \subseteq \omega, y \in L^{\omega+\omega}$. Then there is a λ such that $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda+1)$ and $y \in L^{\omega+\omega}(\lambda)$ and a formula $P_8(v_0, v_1)$ such that* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), (\forall v_1)(\exists! v_0)(v_0 \in \omega$ & $P_8(v_0, v_1)))$.

Proof. Take λ, P, as in Lemma 2.17. Note that $L^{\omega+\omega}(\lambda)$ satisfies the hypotheses of Lemma 2.11. Using the P_5 of Lemma 2.11, take $P_8(v_0, v_1)$ to be $(\exists v_2)((\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_7(v_0, v_1, v_2)) \;\&\; (\forall v_4)(P_5(v_4, v_1) \to \sim (\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_7(v_0, v_1, v_4))) \;\&\; P_7(v_0, v_1, v_2))$.

Lemma 2.19. *Suppose* $P_9(v_0, v_1)$ *is a formula such that* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), (\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_9(v_0, v_1)))$. *Then* $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$.

Proof. Note that there must be an $(\omega, R) \approx (L^{\omega+\omega}(\lambda), \epsilon)$ such that $R \in L^{\omega+\omega}(\lambda + 1)$. In addition, every set of natural numbers arithmetical in R will be in $L^{\omega+\omega}(\lambda + 1)$. Hence straightforwardly, $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$.

Combining Lemmas 2.17 and 2.18, we immediately have:

Theorem 2. *There are formulae* $\varphi_1(v_0, v_1)$, $\varphi_2(v_0, v_1)$, *and* $\varphi_3(v_0, v_1)$ *in* LST *with only the free variables shown such that for each* $x \subset \omega$, $x \in L^{\omega+\omega}$ *there is a limit ordinal* λ *such that*

1) $x \in L^{\omega+\omega}(\lambda)$

2) $(\forall y \in L^{\omega+\omega}(\lambda))(\exists n)(\mathrm{Def}((L^{\omega+\omega}(\alpha), \epsilon), n, y))$

3) $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$

4) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_1(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) < (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

5) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_2(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) = (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

6) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_3(v_0, v_1), f)$ *if and only if* $f(1) = (\mu n \in \omega)(f(0) \in V(\omega + n))$.

Section 3

In this Section we discuss various refinements of Theorem 1.6 and its Corollary.

We assume familiarity with the hierarchy of numerical formulae with one function parameter ranging over ω^ω.

Definition 3.1. A *towered * structure* is a structure (A, R) such that clauses 1) – 10) of Definition 1.21 hold and in addition, for each Π_1^0 predicate $Q(n, f)$ we have $(\exists n)(n \in A \; \& \sim Q(n, J^\omega(\mathrm{Ch}(\mathrm{Th}(A, R))))) \rightarrow (\exists n)(n \in A \; \& \sim Q(n, J^\omega(\mathrm{Ch}(\mathrm{Th}(A, R)))) \; \& \; (\forall m)(m < n \rightarrow Q(m, J^\omega(\mathrm{Ch}(\mathrm{Th}((A, R))))))$. Define $\mathcal{S}^* = [\mathrm{Ch}(\mathrm{Th}((A, R))) : (A, R)$ is a towered* structure$]$.

Lemma 3.1.1. *$L^{\omega+\omega}$ satisfies that $\mathcal{S}^* \cap L^{\omega+\omega}$ is an element of $B_{\omega+3}$ with recursive code.*

Proof. Routine counting of quantifiers and comparison with the Borel hierarchy.

Lemma 3.1.2. *Suppose (A, R), (B, S) are towered* structures such that* $\mathrm{Ch}(\mathrm{Th}((A, R))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((B, S))))$ *and* $\mathrm{Ch}(\mathrm{Th}((B, S))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((A, R))))$. *Then either* $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S)))$ *or* $(\exists f)(\mathrm{Inj}(f, (A, R)(B, S))$ *and* $(\exists x \in B)(\mathrm{Rng}(f) = [y \in B : y < x]$, *where $<$ is as in (B, S) as in* Definition 3.1 *(which refers back to* Definition 1.21*))), or* $(\exists f)(\mathrm{Inj}(f, (B, S), (A, R))$ *and* $(\exists x \in A)(\mathrm{Rng}(f) = [y \in A : y < x]$, *where $<$ is as in (A, R) in* Definition 1.21*)).*

Proof. This is the analogue to Lemma 1.5.1, and is proved exactly the same way, moticing that, for instance, the K of that proof is defined by a Π_1^0 predicate $Q(n, J^\omega(\mathrm{Ch}((A, R))))$.

Arguing as in Section 1, we have

Theorem 3.1. *$L^{\omega+\omega}$ satisfies "there exists an element $Y \in B_{\omega+3}$, with recursive code, such that $\sim D(Y)$". Hence the assertion in quotes is consistent with* **Z**.

Proof. Consider the game given by $Y \subset 2^\omega$, where $Y = [f \in 2^\omega : \lambda n(f(2n)) \in \mathcal{S}$ & $\lambda n(f(2n+1)) \leq_T \lambda n(f(2n))]$.

Definition 3.2. Define $L^\alpha(0) = V(\omega)$, $L^\alpha(\beta+1) = \mathrm{FODO}((L^\alpha(\beta), \epsilon)) \cap V(\alpha)$, $L^\alpha(\lambda) = \bigcup_{\beta<\lambda} L^\alpha(\beta)$, where λ is a limit ordinal. Define $L^\alpha = [x : (\exists\beta)(x \in L^\alpha(\beta))]$.

For the moment, let us concentrate on the case $\alpha = \omega + 1$.

Now we cannot directly speak of Borel subsets of 2^ω and determinateness within $L^{\omega+1}$. What we do is to consider formulae $P(v_0)$ and associate the sentence P^* which naturally formalizes the assertion that $D([f : f \in 2^\omega$ & $P(f)])$. In particular we shall construct a numerical formula $P(f)$ which is in prenex form and has 5 quantifiers (numerical, of course) such that the corresponding sentence P^* fails in $(L^{\omega+1}, \epsilon)$. Thus we can say that, in the appropriate sense, $L^{\omega+1}$ satisfies that "there is a $Y \in B_5$ with recursive code such that $\sim D(Y)$". However, with L^α, where $\omega + 1 < \alpha$, no such devices of expression are needed.

Lemma 3.2.1. *There are formulae $\psi_1(v_0, v_1)$, and $\psi_2(v_0, v_1)$ in* LST *with only the free variables shown such that for each $x \subset \omega$, $x \in L^{\omega+1}$, there is a limit ordinal λ such that*

1) $x \in L^{\omega+1}(\lambda)$

2) $(\forall y \in L^{\omega+1}(\lambda))(\exists n)(\mathrm{Def}((L^{\omega+1}(\alpha), \epsilon), n, y))$

3) $\mathrm{Th}((L^{\omega+1}(\lambda), \epsilon)) \in L^{\omega+1}(\lambda+2)$

4) $\mathrm{Sat}((L^{\omega+1}(\lambda), \epsilon), \varphi_1(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+1}(\beta)) < (\mu\beta)(f(1) \in L^{\omega+1}(\beta))$

5) $\mathrm{Sat}((L^{\omega+1}(\lambda), \epsilon), \varphi_2((v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+1}(\beta)) = (\mu\beta)(f(1) \in L^{\omega+1}(\beta))$

6) $(\forall x \in L^{\omega+1}(\lambda))(x \subset V(\omega))$.

Proof. The proof is like the proof of Theorem 2. One uses standard pairing and inverse pairing functions on $V(\omega)$ to code everything as a subset of $V(\omega)$.

In the following, we use φ_1, and φ_2 as in the statement of Theorem 3.2.1.

Definition 3.3. A *towered⁻ structure* is a structure (A, R) such that

1) $A \subset \omega$ and the relation $x \sim y \equiv \mathrm{Sat}((A,R), \varphi_2(v_0, v_1), \lambda n(x \text{ if } n = 0;$ $y \text{ if } n \neq 0))$ is an equivalence relation on A

2) the relation $x < y \equiv \mathrm{Sat}((A,R), \varphi_1(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ has that $(\forall x, y \in A)((x < y \,\&\, \sim y < x) \vee (y < x \,\&\, \sim x < y) \vee$ $(x \sim y \,\&\, \sim x < y \,\&\, \sim y < x))$ and $(\forall x, y, z \in A)(((x \sim z \,\&\, x < y)$ $\rightarrow z < y) \,\&\, ((x \sim z \,\&\, y < x) \rightarrow y < z))$, and $<$ has no maximal element

3) $A^0 = [\, i : i \in A \,\&\, (\forall j)(\sim j < i)\,]$, $R^0 = R \restriction A^0$

4) we have $(\forall x \in A)(\forall y)(R(y,x) \rightarrow y \in A^0)$

5) suppose $x \in A$. Then $\mathrm{FODO}(([\, i : i < x\,], R \restriction [\, i : i < x\,])) =$ $[\, z \subset [\, i : i < x\,] : (\exists j)(j < x \vee j \sim x) \,\&\, z = [\, k : R(k,j)\,]\,)]$

6) (A, R) satisfies the axiom of extensionality

7) $(\forall i \in A - A^0)(\mathrm{Def}((A,R), i, 2i))$

8) for some k we have that for all $x \in A$ there exists a prenex formula φ with only free variable v_0 and with only k alterations of quantifiers such that $\mathrm{Sat}((A,R), (\exists !\, v_0)(\varphi) \,\&\, \varphi, \lambda n(x))$

9) for each Π^0_3 predicate $Q(n, f)$ we have $(\exists n)(n \in A \,\&$ $\sim Q(n, \mathrm{Ch}(\mathrm{Th}((A,R))))) \rightarrow (\exists n)(n \in A \,\&\, \sim Q(n, \mathrm{Ch}(\mathrm{Th}((A,R))))$ $\&\, (\forall m)(m < n \rightarrow Q(m, \mathrm{Ch}(\mathrm{Th}((A,R))))))$. Define $\mathcal{S}^- =$ $[\, \mathrm{Ch}(\mathrm{Th}((A,R)) \text{ is a towered}^- \text{ structure}\,]$.

Lemma 3.2.2. $[\, f \in 2^\omega : f$ codes $\mathrm{Th}((A,R))$ for some towered⁻ *structure* $(A,R)\,]$ *is in* B_5 *with recursive code. In other words* $\mathcal{S}^- =$ $[\, f \in 2^\omega : f = \mathrm{Ch}(\mathrm{Th}((A,R)))$ *for some towered⁻ structure* $(A,R)\,]$ *is in* B_5 *with recursive code.*

Proof. We define $f \in \mathcal{S} \equiv P_1(f)$ & $P_2(f)$ & $P_3(f)$ & $P_4(f)$ & $P_5(f)$ & $P_6(f)$ & $P_7(f)$ & $P_8(f)$ & $P_9(f)$, where $P_1(f)$ is '$(\forall x)(\varphi_2(x,x))$ & $(\forall x)(\forall y)(\varphi_2(x,y) \equiv \varphi_2(y,x))$ & $(\forall x)(\forall z)((\varphi_2(x,y)$ & $\varphi_2(y,z)) \to \varphi_2(x,z))$' $\in [i : f(i) = 1]$; $P_2(f)$ is '$(\forall x)(\forall y)((\varphi_1(x,y)$ & $\sim \varphi_1(y,x)) \vee (\varphi_1(y,x)$ & $\sim \varphi_1(x,y)) \vee (\varphi_2(x,y)$ & $\sim \varphi_1(x,y)$ & $\sim \varphi_1(y,x)$ & $(\forall x)(\forall y)(\forall z)(((\varphi_2(x,z)$ & $\varphi_1(x,y)) \to \varphi_1(z,y))$ & $((\varphi_2(x,z)$ & (y,z) & $(y,x) \to \varphi_1(y,z)))$ & $\sim (\exists x)(\forall y)(\varphi_1(y,x) \vee \varphi_2(x,y))$' $\in [i : f(i) = 1]$; $P_3(f)$ is '$(\forall x)(x \in V(\omega) \equiv (\forall y)(\varphi_1(x,y) \vee \varphi_2(x,y)))$ & $(\exists x)(x = V(\omega))$' $\in [i : f(i) = 1]$; $P_4(f)$ is '$(\forall x)(\forall y)(y \in x \to y \in V(\omega))$' $\in [i : f(i) = 1]$; $P_6(f)$ is '$(\forall x)(\forall y)(\forall z)(z \in x \equiv z \in y) \to x = y)$' $\in [i : f(i) = 1]$; $P_7(f)$ is "for each sentence $\exists v_0(\varphi)$ such that $f($'$(\exists ! v_0)(\varphi)$'$) = 1$ we have that for some formula ψ with only the free variable v_0, '$\exists v_0(\varphi$ & $\psi)$ & $(\exists ! v_0)(\psi)$' $\in [i : f(i) = 1]$" & $[F$: 'F' $\in [i : f(i) = 1]$ is a consistent set of sentences in LST"; $P_5(f)$ is "for each formula φ with only the free variable v_1 such that $f($'$(\exists ! v_1)(\psi)$'$) = 1$ we have that '$(\exists v_0)(\exists v_1)(\varphi(v_0)$ & $\psi(v_1)$ & $(\varphi_1(v_1, v_0) \vee \varphi_2(v_1, v_0)))$' $\in [i : f(i) = 1]$ if and only if there exists a formula ψ_1 with free variables $v_2, ..., v_k, v_{k+1}$ such that '$(\exists v_0)(\exists v_1)(\exists v_2) ... (\exists v_k)(\forall v_{k+1})(\varphi(v_0)$ & $\psi(v_1)$ & $\varphi_1(v_2, v_0)$ & ... & $\varphi_1(v_k, v_0)$ & $(v_{k+1} \in v_1 \equiv (\varphi_1(v_{k+1}, v_0)$ & $\psi^*)))$' $\in [i : f(i) = 1]$, where ψ^* is the result of relativizing the quantifiers in ψ to those y with $\varphi_1(y, v_0)$"; $P_8(f)$ is "for some k we have that for all formulae P with only the free variable v_0 such that $f($'$(\exists ! v_0)(P)$'$) = 1$ there is a formula ψ with free variable only v_0 and which is prenex and only has k alterations of quantifiers such that $f($'$(\exists v_0)(P$ & $\psi)$'$) = 1$; $P_9(f)$ is $(\forall k)[(\exists n)(A(n)$ & $\sim Q(k,n,f)) \to (\exists n)(A(n)$ & $\sim Q(k,n,f)$ & $(\forall m)(B(m,n) \to Q(k,m,f)))]$, where Q is a complete Π^0_3 predicate, $A(n)$ is "n is odd or (n is even & $|n/2|$ is P with only free variable v_0 and $f($'$(\exists ! v_0)(P)$'$) = 1$ & $(\forall m < n/2)$ $(\sim(|m|$ has only free variable v_0 and is, say, $Q(v_0)$, and $f($'$(\forall v_0)(Q(v_0) \equiv P(v_0))$ & $(\exists ! v_0)(Q)$'$) = 1)))$", $B(m,n)$ is "$A(m)$ & $A(n)$ & $|m/2|$ is P & $|n/2|$ is Q & '$(\exists v_0)(\exists v_1)(P(v_0)$ & $Q(v_1)$ & $\varphi_1(v_0, v_1))$' $\in [i : f(i) = 1]$".

To show that this is the desired conjunction, we must show that, for the corresponding (A, R) to f as in the proof of Lemma 1.3.1, that (A, R) is a towered$^-$ structure. To do this, one proves by induction on the complexity of a formula F that for all assignments g in (A, R), we have $\mathrm{Sat}((A,R), F, g) \equiv$ '$(\exists v_{i_1})(\exists v_{i_2}) ... (\exists v_{i_j})(G_{i_1}(v_{i_1})$ & ... &

$G_{i_j}(v_{i_j})$ & $F)' \in [\, i : f(i) = 1\,]$, where $G_{i_k}(v_0)$ is $|g(i_k)$ if v_0 is even; $G_{i_k}(v_0)$ is the *canonical* definition of $g(i_k)$ in (A^0, R^0) if $g(i_k)$ is odd; and $v_{i_1}, \ldots, v_{i_j}$ is a complete list of the free variables in F.

Theorem 3.2. $L^{\omega+1}$ *satisfies* "there exists an element $Y \in B_5$, with recursive code, such that $\sim D(Y)$".

Proof. Proceed as in Section 1. The predicate defining the set K of the proof of Lemma 1.5.1 is replaced by a Π^0_3 predicate since one needs to consider $P(n, i, j)$ only for $n = 0, 1$.

We can state an independence result corresponding to Theorem 3.2.

Definition 3.3. We let $Z(2)$ be

1) $(\exists x)(x = V(\omega))$

2) $(\forall y)(y \subset V(\omega))$

3) $(\forall z)(z \in x \equiv z \in y) \to x = y$

4) $x \neq \phi \to (\exists y)(y \in x \;\&\; (\forall z)(z \in x \to z \notin y))$

5) $(\exists y)(\forall z)(z \in y \equiv (\exists w)(z \in w \;\&\; w \in x))$

6) $(\forall x)(\exists y)(\forall z)(z \in y \equiv (F \;\&\; z \in x))$, where F is a formula not containing y free

7) $(\forall x)(\exists y)(P(x, y)) \to (\forall x)(\exists f)([\,n : (\exists k)(f(0, k) = n)\,] = x \;\&$ $(\forall m)(P([\,n : (\exists k)(f(m, k) = n)\,], [\,n : (\exists k)(f(m+1, k) = n)\,])))$, where P is a formula which does not mention f free.

It is well known that $L^{\omega+1}$ satisfies $Z(2)$. The dependent choices principle 7) can be seen to hold using the definable well-ordering of $L^{\omega+1}$. For a discussion of the ramified analytical hierarchy, $L^{\omega+1}$, see Boyd, Hensel, and Putnam [1].

Theorem 3.3. $Z(2)$ *is consistent with* "there exists an element $Y \in B_5$, with recursive code, such that $\sim D(Y)$".

Extensions of these independence results can be obtained for certain stronger theories than **Z**. Rather than give a systematic formulation, we given an example of what can be done.

Definition 3.4. We let $\mathbf{Z}(L)$ be **Z** together with $(\exists x)(\exists \alpha)(\alpha = \Omega^L$ & $x = V(\alpha)$, where Ω^L is the first constructible uncountable ordinal). Naturally, we assume some standard formulation of the constructible hierarchy appropriate to **Z**.

Theorem 3.4. *$\mathbf{Z}(L)$ is consistent with "$(\exists \alpha)(\sim D(\alpha))$".*

Proof. Using the Skolem-Lowenheim theorem, choose β countable such that L^β possesses a well-ordering of type β and no well-ordering of ω of type β and a well-ordering on ω of type any $\alpha < \beta$. That is, β is countable and is Ω in L^β. It is not known whether $(\exists \alpha)(\sim D(\alpha))$ holds in L^β. But instead pass to the generic extension of L^β obtained by adjoining a generic well-ordering y of ω of type β. In this extension we have $\mathbf{Z}(L)$. In addition, we can carry out the independence techniques of this paper using $L^\beta(y)$ instead of L^β, where $L^\beta(y)$ is the same as L^β except that $L^\beta(0) = V(\omega) \cup [y]$. The resulting Borel set will have code recursive in y.

We can turn Theorems 3.1 – 3.4 into proofs of consistency from deteminateness. We make use of the usual way of formalizing the constructible hierarchy within set theories, such as the ones being considered, based on sets of restricted type. This formalization is done by means of the predicate $\mathrm{CHY}^+(x,f)$, which is the same as the $\mathrm{CHY}(x,f)$ of Section 2 except that no type restrictions are placed in the successor case. In addition we shall use $\mathrm{CODE}(f,y)$, $\mathrm{CODE}^+(f,y)$ to mean, respectively, that $(\exists x)(\mathrm{CHY}(x,f)$ & y is coded by $f)$, $(\exists x)(\mathrm{CHY}^+(x,f)$ & y is coded by $f)$. Thus, $L^{\omega+\omega}$ was $[y : (\exists f)(\mathrm{CODE}(f,y))]$, and $L = [y : (\exists f)(\mathrm{CODE}^+(f,y))]$.

Lemma 3.5.1. *The following can be proved respectively, in $\mathbf{Z}(2)$ and in $\mathbf{Z}$ without the power set axiom:* $(\mathrm{CHY}(x,f)$ & $\mathrm{CODE}(f,y)) \to (\exists g)(\mathrm{CHY}^+(x,g)$ & $\mathrm{CODE}^+(g,y))$, $(\mathrm{CHY}(x,f)$ & $\mathrm{CODE}(f,y)$ & $f \in V(\omega+\omega)) \to (\exists g)(\mathrm{CHY}^+(x,g)$ & $\mathrm{CODE}^+(g,y)))$.

Lemma 3.5.2. *Shoenfield's absoluteness theorem,* (see Shoenfield [7]) *is provable in* **Z** *without the power set axiom.*

Theorem 3.5. **Z** *without the power set axiom* + $D(\omega + 3)$ *proves the consistency of* **Z**.

Proof. The assertion that $D(Y)$ holds for all $Y \in B_{\omega+3}$ with recursive code is Σ^1_2 in the analytical hierarchy, and is therefore subject to Shoenfield's theorem. Hence in **Z** without power set + $D(\omega + 3)$ we can prove that every $Y \in B_{\omega+3}$ with recursive code has a constructible winning strategy. Now we can formalize the proof of Theorem 3.1, so that we obtain within **Z** without power set, that $(\exists x)(\exists f)(\exists y)(\mathrm{CHY}^+(x,f)$ & $\mathrm{CODE}^+(f,y)$ & $(\forall g)(\sim \mathrm{CODE}(g,y)))$. Fix such a well-ordering x. Then, arguing in **Z** without power set, we have that all of $L^{\omega+\omega}$ is coded in the f with $\mathrm{CHY}^+(x,f)$. Using this f, we can straightforwardly give a model of **Z** and hence derive the consistency of **Z**.

We may similarly obtain

Theorem 3.6. **Z**(2) + $D(5)$ *proves the consistency of* **Z**(2).

The level of the Borel hierarchy jumps up by one if we want to consider sets of Turing degree.

Theorem 3.7. **Z** *without the power set axiom* + "every Turing set $Y \in B_{\omega+4}$ either contains or is disjoint from a Turing cone" *proves the consistency of* **Z**. **Z**(2) + "every Turing set $Y \in B_6$ contains or is disjoint from a Turing cone" *proves the consistency of* **Z**(2).

In fact Theorems 3.5, 3.6, and 3.7 can be sharpened in the following way: our proofs actually produce specific subsets Y of 2^ω, and so the respective hypotheses may be weakened in the respective theorems by using the respective Y instead of using all Y at the respective level of the Borel hierarchy.

Section 4

Here we wish to mention some possibilities for future research.

What is the formal relation between the questions about the Borel hierarchy studied here and the commonly considered axioms and hypotheses in set theory? At one extreme, as far as we know, even D(5) may not be derivable from Morse-Kelley set theory together with the 2nd-order reflection principle *. At another extreme, it may be that Z together with $(\forall x)$ (if x is a well-ordering on ω then the cumulative hierarchy exists up through x) is sufficient to derive $(\forall \alpha)(D(\alpha))$.

What is the relation between Borel determinateness, (written $(\forall \alpha)(D(\alpha))$), and "every Borel set of Turing degrees contains or is disjoint from a Turing cone?"

It is easily seen that the following can be derived from Borel determinateness: for every Borel $Y \subset 2^\omega \times 2^\omega$ either Y can be uniformized by a Borel function or $[(f,g)\colon (g,f) \notin Y]$ can be uniformized by a Borel function. A Borel function is just a subset, X, of $2^\omega \times 2^\omega$ such that $(\forall f \in 2^\omega)(\exists ! g \in 2^\omega)((f,g) \in X)$. A Borel function X uniformizes Y just in case $(\forall f \in 2^\omega)(\exists ! g)((f,g) \in X \;\&\; (f,g) \in Y)$. In fact, a Y can be found which is continuous. So we have

I. to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a Borel function F which either uniformizes Y or uniformizes $[(f,g)\colon (g,f) \notin Y]$

II. there is an ordinal $\alpha < \Omega$ such that to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a Borel function $F \in B_\alpha$ which either uniformizes Y or uniformizes $[(f,g)\colon (g,f) \notin Y]$

III. to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a continuous function F which either uniformizes Y or uniformizes $[(f,g)\colon (g,f) \notin Y]$

IV. Borel determinateness.

What is the relation between I – IV? Of course we have IV → III →

* D.A.Martin has recently derived D(4) from MK + 2nd-order reflector principle (unpublished).

$II \to I$. It seems reasonable to hope for a mathematician's proof of I, but beware of II! Our results can be seen to carry over to obtain the independence of II from $\mathbf{Z}(L)$ using α-degrees, $\alpha < \Omega$.

References

[1] R.Boyd, G.Hensel and H.Putnam, A recursion-theoretic characterization of the ramified analytical hierarchy, Transactions of the A.M.S. 14 (July 1969).
[2] M.Davis, Infinite Games of Perfect Information. Advances in Game Theory, M.Dresher (ed.) (Princeton, 1964).
[3] D.Gale and F.M.Stewart, Infinite Games with Perfect Information, Contributions to the Theory of Games, vol. 2 (Princeton, 1950).
[4] D.Martin, Measurable Cardinals and Analytic Games, Fundamenta Mathematicae, to appear.
[5] A.Mathias, A Survey of Set Theory, Proceedings of the 1967 U.C.L.A. Conference on Set Theory, to appear.
[6] J.Mycielski, On the Axiom of Determinateness I and II. Fundamenta Mathematicae, volumes 53 and 59.
[7] J.Shoenfield, The Problem of Predicativity, Essays on the Foundations of Mathematics (Magnes Press, Jerusalem, 1961 and North-Holland, Amsterdam, 1962).