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Pre-invex Functions in Multiple Objective Optimization

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1. INTRODUCTION

Elster and Nehse [4] considered a class of functions $f\colon S \to \mathbb{R}^m$, $S \subset \mathbb{R}^n$, for which, if x, $y \in S$ and $0 \leqslant \lambda \leqslant 1$, there is $z \in S$ such that

$$f(z) \leqq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

and called such functions convexlike. If S is a convex set and if f is a convex function, then clearly f is convexlike. Elster and Nehse obtained a saddlepoint optimality condition for convexlike mathematical programs. Hayashi and Komiya [9] also considered convexlike functions and developed a Gordan-type theorem of the alternative involving convexlike functions and in addition considered Lagrangian duality for convexlike programs.

Hanson [7] considered differentiable functions $f\colon S \to \mathbb{R}$ for which there exists an n-dimensional vector function $\eta(x, u)$ such that for all x, $u \in S$

$$f(x) - f(u) \geqslant [\eta(x, u)]' \nabla f(u). \tag{1.2}$$

Such functions were termed invex by Craven [2]. Clearly differentiable convex functions are invex. Hanson [7] showed that if, instead of the usual convexity conditions, the objective function and each of the constraints of a nonlinear programming problem all satisfy (1.2) for the same $\eta(x, u)$, weak

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duality and the sufficiency of the Kuhn–Tucker conditions still hold. Furthermore Craven and Glover [3] (see also Ben Israel and Mond [1] and Martin [11]) showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minima.

More generally, Ben Israel and Mond [1] and Hanson and Mond [8] considered (not necessarily differentable) functions defined on S having the property that there exists an n-dimensional vector function $\eta(x, u)$ such that, for all x, $u \in S$ and $0 \leqslant \lambda \leqslant 1$,

$$f(u + \lambda\eta(x, u)) \leqslant \lambda f(x) + (1-\lambda) f(u) \tag{1.3}$$

and observed that differentiable functions satisfying (1.3) also satisfy (1.2). In view of this observation, functions satisfying (1.3) will be called pre-invex.[1] An m-dimensional vector valued function $f\colon S \to \mathbb{R}^m$ is pre-invex on S (with respect to η) if each of its components is pre-invex on S (with respect to η).

It is implicit in the definition of (1.3) that, for all x, $u \in S$ and $0 \leqslant \lambda \leqslant 1$, $u + \lambda\eta(x, u) \in S$; hence pre-invex functions are convexlike. However, pre-invex functions have some interesting properties that are not generally shared by the wider class of convexlike functions. For example, as for convex functions, every local minimum of a pre-invex function is a global minimum and nonnegative linear combinations of pre-invex functions are pre-invex [14].

As a simple example of a function which is not convex but is pre-invex, consider

$$f(x) = -|x|.$$

Then f is pre-invex with η given by

$$\eta(x, y) = \begin{cases} x - y, & \text{if } y \leqslant 0 \quad \text{and} \quad x \leqslant 0 \\ x - y, & \text{if } y \geqslant 0 \quad \text{and} \quad x \geqslant 0 \\ y - x, & \text{if } y > 0 \quad \text{and} \quad x < 0 \\ y - x, & \text{if } y < 0 \quad \text{and} \quad x > 0. \end{cases}$$

Our purpose in this paper is to detail how and where pre-invex functions can replace convex functions in multiple objective optimization.

2. Preliminaries

The following convention for equalities and inequalities will be used. If x, $y \in \mathbb{R}^n$, then

[1] The authors are grateful to V. Jeyakumar for his coinage of the term "pre-invex."

$$x = y \quad \text{iff} \quad x_i = y_i,\ i = 1, 2, ..., n$$

$$x \leqq y \quad \text{iff} \quad x_i \leqslant y_i,\ i = 1, 2, ..., n$$

$$x \leqslant y \quad \text{iff} \quad x \leqq y, \quad \text{and} \quad x \neq y$$

$$x < y \quad \text{iff} \quad x_i < y_i,\ i = 1, 2, ..., n$$

$x \nleqslant y$ is the negation of $x \leqslant y$.

A scalar valued optimization problem may be expressed as

(P) minimize $f(x)$ subject to $g(x) \leqq 0$.

Here $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}^m$, where $S \subset \mathbb{R}^n$.

As previously mentioned, Hayashi and Komiya [9] extended Gordan's alternative theorem (see [10]) from convex to convexlike functions (their results are for a more general problem than (P), but may be easily specialized), and they also prove saddlepoint and Lagrangian duality theorems. For completeness we state and prove an extension to Gordan's alternative theorem involving pre-invex functions and then state the corresponding saddlepoint and duality theorems which follow in the same manner as in the case for convex functions (see, e.g., Geoffrion [5] and Mangasarian [10]). The saddlepoint and duality theorems will be shown to be special cases of Theorems 4.1, 4.2, and 4.3.

THEOREM 2.1. *Let S be a nonempty set in $\mathbb{R}^n$ and let $f: S \to \mathbb{R}^m$ be a pre-invex function on S (with respect to η). Then either*

$$f(x) < 0 \text{ has a solution } x \in S$$

or

$$p^t f(x) \geqq 0 \qquad \text{for all} \quad x \in S, \text{for some } p \in \mathbb{R}^m,\ p \geqslant 0,$$

but both alternatives are never true.

Proof. Following [10], the proof depends on establishing the convexity of the set $\Lambda = \bigcup \{\Lambda(x) : x \in S\}$, where

$$\Lambda(x) = \{u \in \mathbb{R}^m : u > f(x)\}, \qquad x \in S.$$

Under our assumptions this is immediate for if u_1 and u_2 are in Λ, then for $0 \leqslant \lambda \leqslant 1$,

$$\lambda u_1 + (1-\lambda) u_2 > \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geqq f(x_2 + \lambda \eta(x_1, x_2)). \quad \blacksquare$$

The program (P) will be said to satisfy the generalized Slater constraint qualification if g is pre-invex (with respect to η) and there exists $x_1 \in S$ such that $g(x_1) < 0$.

THEOREM 2.2. *For the problem* (P), *assume that f is pre-invex* (*with respect to η*) *and g is pre-invex* (*with respect to η*), *and that the generalized Slater constraint qualification holds. If* (P) *attains a minimum at $x = x_0 \in S$ then there exists $v_0 \in \mathbb{R}^m$, $v_0 \geqq 0$, such that (x_0, v_0) is a solution of the saddlepoint problem* (*all $x \in S$, $v \in \mathbb{R}^m$, $v \geqq 0$*),

$$\varphi(x_0, v) \leqq \varphi(x_0, v_0) \leqq \varphi(x, v_0), \tag{2.1}$$

where $\varphi(x, v)$ is the Lagrangian $f(x) + v'g(x)$. Moreover, if the condition (2.1) *is satisfied for some x_0 and v_0, then x_0 is a minimum for* (P).

In relation to (P), consider the problem

$$\text{(D)} \quad \text{maximize } \varphi(v) \text{ subject to } v \in \mathbb{R}^m,\ v \geqq 0,$$

where $\varphi(v) = \inf_{x \in S} \{f(x) + v'g(x)\}$.

THEOREM 2.3. *In problem* (P), *assume that f is pre-invex* (*with respect to η*) *and g is pre-invex* (*with respect to η*), *and that the generalized Slater constraint qualification holds. Then* (D) *is a dual to* (P).

Many results for the convex scalar optimization problem have been extended to the multiple objective optimization problem, which may be expressed as

$$\text{(PV)} \quad \text{minimize } f(x) \text{ subject to } x \in X,$$

where $f\colon S \to \mathbb{R}^k$, $S \subset \mathbb{R}^n$, $X \subset \mathbb{R}^n$.

This is the problem of finding the set of efficient or Pareto [12] optimal points for (PV): x_0 is said to be efficient if it is feasible for (PV) and there exists no other feasible point x such that $f(x) \leqslant f(x_0)$.

The concept of proper efficiency given by Geoffrion [6] is a slightly restricted definition of efficiency which eliminates efficient points of a certain anomalous type: x_0 is said to be properly efficient if it is efficient for (PV) and if there exists a scalar $M > 0$ such that, for each i,

$$\frac{f_i(x_0) - f_i(x)}{f_j(x) - f_j(x_0)} \leqslant M$$

for some j such that $f_j(x) > f_j(x_0)$ whenever x is feasible for (PV) and $f_i(x) < f_i(x_0)$; thus unbounded trade-offs between the various $\{f_i\}$ are not allowed. An efficient point that is not properly efficient is said to be improperly efficient.

Under convex assumptions, Geoffrion [6] showed that the properly efficient points of (PV) may be characterized in terms of the solutions to a scalar valued parametric programming problem. In the following sections we will show that Geoffrion's assumptions of convexity may be replaced by pre-invexity. We shall also consider the relevant questions of vector saddlepoints and duality.

3. Proper Efficiency and Pre-invexity

In relation to (PV) consider the following scalar minimization problem:

$$(\mathrm{PV}\lambda) \quad \text{minimize } \lambda' f(x) \text{ subject to } x \in X,$$

where $\lambda \in \Lambda^+ = \{\lambda \in \mathbb{R}^k : \lambda > 0, \sum_{i=1}^k \lambda_i = 1\}$. For convenience we assume that $X \subset S$.

Geoffrion [6] established the following fundamental result:

Theorem 3.1. *Let $\lambda_i > 0$ $(i = 1, 2, ..., k)$ be fixed. If x_0 is optimal in* (PVλ), *then x_0 is properly efficient in* (PV).

Assuming f convex and X convex, Geoffrion also established the converse of Theorem 3.1. This result is based on Gordan's alternative theorem (see [10]). Hence replacing Gordan's alternative theorem with our Theorem 2.1 gives:

Theoem 3.2. *Let f be pre-invex on X (with respect to η). Then x_0 is properly efficient in* (PV) *if and only if x_0 is optimal in* (PVλ).

Additionally, in his "Comprehensive Theorem," Geoffrion established a more complete characterization of proper efficient points for the problem (PV). It is not difficult to see that his assumptions of convexity may be replaced by pre-invexity. Here we restate the problems and Geoffrion's "Comprehensive Theorem," replacing any assumption of convexity with pre-invexity.

Problem 1. Find a point x_0 that is a properly efficient solution of (PV).

Problem 2. Find a point x_0 that is a locally properly efficient solution of (PV).

In Problems 3–5, X is taken to be of the form $X = \{x : g(x) \leqq 0\}$.

Problem 3. Find a feasible point x_0 such that none of the k systems $(i = 1, 2, ..., k)$

$$\begin{aligned} &u' \nabla f_i(x_0) < 0 \\ &u' \nabla f_j(x_0) \leqslant 0 \qquad \text{all } j \neq i \\ &u' \nabla g_j(x_0) \leqslant 0 \qquad \text{all } j \text{ such that } g_j(x_0) = 0 \end{aligned}$$

has a solution $u \in \mathbb{R}^n$.

PROBLEM 4. Find a feasible point x_0, a point $y_0 \in \mathbb{R}^m$, $y_0 \geqq 0$, and a point $\lambda_0 \in \Lambda^+$ such that $y_0^t g(x_0) = 0$ and $\nabla [\lambda_0^t f(x_0) + y_0^t g(x_0)] = 0$.

PROBLEM 5. Find a feasible point x_0, a point $y_0 \in \mathbb{R}^m$, $y_0 \geqq 0$, and a point $\lambda_0 \in \Lambda^+$ such that $y_0^t g(x_0) = 0$ and x_0 achieves the unconstrained minimum of $\lambda_0^t f(x) + y_0^t g(x)$.

PROBLEM 6. Find a point x_0 and a point $\lambda_0 \in \Lambda^+$ such that x_0 is optimal in $(\mathrm{PV}\lambda_0)$.

The following assumptions will be made:

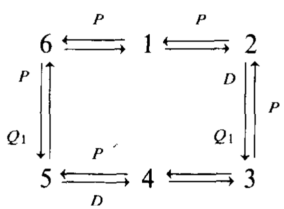
Assumption P. All functions are pre-invex (with respect to η) on X.

Assumption D. All functions are continuously differentiable on X.

Assumption Q_1. The generalized Slater constraint qualification holds.

Assumption Q_2. The Kuhn–Tucker constraint qualification holds [9].

THEOREM 3.3.



The proof of Theorem 3.3 follows in exactly the same manner as that of the "Comprehensive Theorem" in [6], replacing convexity with pre-invexity.

4. VECTOR SADDLEPOINTS AND DUALITY

Here we consider the vector valued optimization problem

$$(\mathrm{PV})' \quad \text{minimize } f(x) \text{ subject to } g(x) \leqq 0,$$

where $f: S \to \mathbb{R}^k$, $g: S \to \mathbb{R}^m$, and the related vector saddlepoint and dual problems.

By studying a natural generalization of the scalar Lagrangian

$$f(x) + y^t g(x)\, e \qquad (e = (1, 1, ..., 1)^t \in \mathbb{R}^k)$$

Tanino and Sawaragi [13] have developed a saddlepoint and duality theory for convex (PV)′, generalizing that of the scalar case. Here we take a slightly different approach (see also [15]) but the results shown have their analogs in [13].

The vector saddlepoint problem is the problem of finding $x_0 \in S$, $y_0 \in \mathbb{R}^m$, $y_0 \geqq 0$, such that

$$f(x_0) + y'g(x_0)\, e \not\geq f(x_0) + y_0' g(x_0)\, e \tag{4.1}$$

$$f(x_0) + y_0' g(x_0)\, e \not\geq f(x) + y_0' g(x)\, e \tag{4.2}$$

for all $x \in S$, $y \in \mathbb{R}^m$, $y \geqq 0$.

The conditions (4.1) and (4.2) are always sufficient for x_0 to be properly efficient for (PV)′ (see [15]). On the other hand, if x_0 is a properly efficient solution of (PV)′, one requires a constraint qualification and convexity to assure the existence of y_0 such that (x_0, y_0) is a solution of (4.1) and (4.2). We now show that this convexity requirement can be weakened to pre-invexity.

THEOREM 4.1. *Let x_0 be a properly efficient solution for* (PV)′; *if the generalized Slater constraint qualification is satisfied, and if f and g are pre-invex* (*with respect to η*), *then there exists $y_0 \geqq 0$ such that (x_0, y_0) is a solution of the saddlepoint problem.*

Proof. Since x_0 is a properly efficient solution for (PV)′ then by Theorem 3.2, x_0 minimizes $\lambda_0' f(x)$ subject to $g(x) \leqq 0$ for some $\lambda_0 \in \Lambda^+$. Since $\lambda_0' f$ is pre-invex (with respect to η) and since the generalized Slater condition is satisfied, by Theorem 3.3, there exists $y_0 \in \mathbb{R}^m$, $y_0 \geqq 0$, such that $y_0' g(x_0) = 0$ and

$$\varphi(x_0, y) \leqslant \varphi(x_0, y_0) \leqslant \varphi(x, v_0), \tag{4.3}$$

where $\varphi(x, y) = \lambda_0'[f(x) + y'g(x)\, e]$. Consequently, if (4.1) was not true then for some $i \in \{1, 2, ..., k\}$

$$f_i(x_0) + y'g(x_0) > f_i(x_0) + y_0' g(x_0)$$

and

$$f_j(x_0) + y'g(x_0) \geqslant f_j(x_0) + y_0' g(x_0) \qquad \text{for all} \quad j \neq i;$$

multiplication by λ_{0i}, $i = 1, 2, ..., k$, and summing over all values of i would then contradict (4.3). A similar argument applied to (4.2) would also result in a contradiction. ∎

We now turn our attention to duality and consider the problem of (see [15])

(D) maximize $\Xi = \{v \in \mathbb{R}^k\colon (\exists \lambda \in \Lambda^+,\ y \geqq 0),\ \lambda' v = \inf_x \{\lambda' f(x) + y'g(x)\}\}$.

(D) is the problem of finding all the extreme points of Ξ (see, e.g., [13]).

THEOREM 4.2. (Weak Duality). *Let x be feasible for* (PV)′ *and let* $v \in \Xi$. *Then* $f(x) - v \nleq 0$.

Proof. For some $\lambda \in \Lambda^+$, $y \geqq 0$, $\lambda' v = \inf_x \{\lambda' f(x) + y' g(x)\}$. Since $g(x) \leqq 0$, $\lambda' f(x) \geqslant \lambda' f(x) + y' g(x) \geqslant \inf_x \{\lambda' g(x) + y' g(x)\} = \lambda' v$. So $\lambda'(f(x) - v) \geqslant 0$ and thus $f(x) - v \nleq 0$. ∎

THEOREM 4.3 (Strong Duality). *Let x_0 be a properly efficient solution of* (PV)′ *and let the generalized Slater constraint qualification be satisfied. Then there is an extreme point $\zeta_0 \in \Xi$ such that $f(x_0) = \zeta_0$.*

Proof. Since x_0 is a properly efficient solution for (PV)′ then x_0 solves the pre-invex problem

$$\text{minimize } \lambda' f(x) \text{ subject to } g(x) \leqq 0$$

for some $\lambda \in \Lambda^+$, by Theorem 3.1. Since the generalized Slater condition is satisfied, from Theorem 3.3, there exists $y_0 \geqq 0$ such that $y_0' g(x_0) = 0$ and for all x

$$\lambda' f(x_0) + y_0' g(x_0) \leqslant \lambda' f(x) + y_0' g(x).$$

Thus $\lambda' f(x_0) \leqslant \inf_x \{\lambda' f(x) + y_0' g(x)\} = \lambda' v$ for some $v \in \mathbb{R}^k$. From weak duality $\lambda' f(x_0) = \lambda' v$. If there was no extreme point $\zeta_0 \in \Xi$ such that $f(x_0) = \zeta_0$ then there would be $\hat{\zeta} \in \Xi$ such that $\hat{\zeta} - f(x_0) \geqslant 0$, $\hat{\zeta} - f(x_0) \neq 0$. Hence for all $\lambda \in \Lambda^+$, $\lambda' \hat{\zeta} > \lambda' f(x_0)$. Since $\hat{\zeta} \in \Xi$ there exist $\hat{\lambda} \in \Lambda^+$ and $\hat{y} \in \mathbb{R}^m$, $\hat{y} \geqq 0$, such that

$$\inf_x \{\hat{\lambda}' f(x) + \hat{y}' g(x)\} = \hat{\lambda}' \hat{\zeta} > \hat{\lambda}' f(x_0) \geqslant \hat{\lambda}' f(x_0) + \hat{y}' g(x_0),$$

which is a contradiction. ∎

As indicated earlier, Theorems 2.2 and 2.3 are special cases of Theorems 4.1, 4.2, and 4.3.

Finally, we direct our attention to problem (PV)′ with differentiable functions.

In relation to (PV)′ consider the problem

$$(\mathrm{D})' \ \text{maximize}\ f(u) + y' g(u)\, e$$

$$\text{subject to } \nabla \lambda' f(u) + \nabla y' g(u) = 0 \tag{4.4}$$

$$y \geqq 0,\ \lambda \in \Lambda^+. \tag{4.5}$$

(D)′ is the problem of finding the properly efficient points of $f + y' g e$ subject to the given constraints. (D)′ was first given in [16] and shown to be dual for (PV)′ under convex hypotheses. We now show that these hypotheses may be weakened to pre-invexity.

THEOREM 4.4 (Weak Duality). *Let x be feasible for* (PV)$'$ *and* (u, λ, y) *feasible* for (D)$'$. *If f and g are pre-invex* (*with respect to* η) *for all feasible* (x, u, λ, y) *then*

$$f(x) \nleq f(u) + y'g(u)\, e.$$

Proof.

$$\begin{aligned}
\lambda'\{f(x) - (f(u) + y'g(u)\, e)\} &= \lambda'(f(x) - f(u)) - y'g(u) && \text{(from (4.5))}\\
&\geqslant \eta(x, u)' \, \nabla \, \lambda'f(u) - y'g(u) && \text{(by pre-invexity of } f)\\
&\geqslant \eta(x, u)' \, \{\nabla \lambda'f(u) + \nabla y'g(u)\} - y'g(x) && \text{(by pre-invexity of } g)\\
&= -y'g(x) && \text{(by (4.4)}\\
&\geqslant 0 && \text{since } g(x) \leqq 0,\ y \geqq 0.
\end{aligned}$$

Thus $f(x) \nleq f(u) + y'g(u)\, e$. ▮

THEOREM 4.5 (Strong Duality). *Let f and g be pre-invex* (*with respect to* η) *for all feasible* (x, u, λ, y) *and let x_0 be a properly efficient solution for* (PV)$'$ *at which a constraint qualification is satisfied. Then there exist* (λ, y) *such that* (x_0, λ, y) *is a properly efficient solution of* (PV)$'$ *and the objective values of* (PV)$'$ *and* (D)$'$ *are equal.*

Proof. Since f and g are pre-invex and x_0 is a properly efficient solution of (PV)$'$ then, by Theorem 2.2, x_0 is optimal in (PVλ)$'$ for some $\lambda \in \Lambda^+$. Since also a constraint qualification is satisfied at x_0, then by Wolfe's duality theorem [17], there is $y \geqq 0$ such that (x_0, λ, y) is optimal in the problem

$$\begin{aligned}
(\mathrm{D}\lambda)' \quad &\text{maximize } \lambda'f(u) + y'g(u)\\
&\text{subject to } \nabla \lambda'f(u) + \nabla y'g(u) = 0\\
&y \geqq 0,\ \lambda \in \Lambda^+,
\end{aligned}$$

and $y'g(x_0) = 0$.

Since (x_0, λ, y) is optimal for (Dλ)$'$, (x_0, λ, y) is properly efficient for (D)$'$, from Theorem 2.1. The optimal values of (PV)$'$ and (D)$'$ are equal since $y'g(x_0) = 0$. ▮

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