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ORIGINAL ARTICLE

Approximate analytic solution of (2 + 1) dimensional coupled differential Burger's equation using Elzaki Homotopy Perturbation Method

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Convergence

Abstract Burger's equation plays a key role in explaining the behavior of nonlinear systems. In this paper, Elzaki Homotopy Perturbation Method (EHPM) was applied to (2 + 1) dimensional coupled differential Burger's equation so as to obtain series and exact solution of two dimensional Burger' equations. The approximate analytic solution of the problem was obtained with small amount of computations. Two numerical examples have been solved to illustrate the accuracy and implementation of the method. The comparison of numerical results shows good agreement obtained by other analytical methods, exact and proposed method. Thus, EHPM can further be applied to solve nonlinear problems arising out of different physical phenomena.

1. Introduction

In the last few decades, a large class of nonlinear equations has been derived and widely applied in various fields of nonlinear natural sciences such as Chemistry, Biology, Mathematics and different branches of Physics such as plasma physics, fluid dynamics, condensed matter physics, nonlinear optics and field theory. The exact solution of nonlinear equations plays an important role in determining the properties and behavior of physical systems, but it is hard to find their exact solutions as compared with linear equations. Burger's equation was formulated by Johannes Martinus Burger's [56,57] and it is also called advection diffusion equation which was derived by qualitative approximation of Navier Stoke's equation. It is one of the most fundamental tool for describing the nonlinear diffusion and dissipation phenomena such as shock wave theory, approximation theory of flow, unsaturated oil, dynamics of soil in water, cosmology and seismology, and nonlinear kinematics wave of debris flow. Burger's equation displays a simple model that explains the communication between reaction apparatus, heat conduction, modeling of dynamics, diffusion transport and acoustic waves. The numerical and analytical techniques can be understood with the help of Burger's equation model. Several numerical methods are used to find the solution of nonlinear equations or system of nonlinear equations, such as Finite element method [4], Finite Difference method [5], Spectral method [6] and Gauss Seidel method [7].

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Besides numerical methods different analytical methods were applied by many researchers to obtain the solution of approximate, numerical and exact Burger's equation which play an important role in explaining the behavior of nonlinear systems, such that Suleman et al. used transformed homotopy method to solve Sine Gordon and Klein Gordon equation [8], Taghizadeh obtained, exact solution of Burger's equation by Homotopy Perturbation method and Reduced differential Transform method [12], Majid used Laplace decomposition method to obtain approximate solution of Burger's equation [13,30], Abdul Wahhab used optimal system and exact solution of two dimensional Burger's equation with Infinite Reynolds number [14], Rashid found numerical solution by Chebyshev-Legendre Pseudo-Spectral Method [15], Ganaie used cubic Hermite Collocation to obtain numerical solution [16], Lai used new Lattice Boltzmann model [17], Goyal and Mehra [18] used HPM and RDTM [12], Laplace decomposition method [13], Optimal system and exact solution of two dimensional Burger's equation using infinite Reynolds number [14], Chebyshev Legendre-Pseudo Spectral method [15], Hermite collocation method [16], Lattice Boltzmann Method for solving Burger's equation [17], fast adaptive method applied by Goyal and Mehra [18], Liao used implicit fourth order compact finite difference scheme for Burger's equation [19], Alharbi Used fully implicit finite difference scheme for two dimensional Burger's equation [20], Sheikholeslami et al., Homotopy perturbation [21,27–29,50,51,70–74], Babolin, He's Homotopy Perturbation Method an effective tool for solving a nonlinear system of two dimensional Volterra Fredholm integral equations [22], Dehgan, solution of coupled Burger's equation using Adomian-Pade' technique [23], Inc used Homotopy Analysis method [24], Biazar obtained numerical Solution of Burger's equation using variational iteration method [25], Gondal and Khan [26], Babolin and Dastani [31], Chowdry and Hashim [32], Xu [33], He [34,41,44,46,48, 53,55,56,63,67], Ariel [35,36,54,57], Creticanin [37], Shahed [38], Rafei and Ganji [39], Siddiqui et al. [40], He [41], Abbas Bandy [42], Ganji [52,64,65], Babazadeh and Domairry [58] and so on. The general two dimensional form of Burger's equation is given as follows

$$\frac{\partial\phi}{\partial t}+\phi\frac{\partial\phi}{\partial x}+\psi\frac{\partial\phi}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial\psi}{\partial t}+\phi\frac{\partial\psi}{\partial x}+\psi\frac{\partial\psi}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right), \tag{2}$$

$$\phi(x,y,0)=\eta_1(x,y),\quad \psi(x,y,0)=\eta_2(x,y),\quad x,y\in\Omega, \tag{3}$$

$$\phi(x,y,t)=\eta_3(x,y,t),\ \psi(x,y,t)=\eta_4(x,y,t)\ \ x,y\in\partial\Omega,t>0, \tag{4}$$

where $\phi(x,y,t)$ and $\psi(x,y,t)$ are velocity components in the x and y directions. Re is the Reynolds numbers and η_i (i = 1, 2, ..., 4) are given function values at specific points.

$\Omega=\{(x,y)/a\leqslant x\leqslant b, a\leqslant y\leqslant b\}$ is a domain on $\partial\Omega$ which is a boundary.

Over the last few years, Homotopy Perturbation Method (HPM) and its modified forms are used to find The analytical solution of many nonlinear problems. In this paper, we used the Elzaki transformation along with homotopy perturbation method using He's polynomial to solve the two dimensional Burger's equation. In Section 2 Elzaki transform and basic idea of HPM are explained. Numerical examples are solved in Section 3 and conclusion is explained in Section 4.

2. Elzaki transformation

Elzaki transform was introduced by Elzaki in [1,2] from the classical Fourier integral and it is used to simplify the process of solving both the ordinary and partial differential equations in the time domain. Like Sumudu Transform, Laplace and Fourier Transform, Elzaki Transform is a mathematical tool for solving differential equations. Mathematical formulation of Elzaki transformation is as follows:

$$A=\{f(t):\exists M,k_1,k_2>0,|f(t)|<Me^{\frac{|t|}{k}}, if\ t\in(-1)^j*[0,\infty)\} \tag{5}$$

For a given function in set A, the constant M must be Finite Number and k_1, k_2 may be finite or infinite. Elzaki Transform is denoted by $E(\cdot)$ and defined by the integral equation.

$$E[f(t)]=T(v)=v\int_0^\infty f(t)\ e^{\frac{-t}{v}}dt,\quad t>0,\ k_1\leqslant v\leqslant k_2. \tag{6}$$

where variable 'v' is used in the transformation to factor the variable 't' in the argument of the function 'f' (see Table 1).

2.1. Basic idea of Homotopy Perturbation Method

Suppose that a partial differential equation is given by Eq. (7)

$$D(\phi)-g(r)=0,\quad r\in\Omega, \tag{7}$$

with the boundary conditions

$$\Gamma\left(\phi,\frac{\partial\phi}{\partial\kappa}\right)=0, \tag{8}$$

where D is a differential operator, $g(r)$ is force function, Γ is boundary operator and Ω is domain. Now D can be further separated into linear and nonlinear parts £(ϕ) and $N(\phi)$ respectively; hence, Eq. (7) becomes

$$£(\phi)+N(\phi)-g(r)=0 \tag{9}$$

By using HPM technique, we make a homotopy $\phi(r,p):\Omega\times[0,1]\to R$ which satisfies homotopy of the form:

$$H(\phi,p)=(1-p)[£(\phi)-£(\phi_o)]+p[D(\phi)-g(r)] \\ =0\quad r\in\Omega \tag{10}$$

Table 1 Elzaki transformation of some functions.

$F(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
t^n	$n!\ t^{n+2}$
e^{at}	$v^2/1-av$
Sinat	$av^3/1+a^2v^2$
cosat	$v^2/1+a^2v^2$
Sinhat	$av^3/1-a^2v^2$
coshat	$v^2/1-a^2v^2$

where $p \in [0,1]$ is known as embedding parameter and ϕ_0 is an initial guess of Eq. (7) from Eq. (10), for $p = 0$ we have: (see Figs. 1–6)

$$H(\phi, 0) = \pounds(\phi) - \pounds(\phi_o) = 0 \tag{11}$$

and for $p = 1$;

$$H(\phi, 1) = D(\phi) - g(r) = 0 \tag{12}$$

The variation of p from 0 to 1 is same as change of $\phi(r, p)$ from ϕ_0 to $\phi(r)$. In topology, this is called deformation, while $\pounds(\phi) - \pounds(\phi_0)$ and $D(\phi) - g(r)$ are called homotopy. By using HPM first, we assume that the solution of Eq. (10) can be written as follows:

$$\phi = \phi_o + p\phi_1 + p^2\phi_2 + p^3\phi_3 \ldots . \tag{13}$$

Now getting $p \to 1$, Eq. (13) yields

$$\phi = \lim_{p\to 1}\phi = \phi_0 + \phi_1 + \phi_2 \ldots \tag{14}$$

In most cases, the series in Eq. (14) is convergent. The unknown $\phi_0, \phi_1, \phi_2, \ldots$ can be found by comparing the coefficients of like powers of p in Eq. (10).

3. Solution of (2 + 1) dimensional Burger's equation using Elzaki Homotopy Perturbation Method

3.1. Example

Consider the two dimensional Burger's equation

$$\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right), \tag{15}$$

$$\frac{\partial\psi}{\partial t} + \phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), \tag{16}$$

Along with initial conditions,

$$\phi(x, y, 0) = x + y, \quad \psi(x, y, 0) = x - y. \tag{17}$$

Apply Elzaki transform to Eqs. (15) and (16).

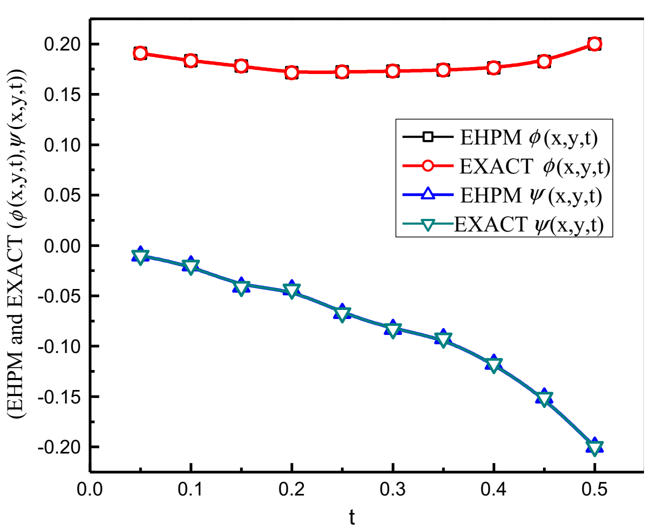

Figure 1 Comparison of EHPM (8th order) and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.2)$.

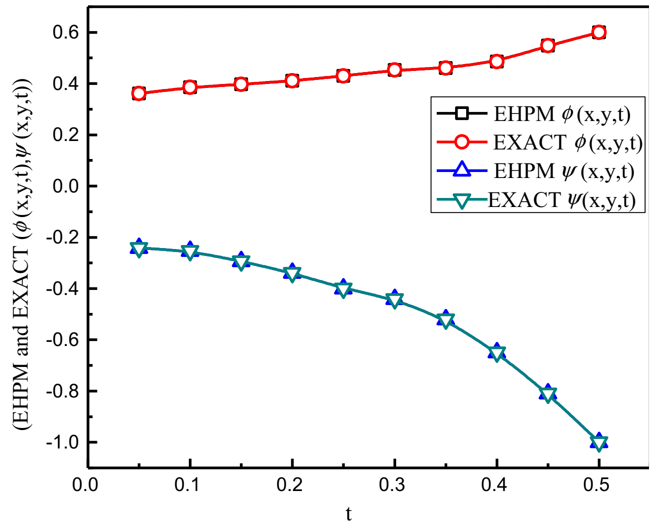

Figure 2 Comparison of EHPM (8th order) and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.1)$.

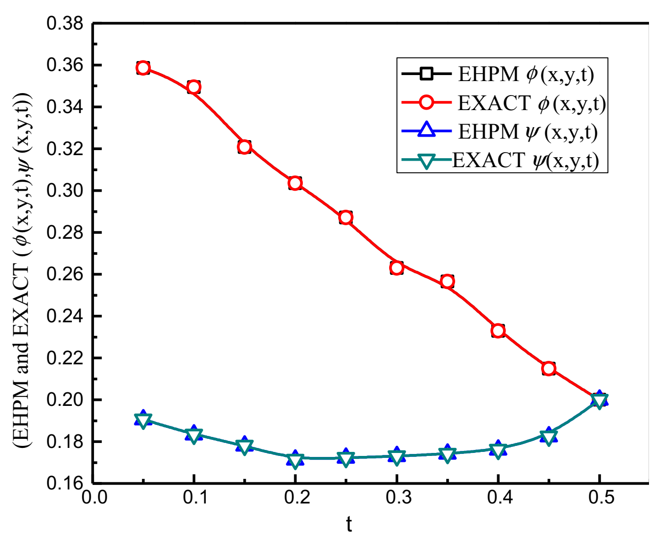

Figure 3 Comparison of EHPM (8th order) and exact solution for $Re = 1$ with $(x, y) = (0.3, 0.2)$.

$$E\left[\frac{\partial\phi}{\partial t}\right] + E\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] = E\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$E\left[\frac{\partial\psi}{\partial t}\right] + E\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] = E\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

$$\frac{1}{v}E[\phi] - v\phi(0) = -E\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] + E\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$\frac{1}{v}E[\psi] - v\psi(0) = -E\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] + E\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

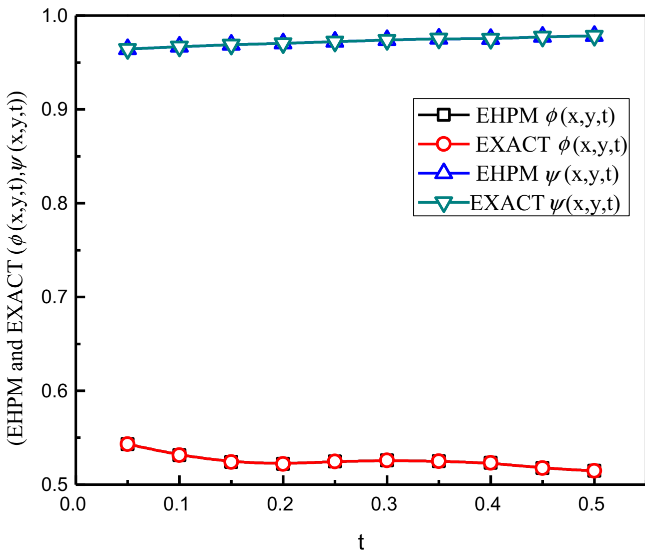

Figure 4 Comparison of EHPM (8th order) and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.4)$.

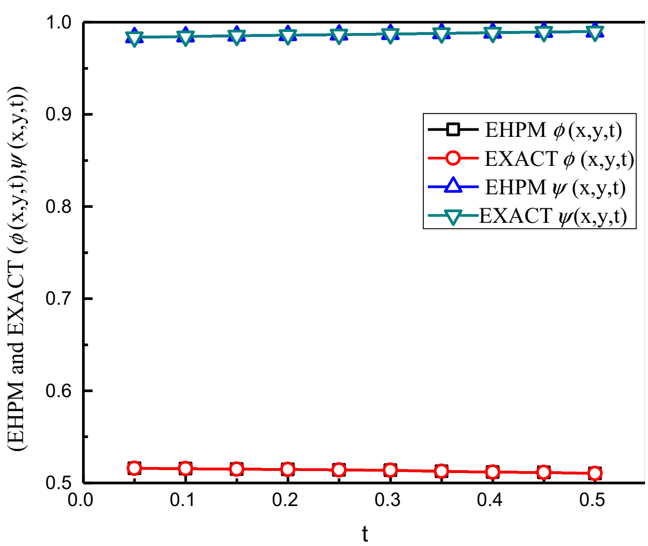

Figure 5 Comparison of EHPM (8th order) and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.6)$.

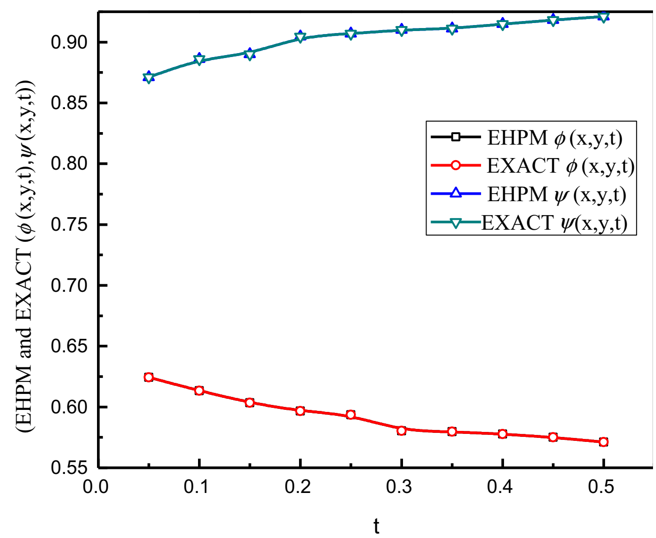

Figure 6 Comparison of EHPM (8th order) and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.8)$.

$$E[\phi] - v^2(x+y) = -vE\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$E[\psi] - v^2(x-y) = -vE\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

$$E[\phi] = v^2(x+y) - vE\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$E[\psi] = v^2(x-y) - vE\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

Now by applying Elzaki inverse transform to the above equations, we have

Table 2 Comparison of EHPM (8th order), Laplace HPM and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.2)$.

t	EHPM $\phi(x,y,t)$	LHPM $\phi(x,y,t)$	Exact $\phi(x,y,t)$	EHPM $\psi(x,y,t)$	LHPM $\psi(x,y,t)$	Exact $\psi(x,y,t)$
0.05	0.190559	0.190559	0.190559	−0.0100305	−0.0100305	−0.0100305
0.10	0.183376	0.183376	0.183376	−0.0200482	−0.0200482	−0.0200482
0.15	0.178001	0.178001	0.178001	−0.0411364	−0.0411364	−0.0411364
0.20	0.171319	0.171319	0.171319	−0.0437843	−0.0437843	−0.0437843
0.25	0.172251	0.172249	0.172249	−0.0671433	−0.0671429	−0.0671433
0.30	0.17311	0.173008	0.173008	−0.0830722	−0.0830717	−0.0830722
0.35	0.174189	0.174185	0.174185	−0.0921531	−0.0921527	−0.0921531
0.40	0.176178	0.176174	0.176174	−0.1176051	−0.1176047	−0.1176051
0.45	0.182480	0.182478	0.182478	−0.1510229	−0.1510226	−0.1510229
0.50	0.200000	0.200000	0.200000	−0.2000000	−0.2000000	−0.2000000

Table 3 Comparison of EHPM (8th order), Laplace HPM and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.1)$.

t	EHPM $\phi(x,y,t)$	LHPM $\phi(x,y,t)$	Exact $\phi(x,y,t)$	EHPM $\psi(x,y,t)$	LHPM $\psi(x,y,t)$	Exact $\psi(x,y,t)$
0.05	0.360916	0.360916	0.360916	−0.241515	−0.241515	−0.241515
0.10	0.385757	0.385757	0.385757	−0.253536	−0.253536	−0.253536
0.15	0.397533	0.397533	0.397533	−0.293665	−0.293665	−0.293665
0.20	0.410314	0.410314	0.410314	−0.338726	−0.338726	−0.338726
0.25	0.430000	0.430000	0.430000	−0.399889	−0.399889	−0.399850
0.30	0.452648	0.452648	0.452643	−0.441565	−0.441565	−0.441534
0.35	0.460697	0.460697	0.460687	−0.520474	−0.520474	−0.520463
0.40	0.485818	0.485818	0.485808	−0.649511	−0.649511	−0.649507
0.45	0.548011	0.548011	0.548001	−0.809921	−0.809921	−0.809916
0.50	0.600000	0.600000	0.600000	−1.000000	−1.000000	−1.000000

Table 4 Comparison of EHPM (8th order), Laplace HPM and exact solution for $Re = 1$ with $(x, y) = (0.3, 0.2)$.

t	EHPM $\phi(x,y,t)$	LHPM $\phi(x,y,t)$	Exact $\phi(x,y,t)$	EHPM $\psi(x,y,t)$	LHPM $\psi(x,y,t)$	Exact $\psi(x,y,t)$
0.05	0.358591	0.358591	0.358591	0.190559	0.190559	0.190559
0.10	0.349396	0.349396	0.349396	0.183376	0.183376	0.183376
0.15	0.320767	0.320767	0.320767	0.178001	0.178001	0.178001
0.20	0.303484	0.303484	0.303484	0.171319	0.171319	0.171319
0.25	0.287159	0.287159	0.287154	0.172255	0.172255	0.172249
0.30	0.262943	0.262943	0.262938	0.173011	0.173011	0.173008
0.35	0.256567	0.256567	0.256561	0.174191	0.174191	0.174185
0.40	0.232949	0.232949	0.232945	0.176189	0.176189	0.176174
0.45	0.214881	0.214881	0.214878	0.182484	0.182484	0.182478
0.50	0.200011	0.200011	0.200000	0.200011	0.200011	0.200000

Table 5 Comparison of EHPM (8th order), Laplace HPM and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.4)$.

t	EHPM $\phi(x,y,t)$	LHPM $\phi(x,y,t)$	Exact $\phi(x,y,t)$	EHPM $\psi(x,y,t)$	LHPM $\psi(x,y,t)$	Exact $\psi(x,y,t)$
0.05	0.543084	0.543084	0.543084	0.964169	0.964169	0.964169
0.10	0.531270	0.531270	0.531270	0.966882	0.966882	0.966882
0.15	0.523919	0.523919	0.523919	0.969061	0.969061	0.969061
0.20	0.521881	0.521881	0.521881	0.970282	0.970282	0.970282
0.25	0.524494	0.524494	0.524494	0.972061	0.972061	0.972061
0.30	0.525717	0.525717	0.525717	0.973832	0.973832	0.973832
0.35	0.524857	0.524857	0.524857	0.975314	0.975314	0.975314
0.40	0.523111	0.523111	0.523111	0.975004	0.975004	0.975004
0.45	0.517457	0.517457	0.517457	0.977345	0.977345	0.977345
0.50	0.514561	0.514561	0.514561	0.978444	0.978444	0.978444

$$\phi(x,y,t) = (x+y) - E^{-1}\left\{vE\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right]\right\} + E^{-1}\left\{vE\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right]\right\}, \quad (18)$$

$$\psi(x,y,t) = (x+y) - E^{-1}\left\{vE\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right]\right\} + E^{-1}\left\{vE\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right]\right\}, \quad (19)$$

We construct a homotopy of the form for Eqs. (18) and (19).

$$\sum_{n=0}^{\infty} p^n\phi_n(x,y,t) = (x+y) - pE^{-1}\left\{vE\left[\sum_{n=0}^{\infty} p^n H_{1n}(\phi) + \sum_{n=0}^{\infty} p^n H_{2n}(\psi,\phi) + \frac{1}{Re}\sum_{n=0}^{\infty} p^n\left((\phi_n)_{xx} + (\phi_n)_{yy}\right)\right]\right\}, \quad (20)$$

$$\sum_{n=0}^{\infty} p^n\psi_n(x,y,t) = (x-y) - pE^{-1}\left\{vE\left[\sum_{n=0}^{\infty} p^n H_{3n}(\phi,\psi) + \sum_{n=0}^{\infty} p^n H_{4n}(\psi) + \frac{1}{Re}\sum_{n=0}^{\infty} p^n\left((\psi_n)_{xx} + (\psi_n)_{yy}\right)\right]\right\}, \quad (21)$$

where

$$\phi(x,y,t) = \sum_{n=0}^{\infty} p^n\phi_n(x,y,t) = \phi_0 + p\phi_1 + p^2\phi_2 + p^3\phi_3 + \dots. \quad (22)$$

$$\psi(x,y,t) = \sum_{n=0}^{\infty} p^n\psi_n(x,y,t) = \psi_0 + p\psi_1 + p^2\psi_2 + p^3\psi_3 + \dots. \quad (23)$$

are recursive relations, now by comparing the like powers of p on both sides of the Eqs. (22) and (23). We get the following equations:

Table 6 Comparison of EHPM (8th order), Laplace HPM and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.6)$.

t	EHPM $\phi(x,y,t)$	LHPM $\phi(x,y,t)$	Exact $\phi(x,y,t)$	EHPM $\psi(x,y,t)$	LHPM $\psi(x,y,t)$	Exact $\psi(x,y,t)$
0.05	0.515813	0.515813	0.515813	0.983475	0.983475	0.983475
0.10	0.515501	0.515501	0.515501	0.984599	0.984599	0.984599
0.15	0.514920	0.514920	0.514920	0.985265	0.985265	0.985265
0.20	0.514586	0.514586	0.514586	0.985975	0.985975	0.985975
0.25	0.514154	0.514154	0.514154	0.986550	0.986550	0.986550
0.30	0.513805	0.513805	0.513805	0.987148	0.987148	0.987148
0.35	0.512496	0.512496	0.512496	0.987885	0.987885	0.987885
0.40	0.511874	0.511874	0.511874	0.988574	0.988574	0.988574
0.45	0.511485	0.511485	0.511485	0.989425	0.989425	0.989425
0.50	0.510123	0.510123	0.510123	0.989788	0.989788	0.989788

Table 7 Comparison of EHPM (8th order), Laplace HPM and exact solution for $Re = 1$ with $(x, y) = (0.2, 0.8)$.

t	EHPM $\phi(x,y,t)$	LHPM $\phi(x,y,t)$	Exact $\phi(x,y,t)$	EHPM $\psi(x,y,t)$	LHPM $\psi(x,y,t)$	Exact $\psi(x,y,t)$
0.05	0.624295	0.624295	0.624295	0.871045	0.871045	0.871045
0.10	0.613239	0.613239	0.613239	0.886077	0.886077	0.886077
0.15	0.603384	0.603384	0.603384	0.890016	0.890016	0.890016
0.20	0.596640	0.596640	0.596640	0.904650	0.904650	0.904650
0.25	0.593446	0.593446	0.593446	0.906875	0.906875	0.906875
0.30	0.580350	0.580350	0.580350	0.909985	0.909985	0.909985
0.35	0.579688	0.579688	0.579688	0.911253	0.911253	0.911253
0.40	0.577634	0.577634	0.577634	0.914864	0.914864	0.914864
0.45	0.575086	0.575086	0.575086	0.918098	0.918098	0.918098
0.50	0.571153	0.571153	0.571153	0.920987	0.920987	0.920987

$$\phi_0(x,y,t) = x + y, \quad (24)$$

$$\psi_0(x,y,t) = x - y, \quad (25)$$

$$\phi_1(x,y,t) = -E^{-1}\left\{vE\left[H_{10}(\phi) + H_{20}(\phi,\psi) + \frac{1}{Re}\left((\phi_0)_{xx} + (\phi_0)_{yy}\right)\right]\right\}, \quad (26)$$

$$\psi_1(x,y,t) = -E^{-1}\left\{vE\left[H_{30}(\phi,\psi) + H_{40}(\psi) + \frac{1}{Re}\left((\psi_0)_{xx} + (\psi_0)_{yy}\right)\right]\right\}, \quad (27)$$

$$\phi_2(x,y,t) = -E^{-1}\left\{vE\left[H_{11}(\phi) + H_{21}(\phi,\psi) + \frac{1}{Re}\left((\phi_1)_{xx} + (\phi_1)_{yy}\right)\right]\right\}, \quad (28)$$

$$\psi_2(x,y,t) = -E^{-1}\left\{vE\left[H_{31}(\phi,\psi) + H_{41}(\psi) + \frac{1}{Re}\left((\psi_1)_{xx} + (\psi_1)_{yy}\right)\right]\right\}, \quad (29)$$

$$\phi_3(x,y,t) = -E^{-1}\left\{vE\left[H_{12}(\phi) + H_{22}(\phi,\psi) + \frac{1}{Re}\left((\phi_2)_{xx} + (\phi_2)_{yy}\right)\right]\right\}, \quad (30)$$

$$\psi_3(x,y,t) = -E^{-1}\left\{vE\left[H_{32}(\phi,\psi) + H_{42}(\psi) + \frac{1}{Re}\left((\psi_2)_{xx} + (\psi_2)_{yy}\right)\right]\right\}, \quad (31)$$

$$\vdots$$

$$\phi_{n+1}(x,y,t) = -E^{-1}\left\{vE\left[H_{1n}(\phi) + H_{2n}(\phi,\psi) + \frac{1}{Re}\left((\phi_n)_{xx} + (\phi_n)_{yy}\right)\right]\right\}, \quad (32)$$

$$\psi_{n+1}(x,y,t) = -E^{-1}\left\{vE\left[H_{3n}(\phi,\psi) + H_{4n}(\psi) + \frac{1}{Re}\left((\psi_n)_{xx} + (\psi_n)_{yy}\right)\right]\right\}, \quad (33)$$

By solving the above equations we get the values of $\phi_0, \phi_1, \phi_2, \ldots, \phi_n$ and $\psi_0, \psi_1, \psi_2, \ldots, \psi_n$, where H $1n(\phi), H_{2n}(\psi,\phi), H_{3n}(\phi,\psi)$ and $H_{4n}(\psi)$ are the He's polynomial, which can be obtained as follows:

$$H_{10}(\phi) = (\phi_0)(\phi_0)_x, \quad (34)$$

$$H_{20}(\phi,\psi) = (\psi_0)(\phi_0)_y, \quad (35)$$

$$H_{30}(\phi,\psi) = (\phi_0)(\psi_0)_x, \quad (36)$$

$$H_{40}(\psi) = (\psi_0)(\psi_0)_y, \quad (37)$$

$$H_{11}(\phi) = (\phi_0)(\phi_1)_x + (\phi_1)(\phi_0)_x, \quad (38)$$

$$H_{21}(\phi,\psi) = (\psi_0)(\phi_1)_y + (\psi_1)(\phi_0)_y, \quad (39)$$

$$H_{31}(\phi,\psi) = (\phi_0)(\psi_1)_x + (\phi_1)(\psi_0)_x, \quad (40)$$

$$H_{41}(\psi) = (\psi_0)(\psi_1)_y + (\psi_1)(\psi_0)_y, \quad (41)$$

$$H_{12}(\phi) = (\phi_0)(\phi_2)_x + (\phi_1)(\phi_1)_x + (\phi_2)(\phi_0)_x, \quad (42)$$

$$H_{22}(\phi,\psi) = (\psi_0)(\phi_2)_y + (\psi_1)(\phi_1)_y + (\psi_2)(\phi_0)_y, \quad (43)$$

$$H_{32}(\phi,\psi) = (\phi_0)(\psi_2)_x + (\phi_1)(\psi_1)_x + (\phi_2)(\psi_0)_x, \quad (44)$$

$$H_{42}(\psi) = (\psi_0)(\psi_2)_y + (\psi_1)(\psi_1)_y + (\psi_2)(\psi_0)_y, \quad (45)$$

After finding the values of He's polynomials, we will find the values of ϕ_i and ψ_i with the help of He's polynomials by setting the value of Reynolds number as $Re = 1$.

$$\phi_0 = x + y, \quad (46)$$
$$\psi_0 = x - y, \quad (47)$$
$$\phi_1 = -2xt, \quad (48)$$
$$\psi_1 = -2yt, \quad (49)$$
$$\phi_2 = 2xt^2 + 2yt^2, \quad (50)$$
$$\psi_2 = 2xt^2 - 2yt^2, \quad (51)$$
$$\phi_3 = 4yt^4 - 4xt^3, \quad (52)$$
$$\psi_3 = 4xt^4 - 4yt^3, \quad (53)$$

Therefore, solution of the Burger's model obtained by homotopy perturbation method along with Elzaki transform is given as follows:

$$\begin{aligned} u(x,y,t) &= \sum_{n=0}^{\infty}\phi_n(x,y,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots \\ &= x + y - 2xt + 2xt^2 + 2yt^2 + 4yt^4 - 4xt^3 + \cdots, \\ &= x(1 + 2t^2 + 4t^4 + 8t^6 + \cdots) - 2xt(1 + 2t^2 + 4t^4 \\ &\quad + 8t^6 + \cdots) + y(1 + 2t^2 + 4t^4 + 8t^6 + \cdots), \\ &= \frac{x + y - 2xt}{1 - 2t^2}, \end{aligned} \quad (54)$$

$$\begin{aligned} v(x,y,t) &= \sum_{n=0}^{\infty}\psi_n(x,y,t) = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \cdots \\ &= x - y - 2yt + 2xt^2 - 2yt^2 + 4xt^4 - 4yt^3 + \cdots \\ &= x(1 + 2t^2 + 4t^4 + \cdots) - y(1 + 2t^2 + 4t^4 + \cdots) \\ &\quad - 2yt(1 + 2t^2 + 4t^4 + \cdots), \\ &= \frac{x - y - 2yt}{1 - 2t^2}, \end{aligned} \quad (55)$$

This is the exact solution of two dimensional Burger's models.

3.2. Example

Consider the problem in Eqs. (14) and (15) with the initial conditions

$$\phi(x,y,0) = \sin\pi x + \cos\pi y, \quad \psi(x,y,0) = x + y. \quad (56)$$

Applying Elzaki transform to both sides of the Eqs. (14) and (15) we get:

$$E\left[\frac{\partial\phi}{\partial t}\right] + E\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] = E\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$E\left[\frac{\partial\psi}{\partial t}\right] + E\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] = E\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

$$\frac{1}{v}E[\phi] - v\phi(0) = -E\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] + E\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$\frac{1}{v}E[\psi] - v\psi(0) = -E\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] + E\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

$$E[\phi] - v^2(\sin\pi x + \cos\pi y) = -vE\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right],$$

$$E[\psi] - v^2(x + y) = -vE\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right],$$

$$E[\phi] = v^2(\sin\pi x + \cos\pi y) - vE\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right], \quad (57)$$

$$E[\psi] = v^2(x + y) - vE\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y}\right] + vE\left[\frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right]. \quad (58)$$

After applying inverse Elzaki transform on both sides of Eqs. (57) and (58), we have the following equations:

$$\phi(x,y,t) = (\sin\pi x + \cos\pi y) - E^{-1}\left\{vE\left[\phi\frac{\partial\phi}{\partial x} + \psi\frac{\partial\phi}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\right]\right\}, \quad (59)$$

$$\psi(x,y,t) = (x - y) - E^{-1}\left\{vE\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right]\right\}, \quad (60)$$

we construct a homotopy for the following system of the equation of the form:

$$\sum_{n=0}^{\infty}p^n\phi_n(x,y,t) = (\sin\pi x + \cos\pi y) - pE^{-1}\left\{vE\left[\sum_{n=0}^{\infty}p^n H_{5n}(\phi) + \sum_{n=0}^{\infty}p^n H_{6n}(\psi,\phi) + \frac{1}{Re}\sum_{n=0}^{\infty}p^n\left((\phi_n)_{xx} + (\phi_n)_{yy}\right)\right]\right\}, \quad (61)$$

$$\sum_{n=0}^{\infty}p^n\psi_n(x,y,t) = (x + y) - pE^{-1}\left\{vE\left[\sum_{n=0}^{\infty}p^n H_{7n}(\phi,\psi) + \sum_{n=0}^{\infty}p^n H_{8n}(\psi) + \frac{1}{Re}\sum_{n=0}^{\infty}p^n\left((\psi_n)_{xx} + (\psi_n)_{yy}\right)\right]\right\}, \quad (62)$$

where

$$\phi(x,y,t) = \sum_{n=0}^{\infty}p^n\phi_n(x,y,t) = \phi_0 + p\phi_1 + p^2\phi_2 + p^3\phi_3 + \cdots \quad (63)$$

$$\psi(x,y,t) = \sum_{n=0}^{\infty}p^n\psi_n(x,y,t) = \psi_0 + p\psi_1 + p^2\psi_2 + p^3\psi_3 + \cdots \quad (64)$$

By solving the above equations we get the values of $\phi_0, \phi_1, \phi_2, \ldots, \phi_n$ and $\psi_0, \psi_1, \psi_2, \ldots, \psi_n$, where $H_{5n}(\phi), H_{6n}(\psi,\phi), H_{7n}(\phi,\psi)$ and $H_{8n}(\psi)$ are the He's polynomials. Solve the above equations to obtain the value of He's polynomial and with the help of He's polynomial we can find out the values of $\phi_0, \phi_1, \phi_2, \ldots, \phi_n$ and $\psi_0, \psi_1, \psi_2, \ldots, \psi_n$, respectively.

$$
\begin{aligned}
\phi_0 &= \sin\pi x + \cos\pi y,\\
\psi_0 &= x + y,\\
\phi_1 &= -[(\sin\pi x + \cos\pi y)(\pi\cos\pi x - \pi\sin\pi y)]t,\\
\psi_1 &= -[(x + y + \sin\pi x + \cos\pi y + \pi(x+y)\cos\pi x)]t\\
\phi_2 &= \begin{bmatrix} (2\pi^3\cos(\pi x + \pi y) - \pi(\sin\pi y)(x + y + \cos\pi y + \sin\pi x + \pi(x+y)\cos\pi x)) \\ +\pi(\cos\pi x)(\cos\pi y + \sin\pi x)(\pi\cos\pi x - \pi\sin\pi y) + (\cos\pi y + \sin\pi x) \\ (\pi^2\cos 2\pi x - \pi^2\sin(\pi x + \pi y)) + (x+y)(-\pi^2\sin(\pi x + \pi y) - \pi^2\cos 2\pi y) \end{bmatrix}\frac{t^2}{2},\\
\psi_2 &= \begin{bmatrix} (x+y)(\pi\sin\pi y - \pi\cos\pi x - 1) - \pi^2\cos\pi y - (\cos\pi y + \sin\pi x)(\pi\cos\pi x - \pi\sin\pi y) \\ -(x + y + \cos\pi y + \sin\pi x + \pi(x+y)\cos\pi x) - (3\pi^2\sin\pi x + \pi^3 x\cos\pi x + \pi^3 y\cos\pi x) \\ +(\cos\pi y + \sin\pi x)(\pi^2 x\sin\pi x - 2\pi\cos\pi x + \pi^2 y\sin\pi x - 1) \end{bmatrix}\frac{t^2}{2},\\
&\vdots
\end{aligned}
$$

The few terms approximate solution $u(x, y, t)$ and $v(x, y, t)$ of Burger's equation obtained through Elzaki Homotopy Perturbation Method:

$$
\begin{aligned}
u(x,y,t) &= \phi_0 + \phi_1 + \phi_2 + \ldots.\\
&= (\sin\pi x + \cos\pi y) - [(\sin\pi x + \cos\pi y)(\pi\cos\pi x - \pi\sin\pi y)]t\\
&\quad + \begin{bmatrix} 2\pi^3\cos(\pi x + \pi y - \pi(\sin\pi y)(x + y + \cos\pi y + \sin\pi x + \pi(x+y)\cos\pi x) \\ +\pi(\cos\pi x)(\cos\pi y + \sin\pi x)(\pi\cos\pi x - \pi\sin\pi y) + (\cos\pi y + \sin\pi x) \\ (\pi^2\cos 2\pi x - \pi^2\sin(\pi x + \pi y)) + (x+y)(-\pi^2\sin(\pi x + \pi y) - \pi^2\cos 2\pi y) \end{bmatrix}\frac{t^2}{2}
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
v(x,y,t) &= \psi_0 + \psi_1 + \psi_2 + \ldots.\\
&= (x+y) - [(x + y + \sin\pi x + \cos\pi y + \pi(x+y)\cos\pi x)]t\\
&\quad + \begin{bmatrix} ((x+y)(\pi\sin\pi y - \pi\cos\pi x - 1)) - \pi^2\cos\pi y - (\cos\pi y + \sin\pi x)(\pi\cos\pi x - \pi\sin\pi y) \\ -(x + y + \cos\pi y + \sin\pi x + \pi(x+y)\cos\pi x) - (3\pi^2\sin\pi x + \pi^3 x\cos\pi x + \pi^3 y\cos\pi x) \\ +(\cos\pi y + \sin\pi x)(\pi^2 x\sin\pi x - 2\pi\cos\pi x + \pi^2 y\sin\pi x - 1) \end{bmatrix}\frac{t^2}{2}
\end{aligned} \tag{66}
$$

Elzaki Homotopy Perturbation Method for the above examples showed a good agreement with the results. The Tables 2–7 show a comparison of Exact, EHPM, Laplace HPM and HPM for different values of Reynolds number. Numerical study shows that the EHPM can be used as a replacement tool for solving nonlinear problems.

4. Conclusions

The main objective of the present study was to obtain the series as well as exact solution of two dimensional Burger's equations by applying Elzaki Homotopy Perturbation Method. The numerical results showed the convergence of Elzaki homotopy perturbation method (EHPM). The results of EHPM were compared with those of Laplace homotopy perturbation method, homotopy perturbation method and exact solution, and excellent agreement is observed between exact and EHPM solution. Therefore, EHPM can be further applied to solve nonlinear problems arising out of different physical phenomena.

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