JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **91**, 39–46 (1983)

Inversion of Nonlinear Stochastic Operators*

G. ADOMIAN

Center for Applied Mathematics, University of Georgia, Athens, Georgia 30602

AND

R. RACH

Hughes Aircraft Company, Electron Dynamics Division, Torrance, California 90509

The operator-theoretic method (Adomian and Malakian, *J. Math. Anal. Appl.* **76**(1), (1980), 183–201) recently extended Adomian's solutions of nonlinear stochastic differential equations (G. Adomian, Stochastic Systems Analysis, *in* "Applied Stochastic Processes," Nonlinear Stochastic Differential Equations, *J. Math. Anal. Appl.* **55**(1) (1976), 441–452; On the modeling and analysis of nonlinear stochastic systems, *in* "Proceeding, International Conf. on Mathematical Modeling," Vol. 1, pp. 29–40) to provide an efficient computational procedure for differential equations containing polynomial, exponential, and trigonometric nonlinear terms $N(y)$. The procedure depends on the calculation of certain quantities A_n and B_n. This paper generalizes the calculation of the A_n and B_n to much wider classes of nonlinearities of the form $N(y, y',...)$. Essentially, the method provides a systematic computational procedure for differential equations containing any nonlinear terms of physical significance. This procedure depends on a recurrence rule from which explicit general formulae are obtained for the quantities A_n and B_n for any order n in a convenient form. This paper also demonstrates the significance of the iterative series decomposition proposed by Adomian for linear stochastic operators in 1964 and developed since 1976 for nonlinear stochastic operators. Since both the nonlinear and stochastic behavior is quite general, the results are extremely significant for applications. Processes need not, for example, be limited to either Gaussian processes, white noise, or small fluctuations.

I. INTRODUCTION

A series of papers by Adomian *et al.* [1–11] have focused on the statistical solution of linear and nonlinear, deterministic or stochastic, differential equations by adaptations of his iterative series method. This

* Presented at the Third International Conference on Mathematical Modeling, University of Southern California, July 29–31, 1981.

0022-247X/83/010039–08\$03.00/0

paper further generalizes these methods and facilitates the inversion of nonlinear stochastic differential operators. Also, we point out the wide applicability of these methods to natural and technological systems, and emphasize their application to nonlinear deterministic as well as nonlinear stochastic differential equations.

Dynamical systems in many areas of scientific research are naturally characterized by random fluctuations and nonlinear behavior. Examples in nuclear reactors, plasmas, propagation, control, signal processing, physiology, economics, etc. occur readily. A few lesser known examples of random and nonlinear growth processes can be observed in the dendritic patterns of snowflakes, the spiral structures within some galaxies, and the traveling wave tube amplifier. In the TWT, for example, the electrons and the RF field strongly interact in a stochastic and nonlinear transfer of energy. These examples help to illustrate that for many physical systems, strongly nonlinear behavior and random processes representing violent fluctuations are the dominant structural features. Modeling of such dynamical systems leads immediately to nonlinear stochastic differential equations. Although the significance of modeling nonlinear and stochastic behavior was recognized long ago, the prevalent methods for solving such equations have not been satisfactory. Most authors have employed linearization, perturbation, and averaging techniques, which often distort the physical model and can create serious discrepancy between the mathematical approximate solution and the actual physical solution.

In 1960 Bellman pointed out the need for a practical method to determine the solution for nonlinear and stochastic differential equations in a computationally tractable form. Further stimulated by a previous applied problem in radar systems design involving random sampling of random processes, in 1961 Adomian formulated his theory of linear stochastic operators and developed the concept of stochastic Green's functions to compute the solution statistics of linear stochastic differential equations. By 1964 he had developed the technique of iterative series decomposition for linear stochastic operators. In 1976 this iterative series method was generalized to include nonlinear stochastic differential equations [3]. Consequent research (at the Center for Applied Mathematics) has developed several extensions, which hopefully will represent the goal that Bellman foresaw in 1960.

Recent extensions for determining the solution process and computing its statistical measures are again concerned with solving general classes of nonlinear stochastic differential equations. The equation considered is $\mathscr{F}y = x$, where x represents a stochastic process and $\mathscr{F}$ is a possibly nonlinear, possibly stochastic, differential operator. We partition $\mathscr{F}$ into linear and nonlinear components represented by $\mathscr{L}$ and $\mathscr{N}$, respectively, and then further partition $\mathscr{L}$ and $\mathscr{N}$ into deterministic and stochastic components, $\mathscr{L} = L + \mathscr{R}$ and $\mathscr{N} = N + \mathscr{M}$. The term y is the solution process whose statistical measures are sought. This leads to $\mathscr{F}y = Ly + \mathscr{R}y + Ny + \mathscr{M}y = x$, where L is linear deterministic and invertible. The general solution process $y = \mathscr{F}^{-1}x$ is decomposed into $y = \sum_{n=0}^{\infty} y_n$ or equivalently $y = \sum_{n=0}^{\infty} \mathscr{F}_n^{-1}x$ with $y_0 = \mathscr{F}_0^{-1}x = L^{-1}x$ (the homogeneous solution is added) and $\mathscr{F}_{n+1}^{-1}x = -L^{-1}[\mathscr{R}\mathscr{F}_n^{-1}x + A_n + B_n]$. The A_n and B_n are the expansion coefficients for analytic expansions in a parameter λ for the nonlinear deterministic term Ny and the nonlinear stochastic term $\mathscr{M}y$, respectively [1, 2]. Then the desired statistical measures of the solution process are computed by taking the appropriate ensemble averages. The extension provided by the inverse operator method [1] over the previous work [3] depends largely on a modified iterative decomposition of the nonlinear component $\mathscr{N}y$ ($\mathscr{N}$ indicates a stochastic component). Calculation of the terms of the modified iterative series depends on a recurrence rule by which we can express the terms of the modified series explicitly in terms of the original iterative series decomposition for linear stochastic operators. Because of the importance of direct calculation of these expansion coefficients, the remainder of this paper is focused upon their systematic computation.

II. Expansion Coefficients for Simple Nonlinear Operators

Consider the nonlinear deterministic component Ny. This nonlinear function of y was called $f(y)$ by Adomian and Malakian. The solution process is assumed to be a decomposition $y = \sum_{i=0}^{\infty} y_i$ and parametrized as $\sum_{i=0}^{\infty} \lambda^i y_i$ only for convenience in collecting terms; λ is *not* a perturbation parameter. Therefore, $f(y) = f(y(\lambda)) = \sum_{n=0}^{\infty} A_n \lambda^n$, where $f(y)$ is a nonlinear function to be specified and assumed to be analytic in λ. This analytic parametrization is a crucial step of the inverse operator method in extending Adomian's methods to include exponential and trigonometric nonlinearities as well as polynomial nonlinearities [3, 7]. We call such nondifferential operators *Ny simple nonlinear operators.* Further research [11] has also considered operators of the form $N(y, y', \ldots, y^{(n)})$ including product nonlinearities.

The analytic parametrization [1] decomposes the inverse operator and the nonlinear components and expresses them as analytic expansions in λ. Thus,

$$\mathscr{F}^{-1}x = \sum_{n=0}^{\infty} \lambda^n \mathscr{F}_n^{-1}x, \qquad Ny = \sum_{n=0}^{\infty} A_n \lambda^n,$$

$$\mathscr{M}y = \sum_{n=0}^{\infty} B_n \lambda^n \qquad (B_n \text{ stochastic}).$$

The term Ny could be y^2 or e^y or similar terms for which the A_n are easily found by previous methods. The complete solutions of the corresponding nonlinear stochastic differential equations have been given [1, 2, 4, 7].

In the resulting series for the solution $y = \sum_{n=0}^{\infty} \lambda^n y_n$, each y_n is calculable in terms of the proceeding y_{n-1}, thus

$$y_1 = y_1(y_0), \qquad y_2 = y_2(y_0, y_1), \qquad y_3 = y_3(y_0, y_1, y_2), \ldots,$$

because

$$A_0 = A_0(y_0), \qquad A_1 = A_1(y_0, y_1), \qquad A_2 = A_2(y_0, y_1, y_2), \ldots .$$

The resulting system of equations is completely calculable since $y_0 = L^{-1}x$ is known and a natural statistical separability results avoiding the usual truncations. Let us now consider the determination of the A_n by implicit differentiations. In the reference work, [1], coefficients were obtained for the first several terms for a number of examples but no specific expression for the general A_n was provided. Such an expression can be written immediately as

$$A_n = (1/n!)(d^n/d\lambda^n) f(y(\lambda))|_{\lambda=0}.$$

Our goal now is a systematic computation scheme with explicit instead of implicit differentiations and interpretation of the results in terms of Adomian's series for the solution process. Let $D = d/d\lambda$ and suppress the arguments for simplicity writing

$$A_n = (1/n!)\, D^n f|_{\lambda=0}. \tag{1}$$

For systematic computation of the A_n note $D = d/d\lambda = (dy/d\lambda)(d/dy)$ since $f = f(y)$ and $y = y(\lambda)$. Then each $D^n f$ is evaluated at $\lambda = 0$ and divided by $(n!)$. Since $y(\lambda) = y_0 + \lambda y_1 + \lambda^2 y_2 + \cdots$, we can write

$$(d^n/d\lambda^n)\, y(\lambda)|_{\lambda=0} = n!\, y_n. \tag{2}$$

Also a convenient abbreviation will be

$$(d^n/dy^n) f[y(\lambda)]|_{\lambda=0} = h_n(y_0). \tag{3}$$

Now consider $D^n f$ in (1). A significant observation, apparent from the computed coefficients A_0, A_1, A_2, A_3 for the specific forms of certain $N(y)$ which were calculated previously is that the $D^n f$ involve $d^i f/dy^i$ multiplied by polynomials in $d^i y/d\lambda^i$. Thus,

$$\begin{aligned} D^1 f &= (df/dy)(dy/d\lambda), \qquad (4)\\ D^2 f &= (d^2 f/dy^2)(dy/d\lambda)^2 + (df/dy)(d^2 y/d\lambda^2)\\ D^3 f &= (d^3 f/dy^3)(dy/d\lambda)^3 + 3(d^2 f/dy^2)(dy/d\lambda)(d^2 y/d\lambda^2)\\ &\quad + (df/dy)(d^3 y/d\lambda^3). \end{aligned}$$

Let us write for $n \geqslant 0$

$$D^n f = \sum_{i=1}^{n} c(i, n)(d^i f/dy^i), \tag{5}$$

where $c(i, n)$ for $1 \leqslant i \leqslant n$ represents the ith coefficient. These coefficients are calculable from a recurrence relation

$$c(i, j) = (d/d\lambda)\{c(i, j-1)\} + (dy/d\lambda)\{c(i-1, j-1)\} \tag{6}$$

for $1 \leqslant i, j \leqslant n$, where $c(i, j)$ is the ith coefficient for $D^j f$ knowing $c(0, 0) = 1$ and $c(1, 0) = 0$. The latter is true because $c(i, j) \equiv 0$ for $i > j$. Coefficient $c(0, 0) = 1$ since it comes from comparison of $D^0 f = c(0, 0)(d^0 f/dy^0) = f$, i.e., $A_0 = f|_{\lambda=0} = f(y_0)$.

Towards a planned computer program development that will extend the work of Elrod [9] to the nonlinear case, some convenient notation will be introduced now.

$$\Psi(i, j) = (d^i y/d\lambda^i)^j, \qquad F(i) = d^i f/dy^i. \tag{7}$$

These quantities are explicit derivatives and our goal is to express the expansion coefficients explicitly. Then (5) is written

$$D^n f = \sum_{i=1}^{n} c(i, n) F(i) \tag{8}$$

for $n \geqslant 1$, where the $c(i, n)$ are to be calculated from the recurrence relation. The results have been calculated from $c(0, 0)$ up to $c(10, 10)$, but are not included because of length. (These will be published along with computer results at a later time.) From (8) we can write, for example,

$$D^3 f = c(1, 3) F(1) + c(2, 3) F(2) + c(3, 3) F(3)$$

which immediately leads to

$$\begin{aligned} D^3 f &= \Psi(3, 1) F(1) + 3\Psi(1, 1) \Psi(2, 1) F(2) + \Psi(1, 3) F(3) \\ &= (d^3 y/d\lambda^3)(df/dy) + 3(dy/d\lambda)(d^2 y/d\lambda^2)(d^2 f/dy^2) \\ &\quad + (dy/d\lambda)^3 (d^3 f/dy). \end{aligned}$$

The results will, of course, check with those previously found by Adomian and Malakian [1, 2].

Using Eq. (8) we can now calculate the $D^n f$ and then, from (1), the A_n are easily determined. Thus (using the convenient abbreviation in (3)),

$$A_0 = (1/0!)\, D^0 f|_{\lambda=0} = f(y_0) = h_0(y_0),$$
$$A_1 = h_1(y_0)\, y_1,$$
$$A_2 = 1/2\{h_2(y_0)\, y_1^2 + 2h_1(y_0)\, y_2\},$$
$$A_3 = 1/6\{h_3(y_0)\, y_1^3 + 6h_2(y_0)\, y_1 y_2 + 6h_1(y_0)\, y_3\},$$
$$A_4 = (1/4!)\{h_4(y_0)\, y_1^4 + 12h_3(y_0)\, y_1^2 y_2 + h_2(y_0)[12y_2^2 + 24y_1 y_3] + 24h_1(y_0)\, y_4\}.$$

The expansion coefficients have been calculated up to A_{10}. (These again are not included because of length and are to be published with the previously mentioned coefficients and computer results.) We observe that when $f(y)$ is chosen simply to be y—resulting in a linear case—the coefficients A_i become the y_i and we get precisely the first author's solution for the linear case.

II. Expansion Coefficients for Differential Nonlinear Operators

Consider the nonlinear operator $Ny = \Gamma(y, y^{(1)}, \ldots, y^{(n)})$. We assume Γ is analytic in λ and $y, y^{(1)}, \ldots, y^{(n)}$ are also analytic in λ. We are concerned with two important subcases of the *differential nonlinear operator* N which are:

(1) sum of nonlinear functions of the time derivatives of y, with each nonlinear function dependent on a single derivative

$$Ny = \sum_{i=0}^{n} N_i y = \sum_{i=0}^{n} f_i(y^{(i)});$$

(2) a sum of products of nonlinear functions of y, each dependent on a single derivative. As an example consider $\Gamma(y, y') = y^2 y'^3$.

Obviously, if $Ny = \Gamma(y)$, we have the simple nonlinearity for which we have previously found expansion coefficients and we must obtain identical results for this limiting case. We, therefore, define the expansion coefficients A_m for the general differential nonlinear operator Ny to be

$$A_m = (1/m!)\, D^m\{\Gamma(y, y', \ldots, y^{(n)})\}|_{\lambda=0},$$

where $y, y', \ldots, y^{(n)}$ are assumed analytic functions of λ.

Case I. The first subcase of our general class was specified by $Ny = \Gamma(y, y', \ldots, y^{(n)}) = \sum_{i=0}^{n} f_i(y^{(i)})$ which we shall call a sum decomposition [3]. Our expansion coefficients are given by

$$A_m = \sum_{i=0}^{n} [(1/m!)\, D^m f_i(y^{(i)})|_{\lambda=0}] = \sum_{i=0}^{n} A_{im}, \tag{10}$$

because $\Gamma(y, y', ..., y^{(n)}) = \sum_{m=0}^{\infty} \lambda^m A_m = \sum_{i=0}^{n} f_i(y^{(i)})$ and each $f_i(y^{(i)}) = \sum_{m=0}^{\infty} \lambda^m A_{im}$. This leads to $\Gamma(y, y', ..., y^{(n)}) = \sum_{i=0}^{n} \sum_{m=0}^{\infty} \lambda^m A_{im} = \sum_{m=0}^{\infty} [\sum_{i=0}^{n} A_{im}] \lambda^m$, therefore $A_m = \sum_{i=0}^{n} A_{im}$.

Case II. The second subcase, product decomposition of the nonlinear operator [12], decomposes $Ny = \Gamma(y, y', ..., y^{(n)})$ into a sum of products. Let us first take pairwise products such as $\Gamma(y, y') = f_0(y) f_1(y') = \prod_{i=0}^{1} f_i(y^{(i)})$ so we can consider nonlinearities such as $y^2 y'^3$. Now the expansion coefficients are given by

$$\begin{aligned} A_m &= (1/m!)\, D^m \{f_0(y^{(0)}(\lambda)) f_1(y^{(1)}(\lambda))\}|_{\lambda=0} \\ &= (1/m!) \sum_{k=0}^{m} \binom{m}{k} [D^{m-k} f_0(y^{(0)}(\lambda))] \cdot [D^k f_1(y^{(1)}(\lambda))]|_{\lambda=0}. \end{aligned}$$

This leads to

$$A_m = \sum_{k=0}^{m} A_{0,m-k} A_{1,k} \tag{11}$$

since

$$\Gamma(y, y') = \sum_{m=0}^{\infty} A_m \lambda^m = \prod_{i=0}^{1} f_i(y^{(i)})$$

and as before $f_i(y^{(i)}(\lambda)) = \sum_{m=0}^{\infty} A_{im} \lambda^m$. This implies $\Gamma(y, y') = (\sum_{m=0}^{\infty} A_{0m} \lambda^m)(\sum_{m=0}^{\infty} A_{1m} \lambda^m) = \sum_{m=0}^{\infty} [\sum_{k=0}^{m} A_{0,m-k} A_{1,k}] \lambda^m$ which also gives us the result of (11). An extended Leibnitz rule in terms of multinomial coefficients can handle products of n factors, that is $\Gamma(y, y', ..., y^{(n)}) = \prod_{i=0}^{n} f_i(y^{(i)})$. We do not write out all of these possibilities here. Computer programming is now in progress.

Essentially we are concerned with nonlinear functions of physical significance, i.e., functions defined by uniformly and stochastically convergent power series, including a Fourier series: $Ny = f(y) = \sum_{i=1}^{\infty} (A_i \cos iy + B_i \sin iy)$ where $f(y)$ is convergent. Since both the stochastic behavior is quite general, i.e., not limited to small fluctuations or white noise (and is considered in general to be non-Gaussian) and the nonlinear behavior is quite general, the problem has been adequately dealt with and can be generalized to the case of partial differential equations [10]. This work has further been extended to systems of nonlinear stochastic differential equations and will be discussed elsewhere. The techniques presented in this paper have solved such specific applied problems as the anharmonic oscillator and the Duffing and Van der Pol oscillators, quickly and naturally. (To appear.)

References

1. G. Adomian and K. Malakian, Operator theoretic solutions of stochastic systems, *J. Math. Anal. Appl.* **76** (1) (1980), 183–201.
2. G. Adomian, Stochastic systems analysis, *in* "Applied Stochastic Processes" (G. Adomian, Ed.), Academic Press, New York, 1980.
3. G. Adomian, Nonlinear stochastic differential equations, *J. Math. Anal. Appl.* **55** (1) (1976), 441–452.
4. G. Adomian, On the modeling and analysis of nonlinear stochastic systems, *in* "Proceedings of the International Conference on Mathematical Modeling," Vol. 1, pp. 29–40, (Avula, Bellman, Luke, and Rigler, Eds.), University of Missouri, 1980.
5. R. E. Bellman and G. Adomian, The stochastic riccati equation, *J. Nonlinear Anal. Theory Meth. Appl.* **4** (6) (1980), 1131–1133.
6. G. Adomian and K. Malakian, Stochastic analysis, *Internat. J. Math. Modeling* **1** (3) (1980), 211–235.
7. G. Adomian and L. H. Sibul, Symmetrized solutions for nonlinear stochastic differential equations, *Internat. J. Math. Math. Sci.* **4** (3) (1981), 529–542.
8. G. Adomian and R. Ambartzumian, "Some Remarks on Stochastic Transformations," *Sov. J. Contemp. Math. Anal.* Allerton Press, Vol. XVI, No. 1, 1981, 25–30.
9. M. Elrod, "Numerical Solution of Linear Stochastic Differential Equations," Ph. D. Dissertation, University of Georgia, Center for Applied Mathematics, 1973.
10. G. Adomian and K. Malakian, Inversion of stochastic partial differential operators—the linear case, *J. Math. Anal. Appl.* **77** (2) (1980), 505–512.
11. G. Adomian, "On Product Nonlinearities in Stochastic Differential Equations," *Appl. Math. Comput.* **8** (1981), 79–82.