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Note

On the Roman domination number of a graph

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ABSTRACT

A Roman dominating function of a graph G is a labeling $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(G)$ of G is the minimum of $\sum_{v\in V(G)} f(v)$ over such functions. A Roman dominating function of G of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$. Cockayne et al. [E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi, S.T. Hedetniemi, On Roman domination in graphs, Discrete Math. 278 (2004) 11–22] posed the following question: What can we say about the minimum and maximum values of $|V_0|, |V_1|, |V_2|$ for a γ_R-function $f = (V_0, V_1, V_2)$ of a graph G? In this paper we first show that for any connected graph G of order $n \geq 3$, $\gamma_R(G) + \frac{\gamma(G)}{2} \leq n$, where $\gamma(G)$ is the domination number of G. Also we prove that for any γ_R-function $f = (V_0, V_1, V_2)$ of a connected graph G of order $n \geq 3$, $|V_0| \geq \frac{n}{5} + 1$, $|V_1| \leq \frac{4n}{5} - 2$ and $|V_2| \leq \frac{2n}{5}$.

1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). The order $|V|$ of G is denoted by n. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The degree of v is $d(v) = |N(v)|$. The minimum degree of G is denoted by $\delta(G)$ (briefly δ). The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The S-*private neighbors* of a vertex v of S are the vertices of $N[v] \setminus N[S \setminus \{v\}]$. The vertex v is its own private neighbor if it is isolated in S. The other private neighbors are *external*, i.e., belong to $V \setminus S$. We call k-path (k-cycle, respectively) a path (cycle) of G of order k and P_k (C_k) an induced k-path (k-cycle). The corona HoK_1 of a graph H is obtained by attaching one pendent edge at each vertex of H.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of minimum cardinality of G is called a $\gamma(G)$-set. Ore proved that every graph of minimum degree $\delta \geq 1$ satisfies $\gamma(G) \leq n/2$ and the following theorem gives the characterization of the extremal graphs.

Theorem A ([3]). *For a connected graph G with order $n \geq 2$, $\gamma(G) = n/2$ if and only if G is the cycle C_4 or the corona HoK_1 of a connected graph H.*

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [6,7] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of a RDF is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight

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of a RDF on G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. For a more thorough treatment of domination parameters and for terminology not presented here see [4,8].

It is known that $\gamma_R(G) \leq 2\gamma(G)$ for every graph G [2]. If $\delta(G) \geq 2$ and $n \geq 8$, then $\gamma(G) \leq 2n/5$ [5], thus implying $\gamma_R(G) + \frac{\gamma(G)}{2} \leq \frac{5\gamma(G)}{2} \leq n$. But if $\delta(G) = 1$, then $\gamma(G)$ can be as large as $n/2$ and we can only deduce from $\gamma_R(G) \leq 2\gamma(G)$ that $\gamma_R(G) + \frac{\gamma(G)}{2} \leq 5n/4$. The main purpose of this paper is to prove in Section 2 that the inequality $\gamma_R(G) + \frac{\gamma(G)}{2} \leq n$ also holds in graphs with minimum degree 1. The technique we use in Theorem 2 also gives another proof of the following result, already obtained by Chambers, Kinnersley, Prince and West.

Theorem B *([1]). If G is a connected n-vertex graph, then $\gamma_R(G) \leq 4n/5$, with equality if and only if G is C_5 or is the union of $\frac{n}{5}P_5$ with a connected subgraph whose vertex set is the set of centers of the components of $\frac{n}{5}P_5$.*

Section 3 is related to a problem posed by Cockayne et al. in [2]:

Problem 1. What can we say about the minimum and maximum values of $|V_0|, |V_1|, |V_2|$ for a γ_R-function $f = (V_0, V_1, V_2)$ of a graph G?

In Theorem 3 we present an answer to this question.

2. Bound on the sum $\gamma_R(G) + \frac{\gamma(G)}{2}$

The following definitions will provide the extremal families for Theorems 2 and 3.

Definition 1. – $\mathcal{F}$ is the family of graphs obtained from a connected graph H by identifying each vertex of H with the central vertex of a path P_5 or with an internal vertex of a path P_4 where the $|V(H)|$ paths are vertex-disjoint.
– $\mathcal{G}$ is the family of graphs of $\mathcal{F}$ such that each vertex of H is identified with the central vertex of a P_5.
– $\mathcal{G}'$ is the family of graphs of $\mathcal{G}$ constructed from a graph H having a vertex of degree $|V(H)| - 1$.

Theorem 2. *For any connected graph G of order $n \geq 3$,*

(a) $\gamma_R(G) + \frac{\gamma(G)}{2} \leq n$ *with equality if and only if G is C_4, C_5, C_4oK_1 or G belongs to $\mathcal{F}$.*
(b) [1] $\gamma_R(G) \leq \frac{4n}{5}$ *with equality if and only if G is C_5 or belongs to $\mathcal{G}$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function such that $|V_2|$ is maximum. It is proved in [2] that for such a function no edge exists between V_1 and V_2 and every vertex v of V_2 has at least two V_2-private neighbors, one of them can be v itself if it is isolated in V_2 (true for every $\gamma_R(G)$-function), the set V_1 is independent and every vertex of V_0 has at most one neighbor in V_1. Moreover we add the condition that the number $\mu(f)$ of vertices of V_2 with only one neighbor in V_0 is minimum. Suppose that $N_{V_0}(v) = \{w\}$ for some $v \in V_2$. Then the partition $V_0' = (V_0 \setminus \{w\}) \cup \{v\} \cup N_{V_1}(w), V_1' = V_1 \setminus N_{V_1}(w), V_2' = (V_2 \setminus \{v\}) \cup \{w\}$ is a Roman dominating function f' such that $\omega(f') = \omega(f) - 1$ if $N_{V_1}(w) \neq \emptyset$, or $\omega(f') = \omega(f)$, $|V_2'| = |V_2|$ but $\mu(f') < \mu(f)$ if $N_{V_1}(w) = \emptyset$ since then, G being connected of order at least 3, w is not isolated in V_0. Therefore every vertex of V_2 has at least two neighbors in V_0. Let A be a largest subset of V_2 such that for each $v \in A$ there exists a subset A_v of $N_{V_0}(v)$ such that the sets A_v are disjoint, $|A_v| \geq 2$ and $\bigcup_{v \in A} A_v = \bigcup_{v \in A} N_{V_0}(v)$. Note that A_v contains all the external V_2-private neighbors of v. Let $A' = V_2 \setminus A$.

Case 1. $A' = \emptyset$. In this case $|V_0| \geq 2|V_2|$ and $|V_1| \leq |V_0|$ since every vertex of V_0 has at most one neighbor in V_1.

Since V_0 is a dominating set of G and $|V_2| \leq |V_0|/2$ we have

$$\gamma_R(G) + \frac{1}{2}\gamma(G) \leq |V_1| + 2|V_2| + \frac{1}{2}|V_0| \leq |V_1| + |V_2| + |V_0| = n.$$

If $\gamma_R(G) + \frac{1}{2}\gamma(G) = n$ then $|V_0| = 2|V_2|$ and V_0 is a $\gamma(G)$-set.

On the other hand,

$$5\gamma_R(G) = 5|V_1| + 10|V_2| = 4n - 4|V_0| + |V_1| + 6|V_2| = 4n - 3(|V_0| - 2|V_2|) - (|V_0| - |V_1|) \leq 4n.$$

Hence $\gamma_R(G) \leq \frac{4n}{5}$ and if $\gamma_R(G) = \frac{4n}{5}$, then $|V_0| = 2|V_2|$ and $|V_0| = |V_1|$.

Case 2. $A' \neq \emptyset$.

Let $B = \bigcup_{v \in A} A_v$ and $B' = V_0 \setminus B$. Every vertex x in A' has exactly one V_2-private neighbor x' in V_0 and $N_{B'}(x) = \{x'\}$ for otherwise x could be added to A. This shows that

$$|A'| = |B'|. \tag{1}$$

Moreover since $|N_{V_0}(x)| \geq 2$, each vertex $x \in A'$ has at least one neighbor in B. Let $x_B \in B \cap N_{V_0}(x)$ and let x_A be the vertex of A such that $x_B \in A_{x_A}$. The vertex x_A is well defined since the sets A_v with $v \in A$ form a partition of B.

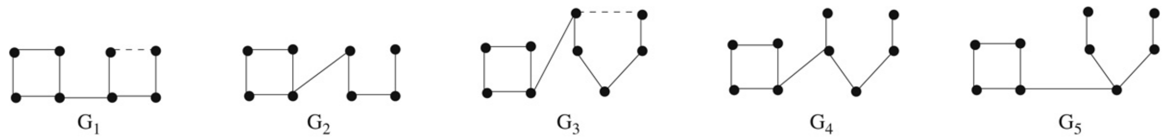

Fig. 1.

Claim 1. *$|A_{x_A}| = 2$ for each $x \in A'$ and each $x_B \in B \cap N_{V_0}(x)$.*

Proof of Claim 1. If $|A_{x_A}| > 2$, then by putting $A'_{x_A} = A_{x_A} \setminus \{x_B\}$ and $A_x = \{x', x_B\}$ we can see that $A_1 = A \cup \{x\}$ contradicts the choice of A. Hence $|A_{x_A}| = 2$, x_A has a unique external V_2-private neighbor x'_A and $A_{x_A} = \{x_B, x'_A\}$. Note that the vertices x_A and x are isolated in V_2 since they must have a second V_2-private neighbor. □

Claim 2. *If $x, y \in A'$ then $x_B \neq y_B$ and $A_{x_A} \neq A_{y_A}$.*

Proof of Claim 2. Let x' and y' be respectively the unique external V_2-private neighbors of x and y. Suppose that $x_B = y_B$, and thus $x_A = y_A$. The function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x_B) = 2, g(x) = g(y) = g(x_A) = 0, g(x'_A) = g(x') = g(y') = 1$ and $g(v) = f(v)$ otherwise, is a RDF of G of weight less that $\gamma_R(G)$, a contradiction. Hence $x_B \neq y_B$. Since $\{x'_A, x_B\} \subseteq A_{x_A}$ and $|A_{x_A}| = 2$, the vertex y_B is not in A_{x_A}. Therefore $A_{y_A} \neq A_{x_A}$. □

Let $A'' = \{x_A | x \in A' \text{ and } x_B \in B \cap N_{V_0}(x)\}$ and $B'' = \bigcup_{v \in A''} A_v$. By Claims 1 and 2,

$$|B''| = 2|A''| \quad \text{and} \quad |A''| \geq |A'|. \tag{2}$$

Let $A''' = V_2 \setminus (A' \cup A'')$ and $B''' = \bigcup_{v \in A'''} A_v = V_0 \setminus (B' \cup B'')$. By the definition of the sets A_v,

$$|B'''| \geq 2|A'''|. \tag{3}$$

Claim 3. *If $x \in A'$ and $x_B \in B \cap N_{V_0}(x)$, then x', x_B and x'_A have no neighbor in V_1. Hence B''' dominates V_1.*

Proof of Claim 3. Let w be a vertex of V_1. If w has a neighbor in $B' \cup B''$, let $g : V(G) \to \{0, 1, 2\}$ be defined by
$g(x'_A) = 2, g(w) = g(x_A) = 0, g(v) = f(v)$ otherwise if w is adjacent to x'_A,
$g(x') = 2, g(w) = g(x) = 0, g(v) = f(v)$ otherwise if w is adjacent to x',
$g(x_B) = 2, g(w) = g(x_A) = g(x) = 0, g(x'_A) = g(x') = 1, g(v) = f(v)$ otherwise if w is adjacent to x_B.
In each case, g is a RDF of weight less than $\gamma_R(G)$, a contradiction. Therefore $N(w) \subseteq B'''$. □

We are now ready to establish the two parts of the theorem.

(a) By Claim 3, $B''' \cup A' \cup A''$ is a dominating set of G. Therefore, by (1)–(3),

$$\begin{aligned}\gamma(G) &\leq |B'''| + |A'| + |A''| \\ &\leq |B'''| + |B''| \\ &\leq (2|B'''| - 2|A'''|) + (2|B''| - 2|A''|) + (2|B'| - 2|A'|).\end{aligned}$$

Hence $\gamma(G) \leq 2|V_0| - 2|V_2|$ and $\gamma_R(G) + \frac{1}{2}\gamma(G) \leq (|V_1| + 2|V_2|) + (|V_0| - |V_2|) = n$.

If $\gamma_R(G) + \frac{1}{2}\gamma(G) = n$ then $|A''| = |A'|$, $|B'''| = 2|A'''|$ and $B''' \cup A' \cup A''$ is a $\gamma(G)$-set. We note that these conditions of equality include that of Case 1 since in Case 1, $A' = A'' = \emptyset$, $A''' = V_2$ and $B''' = V_0$.

The first condition, $|A''| = |A'|$, implies that each vertex x of A' has exactly one neighbor x_B in B. Hence the vertices of $A' \cup A'' \cup B' \cup B''$ (in Case 2) can be partitioned into 5-paths $x'xx_Bx_Ax'_A$ with $x' \in B'$, $x \in A'$, $x_A \in A''$ and $x_B, x'_A \in B''$. The second condition, $|B'''| = 2|A'''|$, implies that $|A_v| = 2$ for each $v \in A$. Hence the vertices of $A''' \cup B'''$ can be partitioned into 3-paths v_1vv_2 with $v \in A'''$ and $v_1, v_2 \in B'''$. The third condition, $A' \cup A'' \cup B'''$ is a $\gamma(G)$-set, implies that for each $v \in A'''$, at least one of v_1, v_2 has a neighbor in V_1 for otherwise $(A' \cup A'' \cup B''' \cup \{v\}) \setminus \{v_1, v_2\}$ is a dominating set of order $\gamma(G) - 1$. Hence the vertices of $A''' \cup B''' \cup V_1$ are partitioned into 4-paths wv_1vv_2 or 5-paths $w_1v_1vv_2w_2$.

In this partition of V, each 4-path and 5-path has exactly two vertices in the $\gamma(G)$-set $A' \cup A'' \cup B'''$ and contributes for 2 to $\gamma(G)$. Each 4-path has one vertex in V_2, one in V_1 and two in V_0 and contributes for 3 to $\gamma_R(G)$. Each 5-path has either two vertices in V_2 and three in V_0 or one vertex in V_2, two in V_0 and two in V_1, and thus contributes for 4 to $\gamma_R(G)$. Hence each 4-path and 5-path of the partition induces in G a P_4, C_4, P_5 or C_5.

If a 4-path induces a C_4 and $G \neq C_4$, then there exists an edge between the C_4 and a 4-path or 5-path. The contribution of the two paths to $\gamma(G)$ and $\gamma_R(G)$ should be respectively 4 and 6 or 7. Fig. 1, where the dotted lines may exist or not, shows that this is impossible because $\gamma(G_i) = 3$ for $1 \leq i \leq 4$ and $\gamma_R(G_5) = 6 < 7$. Similarly Fig. 2 shows that it is not possible that a 5-path induces a C_5 and $G \neq C_5$ because $\gamma(G_1) = 3 < 4$, $\gamma_R(G_2) = 6 < 7$ and $\gamma_R(G_i) = 7 < 8$ for $3 \leq i \leq 5$.

We suppose that G is different from C_4 and C_5. The set of the $k \geq 0$ P_4's of the partition induces a subgraph J such that $\gamma(J) = 2k = |V(J)|/2$. By Theorem A, each component of J is the corona of a connected graph. Thus all the endvertices of

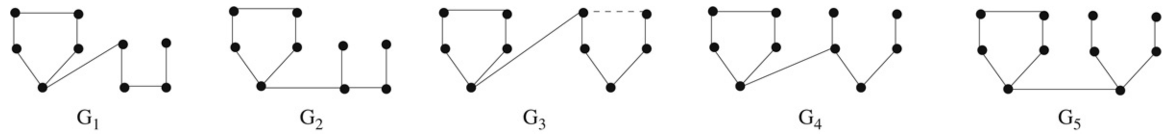

Fig. 2.

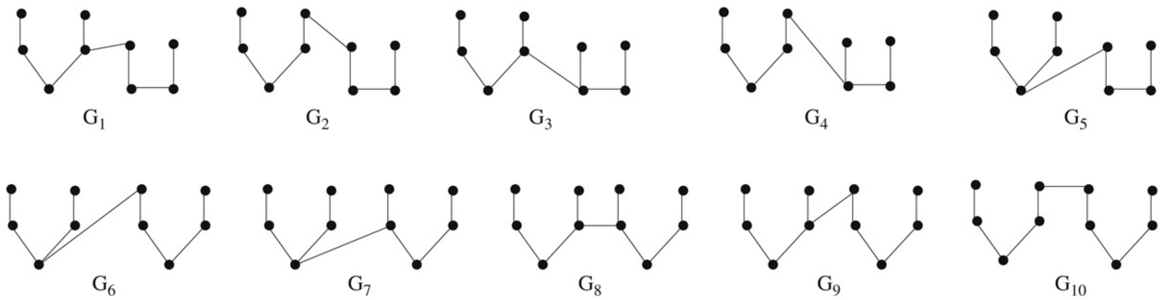

Fig. 3.

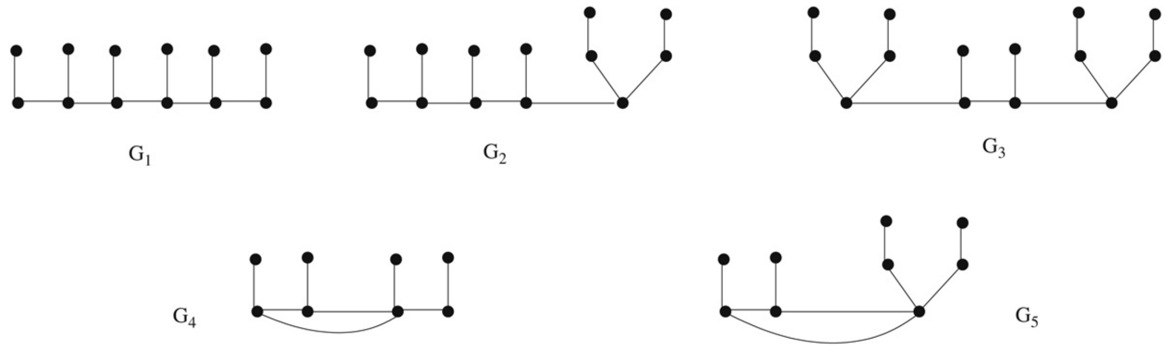

Fig. 4.

the P_4's have degree 1 in J and every edge between two P_4's joins vertices of degree 2 in each P_4. Similarly Fig. 3 shows that the only possibility for the extremity of an edge between a P_5 and a P_4 or P_5 is to be the central vertex of a P_5 or a vertex of degree 2 of a P_4 (because $\gamma(G_i) = 3 < 4$ for $1 \le i \le 2$, $\gamma_R(G_i) = 6 < 7$ for $3 \le i \le 5$ and $\gamma_R(G_i) = 7 < 8$ for $6 \le i \le 10$).

Finally Fig. 4 shows that the two internal vertices of a P_4 cannot be adjacent to vertices of two different P_4 or P_5, nor to the same internal vertex of another P_4, nor to the central vertex of a P_5 (because $\gamma_R(G_1) = 8 < 9$, $\gamma_R(G_2) = 9 < 10$, $\gamma_R(G_3) = 10 < 11$, $\gamma_R(G_4) = 5 < 6$, $\gamma_R(G_5) = 6 < 7$). Therefore either G contains two P_4's $xyzt$, $x'y'z't'$ together with the edges yy', zz', or G consists of paths P_4 and P_5 joined by edges between the central vertex of each P_5 and one internal vertex of each P_4. Since G is connected, $G = C_4 \text{o} K_1$ in the first case and G belongs to Family $\mathcal{F}$ in the second case.

Conversely, it is easy to check that each of $C_4, C_5, C_4\text{o}K_1$ satisfies $\gamma_R(G) + \frac{1}{2}\gamma(G) = n$. Let now G be a graph of $\mathcal{F}$ composed of k_1 paths P_4 and k_2 paths P_5. Then $\gamma(G) = 2k_1 + 2k_2$, $\gamma_R(G) = 2k_1 + 3k_2$ and $\gamma_R(G) + \frac{1}{2}\gamma(G) = 4k_1 + 5k_2 = n$.

(b) By Claim 3 and since each vertex of V_1 has at most one neighbor in V_0, $|V_1| \le |B'''|$. Using this inequality and (1)–(3) we get

$$\begin{aligned}
5\gamma_R(G) &= 5|V_1| + 10|V_2| \\
&= 4n - 4|V_0| + |V_1| + 6|V_2| \\
&\le 4n - 4|B'| - 4|B''| - 4|B'''| + |B'''| + 6|A'| + 6|A''| + 6|A'''| \\
&\le 4n + 2(|A'| - |A''|) + 3(2|A'''| - |B'''|) \\
&\le 4n.
\end{aligned}$$

Hence $\gamma_R(G) \le \frac{4n}{5}$.

If $\gamma_R(G) = \frac{4n}{5}$ then $|A''| = |A'|$, $|B'''| = 2|A'''|$ and $|V_1| = |B'''|$. We note that these conditions of equality include that of Case 1.

The first two conditions of equality, $|A''| = |A'|$, $|B'''| = 2|A'''|$, are the same as in Part (a) and imply that $V_2 \cup V_0$ can be partitioned into 5-paths and 3-paths. The third condition, $|V_1| = |B'''|$, implies that the edges between B''' and

V_1 form a matching covering B''' and V_1. Thus the 3-paths partitioning $A''' \cup B'''$ can be prolonged to 5-paths partitioning $A''' \cup B''' \cup V_1$. Hence V is partitioned into 5-paths, each of them contributes for 4 to $\gamma_R(G)$ and thus induces P_5 or C_5 in G. Also the configurations shown by G_3, G_4, G_5 in Fig. 1 and G_6 to G_{10} in Fig. 2, for which the global contribution of the two 5-paths to $\gamma_R(G)$ is too small, are impossible. Therefore $G \in \{C_5\} \cup \mathcal{G}$.

Conversely, every graph G in $\{C_5\} \cup \mathcal{G}$ obviously satisfies $\gamma_R(G) = \frac{4n}{5}$. □

3. Bounds on $|V_0|$, $|V_1|$ and $|V_2|$ for a $\gamma_R(G)$-function (V_0, V_1, V_2)

Theorem 3. *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R(G)$-function of a connected graph G of order $n \geq 3$. Then*

1. $1 \leq |V_2| \leq \frac{2n}{5}$ *and a graph G admits a $\gamma_R(G)$-function such that $|V_2| = \frac{2n}{5}$ if and only if G belongs to $\mathcal{G} \cup \{C_5\}$.*
2. $0 \leq |V_1| \leq \frac{4n}{5} - 2$ *and a graph G admits a $\gamma_R(G)$-function such that $|V_1| = \frac{4n}{5} - 2$ if and only if G belongs to $\mathcal{G}' \cup \{C_5\}$.*
3. $\frac{n}{5} + 1 \leq |V_0| \leq n - 1$ *and a graph G admits a $\gamma_R(G)$-function such that $|V_0| = \frac{n}{5} + 1$ if and only if G belongs to $\mathcal{G}' \cup \{C_5\}$.*

Proof. By Theorem B, $|V_1| + 2|V_2| \leq 4n/5$.

1. If $V_2 = \emptyset$, then $V_1 = V$ and $V_0 = \emptyset$. The RDF $(0, n, 0)$ is not minimum since $|V_1| + 2|V_2| > 4n/5$. Hence $|V_2| \geq 1$. On the other hand, $|V_2| \leq 2n/5 - |V_1|/2 \leq 2n/5$.

If $|V_2| = 2n/5$, then $4n/5 \leq |V_1| + 2|V_2| = \gamma_R(G) \leq 4n/5$. Therefore $\gamma_R(G) = 4n/5$ and by Theorem B, G is C_5 or belongs to $\mathcal{G}$. Conversely define the function f by giving the value 2 to the vertices adjacent to leaves when $G \in \mathcal{G}$ and to two non-adjacent vertices when $G = C_5$, and the value 0 to the other vertices. Then f is a $\gamma_R(G)$-function with $|V_2| = 2n/5$.

2. Since $|V_2| \geq 1$, $|V_1| \leq 4n/5 - 2|V_2| \leq 4n/5 - 2$.

If $|V_1| = 4n/5 - 2$, then $4n/5 \leq |V_1| + 2|V_2| = \gamma_R(G) \leq 4n/5$. Therefore $\gamma_R(G) = 4n/5$, i.e., $G \in \{C_5\} \cup \mathcal{G}$, and $|V_2| = 1$. When $G \in \mathcal{G}$, let G be obtained by identifying each vertex of a graph H with the central vertex of a P_5 and let $V_2 = \{x\}$. Then $V_0 = N(x)$, $V_1 = V \setminus N[x]$ and $4n/5 - 2 = |V_1| = n - d(x) - 1$. Hence $d(x) = n/5 + 1$. The unique vertex x of V_2 belongs to H and must be adjacent to all the other vertices of H. Therefore $G \in \{C_5\} \cup \mathcal{G}'$.

Conversely if $G \in \mathcal{G}'$, the function f defined by $f(x) = 2$ for some vertex x of H of degree $|V(H)| - 1$, $f(v) = 0$ for $v \in N(x)$ and $f(v) = 1$ elsewhere is a $\gamma_R(G)$ function with $|V_1| = 4n/5 - 2$. Similarly, there exists a $\gamma_R(C_5)$ function with $|V_2| = 1$ and $|V_1| = 2 = 4n/5 - 2$.

3. The upper bound comes from $|V_0| \leq n - |V_2| \leq n - 1$. For the lower bound, adding side by side $2|V_0| + 2|V_1| + 2|V_2| = 2n$, $-|V_1| - 2|V_2| \geq -4n/5$ and $-|V_1| \geq -4n/5 + 2$ gives $2|V_0| \geq 2n/5 + 2$. Therefore $|V_0| \geq n/5 + 1$.

If $|V_0| = n/5 + 1$ then $|V_1| = 4n/5 - 2$ and thus $G \in \{C_5\} \cup \mathcal{G}'$. Conversely if $G \in \mathcal{G}'$ then the $\gamma_R(G)$ function described in Part 2 is such that $|V_0| = d(x) = |H| + 1 = n/5 + 1$. Also for the $\gamma_R(C_5)$-function with $|V_2| = 1$, we have $|V_0| = 2 = n/5 + 1$.

Note that the lower bounds 1 and 0 on $|V_2|$ and $|V_1|$ and the upper bound $n - 1$ on $|V_0|$ cannot be improved since they are attained by stars. □

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