Available online at www.sciencedirect.com

SCIENCE DIRECT

Topology and its Applications 149 (2005) 149–160

Topology and its Applications

www.elsevier.com/locate/topol

Coronas of balleans

I.V. Protasov

Department of Cybernetics, Kyiv University, Volodimirska 64, Kyiv GSP, Ukraine

Received 23 December 2003; accepted 13 September 2004

Abstract

We introduce balleans as asymptotical counterparts of uniform topological spaces. Using slowly oscillating functions, for every ballean we define two compact spaces: corona and binary corona. These spaces can be considered as generalizations of the Higson's coronas of metric spaces and the spaces of ends of groups, respectively. We consider some balleans related to an infinite group and prove some results concerning their coronas. At the end we apply these results to describe the compact right-zero semigroups which are continuous homomorphic images of G^*, the reminder of the Stone–Čech compactification of discrete group G.

MSC: 54D35; 54E15

Keywords: Ball structures; Ballean; Slowly oscillating function; Corona; Binary corona; Stone–Čech compactification

1. Ball structures and balleans

A *ball structure* is a triple $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathbb{B}$, P is called the *set of radiuses*.

E-mail address: protasov@unicyb.kiev.ua (I.V. Protasov).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2004.09.005

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x,\alpha) = \{y \in X\colon\ x \in B(y,\alpha)\}, \qquad B(A,\alpha) = \bigcup_{a \in A} B(a,\alpha).$$

A ball structure $\mathbb{B} = (X, P, B)$ is called *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x,\alpha') \subseteq B(x,\alpha), \qquad B(x,\beta') \subseteq B^*(x,\beta).$$

A ball structure $\mathbb{B} = (X, P, B)$ is called *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x,\alpha) \subseteq B^*(x,\alpha'), \qquad B^*(x,\beta) \subseteq B(x,\beta').$$

A ball structure $\mathbb{B} = (X, P, B)$ is called *lower multiplicative* if, for any $\alpha, \beta \in P$ there exists $\gamma \in P$ such that, for every $x \in X$,

$$B\big(B(x,\gamma),\gamma\big) \subseteq B(x,\alpha) \cap B(x,\beta).$$

A ball structure $\mathbb{B} = (X, P, B)$ is called *upper multiplicative* if, for any $\alpha, \beta \in P$ there exists $\gamma \in P$ such that, for every $x \in X$,

$$B\big(B(x,\alpha),\beta\big) \subseteq B(x,\gamma).$$

Let $\mathbb{B} = (X, P, B)$ be a lower symmetric, lower multiplicative ball structure. Then the family

$$\left\{\bigcup_{x \in X} B(x,\alpha) \times B(x,\alpha)\colon\ \alpha \in P\right\}$$

is a fundamental system of entourages for some (uniquely determined) uniform topological space. On the other hand, if X is a uniformity $\mathcal{U} \subseteq X \times X$, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x,U) = \{y \in X\colon\ (x,y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure is a *ballean* if $\mathbb{B}$ is upper symmetric and upper multiplicative.

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f: X_1 \to X_2$ is called a *$\prec$-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f\big(B_1(x,\alpha)\big) \subseteq B_2\big(f(x),\beta\big).$$

A mapping $f: X_1 \to X_2$ is called a *$\succ$-mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that, for every $x \in X_1$,

$$B_2\big(f(x),\beta\big) \subseteq f\big(B_1(x,\alpha)\big).$$

A bijection $f: X_1 \to X_2$ is called an *isomorphism* between $\mathbb{B}_1$ and $\mathbb{B}_2$ if f is a $\prec$-mapping and f is a $\succ$-mapping.

Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be balleans with common support X. We say that $\mathbb{B}_1 \prec \mathbb{B}_2$ if the identity mapping $\mathrm{id}: X \to X$ is a $\prec$-mapping of $\mathbb{B}_1$ to $\mathbb{B}_2$. If $\mathbb{B}_1 \prec \mathbb{B}_2$ and $\mathbb{B}_2 \prec \mathbb{B}_1$ we say that $\mathbb{B}_1$, $\mathbb{B}_2$ are equivalent.

A property $\mathcal{P}$ of balleans is called a *ball property* if any ballean isomorphic to a ballean with property $\mathcal{P}$ also has that property $\mathcal{P}$. Now we define some basic ball properties.

Let $\mathbb{B} = (X, P, B)$ be a ballean, $x, y \in X$. We say that x, y are *connected* if there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. A subset $Y \subseteq X$ is called *connected* if any two elements from Y are connected. Note that connectedness is an equivalence relation on X, so X disintegrates into connected components. A ballean is called *connected* if its support is connected.

A subset $Y \subseteq X$ is called *bounded* if there exists $x \in X$, $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. We say that $\mathbb{B}$ is *bounded* if its support is bounded. Let $\mathbb{B}$ be connected, $x_0 \in X$, $Y \subseteq X$. Then Y is bounded if and only if there exists $\alpha \in P$ such that $Y \subseteq B(x_0, \alpha)$.

For an arbitrary ballean $\mathbb{B} = (X, P, B)$, we define a reordering $\leqslant$ on the set P by the rule: $\alpha \leqslant \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$.

A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leqslant \beta$. The *cofinality* cf $\mathbb{B}$ of $\mathbb{B}$ is the minimal cardinality of cofinal subsets of P.

Let (X, d) be a metric space, $\mathbb{R}^+ = \{\alpha \in \mathbb{R}\colon \alpha \geqslant 0\}$. Given any $x \in X$, $r \in \mathbb{R}^+$, we put

$$B_d(x, r) = \bigl\{y \in X\colon d(x, y) \leqslant r\bigr\}.$$

The ballean $\mathbb{B}(X, d) = (X, \mathbb{R}^+, B_d)$ is called a *metric* ballean. We say that a ballean $\mathbb{B}$ is metrizable if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, d)$ for some metric space (X, d). By [9], a ballean $\mathbb{B}$ is metrizable if and only if $\mathbb{B}$ is connected and cf $\mathbb{B} \leqslant \aleph_0$.

Formally, the notion of ballean is an asymptotic duplicate of the notion of uniform topological space. It is well known [3, Chapter 8] that every uniform topological space can be approximated by metric spaces. Now we describe the ballean analogue of such an approximation.

Let $\{\mathbb{B}_\lambda = (X_\lambda, P, B_\lambda)\colon \lambda \in I\}$ be a family of balleans with pairwise disjoint supports and common set of radiuses and let $X = \bigcup_{\lambda \in I} X_\lambda$. For every $x \in X$, $x \in X_\lambda$ and every $\alpha \in P$, we put $B(x, \alpha) = B_\lambda(x, \alpha)$. The balleans $\mathbb{B} = (X, P, B)$ is called a *disjoint union* of the family $\{\mathbb{B}_\lambda\colon \lambda \in I\}$. A ballean is called *pseudometrizable* if it is a disjoint union of metrizable balleans.

Let $\{\mathbb{B}_\lambda = (X, P_\lambda, B_\lambda)\colon \lambda \in I\}$ be a family of balleans with common support. Suppose that, for any $\lambda_1, \lambda_2 \in I$, there exists $\lambda \in I$ such that $\mathbb{B}_{\lambda_1} \prec \mathbb{B}_\lambda$, $\mathbb{B}_{\lambda_2} \prec \mathbb{B}_{\lambda_s}$. For every $\lambda \in I$, we choose a copy $P'_\lambda = f_\lambda(P_\lambda)$ such that the family $\{P'_\lambda\colon \lambda \in I\}$ if disjoint. Put $P = \bigcup_{\lambda \in I} P'_\lambda$. For any $x \in X$, $\beta \in P$, $\beta \in P_\lambda$, we put $B(x, \beta) = B_\lambda(x, f_\lambda^{-1}(\beta))$. The ballean $\mathbb{B} = (X, P, B)$ is called an *inductive limit* of the family $\{B_\lambda\colon \lambda \in I\}$.

By [10], every ballean is isomorphic to the inductive limit of some family of pseudometrizable balleans.

Now we describe a ballean analogue of normality. Let $\mathbb{B} = (X, P, B)$ be a ballean. We say that the subsets Y, Z of X are *asymptotically disjoint* (and write $Y \perp Z$) if, for every $\alpha \in P$, there exists a bounded subset $U_\alpha \subseteq X$ such that

$$B(Y \setminus U_\alpha, \alpha) \cap B(Z \setminus U_\alpha, \alpha) = \emptyset.$$

We say that Y, Z are *asymptotically separated* (and write $Y \amalg Z$) if, for every $\alpha \in P$, there exists a bounded subset $U_\alpha \subseteq X$ such that, for every $\beta \in P$,

$$B(Y \setminus U_\alpha, \alpha) \cap B(Z \setminus U_\beta, \beta) = \emptyset.$$

A ballean $\mathbb{B}$ is called *normal* if, for all subsets Y, Z of X, $Y \perp Z$ implies $Y \amalg Z$.

To formulate the balleans counterparts of Urysohn's lemma and the Tietze–Urysohn theorem we need the following definition.

Let $\mathbb{B} = (X, P, B)$ be a ballean and let $(Y, \mathcal{U})$ be a uniform topological space. A mapping $h : X \to Y$ is called *slowly oscillating* if, for every entourage $U \in \mathcal{U}$ and every $\alpha \in P$, there exists a bounded subset V of X such that, for every $x \in X \setminus V$,

$$h\big(B(x,\alpha)\big) \times h\big(B(x,\alpha)\big) \subseteq U.$$

If $Y = \mathbb{R}$ with the uniformity determined by standard metric, then $h : X \to \mathbb{R}$ is slowly oscillating if and only if, for every $\varepsilon > 0$ and every $\alpha \in P$, there exists a bounded subset V of X such that, for every $x \in X \setminus V$,

$$\operatorname{diam} h\big(B(x,\alpha)\big) < \varepsilon,$$

where $\operatorname{diam} A = \sup\{|a - b|:\ a, b \in A\}$.

Let $\mathbb{B} = (X, P, B)$ be a normal ballean and let Y_0, Y_1 be disjoint and asymptotically disjoint subset of X. By [11, Theorem 2.1], there exists a slowly oscillating function $h : X \to [0, 1]$ such that $h|_{Y_0} \equiv 0$, $h|_{Y_1} \equiv 1$.

By [11, Theorem 2.2], a ballean $\mathbb{B}$ is normal if and only if, for every subset $Y \subseteq X$ and every bounded slowly oscillating function $h : Y \to \mathbb{R}$, there exists a bounded slowly oscillating function $g : X \to \mathbb{R}$ such that $g|_Y = h$.

The notion of ball structures and balleans were motivated by combinatorics [1]. Similar notions were defined and investigated in asymptotic topology [2]. We describe the most general of them.

A set X is called a *coarse space* [8] if there is a distinguished collection $\mathcal{E}$ of subsets of product $X \times X$ called *entourages* such that:

- Any finite union of entourages is contained in an entourage.
- The union of all entourages is the entire space $X \times X$.
- The *inverse* of an entourage M

  $$M^{-1} = \big\{(y, x) \in X \times X:\ (x, y) \in M\big\}$$

  is contained in an entourage.
- The *composition* of entourages M_1 and M_2

  $$M_1 M_2 = \big\{(x, z) \in X \times X:\ (x, y) \in M_1,\ (y, z) \in M_2 \text{ for some } y \in X\big\}$$

  is contained in an entourage.

Every coarse space $(X, \mathcal{E})$ can be considered as the connected ballean $(X, \mathcal{E}, B)$, where $B(x, E) = \{y:\ (x, y) \in E\} \cup \{x\}$, $E \in \mathcal{E}$. On the other hand, every connected ballean (X, P, B) can be considered as the coarse space $(X, \mathcal{E})$, where $\mathcal{E} = \{\bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha):\ \alpha \in P\}$.

2. Coronas

Fix a ballean $\mathbb{B} = (X, P, B)$, endow X with the discrete topology and consider the Stone–Čech compactification βX of X. We take the points of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters. For every subset $A \subseteq X$, we put $\overline{A} = \{q \in \beta X\colon A \in q\}$. The topology of βX can be defined by stating that the family $\{\overline{A}\colon A \subseteq X\}$ is a base for the open sets. For every filter φ on X, the subset $\bar{\varphi} = \bigcap\{\overline{A}\colon A \in \varphi\}$ is closed in βX, and, for every nonempty closed subset $K \subseteq \beta X$, there exists a filter φ on X such that $K = \bar{\varphi}$. Let Y be a compact Hausdorff space. For every mapping $f : X \to Y$, denote by f^β the Stone–Čech extension of f onto βX.

Denote by $X^\sharp$ the set of all ultrafilters r on X such that every $R \in r$ is unbounded in $\mathbb{B}$, and put $X^\flat = \beta X \setminus X^\sharp$. Clearly, $X^\sharp$ is a closed subspace of βX.

Given any $r, q \in X^\sharp$, we say that r, q are *parallel* (and write $r \parallel q$) if there exists $\alpha \in P$ such that, for every $R \in r$, we have $B(R, \alpha) \in q$. By [11, Lemma 4.1], $\parallel$ is an equivalence on $X^\sharp$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\sharp \times X^\sharp$) equivalence on $X^\sharp$ such that $\parallel \subseteq \sim$. By [3, Theorem 3.2.11], the quotient $X^\sharp/\sim$ is compact Hausdorff space. It is called the *corona* of $\mathbb{B}$ and is denoted by $\nu(\mathbb{B})$. To clarify the virtual equivalence $\sim$, we use the following two observations.

- *If $r, q \in X^\sharp$ and $r \| q$, then, for every slowly oscillating function $h : X \to [0, 1]$, we have $h^\beta(r) = h^\beta(q)$.*

Indeed, pick $\alpha \in P$ such that, for every $R \in r$, we have $B(R, \alpha) \in q$. Let ε be an arbitrary positive real number. We put

$$R_\varepsilon = \left\{x \in X\colon \left|h(x) - h^\beta(r)\right| < \varepsilon\right\}$$

and note that $R_\varepsilon \in r$. Since h is slowly oscillating, there exists a bounded subset V of X such that, for every $x \in X \setminus V$, we have $\operatorname{diam} h(B(x, \alpha)) < \varepsilon$. Then $R_\varepsilon \setminus V \in r$, $B(R_\varepsilon \setminus V, \alpha) \in q$ and $|h(x) - h^\beta(r)| < 2\varepsilon$. It follows that $h^\beta(q) = h^\beta(r)$.

- *Let $\mathbb{B}$ be connected and let $h : X \to [0, 1]$ be a function such that $h^\beta(r) = h^\beta(q)$ for any two parallel ultrafilters r, q. Then h is slowly oscillating.*

Suppose the contrary. Since $\mathbb{B}$ is connected, the family $\Im$ of all bounded subsets of X is closed under finite unions, so $\Im$ is directed by inclusion. Choose $\alpha \in P$ and $\varepsilon > 0$ such that, for every $F \in \Im$, there exists $x(F) \in X \setminus F$ such that, $\operatorname{diam} h(B(x(F), \alpha)) > \varepsilon$. For every $F \in \Im$, we take $y(F) \in B(x(F), \alpha))$ such that $|h(x(F)) - h(y(F))| > \varepsilon$. Then we get two nets $\{x(F)\colon F \in \Im\}$ and $\{y(F)\colon F \in \Im\}$.

Endow $\Im$ with the discrete topology and fix an arbitrary ultrafilter $p \in \beta\Im$ such that $\{H \in \Im\colon F \subseteq H\} \in p$ for every $F \in \Im$. Let $f_1 : \Im \to X$, $f_2 : \Im \to X$ be the mappings defined by $f_1(F) = x(F)$, $f_2(F) = y(F)$. We put $r = f_1^\beta(p)$, $q = f_2^\beta(p)$. Then $r \parallel q$ but $|h^\beta(r) - h^\beta(q)| \geqslant \varepsilon$, a contradiction.

The following example, suggested by the referee, shows that the connectedness assumption cannot be omitted in the second observation.

Let $\mathbb{B}_1 = (X_1, P, B_1)$, $\mathbb{B}_2 = (X_2, P, B_2)$ be connected balleans such that $\mathbb{B}_1$ is bounded, $\mathbb{B}_2$ is unbounded, $|X_1| > 1$ and $X_1 \cap X_2 = \emptyset$. Let $\mathbb{B} = (X, P, B)$ be the disjoint union of $\mathbb{B}_1$ and $\mathbb{B}_2$. We take two points $x_1 \in X_1$, $x_2 \in X_2$ and define the function $h : X \to [0,1]$ by the rule: $h(x_1) = h(x_2) = 1$ and $h(x) = 0$ for every $x \in X \setminus \{x_1, x_2\}$. Since every bounded subset of X is contained either in X_1 or in X_2, h is not slowly oscillating. On the other hand, $X^\sharp = X_1^\sharp$. It follows that, for any $r, q \in X^\sharp$ (in particular, for any parallel ultrafilters $r, q \in X^\sharp$), we have $h^\beta(r) = 0$, $h^\beta(q) = 0$ so $h^\beta(r) = h^\beta(q)$.

Proposition 1. *Let $\mathbb{B} = (X, P, B)$ be a connected ballean, $q, r \in X^\sharp$. Then $q \sim r$ if and only if $h^\beta(q) = h^\beta(r)$ for every slowly oscillating function $h : X \to [0,1]$.*

Proof. Let us consider the closed equivalence $\sim_{[0,1]}$ on $X^\sharp$ defined by the rule: $r \sim_{[0,1]} q$ if and only if $h^\beta(r) = h^\beta(q)$ for every slowly oscillating function $h : X \to [0,1]$. We have to prove that $\sim = \sim_{[0,1]}$. By the above observations, $\parallel \subseteq \sim_{[0,1]}$, so $\sim \subseteq \sim_{[0,1]}$.

To show the reverse inclusion we put $Y = X^\flat \cup \nu(\mathbb{B})$ and define the topology on Y as follows. If $y \in X^\flat$ then a subset $U \subseteq Y$ is a neighborhood of y if and only if U contains a neighborhood of y in $X^\flat$ as a subspace of βX. Assume that $y \in \nu(X)$. Since y is a closed subset of βX, there exists a filter φ on X such that $y = \overline{\varphi}$. Then a subset $W \subseteq Y$ is a neighborhood of y if and only if there exist a neighborhood V of y in $\nu(\mathbb{B})$ and an element $F \in \varphi$ such that $V \cup \{z \in X^\flat\colon\ F \in z\} \subseteq W$. It is easy to verify that Y is a compact Hausdorff space and $X \subseteq X^\flat$ is a dense subset of Y.

Now suppose that $r, q \in X^\sharp$ and $[r] \neq [q]$ where $[r] \in \nu(\mathbb{B})$, $[q] \in \nu(\mathbb{B})$ are the corresponding $\sim$-equivalence classes. Then there exists a continuous function $f : Y \to [0,1]$ such that $f([r]) = 0$, $f([q]) = 1$. Put $h = f|_X$ and note that $h^\beta(t) = f([t])$ for every $t \in X^\sharp$. It follows that h is slowly oscillating and $h^\beta(r) \neq h^\beta(q)$. $\square$

If a ballean $\mathbb{B}$ is normal and connected we can go far in the clarification of the equivalence $\sim$. By [11, Lemma 4.2], $r \sim q$ if and only if, for any $R \in r$, $Q \in q$ there exists $\alpha \in P$ such that $B(R, \alpha) \cap B(Q, \alpha)$ is unbounded. Hence, r, q are non-equivalent if and only if there exist $R \in r$, $Q \in q$ such that $R \perp Q$. It should be remarked that the connectedness assumption is missing in the formulation of Lemma 4.2 of [11].

Let (X, d) be a metric space and let R, Q be unbounded subsets of X such that $R \perp Q$. By [11, Example 2.3], there exists a continuous slowly oscillating function $h : X \to [0,1]$ such that $h|_R \equiv 0$, $h|_Q \equiv 1$. In view of Proposition 1, for $r, q \in X^\sharp$, we have $r \sim q$ if and only if $h^\beta(r) = h^\beta(q)$ for every continuous slowly oscillating function $h : X \to [0,1]$.

A metric space (X, d) is called perfect if every ball $B_d(x, r)$ is compact. It is worth mentioning that the category of metric spaces (with the appropriate morphisms) is the main subject of large scale topology [2].

Now let (X, d) be a perfect metric space and let $S(X)$ be the set of all continuous slowly oscillating functions $h : X \to [0,1]$. Put $f = \prod_{h \in S(X)} h$ and note that f is an embedding of X into $[0,1]^{S(X)}$. Following [2, §6], we identify X with $f(X)$. The closure of $f(X)$ in $[0,1]^{S(X)}$ is called the *Higson's compactification* of X, and the remainder $\overline{f(X)} \setminus f(X)$ is called the *Higson's corona* of X. Denote by X_{disc} the set X with the discrete topology and put $\tilde{f} = f \circ \mathrm{id}$, where $\mathrm{id} : X_{\mathrm{disc}} \to X$ is the identity mapping. Then $\overline{f(X)}$ can be identified with the quotient $\beta X_{\mathrm{disc}}/{\approx}$, where $r \approx q$ if and only if $\tilde{f}^\beta(r) = \tilde{f}^\beta(q)$. Clearly, $\tilde{f}^\beta(X^\sharp) \subseteq$

$\overline{f(X)}\backslash f(X)$. Since (X,d) is perfect, we have $f^\beta(X^\flat) = f(X)$, so $\tilde{f}^\beta(X^\sharp) = \overline{f(X)}\backslash f(X)$. Let $r, q \in X^\sharp$. By above remark, $r \sim q$ if and only if $r \approx q$. Hence, the Higson's corona $\overline{f(X)}\backslash f(X)$ can be identified with the corona $\nu(\mathbb{B}(X,d))$ of the ballean $\mathbb{B}(X,d)$.

Now we define a more coarse corona of a ballean $\mathbb{B} = (X, P, B)$ using slowly oscillating functions taking values 0, 1.

We say that the ultrafilters $r, q \in X^\sharp$ are *binary equivalent* (and write $r \sim_{\{0,1\}} q$) if $h^\beta(r) = h^\beta(q)$ for every slowly oscillating function $h : X \to \{0,1\}$. Clearly, $\sim_{\{0,1\}}$ is a closed equivalence on $X^\sharp$. The quotient $X^\sharp/\sim_{\{0,1\}}$ is called the *binary corona* of $\mathbb{B}$, it is denoted by $\varepsilon(\mathbb{B})$ and the elements of $\varepsilon(\mathbb{B})$ are called the *ends* of $\mathbb{B}$.

A subset A of X is called *almost invariant* if $B(A,\alpha)\backslash A$ is bounded for every $\alpha \in P$. We use the following observations.

- *Every bounded subset is almost invariant.*
- *If $A \subseteq X$ is almost invariant, then $A \setminus X$ is almost invariant.*

We use the proof suggested by the referee. Let $\alpha \in p$ be given. Pick $\alpha' \in P$ such that, for all $x \in X$, $B(x,\alpha) \subseteq B^*(x,\alpha')$. Pick $\delta \in P$ and $z \in P$ such that $B(A,\alpha')\backslash A \subseteq B(z,\delta)$. Pick $\gamma \in P$ such that, for all $x \in X$, $B(B(x,\delta),\alpha) \subseteq B(x,\gamma)$. We claim that $B(X \setminus A,\alpha) \setminus (X \setminus A) \subseteq B(z,\gamma)$. To see this, let $y \in B(X \setminus A,\alpha) \setminus (X \setminus A)$ and note that $y \in A$. Pick $x \in X \setminus A$ such that $y \in B(x,\alpha)$. Then $y \in B^*(x,\alpha')$ so $x \in B(y,\alpha')$. Therefore $x \in B(A,\alpha')\backslash A$, so $B(x,\alpha) \subseteq B(z,\gamma)$ and hence $y \in B(z,\gamma)$.

- *If a function $h : X \to \{0,1\}$ is slowly oscillating then $h^{-1}(0)$ is almost invariant.*

Put $A = h^{-1}(0)$ and let $\alpha \in P$ be given. Choose a bounded subset V of X such that, for every $x \in X \setminus V$, $\operatorname{diam} h(B(x,\alpha)) < 1$. Then $B(A\backslash V,\alpha) \subseteq A$, $B(A,\alpha) \subseteq A \cup B(V,\alpha)$ so $B(A,\alpha) \setminus A$ is bounded and hence A is almost invariant.

- *Let $h : X \to \{0,1\}$ be a function such that $f^{-1}(0)$ is almost invariant. Then h is slowly oscillating.*

We put $A = h^{-1}(0)$ and assume that $\mathbb{B}$ is connected. Let $\alpha \in P$ be given. Since A and $X \setminus A$ are almost invariant, the subsets $V_1 = B(A,\alpha) \setminus A$, $V_2 = B(X \setminus A,\alpha) \setminus (X \setminus A)$ are bounded. By connectedness of $\mathbb{B}$, $V_1 \cup V_2$ is bounded. Put $U = B^*(V_1 \cup V_2,\alpha)$ and note that U is bounded. If $x \in A \setminus U$ then $B(x,\alpha) \cap V_1 = \emptyset$ so $B(x,\alpha) \subseteq A$ and $\operatorname{diam} h(B(x,\alpha)) = 0$. Analogously, if $x \in X \setminus A$ then $\operatorname{diam} h(x,\alpha) = 0$. Therefore h is slowly oscillating.

Now let $\mathbb{B}$ be the disjoint union of the family $\{\mathbb{B}_\alpha = (X_\alpha, P, B_\alpha)\colon \lambda \in I\}$ of connected balleans. Since every bounded subset of X is contained in some subset X_λ, there exists $\lambda_0 \in I$ such that, for every $\lambda \in I$, $\lambda \neq \lambda_0$, either $X_\lambda \subseteq A$ or $X_\lambda = X\backslash A$. By the above paragraph, h is slowly oscillating.

Proposition 2. *Let $\mathbb{B} = (X, P, B)$ be a ballean, $r, q \in X^\sharp$. Then r, q are binary equivalent if and only if, for every almost invariant subset $A \subseteq X$, $A \in r$ implies $A \in q$.*

Proof. Assume that r, q are binary equivalent, A be an almost invariant subset of X and $A \in r$. Take the slowly oscillating function $h : X \to \{0, 1\}$ defined by the rule $h|_A \equiv 0$, $h|_{X \setminus A} \equiv 1$. Since $h^\beta(r) = h^\beta(q)$ we get $A \in q$.

Assume that r, q are not binary equivalent and take a slowly oscillating function $h : X \to \{0, 1\}$ such that $h^\beta(r) \neq h^\beta(q)$. Let $h^\beta(r) = 0$, $h^\beta(q) = 1$. Then $h^{-1}(0)$ is almost invariant, $h^{-1}(0) \in r$ but $h^{-1}(0) \notin q$. $\square$

In view of Proposition 2 we can identify $\varepsilon(\mathbb{B})$ with the set E of all maximal filters in the family $\mathcal{A}$ of all unbounded almost invariant subsets of X endowed with the topology defined by the family $\{\{\varphi \in E\colon\ A \in \varphi\}\colon\ A \in \mathcal{A}\}$ as a base for the open sets. In particular, this identification shows that $\varepsilon(\mathbb{B})$ is zero-dimensional.

To motivate the end-terminology we consider a discrete group G. A subset $A \subseteq G$ is called almost invariant if $Ag \setminus A$ is finite for every $g \in G$. Denote by $E(G)$ the set of all maximal filters in the family of all infinite almost invariant subsets of G. Then $E(G)$ is the reminder of the Freudental–Hopf compactification of G and every element of $E(G)$ is called an end of G (for this approach to definitions of ends see [7]). In the next section we define the ballean $\mathbb{B}_r(G, \aleph_0)$ with the support G such that $\varepsilon(\mathbb{B}_r(G, \aleph_0)) = E(G)$. Thus, the binary corona of ballean can be considered as a generalization of the space of ends of group.

We conclude this section with the following example, showing that the connectedness assumption cannot be omitted in Proposition 1.

Let (X_n, d_n), $n = 1, 2, \ldots,$ be metric spaces such that $X_n = \{y_n, z_n\}$, $d_n(y_n, z_n) = n$ and, for $n \neq m$, $X_n \cap Y_m = \emptyset$. Let $\mathbb{B} = (X, \mathbb{R}^+, B)$ be the disjoint union of the family $\{\mathbb{B}(X_n, d_n)\colon\ n = 1, 2, \ldots\}$ of metric balleans. If $h : X \to [0, 1]$ is a slowly oscillating function, then there exists m such that $h(y_n) = h(z_n)$ for all $n \neq m$. It follows that $h^\beta(r) = h^\beta(q)$ for any two ultrafilters $r, q \in X^\sharp$.

On the other hand, let $r, q \in X^\sharp$ and $r \| q$. Pick $\alpha \in \mathbb{R}^+$ such that, for every $R \in r$, $B(R, \alpha) \in q$. Since $B(R, \alpha) \setminus R$ is finite, we have $R \in q$ and $r = q$. Hence $\| = \sim$ and $\nu(\mathbb{B}) = X^\sharp$.

3. Group balleans

Let G be an infinite group with identity e, κ be an infinite cardinal such that $\kappa \leqslant |G|$. Denote by $\Im(G, \kappa)$ the family $\{F \subseteq G\colon\ |F| < \kappa,\ e \in F\}$ and, for any $g \in G$, $F \in \Im(G, \kappa)$, put

$$B_l(g, F) = gF, \qquad B_r(g, F) = Fg.$$

Thus, we get two balleans

$$B_l(G, \kappa) = \bigl(G, \Im(G, \kappa), B_l\bigr), \qquad B_r(G, \kappa) = \bigl(G, \Im(G, \kappa), B_r\bigr).$$

Note that the mapping $x \mapsto x^{-1}$ is an isomorphism between $\mathbb{B}_l(G, \kappa)$ and $\mathbb{B}_r(G, \kappa)$.

Proposition 3. *For every infinite group G of regular cardinality, the cardinality of corona $\nu(\mathbb{B}_r(G, |G|))$ is $2^{2^{|G|}}$.*

Proof. Let $G = \{g_\alpha\colon \alpha < |G|\}$, $g_0 = e$. For every $\alpha < |G|$, we put $F_\alpha = \{g_\beta\colon \beta \leqslant \alpha\}$. Then we can construct inductively the subset $X = \{x_\alpha\colon \alpha < |G|\}$ of G such that $F_\alpha x_\alpha \cap F_\beta x_\beta = \emptyset$ for all $\alpha > \beta$. Since $|G|$ is regular, for every $F \in \Im(G, |G|)$, there exists $\alpha < |G|$ such that $F \subseteq F_\alpha$. Hence, there exists a subset V of cardinality $< |G|$ such that $Fx \cap Fy = \emptyset$ for any two distinct elements $x \in X \setminus V$, $y \in X \setminus V$. It follows that any two disjoint subsets of X of cardinality $|G|$ are asymptotically disjoint. Now consider the family $\mathcal{U}$ of all ultrafilters r on G such $X \in r$ and $|R| = |X|$ for every $R \in r$. Clearly, $|\mathcal{U}| = 2^{2^{|G|}}$. Take any two distinct ultrafilters $r, q \in \mathcal{U}$, choose $R \in r$, $Q \in q$ such that $R \cap Q = \emptyset$. Then $R \perp Q$. By [11, Proposition 1.1], the ballean $\mathbb{B}_r(G, |G|)$ is normal. Hence, there exists a slowly oscillating function $h : G \to [0, 1]$ such that $h^\beta(r) \neq h^\beta(q)$. It follows that r, q define distinct elements $[r]$, $[q]$ of $\nu(\mathbb{B}_r(G, |G|))$. $\square$

Proposition 4. *Let G be an Abelian group and let κ be an infinite regular cardinal such that $\kappa < |G|$.Then the corona $\nu(\mathbb{B}_r(G, \kappa))$ is a singleton.*

Proof. It suffices to show that every slowly oscillating function $h : G \to [0, 1]$ is constant at infinity. More precisely, there exists $c \in [0, 1]$ such that, for every $\varepsilon > 0$, there exists a subset V of G such that $|V| < \kappa$ and $|h(x) - c| < \varepsilon$ for every $x \in G \setminus V$.

We prove the following auxiliary statement. Let X be a subset of G such that $|X| = \kappa$. Then there exists a subgroup H of G such that $X \subseteq H$, $|H| = \kappa$ and the restriction $h|_{Hg}$ is constant for every $g \in G \setminus H$. Let $X = \{x_\alpha\colon \alpha < \kappa\}$. Put $H_0 = \{x_0\}$, $F_0 = \emptyset$. Suppose that, for some ordinal $\beta < \kappa$, we have chosen the subsets $\{H_\alpha\colon \alpha < \beta\}$ and $\{F_\alpha\colon \alpha < \beta\}$ of cardinality $< \kappa$. If β is a limit ordinal, we put $H_\beta = \{x_\beta\} \cup \bigcup_{\alpha<\beta} H_\alpha$, $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$. Since κ is regular we have $|H_\beta| < \kappa$, $|F_\beta| < \kappa$. If β is a non-limit ordinal, we choose the limit ordinal β_0 and the natural number n such that $\beta = \beta_0 + n$. Put $W = \{x_\beta\} \cup H_{\beta_0+n-1} \cup F_{\beta_0+n-1}$. Clearly, $|W| < \kappa$. Denote by H_β the set of all elements of G which can be written as the group words of length $\leqslant n$ in the alphabet W. Since h is slowly oscillating and $|H_\beta| < \kappa$, there exists a subset F_β of G such that $|F_\beta| < \kappa$, $F_{\beta_0+n-1} \subseteq F$ and, for every $x \in G \setminus F_\beta$,

$$\operatorname{diam} h(H_\beta x) < \frac{1}{n}.$$

After κ steps we put $H = \bigcup_{\alpha<\kappa} H_\alpha$. By the construction we conclude that H is a subgroup of G, $|X| \subseteq H$, $|H| = \kappa$ and $F_\alpha \subseteq H$ for every $\alpha < \kappa$. Now let $y \in H$, $y' \in H$ and $g \in G \setminus H$. Take an arbitrary $\varepsilon > 0$ and choose the limit ordinal $\beta_0 < \kappa$ and the natural number n such that $\frac{1}{n} < \varepsilon$ and $y \in H_{\beta_0+n-1}$, $y' \in H_{\beta_0+n-1}$. Put $\beta = \beta_0 + n$. Since $y \in H_\beta$, $y' \in H_\beta$ and $g \notin F_\beta$, we have

$$\left|h(yg) - h(y'g)\right| < \frac{1}{n} < \varepsilon.$$

It follows that $h|_{Hg}$ is constant.

At last, suppose that h is not constant at infinity. Then there exist $\varepsilon > 0$ and injective κ-sequences $\langle y_\alpha\rangle_{\alpha<\kappa}$ and $\langle z_\alpha\rangle_{\alpha<\kappa}$ such that $|h(y_\alpha) - h(z_\alpha)| > \varepsilon$ for every $\alpha < \kappa$. Put $X = \{y_\alpha, z_\alpha\colon \alpha < \kappa\}$. By the auxiliary statement, there exists a subgroup H of G such that $X \subseteq H$, $|H| = \kappa$ and the restriction $h|_{Hg}$ is constant for every $g \in G \setminus H$. Fix an arbitrary

$g_0 \in G \setminus H$ and put $Y = \{e, g_0\}$. Since h is slowly oscillating, there exists a subset U of G such that $|U| < \kappa$ and, for every $x \in G \setminus U$,

$$\operatorname{diam} h(Yx) < \frac{\varepsilon}{2}.$$

Choose $\alpha < \kappa$ such that $y_\alpha \notin U$, $z_\alpha \notin U$. Then we have

$$\left|h(y_\alpha) - h(z_\alpha)\right| \leqslant \left|h(y_\alpha) - h(g_0 y_\alpha)\right| + \left|h(g_0 y_\alpha) - h(g_0 z_\alpha)\right| + \left|h(z_\alpha) - h(g_0 z_\alpha)\right|.$$

Since G is Abelian, $g_0 y_\alpha = y_\alpha g_0$, $g_0 z_\alpha = z_\alpha g_0$. Since $y_\alpha \in H$, $z_\alpha \in H$ and $g_0 \in G \setminus H$, we have $h(y_\alpha g_0) = h(z_\alpha g_0)$. Hence, $|h(y_\alpha) - h(z_\alpha)| < \varepsilon$ and we get a contradiction to the choice of $\langle y_\alpha \rangle_{\alpha<\kappa}$, $\langle z_\alpha \rangle_{\alpha<\kappa}$. □

The above proposition remains true (with only slight modification of the proof) under some weaker assumptions instead of commutativity of G. In particular, it is true if either the center $\{x \in G\colon xg = gx \text{ for every } g \in G\}$ of G is of cardinality $\geqslant \kappa$ or every subgroup of G of cardinality κ is contained in some invariant subgroup of cardinality κ. On the other hand, every free group of rank > 1 has infinitely many ends. It follows that $\nu(\mathbb{B}_r(F, \aleph_0))$ is infinite so Proposition 4 is not true for F.

4. Applications to βG

Let G be a discrete group, βG be the Stone–Čech compactification of G, $G^* = \beta G \setminus G$. Using the universal property of the Stone–Čech compactification, the group multiplication on G can be extended to βG in such a way that, for every $r \in \beta G$, the right shift $x \mapsto xr$ is continuous, and, for every $g \in G$, the left shift $x \mapsto gx$ is continuous. Formally, the product rq of the ultrafilters $r, q \in \beta G$ is defined by the rule: given any subset A of G,

$$A \in rq \iff \left\{g \in G\colon g^{-1}A \in q\right\} \in r.$$

For more information about the compact right topological semigroup βG and its combinatorial applications see [5].

In what follows we suppose that G is infinite and κ is an infinite cardinal such that $\kappa \leqslant |G|$. We put $G^{(\kappa)} = \{r \in \beta G\colon |R| \geqslant \kappa \text{ for every } R \in r\}$ and note that the subsemigroup $G^{(\kappa)}$ of βG coincides with the set $G^\sharp$ of all unbounded ultrafilters in the ballean $\mathbb{B}_r(G, \kappa)$. If $\kappa = \aleph_0$ then $G^{(\kappa)} = G^*$. If $\kappa = |G|$ we use the notation G^{uni} instead of $G^{(\kappa)}$.

If $r \in G^{(\kappa)}$, $g \in G$ then the ultrafilters r, gr are parallel in the ballean $\mathbb{B}_l(G, \kappa)$. If $\kappa = \aleph_0$ then $r, q \in G^*$ are parallel in $\mathbb{B}_r(G, \kappa)$ if and only if $q = gr$ for some element $g \in G$. It follows that every element of $\nu(\mathbb{B}_r(G, \kappa))$ is a closed left ideal of βG.

A semigroup S is called right-zero if $xy = y$ for all $x, y \in S$. In what follows we assume that $\nu(\mathbb{B}_r(G, \kappa))$ is endowed with the structure of a right-zero semigroup.

The above observation shows that the factor-mapping of $G^{(\kappa)}$ to $\nu(\mathbb{B}_r(G, \kappa))$ is a homomorphism. The next proposition states that $\nu(\mathbb{B}_r(G, \aleph_0))$ is the maximal continuous right-zero homomorphic image of G^*.

Proposition 5. *If a compact right-zero semigroup S is a continuous homomorphic image of G^*, then S is a continuous homomorphic image of $\nu(\mathbb{B}_r(G, \aleph_0))$.*

Proof. We use the following observation. If the semigroup G^* is partitioned into closed left ideals of G^* then every member I of the partition is a left ideal of βG. It suffices to show that $xI \subseteq I$ for every $x \in G$. Suppose the contrary and choose $g \in G$, $r \in I$ such that $gr \notin I$. Choose the element J of the partition such that $gr \in J$. Since J is a left ideal of G^* we have $r(gr) \in J$ and get a contradiction to $(rg)r \in I$.

Now let f be a continuous homomorphism of G^* onto S. By definition of $\nu(\mathbb{B}_r(G, \aleph_0))$, it suffices to show that, for every $r, q \in G^*$, $f(r) = f(q)$. Since S is a right-zero semigroup, every element of the partition $\{f^{-1}(s)\colon s \in S\}$ of G^* is a left ideal of G^*. Hence, $f^{-1}(s)$ is a left ideal of βG for every $s \in S$. Since $r \parallel q$, there exists $g \in G$ such that $q = gr$, so q, r belong to the same member of the partition and $f(q) = f(r)$. □

The same argument shows that a function $h: G \to [0, 1]$ is slowly oscillating in the ballean $\mathbb{B}_r(G, \aleph_0)$ if and only if the restriction h^* of h^β to G^* is a homomorphism of G^* to the right-zero semigroup $[0, 1]$.

Proposition 6. *If a compact zero-dimensional right-zero semigroup S is a continuous homomorphic image of G^*, then S is a continuous homomorphic image of the binary corona $\varepsilon(\mathbb{B}_r(G, \aleph_0))$.*

Proof. Let f be a continuous homomorphism of G^* onto S. It suffices to show that $f(r) = f(q)$ for any two binary equivalent ultrafilters $r, q \in G^*$. Suppose the contrary and choose the binary equivalent ultrafilters $r, q \in G^*$ such that $f(r) \neq f(q)$. Since S is zero-dimensional, there exists a continuous mapping $f': S \to \{0, 1\}$ such that $f'(f(r)) \neq f'(f(q))$. We put $\varphi = f'f$ and note that φ is a continuous homomorphism of G^* to the right-zero semigroup $\{0, 1\}$. Then we take a mapping $h: G \to \{0, 1\}$ such that the restriction h^* of h^β to G^* coincides with φ. Then h is slowly oscillating in $\mathbb{B}_r(G, \aleph_0)$ and $h^\beta(r) \neq h^\beta(q)$. Thus we get a contradiction to the assumption that r, q are binary equivalent. □

We conclude the paper with some illustrations of our considerations.

- By Propositions 3 and 5, for every infinite group G of regular cardinality, there exists a compact right-zero semigroup of cardinality $2^{2^{|G|}}$ which is a continuous homomorphic image of G^{uni}. On the other hand, if G is an uncountable Abelian group, by Propositions 4 and 5, the only continuous right-zero homomorphic image of G^* is a singleton.
- A group G is called *locally finite* if every finite subset of G is contained in some finite subgroup.
  By [11, Lemma 4.3], if G is either uncountable or a countable locally finite group, then $\nu(\mathbb{B}_r(G, |G|)) = \varepsilon(\mathbb{B}_r(G, |G|))$. By [6], every uncountable locally finite group G has one end so $\varepsilon(\mathbb{B}_r(G, \aleph_0))$ is a singleton in this case.
- By the Freudental–Hopf theorem (see [4, Chapter 2]), every infinite group G has one, two or infinitely many ends. In view of Proposition 6, this theorem describes all possible finite right-zero continuous homomorphic image of G^*. This is a step to the following general problem.

Given an infinite group G, determine all finite semigroups which are continuous homomorphic images of G^*.

The first step is this direction was done in [12]. A finite group F is a continuous homomorphic image of G^* if and only if F is a homomorphic image of G.

Acknowledgement

I would like to thank the referee for numerous corrections in the previous version of the paper.

References

[1] T. Banakh, I. Protasov, Ball Structures and Colorings of Graphs and Groups, Mathematical Studies Monograph Series, vol. 11, VNTL, Lviv, 2003.
[2] A. Dranishnikov, Asymptotic topology, Russian Math. Surveys 55 (2000) 1085–1129; Russian original in: Uspekhi Mat. Nauk 55 (6) (2000) 71–116.
[3] R. Engelking, General Topology, PWN, Warszawa, 1985.
[4] P. Harpe, Topics in Geometrical Groups Theory, University Chicago Press, Chicago, 2000.
[5] N. Hindman, D. Strauss, Algebra in the Stone–Cech Compactification—Theory and Applications, de Gruyter, Berlin, 1998.
[6] D.F. Holt, Uncountable locally finite groups have one end, Bull. London Math. Soc. 13 (1981) 557–560.
[7] C.H. Houghton, Ends of groups and the associated first cohomology groups, J. London Math. Soc. 6 (1972) 81–92.
[8] P.D. Mitchener, Coarse homology theories, Algebraic Geom. Topology 1 (2001) 271–297.
[9] I. Protasov, Metrizable ball structures, Algebra Discrete Math. 1 (2002) 129–141.
[10] I. Protasov, Uniform ball structures, Algebra Discrete Math. 1 (2003) 93–102.
[11] I. Protasov, Normal ball structures, Mat. Stud. 20 (2003) 3–16.
[12] I. Protasov, J. Pym, D. Strauss, A lemma on extending functions into F-spaces and homomorphisms between Stone–Čech remainders, Topology Appl. 105 (2000) 209–229.