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A Noncooperative Theory of Coalitional Bargaining

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We explore a sequential offers model of n-person coalitional bargaining with transferable utility and with time discounting. Our focus is on the efficiency properties of stationary equilibria of strictly superadditive games, when the discount factor δ is sufficiently large; we do, however, consider examples of other games where subgame perfectness alone is employed.

It is shown that delay and the formation of inefficient subcoalitions can occur in equilibrium, the latter for some or all orders of proposer. However, efficient stationary equilibrium payoffs converge to a point in the core, as $\delta \to 1$. Strict convexity is a sufficient condition for there to exist an efficient stationary equilibrium payoff vector for sufficiently high δ. This vector converges as $\delta \to 1$ to the egalitarian allocation of Dutta and Ray (1989).

1. INTRODUCTION

We explore an extensive-form approach to modelling n-person coalitional bargaining situations. Our formulation assumes transferable utility and perfect information, and represents a natural generalization of Rubinstein (1982). Like his and other alternating-offer models, our analysis implies a commitment to the offer by the proposer until all respondents have accepted or rejected the offer. However, we preclude *all* other forward commitments, whether by individuals or by coalitions.[1]

We limit our analysis to *stationary* perfect equilibria of the coalitional bargaining game. The main justification for this restriction is a result proved in Chatterjee, Dutta, Ray and Sengupta (1990) which showed that *all* efficient and individually rational allocations can be supported as perfect equilibria for high enough discount factors, if history-dependent strategies are permitted. In contrast, the predictions of the model are drastically sharpened by invoking stationarity. It can also be argued that stationary equilibria constitute a computationally simple class, and that, if "learning" or "teaching" is important in a given economic context, the way to capture these phenomena is through enriching the model, rather than through adjusting the solution concept. Moreover, like Selten (1981), Gul (1989) and others, we find stationary equilibria analytically tractable.

1. Gul (1989) analyses an extensive form in which such commitments are implicitly possible. Selten (1981) and Binmore (1985) contain alternative formulations of characteristic function bargaining. See also Harsanyi (1963), Bennett (1988) and Selten and Wooders (1990).

(We realise, of course, that it is dfficult to put forward a logically complete case for excluding non-stationary equilibria in general.)

Our main focus is on the *efficiency* of bargaining outcomes, and on the limiting properties of such outcomes as discounting vanishes. In strictly superadditive games, efficiency is equivalent to the twin requirements of (a) no delay, and (b) formation of the grand coalition. However, we present a set of examples (Examples 1-3) showing that stationary equilibria may violate either of the two requirements of efficiency mentioned above. These examples are remarkable given the environment of perfect information.[2] They are a consequence of the intrinsic coalitional structure of our model.

Section 3 of the paper presents a characterization of stationary equilibria without delay (Proposition 2). We also provide a sufficient condition that guarantees the existence of stationary equilibria without delay. (Proposition 3).

Section 4 then examines the conditions under which a stationary equilibrium will be efficient. Our main interest is in efficiency "in the limit" as discounting vanishes.

Our bargaining game is defined relative to a *protocol*, that is, an order of proposers and respondents defined for each coalition. We first analyze the issue of efficiency for *all* protocols. We show (Proposition 4) that such a feature is equivalent to the requirement that the game (N, v) satisfies $v(S)/|S| \leqq v(N)/|N|$ for coalitions S. It is easy to see that such games do not have an intrinsic coalitional nature since the grand coalition has higher average worth than every subcoalition. In other words, games where subcoalitions "matter" must exhibit inefficiency for *some* protocol. In deed, the requirement of efficiency for even *one* order of proposers is quite demanding, an observation implied by Proposition 5. It shows that any limit of efficient stationary equilibrium payoff vectors in a strictly superadditive game must yield a payoff vector which is in the *core* of the characteristic function game. This of course immediately implies that a stationary equilibrium can *never* be efficient for high enough discount factors if the characteristic function game has an empty core. Moreover, even if the game has a non-empty core, we show that a game might exhibit inefficiency for *every* protocol (Example 2). Proposition 6 states a sufficient condition for such a phenomenon to occur in general.

In Section 5, we turn to sufficient conditions ensuring efficiency for *some* protocol. Proposition 7 shows that if a game is *strictly convex*, then for high enough discount factors, every stationary equilibrium involves no delay, and there is some protocol under which the stationary equilibrium is efficient.

Finally, we show (Proposition 8) that the (unique) limit payoff vector Lorenz-dominates every other core allocation.[3] We view such a limit result as an additional contribution to the well-known programme of "achieving" cooperative game-theoretic concepts via the play of a strategic game.

2. THE EXTENSIVE FORM

Let (N, v) be a characteristic function game with $|N| = n$. $\mathcal{S}$ is the set of all coalitions. A *protocol* is an ordering of players, one ordering for each coalition. A *proposal* is a pair (S, y) with $S \in \mathcal{S}$ and $\sum_{i \in S} y_i = v(S)$.

2. Of course, in incomplete information models, one can have efficiency failure (see Chatterjee and W. Samuelson (1983), Myerson and Satterthwaite (1988), Admati and Perry (1987)). In complete information models that admit multiple subgame perfect equilibrium paths, delay can be generated by using the multiplicity of paths to "punish" deviators. In our paper, however, it may arise even in an essentially unique equilibrium (given the order of proposers), since the reasons for its occurrence have to do with the coalitional structure and not with the multiplicity of equilibria.

3. In fact, this limit allocation is the *egalitarian allocation* of Dutta and Ray (1989).

The game proceeds as follows. The first player in N under the protocol makes a proposal (S, y). Players in S respond sequentially according to the protocol by saying "Yes" or "No". The first "rejector" becomes the new proposer. If no member of S rejects the proposal, then they leave the game (with allocation y) and the game continues with no lapse in time, with player set $N-S$.

We assume that the formulation of a counterproposal takes one unit of time. Let $\delta<1$ be the common discount factor applied to this time period.

Each proposal, all acceptances of the proposal and a rejection (if any) count as separate *stages* of the game. A *k-history* h_k, for $k \geqq 2$, is a complete listing of the previous $k-1$ stages. A (pure) strategy for player i assigns proposals at every h_k at which i is a proposer and responses to ongoing proposals at every h_k at which i is a responder. Noting that player i's payoff is $\delta^{t-1}x$ if i receives x in period t, we have a full game-theoretic formulation. One may define in the standard way the notion of a (subgame) perfect (Nash) equilibrium for this game.

A strategy for player i is *stationary* if the decisions assigned to player i are independent of all features of h_k except possibly the set of players remaining in the game and the ongoing proposal (if any). A *stationary equilibrium* is a perfect equilibrium with each player employing a stationary strategy.

Throughout this paper, we consider only stationary equilibria. The main reason for this is the following proposition, which is a generalization of a result due to Herrero (1985) and Shaked (1986).

Say that a bargaining game (N, v) is *strictly superadditive* if for any S, $T \in \mathscr{S}$ with $S \cap T = \emptyset$, $v(S \cup T) > v(S) + v(T)$.

Proposition 0. *Let* (N, v) *be a strictly superadditive bargaining game with* $|N| \geqq 3$. *Then there exists* $\delta^* \in (0, 1)$ *such that for each allocation* $\mathbf{x}$ *with* $x_i \geqq v(i)$ *for all* i *and each* $\delta \in (\delta^*, 1)$, *there exists a perfect equilibrium with the outcome* $\mathbf{x}$.

Proof. See Chatterjee, Dutta, Ray and Sengupta (1990). ||

This proposition reflects a problem common to much of dynamic game theory. With history-dependent strategies, one can support a plethora of outcomes. It is difficult, moreover, to make a logically convincing argument to rule out such equilibria.

3. STATIONARY EQUILIBRIUM AND DELAY

Fix any stationary equilibrium. Consider a history such that i has to propose at player set S. By stationarity, i, and everyone else who may move after him on the equilibrium of this subgame, behave the same way, irrespective of the history of arrival at player set S. Consequently, we may write $x_i(S, \delta)$ as the subgame equilibrium payoff to i when i is the proposer with S being the player set remaining. For each i, define $y_i(S, \delta) \equiv \delta x_i(S, \delta)$. Then it should be clear that if some individual j were to receive a proposal $(T, \mathbf{z})$ at player set S, with the property that $z_k \geqq y_k(S, \delta)$ for all $k \in T$ that are yet to respond to this proposal (including j, of course), the equilibrium response of j must be to accept. On the other hand, if the above inequality holds for all k who are to respond after j, but *not* for j, then his response will be to reject the proposal.[4] We will therefore refer to $y(S, \delta) \equiv (y_i(S, \delta))$ as the *equilibrium response vector* at the player set S.

We start with a lemma that holds for any equilibrium response vector.

4. More than this cannot be asserted at this stage. In particular, it is *not* true that j will accept any proposal that yields him more than $y_j(S, \delta)$. We will return to this issue below.

Lemma 1. *Fix a stationary equilibrium and a history that ends with the player set S. Let $y(S, \delta)$ be the equilibrium response vector. Then, for all $i \in S$, we have*

$$y_i(S, \delta) \geqq \delta \max_{i \in T \subseteq S} [v(T) - \textstyle\sum_{j \in T-i} y_j(S, \delta)]. \tag{1}$$

with the property that (1) holds with equality for all i whose equilibrium strategy as a proposer is to make an acceptable proposal at the player set S.

Proof. Using the discussion above, it is clear that if i has to propose, he can guarantee acceptance by choosing any coalition T such that $i \in T$ and by offering $y_j(S, \delta)$ to every $j \neq i$ in T. This implies that $x_i(S, \delta) \geqq \max_{i \in T \subseteq S} [v(T) - \sum_{j \in T-i} y_j(S, \delta)]$, and using the definition of $y_i(S, \delta)$, we obtain (1).

If (1) holds with *strict* inequality, this means that i has made a proposal (T, z) such that $z_j < y_j(S, \delta)$ for some $j \in S$. Consider the protocol restricted to T and look at the last player in the order for which this inequality holds. This player must reject the proposal, by the discussion above. Consequently, i must make an equilibrium proposal that is unacceptable, and this completes the proof of the lemma. ||

From Lemma 1, it follows that at the player set N, if for some i, $y_i(N, \delta) > \delta[v(T) - \sum_{j \in T-i} y_j(T, \delta)]$ for all $T \subseteq N$, then player i must make an unacceptable offer causing delay along the equilibrium path. The following example demonstrates that there are bargaining games where it pays a player to make such an unacceptable offer in equilibrium.

Example 1 (*Delay*). This example is due to Bennett and van Damme (1988). $N = \{1, 2, 3, 4\}$, $v(\{1, j\}) = 50, j = 2, 3, 4$; $v(\{i, j\}) = 100$, $i, j = 2, 3, 4$ and $v(S) = 0$ for all other $S \in \mathcal{S}$.

We claim that for some player starting the game, this example will yield *equilibrium* delay for all discount factors sufficiently close to unity. We prove this by contradiction. Suppose not. Then for all i, (1) holds with equality. Imposing this, we see that $y_i(N, \delta) = \delta 100/(1+\delta)$ for $i \neq 1$ and $y_1(N, \delta) = \delta[50 - (100\delta/(1+\delta)]$.

So if player 1 starts, he gets $y_1(N, \delta)/\delta$, which dwindles to zero as $\delta \to 1$.

Consider, now, that 1 deviates by picking the coalition {1, 2} and offering 0 to player 2. 2 will certainly reject the offer. For in the next period, 2 can obtain $y_2(N, \delta)/\delta$ by proposing, say, the coalition {2, 3} and offering 3 $y_3(N, \delta)$. This coalition then leaves the game.

Player 4 thus has no option but to share a payoff of 50 with Player 1, whose payoff now approaches 25 as δ goes to 1. So for large δ, player 1 has a worthwhile deviation, a contradiction.[5]

We will be interested in bargaining games for which stationary equilibria do not exhibit delay. This is our task in the next section.

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5. One can check that for this particular example, delay occurs in *any* perfect equilibrium (not necessarily stationary) for a range of discount factors, whenever 1 starts the game.

3.1. *No-delay stationary equilibrium*

An interesting class of stationary equilibria that will play an important role in our analysis has the property that after every history, the player who has to make a proposal makes an acceptable proposal. We will call such an equilibrium a *no-delay stationary equilibrium.*[6]

By Lemma 1, characterizing a no-delay stationary equilibrium is then equivalent to obtaining a solution to the following class of simultaneous equations. For any coalition $S \in \mathcal{S}$ and discount factor $\delta \in (0, 1)$, let $\boldsymbol{m}(S, \delta) = (m_i(S, \delta))_{i \in S}$ be the solution to

$$y_i(S, \delta) = \delta \max_{T \subseteq S} v(T) - \sum_{j \in T-i} y_j(S, \delta)]. \tag{2}$$

One can prove the following.

Proposition 1. *For each $S \in \mathcal{S}$ and $\delta \in (0, 1)$, $\boldsymbol{m}(S, \delta)$ exists and is unique.*

Proof. We first establish existence. Let $B \equiv [0, \max_{S \in \mathcal{S}} v(S)]^n$. Define $\phi : B \to B$ by $\phi_i(\boldsymbol{m}) \equiv \delta \max_{T \subseteq S} \{v(T) - \sum_{j \in T-i} m_j\}$, for $\boldsymbol{m} \in B$.

Note that $\phi(\boldsymbol{m}) \in B$. By the maximum theorem, ϕ is continuous. By Brouwer's fixed point theorem, the result follows.

To prove uniqueness, we will use the following lemma.

Lemma 2. *For any player set S, consider a vector $\boldsymbol{y}(S, \delta)$ (not necessarily an equilibrium response vector) such that for all $j \in S$, (1) holds. Moreover, for some $i \in S$, suppose that (1) holds with equality. Then for any T that attains the maximum in (1) and for all $j \in T$, we have $y_j(S, \delta) \geqq y_i(S, \delta)$.*

Proof. For player i, we have

$$y_i(S, \delta) = \delta[v(T) - \sum_{j \in T-i} y_j(S, \delta)] \tag{3}$$

and for $j \in T - \{i\}$, we have by supposition,

$$y_j(S, \delta) \geqq \delta[v(T) - \sum_{k \in T-j} y_k(S, \delta)]. \tag{4}$$

Adding $-\delta y_j(S, \delta)$ on both sides of (4) and using (3), one has

$$(1-\delta) y_j(S, \delta) \geqq \delta[v(T) - \sum_{k \in T-j} y_k(S, \delta)] - \delta y_j(S, \delta) = (1-\delta) y_i(S, \delta)$$

Since $\delta \in (0, 1)$, the result follows. ||

We now return to the main proof. Suppose, on the contrary, that there are two distinct solutions $\boldsymbol{m}$ and $\boldsymbol{m}'$ to (1). Let $K = \{i \in S \backslash m_i \neq m_i'\}$ and w.l.o.g choose $k \in K$ with m_k such that

$$m_k = \max\{z \backslash z = m_i \text{ or } m_i' \text{ for } i \in K\} \tag{5}$$

Of course, $m_k > m_k'$. Let $T \subseteq S$ satisfy

$$m_k = \delta[v(T) - \sum_{j \in T-k} m_j] \tag{6}$$

Then from the definition of m_k',

$$m_k' \geqq \delta[v(T) - \sum_{j \in T-k} m_j'] \tag{7}$$

6. One could use a weaker definition: that there must be no delay along the equilibrium path alone, but general results regarding such equilibria appear to be considerably more difficult to obtain.

Now by applying Lemma 2, we have for all $j \in T-k$, $m_j \geqq m_k$. So, if $m'_j > m_j$ for any such j, the definition of k in (5) is contradicted. Thus for all $j \in T-k$, we have $m'_j \leqq m_j$. Combining (6) and (7), we therefore obtain $m'_k \geqq m_k$, contradicting (5). ||

Let Σ^i be the collection of all stationary strategies for Player i, of the following kind:

(A) If i proposes after a history with player set S, he chooses a coalition T where $m_i(S, \delta)$ is attained and proposes the allocation

$$\left\{\frac{m_i(S, \delta)}{\delta}, (m_j(S, \delta))_{j \in T-i}\right\}$$

for the coalition.[7]

(B) If i responds to some ongoing proposal (T, z) after a history with set of players $S \in \mathcal{S}$, he accepts the proposal if $z_j \geqq m_j(S, \delta)$ for all $j \in T$ who are yet to respond (including i), and rejects it otherwise.

Let $\Sigma \equiv \prod_{i \in N} \Sigma^i$.

It is easy to see that if any strategy vector $\sigma \in \Sigma$ is an equilibrium, then there is no delay in equilibrium nor in any subgame, and the initial proposer gets $m_i(N, \delta)/\delta$. Indeed, by Lemma 1 and Proposition 1 (the uniqueness of $\boldsymbol{m}(S, \delta)$), the converse is also true. This is easily seen for part (A), using the results just cited.

To see part (B), we will use part (A). Assume that for some proposal (T, z), i is the last respondent. Then clearly (B) is true.[8] Now proceed by induction. Fix an integer $k \geqq 0$. Suppose that (B) is true for all proposals (T, z) and respondents j with m more respondents to follow, where $0 \leqq m \leqq k$. Now consider a proposal (T, z) and a respondent i with $k+1$ respondents to follow. If $z_j \geqq m_j(S, \delta)$ for all j who are yet to respond (including i), it follows easily from the induction hypothesis and Part (A) that i should accept. If not, then either $z_i < m_i(S, \delta)$ or $z_j < m_j(S, \delta)$ for some j to follow i. In the former case, i is better off by rejecting because he can get $m_i(S, \delta)/\delta$ in the next period. In the latter case, too, he is better off by rejecting! For if he accepts, then a later respondent will reject and offer i $m_i(S, \delta)$. If i seizes the initative by rejecting the proposal, he can get $m_i(S, \delta)/\delta$, which is larger.

These observations are collected together in

Proposition 2. *Let σ be a stationary equilbrium. Then σ is a no-delay equilibrium if and only if $\sigma \in \Sigma$.*

We have already seen in Example 1 that $\sigma \in \Sigma$ need not be an equilibrium. We now provide a sufficient condition under which the set of stationary equilibria is exactly Σ.

Condition M. For all $S, T \in \mathcal{S}$ and $\delta \in (0, 1)$, if $T \subset S$, then $m_i(S, \delta) \geqq m_i(T, \delta)$ for $i \in T$.

The reader should observe that (M) is quite independent of superadditivity conditions. It neither implies nor is implied by superadditivity.

7. Σ_i contains more than one element if and only if for some player set S there is more than one $T \subseteq S$ that solves the maximization problem in (2).

8. For the purpose of ensuring an equilibrium, equality must imply acceptance.

Proposition 3. *Suppose that condition M holds for the bargaining game (N, v). Then the set of stationary equilibria is exactly Σ. That is, no stationary equilibrium involves delay in equilibrium in any subgame.*

Proof. We first check that any strategy $\sigma \in \Sigma$ is indeed an equilibrium. Pick $i \in N$ and let all players in $j \in N-i$ use the strategies prescribed by Σ^j. Consider a subgame with player set S with i as the proposer. Note that by playing according to Σ^i, player i can guarantee himself a payoff of $m_i(S, \delta)/\delta$. If he makes an unacceptable offer today, he will receive a present value of $m_i(T, \delta)$, for some $T \subseteq S$. Since $\delta < 1$, by Condition (M), this payoff is strictly less. Thus i will make an acceptable offer, which, given the strategies of the other players, must agree with Σ^i. To check responses, proceed exactly as in the discussion preceding Proposition 2.

Next, we show that every stationary equilibrium belongs to Σ. Consider a stationary equilibrium. Consider a history ending with player set S. Let $y(S, \delta)$ be the equilibrium response vector. It will suffice to show that $y(s, \delta) = \mathbf{m}(S, \delta)$. This is clearly true if $|S| = 1$. Suppose, for some integer $1 \leqq K < n$, that this is true for all S with $|S| \leqq K$. Now if the hypothesis is false for some S with $|S| = K+1$, then, using Lemma 1, $J \equiv \{j \in S \backslash m_j(S, \delta) < y_j(S, \delta)\}$ must be non empty. Pick $i^* \in J$ such that $m_{i^*}(S, \delta) \geqq m_j(S, \delta)$ for all $j \in J$.

By condition (M), we have $m_{i^*}(S, \delta) \geqq m_{i^*}(S', \delta)$ for all $S' \subseteq S$ such that $i^* \in S'$. Thus by the induction hypothesis, at the player set S, i^* must be making an acceptable proposal (T, z), where $z_j = y_j(S, \delta)$ for $j \in T - i^*$ and $z_{i^*} = y_{i^*}(S, \delta)/\delta$. Now, $T = (T \cap J') \cup (T - J')$, where $J' \equiv \{j \in S \backslash m_j(S, \delta) > m_{i^*}(S, \delta)\}$. By Lemma 2, we have for all $j \in T - J'$,

$$y_j(S, \delta) \geqq y_{i^*}(S, \delta) > m_{i^*}(S, \delta) \geqq m_j(S, \delta). \tag{8}$$

Now consider $j \in J'$. Let T' be a set that attains the maximum in (2) (for $i = j$). Then, by Lemma 2 and the fact that $j \in J'$, $m_k(S, \delta) \geqq m_j(S, \delta) > m_i(S, \delta)$ for all $k \in T'$. Using (1) for $i = j$, we have, therefore,

$$\begin{aligned} y_j(S, \delta) \geqq \delta[v(T') - \textstyle\sum_{k \in T'-j} y_k(S, \delta)] &\geqq \delta[v(T') - \textstyle\sum_{k \in T'-j} m_k(S, \delta)] \\ &= m_j(S, \delta) \end{aligned}$$

On the other hand, since $m_j(S, \delta) > m_{i^*}(S, \delta)$, we have from the definition of i^* that $y_j(S, \delta) \leqq m_j(S, \delta)$. Combining, we see that $y_j(S, \delta) = m_j(S, \delta)$ for all $j \in J'$. Putting this information together with (8), we have

$$y_{i^*} = \delta[v(T) - \textstyle\sum_{k \in T - i^*} y_k] \leqq \delta[v(T) - \textstyle\sum_{k \in T_i^*} m_k] \leqq m_{i^*}$$

contradicting our supposition that i^* is in J. ||

4. STATIONARY EQUILIBRIUM AND EFFICIENCY

An allocation $x \in \mathcal{R}^n$ is *feasible* for (N, v) if there is a partition of N into coalitions $(S_1, \ldots, S_k)$ such that $\sum_{i \in S_j} x_i \leqq v(S_j)$ for all $j = 1, \ldots, k$.

An equilibrium is *efficient* if there is no feasible allocation for (N, v) such that every agent receives a higher utility (relative to the equilibrium payoff) in that allocation.

Our interest in this section is to try to characterize the set of bargaining games for which the resulting bargaining equilibrium is efficient (for high enough discount factors). It should be recalled that an important element of our bargaining game was the specification of the protocol. Consequently, one is also interested in knowing how our efficiency results are related to this particular aspect of the game.

4.1. *Efficiency of stationary equilibria for all protocols*

We start with a definition. Say that a game (N, v) is *dominated by its grand coalition* if for all $S \subseteq N$, we have

$$\frac{v(N)}{|N|} \geqq \frac{v(S)}{|S|}.$$

Remark. Note that the bargaining game analysed by Rubinstein (1982) and its n-person generalization by Herrero (1985), Shaked (1986) and others fall in this category.

We may now state

Proposition 4. *The following statements are equivalent:*[9]

(a) (N, v) *has the property that for every protocol, there is a sequence of discount factors tending to one and a corresponding sequence of efficient stationary equilibria.*
(b) (N, v) *is dominated by its grand coalition.*

Remark. For these games, the limit equilibrium payoff vector is the allocation obtained by equal division of $v(N)$ (see proof below).

Proof. We first show that (b) implies (a). Assume (b). Consider any stationary equilibrium. Let x_i be the payoff to player i from this equilibrium, for any history where N is the player set and i has to propose. For i to obtain this payoff, there must be a history with player set S and i as the proposer with i making an acceptable proposal (T, z), where $z_i = v(T) - \sum_{j \in T-i} z_j \geqq x_i$. By stationarity, the equilibrium response vector at S must be δz_i for i and z_j for $j \neq i$. Applying Lemma 2, we have $z_j \geqq \delta z_i$ for all $j \in T$. Thus,

$$x_i \leqq z_i \leqq \frac{v(T)}{1+\delta(|T|-1)} \leqq \frac{v(N)}{1+\delta(n-1)}$$

with the last inequality holding strictly whenever $T \neq N$ (use (b)).

Thus, i's payoff from any stationary equilibrium is bounded above by $v(N)/[1+\delta(n-1)]$, and this is true of *every* $i \in N$. So any proposer i at the player set N can make the acceptable proposal (N, z'), where $z'_j = \delta v(N)/[1+\delta(n-1)]$ for $j \neq i$ and obtain $v(N)/[1+\delta(n-1)]$. So the infimum of player i's equilibrium payoffs is $\delta v(N)/[1+\delta(n-1)]$. This implies that the equilibrium response vector y at the player set N is $y_j = \delta v(N)/[1+\delta(n-1)]$. Given (b), it then follows that every player makes an acceptable offer forming the grand coalition at the player set N.

Now we show that (a) implies (b). If (a) is true, then (using the finiteness of protocols) there exists $\delta^* \in (0, 1)$ such that for all $\delta \in (\delta^*, 1)$ and all protocols, each person who starts makes a proposal to the grand coalition which is accepted. For $\delta \geqq \delta^*$, using Lemma

9. The equivalence holds in the space of games that admit a stationary equilibrium. We conjecture, though, that for every game, a stationary equilibrium exists.

1 and 2, the equilibrium response vector $y(N, \delta)$ is then given by $y_i(N, \delta) = \delta v(N)/[1+\delta(n-1)]$ and the equilibrium payoff to the proposer is simply $v(N)/[1+\delta(n-1)]$. Since in equilibrium no proposer wants to pick a subcoalition S, we must have

$$\frac{v(N)}{1+\delta(n-1)} \geqq v(S) - \frac{\delta v(N)}{1+\delta(n-1)}(|S|-1).$$

Therefore, for every $S \in \mathscr{S}$,

$$\frac{v(N)}{1+\delta(n-1)} \geqq \frac{v(S)}{1+\delta(|S|-1)}$$

Passing to the limit as $\delta \to 1$ in the above expression, we get (b). ||

Proposition 4 thus shows that the requirement of efficiency for all protocols is extremely demanding. According to our definition, a game is dominated by its grand coalition if the average worth of N is no less than the average worth of any subcoalition. Such games are obviously not ones where subcoalitions matter in any real sense. What happens if we do not insist on such a strong requirement?

4.2. *Efficiency for some protocol*

The negative result of the previous section motivates a weaker inquiry: is it possible to obtain efficiency for at least *one* protocol? We start our analysis by noting a strong property of efficient stationary equilibria.

Proposition 5. *Let (N, v) be strictly superadditive. Suppose that for some protocol, there is a sequence $\delta^k \to 1$ and a corresponding sequence of efficient stationary equilibrium payoff vectors $z(\delta) = (z_i(\delta))_{i \in N}$, converging to z^* as $\delta \to 1$. Then z^* is in the core of the characteristic function (N, v).*[10]

Proof. Since the equilibrium is efficient, it must be that along the equilibrium path, no player makes an unacceptable offer, i.e., there is no delay along the equilibrium path. Moreover, since (N, v) is strictly superadditive, the first proposer (say player 1) must make a proposal to the grand coalition. Therefore, if the equilibrium response vector is $y(N, \delta)$, the equilibrium allocation is given by $z_j(\delta) \equiv y_j(N, \delta)$ for $j \neq 1$ and $z_1(\delta) \equiv y_1(N, \delta)/\delta$. The result now follows by applying Lemma 1, replacing S by N and taking δ to 1. ||

Remark. Proposition 4 does not assert the existence of the limit vector of payoffs, though we conjecture that this limit is always well-defined. For the case of no-delay stationary equilibria, the existence of this limit follows trivially from Lemma 3 below.

An immediate corollary of Proposition 4 is that for high enough discount factors, games with empty cores will *never* possess efficient stationary equilibrium for *any* protocol. Indeed, more is true. We now present an example which shows that even for games with nonempty cores, stationary equilibria are inefficient for *every* protocol.

Example 2 (*The employer-employee game*). $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for all i, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(N) = 1 + \mu$, $0 \leqq \mu < 0{\cdot}5$, $v(23) = \varepsilon$, positive but "small". This game is strictly superadditive, and it has a nonempty core.

10. This result is not true if the game is just superadditive. See Example 2.

It is possible to check the following:

$$m_i(S,\delta)=\begin{cases}\dfrac{\varepsilon\delta}{1+\delta}, & S=\{2,3\}, i\in\{2,3\},\\[2ex] \dfrac{\delta}{1+\delta}, & i\in S,\ S\in(\{1,2\},\{1,3\}),\\[2ex] \dfrac{\delta(1+\mu)}{1+2\delta}, & S=N, i\in N \text{ with } \dfrac{\delta}{1+\delta}<\mu,\\[2ex] \dfrac{\delta}{1+\delta}, & S=N, i\in N \text{ with } \dfrac{\delta}{1+\delta}>\mu.\\[2ex] 0, & S=\{i\}.\end{cases}$$

Using this list, one can observe that condition M is satisfied for ε sufficiently small. Consequently all stationary equilibria are no-delay.

Now we discuss efficiency.[11] If δ is "small", i.e. $\mu>\delta/(1+\delta)$, the grand coalition always forms in stationary equilibrium with the proposer receiving $(1+\mu)/(1+2\delta)$. The outcome is efficient. However, with $\mu<\delta/(1+\delta)$, (as it will be for δ close to 1), the opposite happens. In *no* stationary equilibrium does the grand coalition form: either coalition $\{1,2\}$, or $\{1,3\}$ forms with the the coalition splitting the surplus 1 equally as $\delta\to 1$. The remaining person receives zero. These equilibria are inefficient, though the game has a nonempty core.[12]

The example can be generalized to yield a weaker sufficient condition for inefficiency. The following lemma, which we also use later, will be needed:

Lemma 3. *$\boldsymbol{m}^*(N)\equiv\lim_{\delta\to1}\boldsymbol{m}(N,\delta)$ is well-defined, and for each $i\in N$,*

$$m_i^*(N)=\max_{i\in S\subseteq N}\,[v(S)-\textstyle\sum_{j\in S-i} m_j^*(N)],\tag{9}$$

with the property that for every set S^* that attains the maximum in (9),

$$m_j^*\geqq m_i^*\quad \textit{for all } j\in S^*.\tag{10}$$

The proof of this lemma uses the following two steps. First, one shows that there is a unique vector $\boldsymbol{m}^*(N)$ satisfying (9) and (10). The proof mimics that of Proposition 1. Second, one can easily check that every limit point of $\boldsymbol{m}(N,\delta)$ (as δ goes to 1) must satisfy (9) and (10). We omit the formal details.

An earlier version of our paper (Chatterjee, Dutta, Ray and Sengupta (1987)) shows how the limit vector $\boldsymbol{m}^*(N)$ may be explicitly computed from the parameters of the

11. In a quite different game in which only bilateral trade is allowed, Hendon and Tranaes (1990) find that their equilibrium could involve trading inefficiencies.

12. It might be of independent interest to note that if $\mu=0$, the set of all perfect equilibria in this game coincides with the set of stationary equilibria. For this case, Player 1 cannot get more than $1/(1+\delta)$ in *any* subgame perfect equilibrium, however much he (she) wishes to "teach" Players 2 and 3 and they seek to "learn". Of course, in this case there is no inefficiency, but the limit outcome is not in the core of the game. This shows, by the way, that strict superadditivity is essential for Proposition 5.

model. We use this limit vector to provide a sufficient condition for inefficiency under every protocol:

Proposition 6. *Let (N, v) be a strictly superadditive game such that for every δ, the set of stationary equilibria is exactly Σ. If $\sum_i m_i^*(N) > v(N)$, then there exists $\hat{\delta} < 1$ such that for each $\delta \in (\hat{\delta}, 1)$ and any protocol, each stationary equilibrium is inefficient.*

The proof follows immediately from Lemma 3 and the definition of Σ.

Proposition 6 raises an interesting question: if $\sum_i m_i^*(N) = v(N)$ for a strictly superadditive game, is it true that for *some* protocol, *some* stationary equilibria is efficient 'in the limit'? This would yield a complete characterization. Unfortunately, the answer is no, as the following example demonstrates:

Example 3. $N = \{1, 2, 3\}$, $v(1) = 1$, $v(2) = v(3) = 0$, $v(\{1, 2\}) = 1{\cdot}8$, $v(\{1, 3\} = 1{\cdot}6$, $v(\{2, 3\}) = 0{\cdot}1$, and $v(N) = 2.4$.

This is a strictly superadditive game and condition M holds. Thus all stationary equilibria are no-delay, and by Proposition 2 belong to Σ. Moreover, it is possible to check that

$$\boldsymbol{m}(N, \delta) = (\delta, 1{\cdot}8\delta - \delta^2, 1{\cdot}6\delta - \delta^2)$$

Thus $\sum_i m_i^*(N) = 2{\cdot}4 = v(N)$. However, it can be checked that as $\delta \to 1$, for no player is $m_i(N, \delta)$ attained at the grand coalition N. So we have inefficiency for large discount factors, under every protocol.

In the next section, we continue our search for an interesting class of games which display efficiency for some protocol.

5. EFFICIENCY OF STATIONARY EQUILIBRIA: STRICTLY CONVEX GAMES

In this section, we intend to show that there exists a class of games for which the following holds: for δ close to 1, (i) all stationary equilibria are no-delay and (ii) there exists a protocol for which some stationary equilibrium is efficient.

A game (n, v) is *strictly convex* if for all $S, T \subset N$, with $S - T$ and $T - S$ non-empty, one has

$$v(S \cup T) > v(S) + v(T) - v(S \cap T). \tag{11}$$

Proposition 7. *Let (N, v) be strictly convex. Then there exists $\delta^* \in (0, 1)$ such that for $\delta \geqq \delta^*$, every stationary equilibrium involves no delay. Moreover, there is a protocol and a stationary equilibrium relative to that protocol which is efficient.*

Remarks.

1. Proposition 7 cannot be extended to the class of all convex games (where (11) holds with weak inequality). It can be checked that the game considered in example 3 is convex, but no stationary equilibrium is efficient for high enough discount factors.
2. Even for strictly convex games, the result is not true for all discount factors. In Example 3, the game can be made strictly convex by making $v(N) = 2{\cdot}4 + \varepsilon$, for $\varepsilon > 0$. But as long as ε is not too large, it can be shown that there are intermediate ranges of δ such that efficiency is not obtained for any protocol.

In preparation for the proof of Proposition 7, we define the threshold discount factor δ^*. By (11), there exists $\varepsilon > 0$ such that for all $S, T \in \mathcal{S}$ with $S - T$ and $T - S$ non-empty, we have $v(S \cup T) > v(S) + v(T) - v(S \cap T) + \varepsilon$. Now define $\delta^* \in (0, 1)$ by the condition $(1 - \delta^*)v(N) = \varepsilon/2$. The proof of the proposition will use the following lemmas.

Lemma 4. *Suppose $\delta \in (\delta^*, 1)$. Fix any player set S and let $m_i(S, \delta)$ be attained at S^i, for $i \in S$. Then for all $j, k \in S$ (not necessarily distinct), we have either $S^j \subseteq S^k$ or $S^k \subseteq S^j$.*

Proof. For notational convenience, m_i will denote $m_i(S, \delta)$. Now, if the lemma is false, $S^j - S^k$ and $S^k - S^j$ are both non-empty. Let $S^0 \equiv S^j \cup S^k$ and $S_0 \equiv S^j \cap S^k$. We have

$$\begin{aligned} m_j &\geqq \delta[v(S^0) - \textstyle\sum_{i \in S^0 - j} m_i] \\ &\geqq \delta[v(S^j) - \textstyle\sum_{i \in S^j - j} m_i] + \delta[v(S^k) - \textstyle\sum_{i \in S^k} m_i] - \delta[v(S_0) - \textstyle\sum_{i \in S_0} m_i] + \varepsilon \\ &= m_j + m_k(1 - \delta) - \delta[v(S_0) - \textstyle\sum_{i \in S_0} m_i] + \varepsilon. \end{aligned}$$

Now if S_0 is empty, $\delta[v(S_0) - \sum_{i \in S_0} m_i] = 0$ and is non-empty, by the definition of m_i, we have, for some $l \in S_0$,

$$\delta[v(S_0) - \textstyle\sum_{i \in S_0} m_i] \leqq m_l(1 - \delta) \leqq v(N)(1 - \delta).$$

Combining all this information and using the fact that $\delta \geqq \delta^*$, we have $m_j > m_j - v(N)(1 - \delta) + \varepsilon > m_j$, which is a contradiction. ||

By Lemma 4, for each S, each $i \in S$, and each $\delta \in (\delta^*, 1)$, there is a *unique maximal set* $S^i(\delta)$ that attains the value $m_i(S, \delta)$.

Lemma 5. *For $\delta \geqq \delta^*$, $m_i(S, \delta) \geqq m_j(s, \delta)$ if and only if $S^i(\delta) \subseteq S^j(\delta)$.*

Proof. Drop arguments within parentheses for exposition. The "if" part follows from Lemma 2. To prove "only if", suppose on the contrary that $m_i \geqq m_j$ but that $S^j \subset S^i$ (this is the only possibility, by Lemma 4). Then by Lemma 2, $m_j \geqq m_i$ and so $m_i = m_j$. Now $m_i = \delta[v(S^i) - \sum_{k \in S^i - i} m_k]$. Using $m_i = m_j$ and the fact that $j \in S^i$, this implies that $m_j = \delta[v(S^i) - \sum_{k \in S^i - j} m_k]$. But this contradicts the fact that S^j is the *maximal* set attaining m_j. ||

Lemma 6. *For $\delta \geqq \delta^*$, the condition* (M) *holds for the bargaining game* (N, v).

Proof. Fix S, choose any $k \in S$, and let $m_k(S, \delta)$ be attained at its maximal set $S^k(\delta) \equiv S^k$. For any $T \subset S$, let $m_k(T, \delta)$ be attained at its maximal set $T^k(\delta) \equiv T^k$. Write $S^k \cap T^k = A$ and $T^k - S^k = B$.

Using the strict convexity of (n, v), and writing $m_i(S)$ (resp. $m_i(T)$) for $m_i(S, \delta)$ (resp. $m_i(T, \delta)$), we get

$$\begin{aligned} m_k(S) &\geqq \delta[v(T^k \cup S^k) - \textstyle\sum_{j \in T^k \cup S^k - k} m_j(S)] \\ &> \delta[v(T^k) + v(s^k) - v(A) - \textstyle\sum_{j \in T^k - k} m_j(T)] \\ &\quad + \delta[\textstyle\sum_{j \in B} m_j(T) + \sum_{j \in A - k} m_j(T) - \sum_{j \in S^k - k} m_j(S) - \sum_{j \in B} m_j(S)] \\ &= \delta[v(T^k) - \textstyle\sum_{j \in T^k - k} m_j(T)] + \delta \sum_{j \in B} (m_j(T) - m_j(S)) \\ &\quad + \delta[v(S^k) - \textstyle\sum_{j \in S^k - k} m_j(S)] - \delta[v(A) - \sum_{j \in A - k} m_j(T)] \end{aligned} \tag{12}$$

Now

$$\delta[v(S^k) - \textstyle\sum_{j \in S^k - k} m_j(s, \delta)] = m_k(S, \delta) \tag{13}$$

and, because $k \in A$,

$$\delta[v(A) - \textstyle\sum_{j \in A-k} m_j(T)] \leqq m_k(T) \tag{14}$$

Now for $j \in B$, by Lemma 2, we have $m_j(T) \geqq m_k(T)$. Also, since $B \cap S^k = \emptyset$ and $\delta \geqq \delta^*$, we have by lemma 5, $m_j(S) < m_k(S)$ for all $j \in B$. Therefore

$$\sum_{j \in B} (m_j(T) - m_j(S)) \geqq |B|(m_k(T) - m_k(S)) \tag{15}$$

Using (13), (14) and (15) in (12), we obtain for $\delta \geqq \delta^*$,

$$m_k(S) > m_k(T)$$

which completes the verification of Condition (M). ||

Proof of Proposition 7. Take $\delta \in (\delta^*, 1)$. By Lemma 6 and Proposition 3, every stationary equilibria is no-delay and belongs to σ. Using Lemma 5, there exists a partition $(S_1, S_2, \ldots, S_K)$ of N such that the following holds: if player j belongs to S_j, then $m_j(N, \delta)$ is attained at $\bigcup_{j' \leq j} S_{j'}$. In particular, if $j \in S_K$, then $m_j(N, \delta)$ is attained at N. So any protocol that has any member of S_K as first proposer will yield an efficient, no-delay stationary equilibrium. ||

Given this positive efficiency result for strictly convex games, one is naturally led to ask: what does the limit equilibrium payoff vector look like? We already know that a limit payoff vector is well defined and given by $\boldsymbol{m}^*(N)$.

To characterize this limit, it will be necessary to introduce the concept of *Lorenz domination.* Consider two allocations $\boldsymbol{x}$ and $\boldsymbol{y}$. We will say that $\boldsymbol{x}$ Lorenz dominates $\boldsymbol{y}$ if $\sum_{i=1}^k \hat{x}_i \leqq \sum_{i=1}^k \hat{y}_i$ for all $k \in N$ with strict inequality holding for some k, where $\hat{x}$ and $\hat{y}$ are permutations of $\boldsymbol{x}$ and $\boldsymbol{y}$ in decreasing order.

Lorenz domination represents a partial ordering which has a well known identification with the notion of "greater equality" in payoff distribution.[13]

We can now state our final result.

Proposition 8. *Let (N, v) be a strictly convex game. Suppose that for some protocol, there is a sequence $\delta^k \to 1$ and a corresponding sequence of no-delay stationary equilibria σ^k, such that σ^k is efficient for all k. Then the equilibrium payoff vector converges to a core allocation which Lorenz dominates every other core allocation.*

Remarks.

1. Of course, Proposition 7 ensures that there exists at least one protocol for which the condition of Proposition 8 is met.
2. Using Dasgupta, Sen and Starrett (1973), one can check that this limit equilibrium allocation also maximizes the symmetric Nash product $\prod_{i=1}^n x_i$ subject to the core constraints $v(S) \leqq \sum_{i \in S} x_i$ for all $S \in \mathscr{S}$.

13. See for instance Kolm (1969), Sen (1969), and Dasgupta, Sen and Starrett (1973).

The proposition (and indeed, its proof below) presumes that there exists such a core allocation that Lorenz-dominates every other core allocation. For this we appeal to the following

Fact (Dutta and Ray (1989)). *In convex games, there is a core allocation that Lorenz dominates every other core allocation.*

Proof of Proposition 8. Given the Fact and Propositions 2 and 5, it is sufficient to prove that the limit equilibrium vector $\boldsymbol{m}^*(N)$ is Lorenz undominated by any core allocation. Suppose, contrary to our claim, there exists $\boldsymbol{x}$ in the core of $v(N)$ such that $\boldsymbol{x}$ Lorenz dominates $\boldsymbol{m}^*(N)$. Without loss of generality, write $\boldsymbol{m}^*(N)$ in decreasing order: $m_1^* \geqq m_2^* \cdots \geqq m_n^*$.

Consider the first index i such that $x_i \neq m_i^*$. Using the definition of Lorenz domination, one can show that $x_i^* < m_i^*$, and, moreover, that $x_j \leqq m_j^*$ for all j such that $m_i^* = m_j^*$ (verification of this is straightforward but tedious).

By Lemma 3, we know that there exists S^i such that $m_i^* = v(S^i) - \sum_{j \in S^i - i} m_j^*$, with $m_j^* \geqq m_i^*$ for all $j \in S^i$. But from the previous observation on $\boldsymbol{x}$, we therefore have $x_i < v(S^i) - \sum_{j \in S^i - i} x_j$. This contradicts our supposition that $\boldsymbol{x}$ is a core allocation. ||

6. CONCLUSION

The paper has sought to construct a non-cooperative model of coalitional bargaining under complete information. We have shown that inefficiencies could arise in the form of delay and non-formation of the grand coalition. Moreover, the order in which players move turns out to make a significant difference to the efficiency of the equilibrium.

The limitations of this analysis have to do with the particular extensive form used (a curse common to many bargaining models!) and the use of stationary equilibria. While we personally do not find stationarity unpalatable, we are aware that there is a difference of opinion about this. Further research is needed to determine how this assumption bears up experimentally and also to investigate alternative extensive forms. An extension to NTU games is left for a future paper.

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