Geometry of (Anti-)De Sitter space-time

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Abstract: This work is an introduction to the De Sitter and Anti-de Sitter spacetimes, as the maximally symmetric constant curvature spacetimes with positive and negative Ricci curvature scalar R. We discuss their causal properties and the characterization of their geodesics, and look at the spaces embedded in flat $\mathbb{R}^{p,q}$ spacetimes with an additional dimension. We conclude that the geodesics in these spaces can be regarded as intersections with planes going through the origin of the embedding space, and comment on the consequences.

I. INTRODUCTION

Einstein's general relativity postulates that spacetime is a differential (Lorentzian) manifold of dimension 4, whose Ricci curvature tensor is determined by its mass-energy contents, according to the equations:

$$R_{\mu\lambda} - \frac{1}{2}Rg_{\mu\lambda} + \Lambda g_{\mu\lambda} = \frac{8\pi G}{c^4}T_{\mu\lambda} \qquad (1)$$

where $R_{\mu\lambda}$ is the Ricci curvature tensor, R te Ricci scalar curvature, $g_{\mu\lambda}$ the metric tensor, Λ the cosmological constant, G the universal gravitational constant, c the speed of light in vacuum and $T_{\mu\lambda}$ the energy-momentum tensor.
After the formulation of general relativity, physicists and mathematicians have tried to find exact solutions to Einsteins equations, often relying on the assumption of certain symmetries.
The simplest solutions to these equations are those with the highest degree of symmetry, called maximally symmetric spaces. These can be shown to have constant Ricci scalar R and are uniquely determined by its value [1]. The most common is the well known Minkowski space, with $R = 0$.
The aim of this work is the study of the maximally symmetric space with positive curvature $R > 0$ called de Sitter Space (abbreviated dS), and the space with negative curvature $R < 0$ called Anti-de Sitter space (abbreviated AdS).

II. (ANTI-)DE SITTER SPACE-TIMES

A simple way to understand the n-dimensional De Sitter space (dS_n) is by its embedding in a (n+1)-dimensional flat space $\mathbb{R}^{n,1}$ with Cartesian coordinates $(X^0, X^1, ..., X^n)$ and metric:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + ... + (dX^n)^2. \qquad (2)$$

De Sitter space is then defined as the hyperboloid of radius $a > 0$, the hypersurface with equation:

$$X_\mu X^\mu = -(X^0)^2 + (X^1)^2 + ... + (X^n)^2 = a^2. \qquad (3)$$

In the case of dS_4, introducing the coordinates (T, χ, θ, ϕ) given by:

$$X^0 = a \sinh\left(\frac{T}{a}\right) \qquad \vec{X} = a \cosh\left(\frac{T}{a}\right)\vec{n} \qquad (4)$$

where $\vec{X} = \left(X^1, X^2, X^3, X^4\right)$ and $\vec{n} = (\cos\chi, \sin\chi\cos\theta, \sin\chi\sin\theta\cos\phi, \sin\chi\sin\theta\sin\phi)$ with $T \in (-\infty, \infty)$, $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, then the line element is:

$$ds^2 = -dT^2 + a^2\cosh^2\left(\frac{T}{a}\right)\left[d\chi^2 + \sin^2\chi\, d\Omega_2^2\right] \qquad (5)$$

where the surfaces of constant time $dT = 0$ have metric $dl^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2$, which represents a 3-dimensional sphere S^3 in spherical coordinates. Thus, we can conclude that dS_4 is spatially closed.
Disregarding the coordinate singularities at $\chi = 0, \pi$, $\theta = 0, \pi$ and $\phi = 0, 2\pi$, we have a complete map of dS_4. This map, suppressing two dimensions, allows us to give a visual representation of the space as a one-sheeted 2-dimensional hyperboloid inside a 3-dimensional space (see figure 1).

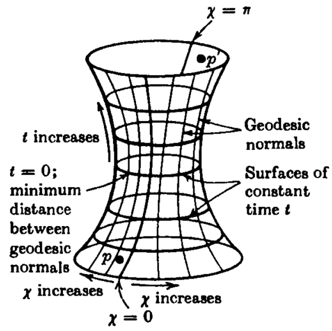

FIG. 1: De Sitter space with coordinates (T, χ, θ, ϕ) with sections of constant T and χ. Reduced model suppressing two dimensions θ and ϕ.

In the reduced model we see that the spatial sections contract to a minimum spatial volume (for $T = 0$) and then re-expand to infinity, given the congruence associated with this frame. Therefore, we say De Sitter space is an expanding space-time for $T > 0$.

Regarding the n-dimensional Anti-de Sitter space (AdS_n), we can embed it in a (n+1)-dimensional flat space $\mathbb{R}^{n-1,2}$ with metric:

$$ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + ... + (dX^n)^2 \quad (6)$$

and then AdS_n is defined as the hyperboloid with equation:

$$X_\mu X^\mu = -(X^0)^2 - (X^1)^2 + (X^2)^2 + ... + (X^n)^2 = -a^2. \quad (7)$$

In the case of AdS_4, introducing the coordinates (t, r, θ, ϕ) given by:

$$X^0 = a \sin\left(\frac{t}{a}\right)\cosh\left(\frac{r}{a}\right) \quad X^1 = a\cos\left(\frac{t}{a}\right)\cosh\left(\frac{r}{a}\right)$$
$$\vec{X} = a\sinh\left(\frac{r}{a}\right)\vec{n} \quad (8)$$

where $\vec{X} = \left(X^2, X^3, X^4\right)$ and $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with coordinates $0 \le t \le 2\pi a$, $r \ge 0$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$, then the line element is:

$$ds^2 = a^2\left(-\cosh^2\left(\frac{r}{a}\right)dt^2 + dr^2 + \sinh^2\left(\frac{r}{a}\right)d\Omega_2^2\right) \quad (9)$$

where the surfaces of constant time $dt = 0$ have metric $dl^2 = dr^2 + \sinh^2\left(\frac{r}{a}\right)d\Omega_2^2$, which represents an hyperbolic space H^3. We conclude that AdS_4 is spatially open.
Disregarding the coordinate singularities at $r = 0$, we have a complete map of AdS_4. Suppressing two dimensions we can give a visual representation like in dS_4 case (see figure 2).

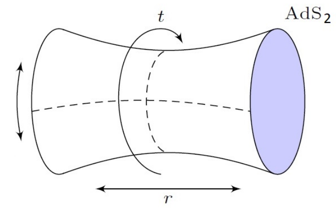

FIG. 2: Anti-de Sitter space with coordinates (t, r, θ, ϕ). Reduced model suppressing two dimensions ϕ and θ.

Contrary to the De Sitter case, in AdS_4 we have a periodic time coordinate, with the points $(0, r, \theta, \phi)$ and $(2\pi a, r, \theta, \phi)$ representing the same point in the space, which leads to physically unrealistic closed timelike curves. As the metric described does not explicitly include the time periodicity in its expression, we can unwrap the time coordinate by extending $t \in (-\infty, +\infty)$, working with the so called Anti-de Sitter universal covering space. We will work with this covering space in the following sections.

Intrinsically, these hypersurfaces are analogous to a sphere in Euclidean space ($X_\mu X^\mu = a^2$) and are therefore usually called pseudospheres.
Both in the de Sitter and anti-de Sitter case, the hyperboloid's radius is related to the curvature Ricci scalar by the expressions:

$$R_{dS_n} = \frac{n(n-1)}{a^2} \quad and \quad R_{AdS_n} = -\frac{n(n-1)}{a^2} \quad (10)$$

which can be computed from the given metrics (5) and (9). Furthermore, these spaces can be regarded as solutions for the Einsteins equations of empty spaces ($T_{\mu\lambda} = 0$) with cosmological constant $\Lambda = (n-2)R/2n$ with the corresponding Ricci scalar (see [1] and [3]).

III. CAUSAL DIAGRAMS

To study the causal properties of the (Anti-)De Sitter space, it is useful to bring infinite coordinate values into finite range. By doing this, we will be able represent the causal structure using the so-called Carter-Penrose diagrams.

For dS_4 with coordinates given in (5) we perform the change of variable $\cosh(T/a) = 1/\cos\eta$ to get the conformal metric:

$$ds^2 = \frac{a^2}{\cos^2\eta}\left(-d\eta^2 + d\chi^2 + \sin^2\chi\, d\Omega_2^2\right) \quad (11)$$

for $-\pi/2 \le \eta \le \pi/2$. To draw the diagram we use the reduced model with only η and χ coordinates. Note that the points $\chi = 0$ and $\chi = \pi$ represent antipodal points in the hyperboloid and opposite poles in the pseudosphere. In this conformal metric null geodesics are represented as straight lines with a slope of 45º(see figure 3).

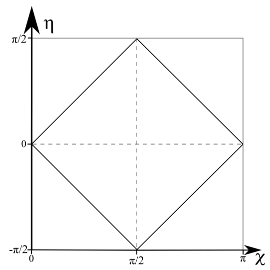

FIG. 3: Penrose diagram for the De Sitter space with conformal metric and null geodesics as 45º lines. Every point inside the diagram corresponds to a S^2 spatial section in the full model.

Due to the relation of the conformal coordinates in dS_4 we have an interesting property of the space. Given a static observer at $\chi = 0$, a photon sent from a timelike observer in its antipodal point $\chi = \pi$ (that follows a null geodesic) will not reach the first observer (or it will reach him for $\eta \to \pi/2$ that is $T \to +\infty$). This means that there are pairs of timelike observers in the space that are not causally connected, in the sense that they are not able to exchange signals.

For AdS_4 with coordinates given in (9) we perform the change of variable $\sinh(r/a) = \tan(\rho/a)$ to get the conformal metric:

$$ds^2 = \frac{a^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho\, d\Omega_2^2\right) \quad (12)$$

for $0 \leq \rho \leq \pi a/2$. To draw the diagram we take the reduced model with only t and ρ coordinates. Moreover, due to the unwrapping of the time coordinate the temporal axis extends for $(-\infty, +\infty)$. In our diagram, once the null geodesic reaches the spatial infinity ($\rho = \pi/2$ or $r \to +\infty$) we assume reflecting boundary conditions and the geodesic bounces off and returns (see figure 4). In addition, points at $t = 0$ and $t = \pi a$ represent antipodal points while at $t = 0$ and $t = 2\pi a$ represent the same point in the hyperboloid.

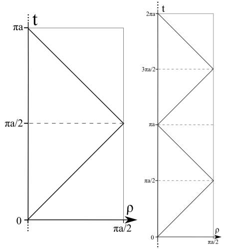

FIG. 4: Partial and $2\pi a-$periodic Penrose diagrams for the Anti-de Sitter space with universal covering and conformal metric, with null geodesics as 45^{o} lines. Every point inside the diagram corresponds to a S^2 spatial section in the full model.

Given the relations between the conformal coordinates, we get an interesting property of AdS_4. For a static observer at $\rho = 0$ a photon emitted from its position will reach spatial infinity $\rho = \pi a/2$ and come back (assuming reflecting boundary) in a finite amount of time.

Aside from the properties deduced by looking at null geodesics, we can wonder how the spaces behave in relation to other geodesics. For flat spacetimes, all points inside the future (past) light cone of an observer can be accessed through a timelike geodesic. In the case of Anti-de Sitter space, we will see that this property is not true by answering the question: can any two points which are connected through a timelike curve also be connected through a timelike geodesic? A related question can be asked for De Sitter spacelike curves and geodesics.

IV. GEODESICS IN ANTI-DE SITTER SPACE

By performing the change of variables $\rho = a\sinh(r/a)$ in (9) we get the static metric:

$$ds^2 = -\left(1+\frac{\rho^2}{a^2}\right)dt^2 + \left(1+\frac{\rho^2}{a^2}\right)^{-1}d\rho^2 + \rho^2 d\Omega_2^2 \quad (13)$$

with $\rho \geq 0$. Using this metric and the Euler-Lagrange equations with the Lagrangian $\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x^\mu}\dot{x^\nu}$, we get the following geodesic equations:

$$\begin{aligned} \ddot{t} + \frac{2\rho}{a^2+\rho^2}\dot{\rho}\dot{t} &= 0 \\ \ddot{\rho} + \frac{\rho}{a^2}\left[\left(1+\frac{\rho^2}{a^2}\right)\dot{t}^2 - \frac{1}{\left(1+\frac{\rho^2}{a^2}\right)}\dot{\rho}^2\right] - & \\ -\rho\left(1+\frac{\rho^2}{a^2}\right)[\dot{\theta} + \sin^2\theta\,\dot{\phi}^2] &= 0 \\ \ddot{\theta} + \frac{2}{\rho}\dot{\rho}\dot{\theta} - \cos\theta\sin\theta\,\dot{\phi}^2 &= 0 \\ \ddot{\phi} + \frac{2}{\rho}\dot{\rho}\dot{\phi} - 2\frac{1}{\tan\theta}\dot{\theta}\dot{\phi} &= 0 \end{aligned} \quad (14)$$

where the dot indicates the derivative with respect to the affine parameter τ. Furthermore, as the Lagrangian used to get the geodesic equations is explicitly independent of the time and angular coordinates t and ϕ, we have that the conjugate momenta $p_t = E$ and $p_\phi = L$ are conserved, that is:

$$\left(1+\frac{\rho^2}{a^2}\right)\dot{t} = E = const. \text{ and } \rho^2\dot{\phi} = L = const. \quad (15)$$

Without loss of generality and due to the symmetries of the space, we can restrict to the equatorial plane $\theta = \pi/2$.

We first take a look at circular timelike geodesics when $\rho = const. \to \dot{\rho} = 0$. In this case, besides the previous constants the geodesic equations are:

$$\ddot{t} = \ddot{\phi} = 0 \text{ and } \frac{\rho}{a^2}\dot{t}^2 - \rho\dot{\phi}^2 = 0. \quad (16)$$

From the second equation we get the relation $\dot{t}/a = \dot{\phi}$. We see the circular geodesics are closed orbits of the form $\phi \propto t/a$ with the same angular velocity $1/a$ regardless of the radial distance. We conclude that, unlike in the case of flat spaces, we can build a rigidly-rotating frame of reference in AdS. Given one such frame with constant angular coordinate $\phi' = \phi - t/a = const.$ then $d\phi^2 = dt^2/a^2$ and from the metric in (13) we get:

$$ds^2 = -\left(1+\frac{\rho^2}{a^2}\cos^2\theta\right)dt^2. \quad (17)$$

We see that observers at rest in the rotating frame have $ds^2 < 0$ and are thus timelike.

On the other hand, we have the following constants of motion for the geodesics (considering $c = 1$):

$$g_{\mu\nu}\dot{x^\mu}\dot{x^\mu} = \begin{cases} \mp 1, & \text{timelike/spacelike} \\ 0, & \text{null} \end{cases} \quad (18)$$

which in our case, and restricting to the radial motion with $L = 0 \to \dot{\phi} = 0$, translates to the following conditions for our geodesics:

$$-\left(1+\frac{\rho^2}{a^2}\right)\dot{t}^2 + \frac{\dot{\rho}^2}{\left(1+\frac{\rho^2}{a^2}\right)} = \begin{cases} \mp 1, & \text{timelike/spacelike} \\ 0, & \text{null} \end{cases} \quad (19)$$

For null radial geodesics, using the constant of motion in (18) we have that the radial equation is:

$$\ddot{\rho} = 0 \tag{20}$$

which gives a linear dependence on proper time.

For timelike radial geodesics, aside from the relations in (15), the equation for the radial coordinate is:

$$\ddot{\rho} + \frac{1}{a^2}\rho = 0 \tag{21}$$

which is the equation of a simple harmonic oscillator with frequency $\omega = 1/a$. Therefore, radial timelike geodesics will be sinusoidal curves which end up returning to $\rho = 0$ for proper time $\tau = \pi a$ (which will be the antipodal point) and again at $\tau = 2\pi a$, that have a maximum finite value for $\rho(\tau)$ depending in the initial conditions (initial position $\rho(0)$ and velocity $\rho'(0)$).
In the spacelike case, we have the same equations with $\omega = -1/a$ so that the geodesics follow hiperbolic functions from a point to infinity.

It is also worth noting that for $L \neq 0$ the constant in (18) includes the term $\dot{\phi}$, so that using the relation in (15) we get the following second-order nonlinear differential equation:

$$\ddot{\rho} + \frac{1}{a^2}\rho - \frac{L^2}{\rho^3} = 0. \tag{22}$$

It is easy to check numerically that the solutions of this equation are still $2\pi a-$periodic. The reason for this will be apparent in the following section.

To summarize the previous results, we have seen that null geodesics are linear on proper time and that both radial and circular timelike geodesics are $2\pi a-$periodic. This fact can be further generalized by looking at the geodesics from the perspective of the embedding flat space.

V. GEODESICS IN THE EMBEDDING FLAT SPACE

Let us recall the definition of the $n-$dimensional (Anti-)de Sitter space as the hyperboloid inside a $(n+1)-$dimensional flat $\mathbb{R}^{(p,q)}$ space. In the case of AdS_n we have the embedding space $\mathbb{R}^{(n-1,2)}$ with metric η_{AB} given by (6). The geodesics in this space can be calculated using the Lagrangian $\mathcal{L} = \frac{1}{2}\eta_{AB}\dot{X}^A\dot{X}^B$. We shall apply the Lagrange multipliers method with the restriction given by the hyperboloid equation (7) to get the Lagrangian for the hyperboloid's geodesics, which is:

$$L = \frac{1}{2}\eta_{AB}\dot{X}^A\dot{X}^B - \lambda(\tau)F(X^A) \tag{23}$$

where $F(X^A) = X_A X^A + a^2 = \eta_{AB}X^A X^B + a^2 = 0$ and λ is the Lagrange multiplier. The geodesic equations are then:

$$\ddot{X}^A + 2\lambda X^A = 0, \text{ for } A = 0, ..., n. \tag{24}$$

To remove the multiplier we take the second derivative of our constrain and get:

$$\ddot{F} = 2(\eta_{AB}\ddot{X}^A X^B + \eta_{AB}\dot{X}^A\dot{X}^B) = 0 \tag{25}$$

where $\eta_{AB}\dot{X}^A\dot{X}^B$ is equal to the constant of motion from (18) under the restriction.

For null geodesics we get $\eta_{AB}\ddot{X}^A X^B = 0$ and for timelike/spacelike geodesics $\eta_{AB}\ddot{X}^A X^B = \pm 1$. By multiplying the geodesic equation by $\eta_{AB}X^B$ and using these relations with the constrain F we get $\lambda = \pm 1/2a^2$ for timelike/spacelike geodesics and $\lambda = 0$ for null, and thus the equations are respectively:

$$\ddot{X}^A \pm \frac{1}{a^2}X^A = 0 \text{ and } \ddot{X}^A = 0. \tag{26}$$

Like in the previous section, we have for timelike geodesics the equation of a simple harmonic oscillator with frequency $\omega = 1/a$, although in this case for all coordinates X^A and general timelike geodesics. Likewise, the expressions for the coordinates X^A are all $2\pi a-$periodic. For spacelike and null geodesics we also have the same result than before but for all coordinates X^A.
From this expression we get that the points in timelike geodesics of AdS_n have a general form given by:

$$X^A = p^A \sin\left(\frac{\tau}{a}\right) + q^A \cos\left(\frac{\tau}{a}\right) \tag{27}$$

where p^A and q^A are constant vectors that due to the constrain F and the orthogonality of the solutions satisfy the conditions:

$$p^A p_A = q^A q_A = a^2 \; and \; p^A q_A = 0. \tag{28}$$

Due to the maximal symmetry of the AdS_n space, given a timelike geodesic G such that $X^A(\tau = 0) = P$ is an arbitrary point of the geodesic, we can find a transformation which makes the coordinates of P equal to $X^0(0) = a$ and $X^i(0) = 0$ for $i = 1, ..., n$. In addition, by performing an appropriate rotation around the X^0 axis we can make the tangent vector of the geodesic at point P (given by $\dot{X}^A(0)$) tangent to the X^1 axis, so that we get $\dot{X}^0(0) = \dot{X}^j(0) = 0$ for $j = 2, ..., n$ and $\dot{X}^1(0) = 1$. With these transformations we find the geodesic in the resulting frame of reference has expression:

$$X^0 = a\cos\left(\frac{\tau}{a}\right), \; X^1 = a\sin\left(\frac{\tau}{a}\right)$$
$$X^j = 0 \text{ for } j = 2, ...n \tag{29}$$

which represents a circle of radius a in the embedding space.
Consequently, we have found that all timelike geodesics of AdS_n can be represented as circles of radius a on an Euclidean plane through the origin (given an appropriate coordinate transformation) which means that they are the intersection of the hyperboloid with planes through the origin of the embedding space. In general, two different timelike geodesics starting from the same point will

only intersect in this point and its antipodal in the hyperboloid, where their planes intersect. Moreover, with this discussion we see that the distinction between radial and circular timelike geodesics is coordinate dependent, as intrinsically they are all intersections with planes in the embedding space.
Likewise, spacelike geodesics are hyperbola branches that are also intersections with planes through the origin.

We now go back to the causal diagram for AdS_4 and represent timelike geodesics. From an origin point at $\rho = 0$, the timelike geodesics are curves that diverge and reconverge to the same point in $\pi a-$periods, and thus the points of these paths are bounded inside the infinite diamonds formed by the null geodesics (see figure 5).

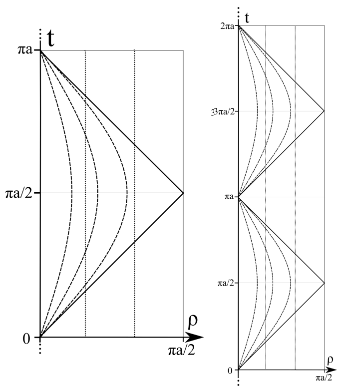

FIG. 5: Partial and $2\pi a-$period Penrose diagrams for the Anti-de Sitter space with universal covering and conformal metric, with null geodesics as 45^o lines and timelike geodesics as dashed lines.

Therefore, the points outside this regions (but inside the causal future) can not be accessed by timelike geodesics although we can construct timelike curves.

Alternatively, the spacelike and timelike geodesics of De Sitter space can also be shown to be the intersections with planes through the origin of the embedding space ([2] and [5]) and thus spacelike geodesics will also be periodic starting in a point and ending in its antipodal (see figure 6). In the same way that the timelike geodesics for AdS, the points that can be accessed by spacelike geodesics in dS are restricted to the diamond shaped area limited by the null geodesics.

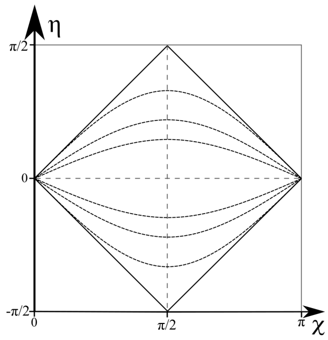

FIG. 6: Penrose diagram for the De Sitter space with conformal metric, null geodesics as 45^o lines and spacelike geodesics as dashed lines.

VI. CONCLUSIONS

To summarize, we have seen that the geometric structure of dS and AdS spaces introduces unusual causal properties that differ from those in flat spaces, like the existence of timelike observers that are not able to exchange signals in dS or the finite time interval of a photon reflected at spatial infinity in AdS.
Furthermore, we have also seen some interesting properties of the geodesics of the spaces, like the existence of rigidly-rotating frames for all radial distances in AdS or the periodic timelike or spacelike geodesics in AdS and dS respectively. These properties become easily apparent by studying the spaces through their embedding in flat $\mathbb{R}^{(p,q)}$ spaces with an additional dimension, where we see that these geodesics are intersections with planes through the origin of the embedding space.

Acknowledgments

This work was done with the help of my adviser Dr. Jaume Garriga, from whom i received all the necessary guides and references to study the subject at hand.
In addition, I would like to thank my family and friends for their continuous support and companionship over these years.

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