Optimal Muting and Load Balancing for eICIC

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***Abstract*—In heterogeneous networks, a macro cell can cause excessive interference to users in co-channel pico cells. In Release 10 of 3GPP's LTE system, enhanced Inter-Cell Interference Coordination (eICIC) has been introduced to reduce the interference experienced by pico users by having the macro mute some of its subframes. This is a tradeoff, because the benefit to pico users comes at the cost of the macro sacrificing a portion of its resources. This tradeoff is also influenced by the determination of which users are offloaded from the macro to the pico. We formulate the joint optimization of the proportion of muted sub-frames and the assignment of users to macro and pico cells as a network-wide utility maximization problem, and obtain an analytical characterization of the optimal solution. We propose an iterative algorithm that successively improves the assignment of users to cells and the muting proportion, and prove its convergence to the optimal solution.**

***Index Terms*—Enhanced Inter-Cell Interference Coordination, EICIC, Almost Blank Subframes, ABS, range extension, utility maximization, load balancing.**

I. INTRODUCTION

In heterogeneous networks, a macro cell causes interference to users in co-channel pico cells. This is aggravated when a *bias* or *range extension* is applied to offload a larger number of users to pico cells. The intention of the range extension is to reduce the load on the macro and maximize the usage of pico cells that are under-loaded relative to the macro. The bias is typically applied at the time of cell selection or handover so that a user would associate with a pico cell even if the received signal from the macro cell were stronger. However, such users that are offloaded due to the range extension would experience excessive interference from the macro. Thus, in order to capture load-balancing gains, some interference mitigation between the macro and pico is necessary.

In Release 10 of 3GPP's Long Term Evolution (LTE) system, enhanced Inter-Cell Interference Coordination (eICIC) has been introduced [1, Section 16.1.5] for this purpose. In eICIC, the interference caused by a macro cell to its underlay pico cells is mitigated by having the macro mute some of its subframes so that pico users experience reduced downlink interference during the muted sub-frames. These are known as *Almost Blank Sub-frames* (ABS) or Low Power Sub-frames (LPS) based on the level of muting (we will generically use the term ABS for all muting levels). Since only the picos get to transmit during the macro's muted sub-frames, this can be considered as *resource partitioning* in time-domain [2]. Pico users can feed back an indication of their channel quality within the muted sub-frames separately from the channel quality within non-muted sub-frames. These *resource restricted measurements* [2] allow the pico cell to transmit at higher spectral efficiency to the pico users during the muted sub-frames. This benefit of increased spectral efficiency for the pico users, however, comes at the cost of the macro users having access to fewer resources due to the macro muting. So in order to ensure that the system as a whole is better off, the proportion of muted sub-frames needs to be carefully chosen. Note that the correct proportion of muted sub-frames to be used depends on the choice of which users are assigned to the picos and which users are assigned to the macros. The larger the number of users assigned to the macro, the greater the cost of the the macro sacrificing resources for muting. Furthermore, another aspect that influences this is the determination by the pico eNB of the users to whom it should allocate the ABS (low-interference) resources, and the users to whom it should allocate the non-ABS (high-interference) resources. This is a function of the underlying resource allocation (or scheduling) method used in the picos. Thus we need to jointly optimize the proportion of muted sub-frames, the assignment of users to macro or pico cells, and the resource allocation within each cell. This joint optimization is the subject of this paper.

Resource allocation to a given set of users within an individual cell is the function of a scheduling algorithm such as the *proportionally fair* (PF) algorithm [3]. PF and its generalization to a gradient-based scheme have been shown to maximize a *utility function* under very general conditions [4], [5], [6]. The utility function quantifies a tradeoff between efficiency and fairness, with PF being a particular instance of the tradeoff given by the logarithmic utility function.

In the multi-cell case, an added dimension is the determination of which users should be assigned to receive allocations from which cell, an aspect that can be considered as *load-balancing*. In the context of networks without eICIC, [7], [8], [9], [10] formulate this problem as utility maximization across multiple cells. [7], [8], [9] recognize a key property, that it is preferable to assign a user resources from a base station that is most attractive according to a certain measure related to the ratio of the user's spectral efficiency from a given base station to the number of users in that base station (for logarithmic utility, or to a shadow price for a more general utility function). This measure can be interpreted as the throughput that a user can expect to achieve if connected to that base station, given the load in that base station. [7] applies this property to associate newly arriving users to base stations, through a heuristic cell-association mechanism that is shown by simulation to approach the performance of the optimal when there are a large number of users. [8] applies this property for successively moving users from one cell to a more attractive one (a procedure that can be practically interpreted as handover). [9] uses this property for an iterative algorithm which successively improves Lagrange multipliers (shadow prices). [7], [9], [10] also observe that in the optimal allocation, some users may have to be allocated resources from more than one cell. [10] solves a relaxed version of the utility optimization, assuming equal resource allocations to users including those that are partially associated to a cell.

Heterogeneous networks with eICIC further require optimal resource partitioning for interference management. Here in addition to deciding (a) which users should be assigned to receive resources from macro cells and pico cells, one also has to decide (b) what the muting proportion (or resource partition) should be, and (c) within each pico cell, which users should receive resource allocations within ABS or non-ABS. [11] demonstrates through simulations that the performance of eICIC is sensitive to the setting of the range extension (which is one type of criterion for deciding (a)) and the muting proportion (b). [12] uses a utility-maximization formulation to address (b) and (c), proposing a dynamic programming method to determine (c) for a given value of the muting proportion, and then finding the optimal muting proportion by exchanging potential utility values corresponding to all possible muting proportions between the base stations. [13] looks primarily at (c), demonstrating through simulations that the use of the proportionally fair scheduler to allocate resources within a pico across ABS and non-ABS gives better performance than a scheme that *a priori* classifies users into cell-center (scheduled only during non-ABS) and cell-edge (scheduled only during ABS), using a criterion related to the difference of the signal strengths received by the user from the macro and pico. [14] addresses (b) and (c) using a Nash-Bargaining Solution approach, which is different from but related to the network utility maximization approach as discussed in [15].

Thus all of these papers address pieces of the overall problem to varying degrees. The main contribution of our paper is a unified approach that addresses all the above aspects, using a joint optimization formulation based on multi-cell utility maximization described in Section II. This can be viewed as a multi-cell generalization of the underlying scheduling/resource allocation problem within each cell, incorporating the multi-cell aspects of load balancing through optimal cell assignment and interference management through optimal resource partitioning. In Section III we obtain the conditions that characterize the optimal assignment of users to macro or pico cells for a given muting proportion. We show how these conditions admit a simple graphical interpretation that provides insight into the case where the optimal solution may involve allocating at least one user resources from more than one cell. We highlight the role played in this determination by the ratio of the user's spectral efficiency to a certain *generalized proportionally fair* (GPF) metric that represents the load of a cell. For the logarithmic utility function, the GPF metric turns out to be equivalent to the number of users, validating an observation in [7], [8], [9], [10]. The rule for the optimal assignment of users to cells also yields the criterion by which a pico cell can determine the users to which it should allocate resources within ABS or within non-ABS (or both), with the ratio of the users' spectral efficiency to the GPF metric again playing a key role. Based on these insights, we obtain a criterion for improving the assignment of users to cells one user at a time. In Section IV, we characterize the conditions for the optimal muting proportion for a given assignment of users to cells, and identify a criterion for improving the muting proportion. The GPF metric again turns out to be instrumental in determining the optimal muting proportion. In Section V, we propose an iterative improvement algorithm that successively uses the criteria for improvement of the assignment of users to cells and the muting proportion, and prove its convergence to the global maximum of the network-wide utility maximization.

Our approach in this paper is purely analytical, since the key results are mathematical in nature. Due to space constraints we omit detailed proofs, instead concentrating on the key insights from the results, and provide high-level outlines of the key arguments used in the proofs.

II. Problem Formulation

Consider a system in which users $1, \cdots, N$ are served by a network of M macro cells and P pico cells. Each cell has a unit resource that it can allocate amongst the users. We assume that all the macros mute (or do not mute) in tandem, i.e. in the same set of sub-frames, so that all macros use the same muting proportion a. We can visualize each pico cell as consisting of two logical sub-cells, one which is active only during ABS (and thus has access to a fraction a of the pico-cell's unit resource), and the other which is active only within non-ABS (and thus has access to remaining fraction $1-a$ of the pico-cell's unit resource). Due to muting, each macro cell can allocate only a fraction $1-a$ of its unit resource. Thus we can consider a collection of $2P+M$ sub-cells, wherein P of the sub-cells (comprising the picos' ABS sub-cells) are active during ABS, and the remaining $P+M$ (comprising the picos' non-ABS sub-cells and the macros) are active during non-ABS. For simplicity, we will simply refer to each of these as just cells rather than sub-cells. User u can achieve a spectral efficiency (i.e. throughput per unit resource) of $R_c(u)$ from cell c. $R_c(u)$ is known through the resource-restricted measurements fed back by the user. We assume the $R_c(u)$ to be given constants (i.e. not time-varying, and not dependent on the resource allocation). For each u, we assume that $R_c(u) > 0$ for at least one c.

For mathematical convenience, we restrict a to be in $[\epsilon_a, 1-\epsilon_a]$ for some $\epsilon_a > 0$. This is not a major limitation, because $\epsilon_a > 0$ can be chosen arbitrarily small.

In a hard-handover system like LTE, a user is allocated resources from only one cell at a time. But since we are treating the ABS resources and the non-ABS resources within a pico cell as two separate logical cells, a user in the pico cell could receive resource allocations from both of these. So in our formulation we will allow the possibility of a user being allocated resources from multiple cells. This relaxation also lets us address evolutions of LTE like Coordinated Multi-Point operation (CoMP) [16], wherein a user could receive allocations from multiple cells. We assume the user devices have appropriate capabilities (including receiving simultaneously from multiple cells if needed), in order to achieve any feasible resource allocation.

Let the fraction of cell c's resources allocated to user u be $\rho_c(u)$. A *serving cell* of user u, i.e. a cell such that $\rho_c(u) > 0$, will be denoted by $s(u)$. We collect the resource allocations $\rho_c(u)$ into a $(2P+M)N$-tuple $\boldsymbol{\rho}$. We say that an allocation $\boldsymbol{\rho}$ is *consistent with muting proportion* a if $\forall c \in \{1, \cdots, 2P+M\}$,

$$\begin{aligned} 0 \leq \rho_c(u) \quad &\leq \quad 1 \text{ for all users } u \\ \sum_{u=1}^{N} \rho_c(u) \quad &\leq \quad \begin{cases} a, & \text{if } c \text{ is an ABS cell} \\ 1-a, & \text{if } c \text{ is a non-ABS cell.} \end{cases} \end{aligned} \tag{1}$$

Define $\mathcal{S}_a$ as the set of all $(\boldsymbol{\rho}, a)$ such that $\boldsymbol{\rho}$ is consistent with a given muting proportion a. The *feasible set* of resource allocations consistent with some muting proportion is then

$$\mathcal{S} = \bigcup_{a \in [\epsilon_a, 1-\epsilon_a]} \mathcal{S}_a. \tag{2}$$

A *cell assignment indicator* is a binary $N(2P+M)$-tuple $\mathbf{z}$ that indicates, for each user u and cell c, whether that user is allowed to get resources in that cell. A *valid* $\mathbf{z}$ indicates, for each user u, at least one allowed cell c wherein the user can achieve non-zero spectral efficiency, i.e. $z_c(u) = 1$ for at least one c such that $R_c(u) > 0$. We say that $\mathbf{z}$ is *consistent with* $\mathbf{z}'$ if, for every c, u such that $z'_c(u) = 0$, we also have $z_c(u) = 0$. A resource allocation $\boldsymbol{\rho}$ is naturally associated with a cell indicator $\tilde{\mathbf{z}} = \zeta(\boldsymbol{\rho})$, satisfying $\tilde{z}_c(u) = 0$ if $\rho_c(u) = 0$, and $\tilde{z}_c(u) = 1$ otherwise. We call $\zeta(\boldsymbol{\rho})$ the *derived cell assignment indicator* of $\boldsymbol{\rho}$. If $\zeta(\boldsymbol{\rho})$ is consistent with a cell assignment indicator $\mathbf{z}'$, then we say that $\boldsymbol{\rho}$ is consistent with $\mathbf{z}'$. If $\boldsymbol{\rho}$ is also consistent with muting proportion a, we say that $(\boldsymbol{\rho}, a)$ is consistent with $\mathbf{z}'$.

For a given valid cell assignment indicator $\mathbf{z}$, define $\mathcal{S}^0(\mathbf{z})$ as the set of all $(\boldsymbol{\rho}, a) \in \mathcal{S}$ that are consistent with $\mathbf{z}$. We can further define $\mathcal{S}^0_a(\mathbf{z}) = \mathcal{S}^0(\mathbf{z}) \cap \mathcal{S}_a$ comprising the allocations consistent with both the cell assignment indicator $\mathbf{z}$ and the muting proportion a. Note that all of $\mathcal{S}$, $\mathcal{S}_a$, $\mathcal{S}^0(\mathbf{z})$, and $\mathcal{S}^0_a(\mathbf{z})$ are closed, bounded, and convex.

The throughput given to user u by resource allocation $\boldsymbol{\rho}$ is

$$T(\boldsymbol{\rho}; u) = \sum_{c=1}^{(2P+M)} \rho_c(u) R_c(u) \tag{3}$$

Define the *system utility* provided by $(\boldsymbol{\rho}, a)$ as

$$\mathcal{U}(\boldsymbol{\rho}, a) = \sum_u w_u U_u(T(\boldsymbol{\rho}; u)), \tag{4}$$

where $U_u(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is the *user utility function* of user u, and $w_u > 0$ is the (given) weight associated with user u. U_u are typically taken to be monotonic increasing and concave. The weights are typically used to differentiate the Quality of Service (QoS) afforded to different users. The system utility does not explicitly depend on a, but we define $\mathcal{U}$ as a function of $(\boldsymbol{\rho}, a)$ to emphasize that we are looking at resource allocations consistent with muting in eICIC. Note that if each $U_u(\cdot)$ is a strictly concave function, then $\mathcal{U}(\cdot)$ is also a strictly concave function over the convex set $\mathcal{S}$. Formally the system utility maximization problem is stated as follows.

Problem: Find $(\boldsymbol{\rho}, a) \in \mathcal{S}$ to maximize $\mathcal{U}$ over $\mathcal{S}$.

Note also that an optimal solution $(\boldsymbol{\rho}, a)$ must satisfy

$$\sum_{u=1}^{N} \rho_c(u) = \begin{cases} a & \text{if } c \text{ is an ABS cell} \\ 1-a & \text{if } c \text{ is a non-ABS cell,} \end{cases} \tag{5}$$

i.e. with equality rather than inequality in (1), otherwise we could increase the allocation $\rho_c(u)$ to some user and increase the system utility.

We specifically consider the family of strictly concave user utility functions $U^\alpha : \mathbb{R}^+ \to \mathbb{R}$ for $\alpha > 0$, where for $x > 0$,

$$U^\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1. \end{cases} \tag{6}$$

This family of utility functions defines various tradeoffs between efficiency and fairness. Smaller values of α skew the tradeoff in favour of efficiency, and larger values in favour of fairness. The case of $\alpha = 1$ (corresponding to $U(x) = \log(x)$) gives *proportional fairness* (PF).

Since $\mathcal{S}$, $\mathcal{S}_a$, and $\mathcal{S}^0_a(\mathbf{z})$ are all closed, bounded, and convex subsets of $\mathbb{R}^n$, and since $\mathcal{U}$ is a concave continuously differentiable function, standard theorems ([17, Theorem 2.41], [17, Theorem 4.16], [18, Proposition 2.1.1, p. 193]) imply that (a) $\mathcal{U}$ attains its maximum on each of these sets, (b) any local maximum is also a global maximum, and (c) if $\mathcal{U}$ is strictly concave, then it has a unique global maximum.

Our approach to the optimization of $\mathcal{U}$ over $\mathcal{S}$ is motivated by the following property.

Proposition 1: If $(\boldsymbol{\rho}, a)$ maximizes $\mathcal{U}$ over both $\mathcal{S}_a$ and $\mathcal{S}^0(\tilde{\mathbf{z}})$ where $\tilde{\mathbf{z}} = \zeta(\boldsymbol{\rho})$, then $(\boldsymbol{\rho}, a)$ maximizes $\mathcal{U}$ over $\mathcal{S}$. □

Proof: The proof works by showing that any change from $(\boldsymbol{\rho}, a)$ to $(\boldsymbol{\rho}', a') \in \mathcal{S}$ within a sufficiently small ϵ-ball around $(\boldsymbol{\rho}, a)$ can be decomposed as a superposition of two changes, one that preserves the muting proportion at a, and another that remains consistent with the cell assignment indicator vector $\tilde{\mathbf{z}}$. Since $(\boldsymbol{\rho}, a)$ is optimal within $\mathcal{S}_a$ by assumption, any small change that preserves the muting proportion at a results in a decrease in the utility. Since $(\boldsymbol{\rho}, a)$ also maximizes $\mathcal{U}$ over $\mathcal{S}^0(\tilde{\mathbf{z}})$ where $\tilde{\mathbf{z}} = \zeta(\boldsymbol{\rho})$, any small change that preserves the cell assignment indicator at $\tilde{\mathbf{z}}$ also results in a decrease in the utility. Thus when these two changes are superposed, the net change also results in a decrease in utility. Since this is true for any change within an ϵ-ball, [18, Proposition 2.1.2] implies that $(\boldsymbol{\rho}, a)$ must be optimal over $\mathcal{S}$. ■

Proposition 1 decomposes the global optimization over $\mathcal{S}$ into two sub-problems: (i) optimal load-balancing, finding the best cell assignment for a fixed muting proportion (i.e. maximizing $\mathcal{U}$ over $\mathcal{S}_a$), and (ii) optimal resource partitioning, finding the best muting proportion for a fixed cell assignment (i.e. maximizing $\mathcal{U}$ over $\mathcal{S}^0(\mathbf{z})$). If $(\boldsymbol{\rho}, a)$ is simultaneously optimal for both then it is also globally optimal over $\mathcal{S}$.

III. Load-Balancing for a fixed muting proportion

We now analyze the maximization of $\mathcal{U}(\boldsymbol{\rho}, a)$ over $\mathcal{S}_a$ keeping the muting proportion fixed at a.

Proposition 2: $(\boldsymbol{\rho}^*, a)$ is optimal over $\mathcal{S}_a$ iff $(\boldsymbol{\rho}^*, a)$ satisfies (5), and there exist non-negative constants $v_c, c = 1, \cdots, 2P+M$, such that for all users u and cells c,

$$\frac{\partial \mathcal{U}}{\partial \rho_c(u)}(\boldsymbol{\rho}^*, a) = R_c(u) w_u \frac{\partial U_u}{\partial T}(T(\boldsymbol{\rho}^*; u)) \begin{cases} = v_c & \text{if } \rho_c^*(u) > 0 \text{, and} \\ \leq v_c & \text{otherwise.} \end{cases} \tag{7}$$

□

Note that if, for some user u, $\rho_c^*(u) > 0$ for multiple cells c, say for cells c_1 and c_2, then we must have

$$w_u \frac{\partial U_u}{\partial T}(T(\boldsymbol{\rho}^*; u)) = \frac{v_{c_1}}{R_{c_1}(u)} = \frac{v_{c_2}}{R_{c_2}(u)}.$$

Similarly, for any user u, if $\rho_{c_1}^*(u) > 0$ and $\rho_{c_2}^*(u) = 0$ for cells c_1, c_2, then we must have

$$w_u \frac{\partial U_u}{\partial T}(T(\boldsymbol{\rho}^*; u)) = \frac{v_{c_1}}{R_{c_1}(u)} \leq \frac{v_{c_2}}{R_{c_2}(u)}.$$

Thus user u will have $\rho_{c^*}^*(u) > 0$ for cell c^* only if

$$c^* = \arg\min_c \frac{v_c}{R_c(u)}. \tag{8}$$

Proof: The proof largely follows the application of [18, Proposition 2.1.2] described in in [18, Example 2.1.2]. ■

A further important sub-problem is maximization of $\mathcal{U}$ over $\mathcal{S}_a^0(\mathbf{z})$, fixing both the muting proportion a and the cell assignment indicator $\mathbf{z}$. The key difference from the optimization over $\mathcal{S}_a$ above is that if a user u is indicated as not assigned to cell c (i.e. $z_c(u) = 0$), that user need not satisfy the inequality part of (7). Thus $\mathbf{z}$ may constrain user u to be assigned to a sub-optimal cell, but user u will get an optimal allocation subject to this constraint.

Proposition 3: $(\boldsymbol{\rho}^*, a)$ is optimal over $\mathcal{S}_a^0(\mathbf{z})$ iff $(\boldsymbol{\rho}^*, a)$ satisfies (5), and there exist non-negative constants v_c, $c \in \{1, \cdots, 2P+M\}$, such that for all users u and cells c,

$$\frac{\partial \mathcal{U}}{\partial \rho_c(u)}(\boldsymbol{\rho}^*, a) = R_c(u) w_u \frac{\partial U_u}{\partial T}(T(\boldsymbol{\rho}^*; u)) \begin{cases} = v_c, & \text{if } \rho_c^*(u) > 0 \\ \leq v_c, & \text{if } \rho_c^*(u) = 0 \text{ and } z_c(u) = 1. \end{cases} \tag{9}$$

□

We note here that given a muting proportion a and a cell assignment indicator $\mathbf{z}$, the resource allocation $\boldsymbol{\rho}$ that achieves this optimum over $\mathcal{S}_a^0(\mathbf{z})$ is achieved by an appropriately designed scheduler. In a single-cell context, the PF scheduler [3] achieves an allocation that maximizes the log utility function for a given set of users within a single cell. The PF scheduler, at each iteration, tries to equalize among the users in the cell a *PF metric* which is derived from the gradient of the log utility function. At the log-utility-maximizing allocation, the PF metrics of all users in the cell will be equalized. This equalized PF metric of the users is the equivalent of the v_c coefficient of (9) in a single-cell context. For more general utility functions like (6), the gradient-based scheduler algorithm [4], [6] behaves similarly, achieving the utility-maximizing allocation by equalizing a *generalized PF metric* (GPF metric) which is the single-cell equivalent of v_c in (9) for the appropriate utility. In the multi-cell case, given a muting proportion a and an assignment $\mathbf{z}$ of users to cells, the same gradient-based scheduler can be operated in each cell c over the users u assigned to that cell (i.e. with $z_c(u) = 1$). In this case, the throughput of a user used in computing the gradient of the utility should be taken to be the total throughput provided to the user across all cells. Thus if there is a user that are assigned to multiple cells (i.e. if for some user u, $z_c(u) = 1$ for more than one cell c), then appropriate coordination or messaging is required so that the scheduling within each cell can take into account the user's total throughput. With this modification, the same gradient-based scheduler achieves the multi-cell utility-maximizing allocation $\boldsymbol{\rho}^*$ for the given muting proportion a and the cell assignment $\mathbf{z}$, i.e. it yields the $(\boldsymbol{\rho}^*, a)$ that maximizes $\mathcal{U}$ over $\mathcal{S}_a^0(\mathbf{z})$ and satisfies (9). The gradient-based scheduler within each cell c would equalize the GPF metric among the users for which $\rho_c^*(u) > 0$, yielding the coefficients v_c in (9).

Denote $\mu_c = a$ if c is an ABS cell, and $\mu_c = 1 - a$ if c is a non-ABS cell. Let $\beta_c(u)$ be the fraction of user u's throughput that is delivered by cell c. Then for the logarithmic utility function, we can show from (9) the identity

$$\mu_c v_c = \sum_u w_u \beta_c(u). \tag{10}$$

Consider the case where all the weights are unity ($w_u = 1$). If each user gets its entire throughput from just one cell (i.e. for each u, $\beta_c(u) = 1$ for some c), then the right hand side is N_c, the number of users getting their throughput from cell c. Thus (10) relates the PF metric v_c to the number of users in the cell. But if for some user u, $0 < \beta_c(u) < 1$ for some c, then that user essentially contributes to v_c as a "fractional user" within cell c, with a fraction $\beta_c(u)$. The following proposition clarifies the conditions under which a user can get its throughput partially from multiple cells, and the criterion that determines whether a user prefers one cell over another.

Proposition 4: Consider $U_u(T) = \log(T)$ and $w_u = 1$ for all u. Suppose $(\boldsymbol{\rho}^*, a)$ maximizes $\mathcal{U}$ over $\mathcal{S}_a$, with the corresponding constants v_c as in Proposition 2. Consider any two cells (without loss of generality, cells 1 and 2). Suppose there are L users u such that $\rho_c^*(u) > 0$ for at least one $c \in \{1, 2\}$. Then the following properties hold.

1) If $\rho_1^*(u) > 0$ then $R_1(u)/R_2(u) \geq v_1/v_2$, and if $\rho_2^*(u) > 0$ then $R_1(u)/R_2(u) \leq v_1/v_2$. That is, users prefer cell 1 to cell 2 based on a *threshold rule*, comparing $R_1(u)/R_2(u)$ to a threshold v_1/v_2.
2) Suppose the L users have distinct values of $R_1(u)/R_2(u)$, and the users are numbered $1, \cdots, L$ in increasing order of $R_1(u)/R_2(u)$. Let $\sigma = \mu_1 v_1 + \mu_2 v_2$.

Then $\mu_2 v_2$ is the abscissa of the intersection of the functions $F_1 : (0,1] \to \mathbb{R}^+$ and $F_2 : (0,1] \to \mathbb{R}^+$ defined below, and v_1/v_2 (the threshold) is the ordinate.

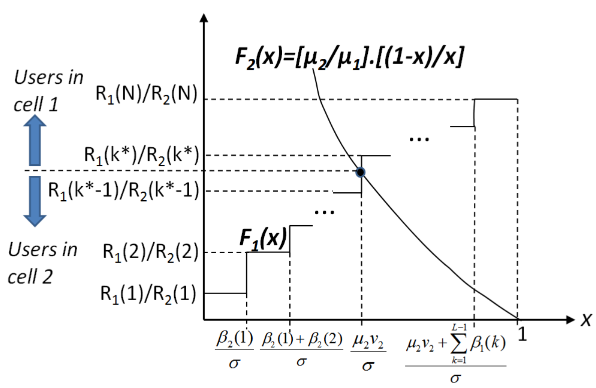
For $i = 1, \cdots, L$, define $F_1(x)$ and $F_2(x)$ as

$$F_1(x) = \frac{R_1(i)}{R_2(i)} \begin{cases} \text{if } x \in \left[0, \frac{\mu_2 v_2}{\sigma}\right] \\ \quad \text{and } x \in \left[\frac{\sum_{k=1}^{i-1} \beta_2(u)}{\sigma}, \frac{\sum_{k=1}^{i} \beta_2(u)}{\sigma}\right] \\ \text{or, if } x \in \left(\frac{\mu_2 v_2}{\sigma}, 1\right] \\ \quad \text{and } x \in \left[\frac{\mu_2 v_2 + \sum_{k=1}^{i-1} \beta_1(u)}{\sigma}, \right. \\ \quad \left. \frac{\mu_2 v_2 + \sum_{k=1}^{i} \beta_1(u)}{\sigma}\right] \end{cases} \tag{11}$$

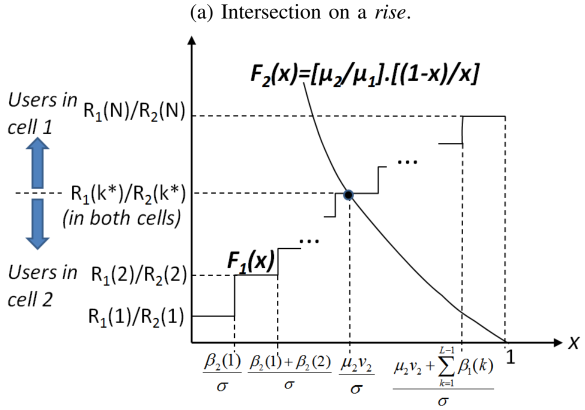
$$F_2(x) = \frac{\mu_2}{\mu_1} \frac{1-x}{x}. \tag{12}$$

□

F_1 is a staircase function which takes values $R_1(u)/R_2(u)$ for the L users. Due to the ordering of the users, the steps of the staircase to the left of $x^* = \mu_2 v_2/\sigma$ correspond to users u for whom $R_1(u)/R_2(u) \le v_1/v_2$, whereas the steps of the staircase to the right of $x^* = \mu_2 v_2/\sigma$ correspond to users u for whom $R_1(u)/R_2(u) > v_1/v_2$. This is graphically illustrated in Fig. 1. The curves for F_1 and F_2 intersect at $x^* = \mu_2 v_2/\sigma$, and $F_2(x^*) = v_1/v_2$. If k^* is the lowest user index such that $R_1(k^*)/R_2(k^*) \ge v_1/v_2$, then all users $u < k^*$ (in the renumbered order) will get zero resources from cell 1, while all users $u \ge k^*$ will get zero resources from cell 2. If the intersection occurs at a *rise* of the staircase function F_1, then there is no user that will receive non-zero resources from both cells. This case is illustrated in Fig. 1a. If the intersection occurs at a *flat* of the function $F_1(\cdot)$, then user k^* (in the renumbered order) gets resources from both cells, as illustrated in Fig. 1b, and for this user, $R_1(k^*)/R_2(k^*) = v_1/v_2$.



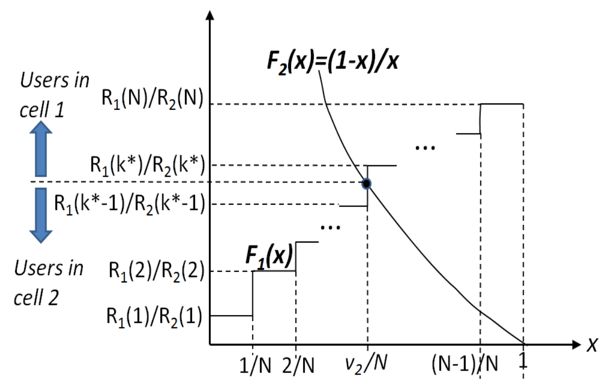
(a) Intersection on a *rise*.



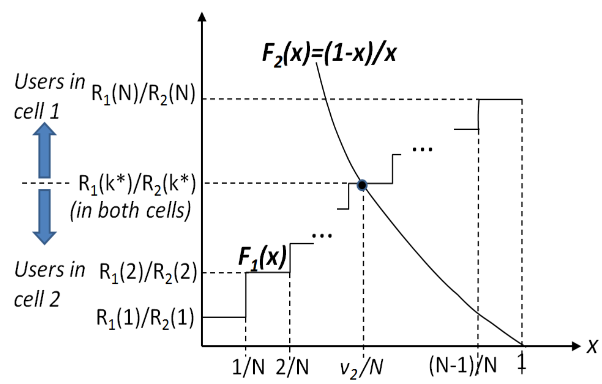
(b) Intersection on a *flat*.

Fig. 1: Graphical illustration of Proposition 4

As a special case of this, suppose there are just two macro cells in the system (no pico cells) with ABS turned off (i.e. $a = 0$), so that $\mu_1 = \mu_2 = 1$ and $L = N$, the total number of users. The behaviour reduces to the illustration of Fig. 2.



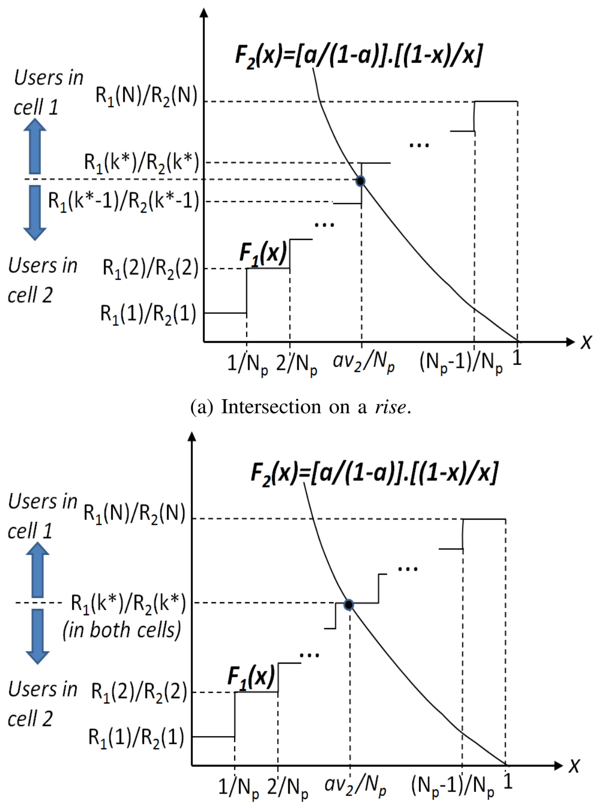
(a) Intersection on a *rise*.



(b) Intersection on a *flat*.

Fig. 2: Two macro cells.

As a second special case, consider a single macro cell with a single pico-cell within the coverage of the macro, with a given muting proportion a. In this case there are three (sub-)cells: the pico's non-ABS sub-cell, the pico's ABS sub-cell, and the macro-cell. Take the cells 1, 2 from Proposition 4 to be the pico's non-ABS sub-cell and ABS sub-cell respectively. At the $(\boldsymbol{\rho}^*, a)$ that optimizes $\mathcal{U}$ over $\mathcal{S}_a$, suppose there are $L = N_p$ users which are allocated non-zero resources within either the pico's ABS sub-cell or the pico's non-ABS sub-cell. Proposition 4 indicates that the determination by the pico cell of which users should receive allocations within the non-ABS resources (i.e. $\rho_1^* > 0$) and which users within the ABS resources (i.e. $\rho_2^* > 0$) also follows a threshold rule with a threshold of v_1/v_2, as illustrated in Fig. 3. Pico users for whom $R_1(u)/R_2(u) > v_1/v_2$ are allocated non-ABS resources, and pico users for whom $R_1(u)/R_2(u) < v_1/v_2$ are allocated ABS resources. A pico user who has $R_1(u)/R_2(u) = v_1/v_2$ (user k^* in Fig. 3b) can get allocations from both ABS and non-ABS resources, . This holds more widely with multiple picos under



(a) Intersection on a *rise*.

(b) Intersection on a *flat*.

Fig. 3: One macro and one pico with a given a - optimal allocation between ABS and non-ABS within the pico.

a macro, and not just with a single pico as in this example.

The threshold rule in Proposition 4 also provides a prescription for improving the cell assignment $\mathbf{z}$. Consider any current serving cell $s(u)$ of user u. Suppose that for some cell c,

$$\frac{R_{s(u)}(u)}{R_c(u)} < \frac{v_{s(u)}}{v_c}, \text{ or, } \frac{R_{s(u)}(u)}{v_{s(u)}} < \frac{R_c(u)}{v_c}. \tag{13}$$

Per Proposition 4, it would be preferable to allocate user u resources from cell c rather than from $s(u)$. Thus to improve the cell assignment $\mathbf{z}$, we should identify a user u and cell c for which (13) holds but $z_c(u) = 0$ (i.e. the current cell assignment indicator $\mathbf{z}$ does not allow user u to get resources from cell c), and then modify the cell assignment by setting $z_c(u) = 1$. Although Proposition 4 restricts to the log utility and $w_u = 1$, we can show that this cell assignment improvement criterion holds more generally as well.

This cell assignment improvement process enables the user to get resources in an additional cell, but does not necessarily prevent the user from also getting resources from its original cell, since it may be optimal to allocate the user resources from both. This can be thought of as a generalization of handover.

IV. Optimal Resource Partition for a given Cell Assignment

We now look at the conditions for optimality of a muting proportion a^* and a corresponding allocation $\boldsymbol{\rho}^*$, keeping the cell assignment indicator $\mathbf{z}$ fixed, i.e. conditions for optimality of $(\boldsymbol{\rho}^*, a^*)$ over $\mathcal{S}^0(\mathbf{z})$.

First note that if $(\boldsymbol{\rho}^*, a^*)$ is optimal over $\mathcal{S}^0(\mathbf{z})$, it has to be optimal over $\mathcal{S}^0_{a^*}(\mathbf{z})$, for which the optimality conditions are given by (9). Then we assert the following.

Proposition 5: For $a^* \in [\epsilon_a, 1-\epsilon_a]$, if $(\boldsymbol{\rho}^*, a^*)$ is optimal over $\mathcal{S}^0_{a^*}(\mathbf{z})$, it is also optimal over $\mathcal{S}^0(\mathbf{z})$ if and only if

$$\begin{aligned} a^* = \epsilon_a \quad &\text{and} \quad \sum_{c \text{ is ABS}} v_c \le \sum_{c \text{ is non-ABS}} v_c, \\ \text{or } a^* \in (\epsilon_a, 1-\epsilon_a) \quad &\text{and} \quad \sum_{c \text{ is ABS}} v_c = \sum_{c \text{ is non-ABS}} v_c, \\ \text{or } a^* = 1-\epsilon_a \quad &\text{and} \quad \sum_{c \text{ is ABS}} v_c \ge \sum_{c \text{ is non-ABS}} v_c. \end{aligned} \tag{14}$$

□

Proof: The proof uses the fact that $(\boldsymbol{\rho}^*, a^*)$ satisfies (9) to show that the condition of optimality given by [18, Proposition 2.1.2] is equivalent to (14). ■

Proposition 5 indicates that $\sum_{c \text{ is ABS}} v_c$ (respectively $\sum_{c \text{ is non-ABS}} v_c$) can be interpreted as a measure of the total loading of the ABS (respectively non-ABS) resources. At the optimal muting proportion, the loading of the ABS resources is balanced by the loading of the non-ABS resources.

We now look at finding a new muting proportion b^* in the appropriate direction while preserving the cell assignment $\mathbf{z}$, and a corresponding allocation $(\boldsymbol{\rho}', b^*) \in \mathcal{S}^0_{b^*}(\mathbf{z})$ which improves the utility over $(\boldsymbol{\rho}, a)$.

Proposition 6: Suppose $(\boldsymbol{\rho}, a)$ maximizes $\mathcal{U}$ over $\mathcal{S}^0_a(\mathbf{z})$. For any muting proportion $b \in [\epsilon_a, 1-\epsilon_a]$, let $\boldsymbol{\rho}^{sc} \in \mathcal{S}^0_b(\mathbf{z})$ be a scaled version of $\boldsymbol{\rho}$, defined as follows.

$$\begin{aligned} \text{If } c \text{ is ABS, } \rho^{sc}_c(u) &= \frac{b}{a}\rho_c(u) \\ \text{If } c \text{ is non-ABS, } \rho^{sc}_c(u) &= \frac{(1-b)}{(1-a)}\rho_c(u) \end{aligned} \tag{15}$$

Then for $U_u(\cdot) = U^\alpha(\cdot)$ defined in (6),

$$\mathcal{U}(\boldsymbol{\rho}^{sc}, b^*) \ge \mathcal{U}(\boldsymbol{\rho}, a)$$

where

$$b^* = \max\left[\epsilon_a, \min\left(1-\epsilon_a, \frac{\frac{a}{1-a}\left(\sum_{c \text{ is ABS}} v_c\right)^{1/\alpha}}{\frac{a}{1-a}\left(\sum_{c \text{ is ABS}} v_c\right)^{1/\alpha} + \left(\sum_{c \text{ is non-ABS}} v_c\right)^{1/\alpha}}\right)\right]. \tag{16}$$

□

Proof: Having determined the direction of increase in Proposition 5, we find the best value of b in that direction based on the scaled resource allocations $\boldsymbol{\rho}^{sc}$. The proof shows that $\mathcal{U}(\boldsymbol{\rho}^{sc}, b)$ is a concave function of b, and b^* is the point at which this is maximized. ■

V. Iterative Algorithm

We now propose an iterative procedure that successively uses the cell assignment improvement condition (13) and the muting proportion improvement (16), and show that this converges to the optimal solution that maximizes the utility. We restrict attention to the family of utility functions (6).

At step 1, start with an arbitrary cell assignment indicator $\mathbf{z}_1$ and arbitrary muting proportion $a_1 \in [\epsilon_a, 1-\epsilon_a]$. Find $\boldsymbol{\rho}_1$ such that $(\boldsymbol{\rho}_1, a_1)$ maximizes $\mathcal{U}$ over $\mathcal{S}^0_{a_1}(\mathbf{z}_1)$ (e.g. by using the gradient-based scheduler). Choose $\lambda_1 \in (0,1)$. For each iteration $i = 1, 2, \cdots$, if there is at least one user u and cell c satisfying

$$\rho_{i,c}(u) = 0 \text{ and } \frac{R_c(u) v_{s_i(u)}}{R_{s_i(u)}(u)} - v_{i,c} > 0, \tag{17}$$

(where $s_i(u)$ is any serving cell of user u at iteration i i.e. $\rho_{s_i(u),i}(u) > 0$), then among such users, pick c, u as follows.

1) Set $m = 1$.
2) Pick any c, u for which
$$\frac{R_c(u) v_{s_i(u)}}{R_{s_i(u)}(u)} - v_{i,c} > \lambda_i^m. \tag{18}$$
If there is no such c, u, set $m = m+1$, and repeat.
3) Set $\lambda_{i+1} = \lambda_i^m$, where m is the value reached in the above step.

The metric in (17) is derived from the cell assignment improvement condition in (13). The above selection ensures that any c, u for which (17) is becoming vanishingly small are ignored if there are other c, u for which the metric is larger. With the chosen c, u, let

$$\begin{aligned} \delta_i^1 &= \frac{R_c(u) v_{s_i(u)}}{R_{s_i(u)}(u)} - v_{i,c} \\ &\quad \text{(or 0 if there was no } c, u \text{ satisfying (17))} \\ \delta_i^2 &= \left| \sum_{c \text{ is ABS}} v_{i,c} - \sum_{c \text{ is non-ABS}} v_{i,c} \right|. \end{aligned} \tag{19}$$

Execute one of Operations 1 and 2 below to go from $(\boldsymbol{\rho}_i, a_i)$ to $(\boldsymbol{\rho}_{i+1}, a_{i+1})$, based on the following rule. If only one of $\delta_i^k, k = 1, 2$ is positive, execute operation k. If both are positive, execute operation $k^* = \arg\max_k \delta_i^k$. If neither is positive, stop.

Operation 1 takes $(\boldsymbol{\rho}_i, a_i)$ to $(\boldsymbol{\rho}_{i+1}, a_{i+1})$ keeping the muting proportion constant ($a_{i+1} = a_i$) but improving the cell assignment. It comprises the following:

1) Set $a_{i+1} = a_i$.
2) Let $\mathbf{z}'$ be identical to $\zeta(\boldsymbol{\rho}_i)$ for all users and cells except setting $z'_c(u) = 1$ for the selected c, u.
3) Find $\boldsymbol{\rho}_{i+1}$ such that $(\boldsymbol{\rho}_{i+1}, a_{i+1})$ maximizes $\mathcal{U}(\cdot)$ over $\mathcal{S}^0_{a_{i+1}}(\mathbf{z}')$ (e.g. using the gradient-based scheduler).
4) Set $\mathbf{z}_{i+1} = \zeta(\boldsymbol{\rho}_{i+1})$.

Operation 2 takes $(\boldsymbol{\rho}_i, a_i)$ to $(\boldsymbol{\rho}_{i+1}, a_{i+1})$ improving the muting proportion while staying consistent with the cell assignment, as follows.

1) Set
$$a_{i+1} = \frac{\frac{a_i}{1-a_i}\left(\sum_{c \text{ is ABS}} v_{c,i}\right)^{\frac{1}{\alpha}}}{\frac{a_i}{1-a_i}\left(\sum_{c \text{ is ABS}} v_{c,i}\right)^{\frac{1}{\alpha}} + \left(\sum_{c \text{ is non-ABS}} v_{c,i}\right)^{\frac{1}{\alpha}}} \tag{20}$$
2) Set $\boldsymbol{\rho}_{i+1}$ such that $(\boldsymbol{\rho}_{i+1}, a_{i+1})$ maximizes $\mathcal{U}(\cdot)$ over $\mathcal{S}^0_{a_{i+1}}(\mathbf{z}_i)$ (e.g. using the gradient-based scheduler).
3) Set $\mathbf{z}_{i+1} = \zeta(\boldsymbol{\rho}_{i+1})$.

Note that with the above iteration steps, for each i, by construction, $(\boldsymbol{\rho}_i, a_i)$ is optimal over $\mathcal{S}^0_{a_i}(\mathbf{z}_i)$.

Proposition 7: The iteration $\{(\boldsymbol{\rho}_i, a_i)\}$converges to $(\boldsymbol{\rho}, a)$ which maximizes $\mathcal{U}$ over $\mathcal{S}$. □

Proof: The key steps are as follows. The sequence of utilities $\{\mathcal{U}(\boldsymbol{\rho}_i, a_i)\}$ is monotonic increasing by construction, since the operation at each iteration improves the utility. Further $\{\mathcal{U}(\boldsymbol{\rho}_i, a_i)\}$ is bounded above by the global maximum and thus converges, say to $\overline{U}$. Now the set $\mathcal{S}$ is compact, so sequence $(\boldsymbol{\rho}_i, a_i)$ in compact set $\mathcal{S}$ has at least one convergent subsequence $(\boldsymbol{\rho}_{i_k}, a_{i_k})$, and its limit point $(\boldsymbol{\rho}, a) \in \mathcal{S}$ since $\mathcal{S}$ is closed. By the continuity of $\mathcal{U}$, we must have $\mathcal{U}(\boldsymbol{\rho}, a) = \overline{U}$. Now we can show that if the limit point $(\boldsymbol{\rho}, a)$ of the convergent subsequence $(\boldsymbol{\rho}_{i_k}, a_{i_k})$ does not optimize $\mathcal{U}$ over $\mathcal{S}_a$ (respectively $\mathcal{S}^0(\tilde{\mathbf{z}})$, where $\tilde{\mathbf{z}} = \zeta(\boldsymbol{\rho})$), then Operation 1 (respectively Operation 2) would give an improvement in the utility that does not asymptotically vanish, contradicting the fact that the sequence of utilities is convergent. Thus the limit point of any convergent subsequence, $(\boldsymbol{\rho}, a)$, must optimize $\mathcal{U}$ over both $\mathcal{S}_a$ as well as $\mathcal{S}^0(\tilde{\mathbf{z}})$. By Proposition 1, $(\boldsymbol{\rho}, a)$ is optimal over $\mathcal{S}$. Since the utility functions (6) are strictly concave, $(\boldsymbol{\rho}, a)$ must be the unique global maximum. Finally we show that every subsequence is convergent, so that the whole iteration sequence converges to $(\boldsymbol{\rho}, a)$. ■

VI. Summary

In this paper, we have addressed the problem jointly optimal load-balancing and resource partitioning for eICIC, using a utility-maximization formulation for optimizing the assignment of users to macro and pico cells and the muting proportion. Proposition 1 decomposes the joint optimization problem by showing that it suffices to find the optimal cell assignment for a given muting proportion, and the optimal muting proportion for a given cell assignment - a solution which simultaneously optimizes both of these will also be globally optimal. Proposition 2 characterizes optimal load-balancing for a given muting proportion with the conditions for optimality of a cell assignment of users. It further helps us identify a criterion for improving the utility by modifying the cell assignment one user at a time while keeping the muting proportion constant. Proposition 5 characterizes the optimal resource partitioning for a given assignment of users to cells with the conditions for optimality of a muting proportion. Proposition 6 provides a criterion for improving the utility by modifying the muting proportion while remaining consistent with the assignment of users to cells. We have proposed an an iterative procedure that successively uses these utility improvement operations and converges to the global optimum.

Along the way we have also identified several key properties. One is the characterization of the rule by which a cell is considered preferable for a given user over another cell. Proposition 4 shows that this behaves like a threshold rule, comparing the ratio of a user's achievable spectral efficiencies in the two cells to a certain threshold. We have shown a graphical interpretation wherein the threshold point is identified as the intersection of a certain step function F_1 (constructed using the ratios of the users' spectral efficiencies in the two cells) with another monotonically decreasing function F_2. This graphical interpretation also illustrates the case where the optimal resource allocation for some user may consist of allocations from multiple cells, depending on whether the interesection F_1 and F_2 occurs on a rise or a flat or a rise of F_1. We pointed out a special case of this for two macro cells (without ABS), and another for the threshold rule between the non-ABS and ABS resources within a pico cell.

In the formulation in this paper, we restricted the muting proportion to $[\epsilon_a, 1 - \epsilon_a]$, where $\epsilon_a > 0$ can be chosen arbitrarily small. We plan to address the case of $\epsilon_a = 0$ in a future paper. We assumed that all macro cells employ the same muting proportion, but we note that the formulation also encompasses typical scenarios where different macros can use different muting proportions. For example, in many deployments a pico cell may be largely overlapped only by a single macro, with only mild interference from other macros. Our results can then be applied considering the set of relevant cells to consist of just a given macro and its underlay picos, allowing each macro to determine its optimal muting proportion separately from other macros. To mitigate residual interference when the macros choose different muting proportions, the set of allowed muting patterns can be designed to have maximal alignment and minimize the mismatch. We have also assumed that the spectral efficiencies of a user relative to each cell are constant and not time-varying. In practice, this is suitable for either slow-time-scale adaptation of the muting proportion and cell assignment by using long-term average measurements, or for fast time-scale adaptation using short-term measurements if facilitated by the network architecture. A time-varying channel can also yield multi-user diversity gain, which was not considered in this paper. Stochastic analysis of our algorithm with time-varying channels is left to future work.

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