A double-auction mechanism for mobile data-offloading markets with strategic agents

K. P. Naveen* and Rajesh Sundaresan†

*Department of Electrical Engineering, Indian Institute of Technology Tirupati, India

†Department of ECE and Robert Bosch Centre for Cyber Physical Systems, Indian Institute of Science, Bangalore, India

Email: naveenkp@iittp.ac.in, rajeshs@iisc.ac.in

***Abstract*—We consider a recently proposed double-auction mechanism for mobile data-offloading. Network operators (users) derive benefit from offloading their traffic to third party WiFi or femtocell network (link-supplier). A link-supplier experiences costs for the additional capacity that he provides. Users and link-supplier (collectively referred to as agents) have their utilities and cost function as private knowledge. A system-designer decomposes the problem into a network problem (with surrogate utilities and surrogate cost functions) and agent problems (one per agent). The surrogate utilities and cost functions are modulated by the agents' bids. Agents' payoffs and costs are then determined by the allocations and prices set by the system designer. So long as the agents do not anticipate the effect of their actions, a competitive equilibrium exists as a solution to the network and agent problems, and this equilibrium optimizes the system utility. This work shows that when the agents are strategic (price-anticipating), the presence of strategic supplying agents drives the system to an undesirable equilibrium with zero participation. This is in stark contrast to the setting when link-suppliers are not strategic where the efficiency loss is at most 34%. The paper then proposes a Stackelberg game modification to alleviate the efficiency loss problem. The system designer first announces the allocation and payment functions. He then invites the supplying agents to announce their bids. He then invites the users to respond to the suppliers' bids. The resulting efficiency loss is characterized in terms of the suppliers' cost functions.**

I. INTRODUCTION

Mobile data offloading is an attractive way to manage growth in mobile-data traffic. Traffic meant for the macrocellular network can be offloaded to already installed third-party Wi-Fi or femtocell networks. This provides an alternative means of network expansion. Wi-Fi access-point operators and femtocell network operators will however expect compensation for allowing macrocellular network traffic through their access points. Studies for secure and seamless offloading are already in place (see [1]–[4]).

In recent work, Iosifidis et al. [5] proposed a double-auction mechanism where a set of mobile network operators (buyers or users in this work) compete for resources from access-point operators (sellers or links in this work). The utilities of the users and costs of the links are private information to the respective parties. A centralized broker collects how much each network operator is willing to pay each access-point operator, collects indications of the costs at each access point, and then determines how much of traffic should be offloaded to each access point and how much each will pay or get. The mobile network operators and the access-point operators then comply. This is a scenario with an asymmetric information structure where (a) the broker is not aware of the actual needs and costs of network and access-point operators, (b) each operator is aware only of his own needs or costs, and (c) all agents are *price-taking* (made precise in the next section). Following Kelly et al. [6], Iosifidis et al. [5] showed that a tâtonnement procedure converges to the system optimal operating point.

Iosifidis et al. [5, p.1635] point out that designing incentive compatible mechanisms for double-auctions which are weakly budget balanced (the broker should not end up subsidizing the mechanism) is 'notoriously hard' and has been done only in certain simplified settings (McAfee auction [7]) or can be computationally intensive. So [5] took a network utility maximization approach and left the analysis of the price-anticipating scenario open [5, Sec VII, p.1646].

We do the following in this paper. (1) We first re-derive the result on efficient allocation when the agents are price-taking, mainly to set up the notation for the next two results. (2) We then analyze the price-anticipating scenario along the lines of Johari et al. [8]. When agents are price-anticipating, they recognize the effect of their bids on the allocation. The appropriate equilibrium notion is a Nash equilibrium. The situation in Johari et al. [8], when mapped to the current offloading setting, would be one where the access-point operators are not strategic. The efficiency loss due to price-anticipating mobile offloading agents is then at most 34%. However, when the access-point agents are also strategic and price-anticipating, the equilibrium is one where the offloading agents prefer not to offload any traffic. The efficiency loss is then 100%. The main message is that the earlier proposed double-auction mechanism of [5] works when agents are price-taking, but not in the more real situation when agents also are price-anticipating. In the latter case, one must look for alternative double-auction mechanisms. (3) We then propose a modified allocation mechanism where the supplying agent bids first and the users bid in response. The allocations and payments are a priori announced functions of the bids. We then characterize the resulting efficiency loss in terms of the supplier's cost function. For instance, for the quadratic link-cost function, the worst-case efficiency loss (with the worst-case taken over

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users' utility functions) is at most 25%.

The paper is organized as follows. In Section II, we discuss the system model and problem definition. In Section III, we discuss the price-taking scenario for the single-link case. In Section IV, we analyze the price-anticipating scenario. As a positive result, in Section V, we discuss our proposed mechanism and characterize the worst-case efficiency loss in terms of the supplying agent's cost function. To focus on the flow of key ideas, we have moved the proofs of our main results to Section VI (other proofs that are along standard lines are omitted due to page limitation). The paper concludes with some remarks in Section VII.

II. System Model and Problem Definition

Consider a scenario where M *users* intend to share the bandwidth of a (single) *link* of capacity $C > 0$ owned by a *link-supplier*. In the context of mobile-data offloading [5], users and link-supplier correspond to mobile-network operators and an access-point operator (e.g., Wi-Fi, femtocell), respectively. The mobile-network operators want to buy a share of the limited bandwidth resource available at the access point to offload their macrocellular traffic, while the access point operator is interested in maximizing his profit. In the double auction terminology [7], users are synonymous to buyers bidding for a share of a resource while the link-supplier is the seller. We refer to the users and the link-supplier collectively as *agents*. The social planner, the entity that designs the mechanism (rules for information transfer, allocation, and payments) is referred to as the *network-manager*,

Let x_m denote the rate requested by user $m = 1, 2, \cdots, M$, and let y_m be the rate the link-supplier is willing to allocate to user m. Thus, $\mathbf{x} = (x_1, x_2, \cdots, x_M)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_M)$ represent the *rate-request* and *rate-allocation* vectors, respectively. Let $y = \sum_m y_m$ denote the aggregate-rate allocated by the link-supplier to all users. For user m, the benefit of acquiring a rate of x_m is represented by a utility function $U_m(x_m)$; we assume that U_m, $m = 1, 2, \cdots, M$, are concave, strictly increasing and continuously differentiable with finite $U'_m(0)$. Similarly, the cost incurred by the link-supplier for accepting to serve an aggregate rate of y is given by $V(y)$, where V is strictly convex, strictly increasing and continuously differentiable. Thus, the system optimal solution is the solution to the optimization problem:

SYSTEM

$$\text{Maximize:} \quad \sum_m U_m(x_m) - V\left(\sum_m y_m\right) \tag{1a}$$

$$\text{Subject to:} \quad \sum_m y_m \leq C \tag{1b}$$

$$x_m \leq y_m \ \forall m \tag{1c}$$

$$x_m \geq 0, y_m \geq 0 \ \forall m. \tag{1d}$$

Continuity of the objective function and compactness of the constraint set imply that an optimal solution $\mathbf{x}^s = (x_1^s, x_2^s, \cdots, x_M^s)$ and $\mathbf{y}^s = (y_1^s, y_2^s, \cdots, y_M^s)$ exists. Further, if U_m are strictly concave then (since V is strictly convex) the solution is unique. Since U_m are strictly increasing in x_m, an optimal solution must satisfy $\mathbf{x}^s = \mathbf{y}^s$. Thus, at optimality, the rate-requests (demand) and the rate-allocations (supply) are matched although the capacity C may not be fully utilized.

A network-manager, however, cannot solve the formulation in (1) without the knowledge of user utilities and the link-cost function. Hence, consider the following mechanism proposed by Iosifidis et al. in [5] for rate allocation. Each user m submits a *bid* $p_m \geq 0$ that denotes the amount he is willing to pay, while the link-supplier communicates signals β_m ($m = 1, 2, \cdots, M$) that implicitly indicate the amounts of bandwidth that he is willing to provide; we refer to $\mathbf{p} := (p_1, p_2 \cdots, p_M)$ and $\boldsymbol{\beta} := (\beta_1, \beta_2, \cdots, \beta_M)$ as the *bids* submitted by the users and the link-supplier, respectively.

The network-manager is responsible for fixing the *prices* μ_m ($m = 1, 2, \cdots, M$) and λ that determines the rate allocation. The prices $\boldsymbol{\mu} := (\mu_1, \mu_2, \cdots, \mu_M)$ and λ are supposed to be the optimal dual variables of the following network problem proposed by Iosifidis et al. in [5]:

NETWORK

$$\text{Maximize:} \quad \sum_m p_m \log(x_m) - \sum_m \frac{y_m^2}{2\beta_m} \tag{2a}$$

$$\text{Subject to:} \quad \sum_m y_m \leq C \tag{2b}$$

$$x_m \leq y_m \ \forall m \tag{2c}$$

$$x_m \geq 0, y_m \geq 0 \ \forall m. \tag{2d}$$

In the NETWORK problem above we choose to use $\boldsymbol{\beta}$ instead of a related $\boldsymbol{\alpha}$ that was used in the original formulation by Iosifidis et al. in [5]; the quantities $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related by $\beta_m = 1/\alpha_m \ \forall m$. Then each β_m is $\mathbb{R}_+$-valued, with values on the positive real line, while each α_m is in general $\mathbb{R}_+ \cup \{+\infty\}$-valued. Moreover, the signals in $\boldsymbol{\beta}$ are directly proportional to the amount of bandwidth the link-supplier is willing to share. For instance, a lower value of β_m implies that the bandwidth shared by the link-supplier with user m is low, and vice versa. In particular, $\beta_m = 0$ implies that the link-supplier is unwilling to share any bandwidth with user m. This will be useful while interpreting the Nash equilibrium bid-vectors (Theorem 2).

The above NETWORK problem is identical to the SYSTEM problem but with the true utility and cost functions replaced by *surrogate* utility and cost functions. In the following, we first review the case when the users and the link-supplier are *price-taking*. This means agents assume prices are given and do not anticipate the effect of their bids on the prices set by the network-manager. See Definition 1 below of a competitive equilibrium. We then proceed to study the more-involved *price-anticipating* scenario. Here the agents recognize that the effective price is based on their bids, anticipate the resulting allocation, payment, and therefore their utility, and act accordingly. The resulting utility functions are new functions of the bids; see Definition 2. Our methodology is similar to Johari et al. [8], but the outcome in the price-anticipating scenario is dramatically negative due to the presence of the strategic link-supplier, as we will soon see. We then propose a remedy via a Stackelberg framework where the link-supplier is a lead player and the users are followers.

III. Price-taking scenario

The sequence of exchanges (between the network-manager and the agents) in the price-taking scenario is as follows:

1) The network-manager initiates the bidding process by fixing the prices $(\boldsymbol{\mu}, \lambda)$.
2) The agents accept the prices and respond by announcing their respective bids, denoted $\mathbf{p}$ and $\boldsymbol{\beta}$.
3) The network-manager allocates a rate of $x_m = p_m/\mu_m$ to user m and receives a payment of p_m. Simultaneously, the link-supplier is asked to allocate a rate of $y_m = \beta_m(\mu_m - \lambda)$ to user m; the total payment made to the link-supplier is $\sum_m \beta_m(\mu_m - \lambda)^2$.

Thus, the pay-off to user m, for bidding p_m, is given by

$$P_m\Big(p_m; \mu_m\Big) = U_m\left(\frac{p_m}{\mu_m}\right) - p_m. \tag{3}$$

Similarly, the pay-off to the link-supplier is given by

$$\begin{aligned} & P_L\Big(\boldsymbol{\beta}; (\boldsymbol{\mu}, \lambda)\Big) \\ &= \; -V\left(\sum_m \beta_m\Big(\mu_m - \lambda\Big)\right) + \sum_m \beta_m\Big(\mu_m - \lambda\Big)^2. \end{aligned} \tag{4}$$

Using the above pay-off functions we characterize the solution as a *competitive equilibrium* which is defined as follows (unless mentioned otherwise, we assume that the agents' bids and the link-supplier's prices are non-negative, i.e., $p_m, \beta_m, \mu_m, \lambda \geq 0 \;\forall m$; also, we use $\mathbf{0}$ to denote the vector of all-zeros of appropriate length):

Definition 1 (Competitive Equilibrium [8], [9]): We say that $(\mathbf{p}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu})$ constitutes a competitive equilibrium if the following conditions hold:

(C1) $P_m(p_m; \mu_m) \geq P_m(\overline{p}_m; \mu_m) \;\forall \overline{p}_m \geq 0, \forall m$

(C2) $P_L\Big(\boldsymbol{\beta}; (\boldsymbol{\mu}, \lambda)\Big) \geq P_L\Big(\overline{\boldsymbol{\beta}}; (\boldsymbol{\mu}, \lambda)\Big) \;\forall \overline{\boldsymbol{\beta}} \geq \mathbf{0}$

(C3) Define $\mathcal{M} = \Big\{m : \mu_m \neq \lambda\Big\}$ and

$$\widehat{C} = \sqrt{\left(\sum_m p_m\right)\left(\sum_{m \in \mathcal{M}} \beta_m\right)}. \tag{5}$$

Then, the following should hold:

(C3-a) For all m,

$$\frac{p_m}{\mu_m} = \beta_m\Big(\mu_m - \lambda\Big); \tag{6}$$

(C3-b) For all $m \in \mathcal{M}$, the equality $\mu_m = \mu$ holds, where

$$\mu = \sum_i p_i \Big/ \min\Big\{C, \widehat{C}\Big\}\;; \tag{7}$$

(C3-c) Furthermore,

$$\lambda = \min\left\{0, \left(1 - \left(\frac{C}{\widehat{C}}\right)^2\right)\frac{\sum_i p_i}{C}\right\}. \tag{8}$$

In the above definition, condition (C1) implies that the users do not benefit by deviating from their equilibrium bids p_m, when the prices $(\lambda, \boldsymbol{\mu})$ set by the network-manager are fixed. Similarly, (C2) implies that the link-supplier has no benefit in deviating from the equilibrium bid-vector $\boldsymbol{\beta}$. Although (C1) and (C2) result in the optimality of the users' and the link-supplier's problem of maximizing their respective pay-offs, these conditions by themselves do not guarantee system-optimal performance. The conditions in (C3) (essentially derived from the optimality conditions for NETWORK) are crucial to guarantee that the prices $(\lambda, \boldsymbol{\mu})$ set by the network-manager are dual optimal for SYSTEM. Condition (C3) along with (C1) and (C2) can then be used to show the optimality of a competitive equilibrium. We summarize this result in the following theorem; in particular, we first prove the existence of a competitive equilibrium, and then derive its optimality property. This theorem is essentially an extension of the result due to Kelly [10] and Kelly et al. [6] (see also [8] and [9]). The main difference that warrants an extension is the presence of the link-supplier as an agent.

Theorem 1: When the agents are price-taking, there exists a competitive equilibrium, i.e., there exist vectors $(\mathbf{p}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu})$ satisfying (C1), (C2) and (C3). Moreover, given a competitive equilibrium $(\mathbf{p}, \boldsymbol{\beta}, \lambda, \boldsymbol{\mu})$, the rate vectors $\mathbf{x}$ and $\mathbf{y}$ defined as $x_m = p_m/\mu_m$ and $y_m = \beta_m(\mu_m - \lambda)$ $(\forall m)$ are optimal for the problem SYSTEM in (1).

The result can be gleaned from the results in [5] though it is not explicitly stated. Our proof of Theorem 1 is a direct one that does not rely on any learning dynamics, but instead is based on standard Lagrangian techniques. We omit the proof due to page limitation.

IV. Price-anticipating scenario

In contrast to the price-taking scenario, agents initiate the bidding process in the price-anticipating scenario. Specifically, the steps are as follows:

1) Agents initiate the bidding process by **simultaneously** announcing their bids, denoted $\mathbf{p}$ and $\boldsymbol{\beta}$.
2) The network-manager sets prices $(\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}), \lambda(\mathbf{p}, \boldsymbol{\beta}))$ where $\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}) = (\mu_1(\mathbf{p}, \boldsymbol{\beta}), \cdots, \mu_M(\mathbf{p}, \boldsymbol{\beta}))$. Note that the above prices are dual optimal for the NETWORK problem in (2).
3) The payments and the rates-allocated are exactly as in the price-taking scenario, but with $(\boldsymbol{\mu}, \lambda)$ replaced by $(\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}), \lambda(\mathbf{p}, \boldsymbol{\beta}))$.

In the following lemma we report the expression for the prices $(\lambda(\mathbf{p}, \boldsymbol{\beta}), \boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}))$. The proof of this result is based on standard Lagrangian techniques; the details are omitted due to page limitation.

Lemma 1: Given any vector $(\mathbf{p}, \boldsymbol{\beta})$ of users' and link-supplier's bids, the prices $(\lambda(\mathbf{p}, \boldsymbol{\beta}), \boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\beta}))$ set by the network-manager are given by

$$\lambda(\mathbf{p}, \boldsymbol{\beta}) \;=\; \begin{cases} 0 & \text{if } \sum_i \sqrt{p_i \beta_i} \leq C \\ f^{-1}_{\mathbf{p}, \boldsymbol{\beta}}(C) & \text{otherwise,} \end{cases} \tag{9}$$

where $f^{-1}_{\mathbf{p},\boldsymbol{\beta}}$ is the inverse of $f_{\mathbf{p},\boldsymbol{\beta}}$ defined as

$$f_{\mathbf{p},\boldsymbol{\beta}}(t) \quad = \quad \sum_i \left(\frac{2p_i}{t + \sqrt{t^2 + 4\frac{p_i}{\beta_i}}} \right), \tag{10}$$

and for $m = 1, 2, \cdots, M$

$$\mu_m(\mathbf{p}, \boldsymbol{\beta}) \quad = \quad \frac{\lambda(\mathbf{p}, \boldsymbol{\beta}) + \sqrt{\lambda(\mathbf{p}, \boldsymbol{\beta})^2 + 4\frac{p_m}{\beta_m}}}{2}. \tag{11}$$

Continuing with the discussion, using the above prices in (3), the pay-offs to the users in the price-anticipating scenario can be expressed as follows for $m = 1, 2, \cdots, M$ (for simplicity, we use $\lambda := \lambda(\mathbf{p}, \boldsymbol{\beta})$):

$$\begin{aligned} Q_m(p_m, \mathbf{p_{-m}}, \boldsymbol{\beta}) &= U_m\left(\frac{p_m}{\mu_m(\mathbf{p}, \boldsymbol{\beta})}\right) - p_m \\ &= \begin{cases} U_m\left(\sqrt{p_m\beta_m}\right) - p_m \text{ if } \sum\limits_i \sqrt{p_i\beta_i} \le C \\ U_m\left(\frac{2p_m}{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}}\right) - p_m \text{ otherwise,} \end{cases} \end{aligned} \tag{12}$$

where $\mathbf{p_{-m}} = (p_1, \cdots, p_{m-1}, p_{m+1}, \cdots, p_M)$ denotes the bids of all users other than m, while $\boldsymbol{\beta}$ is the bid submitted by the link-supplier. Similarly, for the link-supplier we have

$$\begin{aligned} &Q_L(\boldsymbol{\beta}, \mathbf{p}) \\ &= \begin{cases} -V\left(\sum\limits_m \sqrt{p_m\beta_m}\right) + \sum\limits_m p_m \quad \text{if } \sum\limits_i \sqrt{p_i\beta_i} \le C \\ -V(C) + \sum\limits_m \frac{1}{\beta_m}\left(\frac{2p_m}{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}}\right)^2 \quad \text{otherwise,} \end{cases} \end{aligned} \tag{13}$$

The quantity $V(C)$ in the above expression is due to complementary slackness conditions which imply

$$\sum_m \frac{p_m}{\mu_m(\mathbf{p}, \boldsymbol{\beta})} = \sum_m y_m = C \text{ whenever } \lambda > 0.$$

The users and the link-supplier recognize that their bids affect the prices and the allocation. Acting as rational and strategic agents, they now anticipate these prices. The appropriate notion of an equilibrium in this context is the following.

Definition 2 (Nash Equilibrium): A bid vector $(\mathbf{p}, \boldsymbol{\beta})$ is a Nash equilibrium if, for all $m = 1, 2, \cdots, M$, we have

$$\begin{aligned} Q_m(p_m, \mathbf{p_{-m}}, \boldsymbol{\beta}) \quad &\geq \quad Q_m(\overline{p}_m, \mathbf{p_{-m}}, \boldsymbol{\beta}) \ \forall \overline{p}_m \geq 0 \\ Q_L(\boldsymbol{\beta}, \mathbf{p}) \quad &\geq \quad Q_L(\overline{\boldsymbol{\beta}}, \mathbf{p}) \ \forall \overline{\boldsymbol{\beta}} \geq \mathbf{0}. \end{aligned}$$

Observe that when $\sum_i \sqrt{p_i\beta_i} < C$, the link is not fully utilized. In this case the Lagrange multiplier $\lambda = \lambda(\mathbf{p}, \boldsymbol{\beta}) = 0$. Examination of (12) and (13) indicates that the payments made by the users are all passed on to the link-supplier. This may be interpreted as follows: for a given set of payments, the link-supplier bids are such that the link is viewed as a costly resource and the network-manager passes on all his revenue to the link-supplier. The link-supplier is thus assured of this revenue even if his link is not fully utilized. If, on the other hand, the link-supplier's bids are such that $\sum_i \sqrt{p_i\beta_i} > C$, then $\lambda > 0$, and it is clear from (13) that not all the collected revenue is passed on to the link-supplier. Indeed, since $\lambda > 0$, we have

$$\sum_m \frac{1}{\beta_m}\left(\frac{2p_m}{\lambda + \sqrt{\lambda^2 + 4\frac{p_m}{\beta_m}}}\right)^2 \quad < \quad \sum_m p_m$$

where the right-hand side is obtained when $\lambda = 0$. The actions of the link-supplier as a strategic agent creates a situation of conflict and results in the following undesirable equilibrium.

Theorem 2: When the users and the link-supplier are price-anticipating, the only Nash equilibrium is $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ where $p^o_m = 0$ and $\beta^o_m = 0$ for all $m = 1, 2, \cdots, M$.

Thus, in the price-anticipating setting, efficiency loss is 100%, which we interpret as a market break-down. Indeed, at $\boldsymbol{\beta}^o = \mathbf{0}$, the link-supplier is assured an income of $\sum_m p_m$. Given this guaranteed income, he minimizes his cost by supplying zero capacity. The resulting equilibrium is one with the largest possible price of anarchy, and the situation is vastly different from the setting when the link-supplier is not viewed as an agent [8].

V. Price-Anticipation with link as lead player

In view of the break-down of the market when both the users and the link-supplier are simultaneously price anticipating, we design an alternative scheme that involves an additional stage. It proceeds as follows.

1) The link-supplier is asked to announce his bid-vector $\boldsymbol{\beta}$. This information is made available to the users.
2) The users are then asked for their bids $p^{\boldsymbol{\beta}}_m$ ($m = 1, 2, \cdots, M$). Let $\boldsymbol{p}^{\boldsymbol{\beta}} = (p^{\boldsymbol{\beta}}_1, p^{\boldsymbol{\beta}}_2, \cdots, p^{\boldsymbol{\beta}}_M)$.
3) The network-manager then computes the prices $(\boldsymbol{\mu}(\mathbf{p}^{\boldsymbol{\beta}}, \boldsymbol{\beta}), \lambda(\mathbf{p}^{\boldsymbol{\beta}}, \boldsymbol{\beta}))$ by solving NETWORK in (2).
4) The payments and the rates-allocated are exactly as in the price-taking scenario, but with $(\boldsymbol{\mu}, \lambda)$ replaced by $(\boldsymbol{\mu}(\mathbf{p}^{\boldsymbol{\beta}}, \boldsymbol{\beta}), \lambda(\mathbf{p}^{\boldsymbol{\beta}}, \boldsymbol{\beta}))$.

The analysis for this case proceeds as follows. Given a $(\boldsymbol{\beta}, \boldsymbol{p})$, the expression for the prices set by the network-manager are as in Lemma 1. As a result, the expressions for the users' and the link-supplier's pay-off functions are exactly as in (12) and (13), respectively, but with $\mathbf{p}$ replaced by $\mathbf{p}^{\boldsymbol{\beta}}$. Using these pay-off functions, we characterize the solution in the form of Stackelberg equilibrium:

Definition 3 (Stackelberg Equilibrium): A bid vector $(\boldsymbol{\beta}, \mathbf{p}^{\boldsymbol{\beta}})$ is said to constitute a Stackelberg equilibrium if, for all $m = 1, 2, \cdots, M$, we have

$$\begin{aligned} Q_m(p^{\boldsymbol{\beta}}_m, \mathbf{p}^{\boldsymbol{\beta}}_{-\mathbf{m}}, \boldsymbol{\beta}) \quad &\geq \quad Q_m(\overline{p}_m, \mathbf{p}^{\boldsymbol{\beta}}_{-\mathbf{m}}, \boldsymbol{\beta}) \quad \forall \overline{p}_m \geq 0 \\ Q_L(\boldsymbol{\beta}, \mathbf{p}^{\boldsymbol{\beta}}) \quad &\geq \quad Q_L(\overline{\boldsymbol{\beta}}, \mathbf{p}^{\overline{\boldsymbol{\beta}}}) \quad \forall \overline{\boldsymbol{\beta}} \geq \mathbf{0}. \end{aligned}$$

Thus, the bid-vector $\boldsymbol{\beta}$ announced by the link-supplier in step-1 anticipates the user bids $\mathbf{p}^{\boldsymbol{\beta}}$ of step-2. For a given $\boldsymbol{\beta}$, the bids submitted by the users is in anticipation of the prices the network-manager announces in step-3.

For the ease of exposition, we assume that $C = \infty$ so that the capacity constraint is not binding (the case where C is finite can be similarly handled). Thus, recalling (9) and (11), we have $\lambda(\mathbf{p}, \boldsymbol{\beta}) = 0$ and $\mu_m(\mathbf{p}, \boldsymbol{\beta}) = \sqrt{\frac{p_m}{\beta_m}}$. As a result the pay-off functions can be simply expressed as

$$Q_m(p_m, \mathbf{p_{-m}}, \boldsymbol{\beta}) = U_m\left(\sqrt{p_m\beta_m}\right) - p_m \quad (14)$$

$$Q_L(\boldsymbol{\beta}, \mathbf{p}) = -V\left(\sum_m \sqrt{p_m\beta_m}\right) + \sum_m p_m. \quad (15)$$

This simplification will enable us to the emphasize key ideas rather than dwell on the technicalities arising from a finite C.

From (14) we see that the user pay-offs are independent of the bids submitted by the other users. As a result, for a given $\boldsymbol{\beta}$, the unique equilibrium strategy for user-m is given by

$$p_m^{\boldsymbol{\beta}} = \arg\max_{p_m \geq 0} \left(U_m\left(\sqrt{p_m\beta_m}\right) - p_m\right). \quad (16)$$

In Lemma 2 we report the expression for $p_m^{\boldsymbol{\beta}}$ that is obtained by solving (16) (proof is omitted due to page limitation).

Lemma 2: For a given $\boldsymbol{\beta}$ such that $\beta_m > 0$, we have $p_m^{\boldsymbol{\beta}} = r_{\beta_m}^2/\beta_m$ where r_{β_m} is the fixed point of $U_m'(r) = 2r/\beta_m$. In case $\beta_m = 0$ we simply have $p_m^{\boldsymbol{\beta}} = 0$.

We extend the definition of r_{β_m} in the above lemma by defining $r_{\beta_m} = 0$ if $\beta_m = 0$. It is then easy to see that $r_{\beta_m} = \sqrt{p_m^{\boldsymbol{\beta}}\beta_m}$ is the allocation to user m. Using the above result in (15), we compute the optimal $\boldsymbol{\beta}$ that the link-supplier should announce in step-1 as

$$\boldsymbol{\beta}^* \in \mathcal{B}^* = \arg\max_{\boldsymbol{\beta}} \left(-V\left(\sum_m r_{\beta_m}\right) + \sum_m \frac{r_{\beta_m}^2}{\beta_m}\right). \quad (17)$$

For any $\boldsymbol{\beta}^* \in \mathcal{B}^*$ it follows that $(\boldsymbol{\beta}^*, \boldsymbol{p}^{\boldsymbol{\beta}^*})$ constitutes a Stackelberg equilibrium, where the rate allocated to user-m is given by $x_m^{\boldsymbol{\beta}^*} = y_m^{\boldsymbol{\beta}^*} = \sqrt{p_m^{\boldsymbol{\beta}^*}\beta_m^*} = r_{\beta_m^*}$. However, we first need to assert the existence of a solution $\boldsymbol{\beta}^*$:

Lemma 3: Suppose $U_m(\cdot)$ and $V(\cdot)$ satisfy the following: $xU_m'(x) \to \infty$ and $V(x)/x \to \infty$ as $x \to \infty$. Then the set $\mathcal{B}^*$ is nonempty. Hence, under the above assumptions on the utilities and cost function, a Stackelberg equilibrium exists.

Remark: The above assumption excludes cost functions that are asymptotically linear, and utilities such as $\log(1+x)$. However, we note that these assumptions are not too restrictive. Also, note that it is not possible to assert the uniqueness of β^* as it is not clear as to how $r_{\beta_m}^2/\beta_m$ varies as a function of β_m (although it can be shown that r_{β_m} increases with β_m).

Given a Stackelberg equilibrium $(\boldsymbol{\beta}^*, \boldsymbol{p}^{\boldsymbol{\beta}^*})$ the *price-of-anarchy (PoA)* is defined as the ratio of the utility at equilibrium (*Nash utility*) to the system optimum (*social utility*):

$$PoA(\{U_m\}; V) = \frac{\sum_m U_m\left(x_m^{\boldsymbol{\beta}^*}\right) - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\sum_m U_m\left(x_m^s\right) - V\left(\sum_m x_m^s\right)} \quad (18)$$

where x_m^s denotes the social optimum allocation to user m (obtained by solving SYSTEM in (1)). Note that we have emphasized the dependency of the PoA on $(\{U_m\}; V)$ by incorporating these into its notation. Since we are interested in studying the worst case PoA, hereafter our reference to $(\boldsymbol{\beta}^*, \boldsymbol{p}^{\boldsymbol{\beta}^*})$ is with the understanding that $\boldsymbol{\beta}^* \in \mathcal{B}^*$ is uniquely chosen such that the corresponding PoA is the least possible.

A. Bound on the PoA

We now proceed to bound the PoA by drawing a comparison with an alternate system where the user-utilities are replaced by the following linear utility functions: for a given $\boldsymbol{\beta}$, let $\overline{U}_m(x_m) = a_m x_m$ $(m = 1, 2, \cdots, M)$ where $a_m = U'(r_{\beta_m})$; the link-cost function V is retained as it is. We use $\mathcal{G}(\{U_m\}, V)$ and $\mathcal{G}(\{\overline{U}_m\}, V)$ to represent the *original system* and the *system with linear-utilities*, respectively. We then have the following result.

Lemma 4: For any given $\boldsymbol{\beta}$, the equilibrium strategy $\boldsymbol{p}_m^{\boldsymbol{\beta}}$ for the users in game $\mathcal{G}(\{U_m\}, V)$ is also the equilibrium strategy for the users in $\mathcal{G}(\{\overline{U}_m\}, V)$.

The following corollary is an immediate consequence of the above result.

Corollary 1: A Stackelberg equilibrium $(\boldsymbol{\beta}^*, \boldsymbol{p}^{\boldsymbol{\beta}^*})$ for the game $\mathcal{G}(\{U_m\}, V)$ is a Stackelberg equilibrium for the game $\mathcal{G}(\{\overline{U}_m\}, V)$ as well.

We can now bound the PoA incurred in $\mathcal{G}(\{U_m\}, V)$ by that incurred in the game with linear user-utilities $\mathcal{G}(\{\overline{U}_m\}, V)$.

Lemma 5: Let $\{\overline{U}_m\}$ denote the linear user-utilities corresponding to $\boldsymbol{\beta}^*$, i.e., $\overline{U}_m(x_m) = a_m x_m$ where $a_m = U'(r_{\beta_m^*})$. Then, we have $PoA(\{U_m\}; V) \geq PoA(\{\overline{U}_m\}; V)$.

Thus, we see that for any system there exists a corresponding system with linear-utilities such that the latter system yields a lower PoA. As a result, for a fixed link-cost function V, the worst-case PoA can be obtained by minimizing $PoA(\{\widetilde{U}_m\}; V)$ over the set of all linear utility functions $\widetilde{U}_m(x_m) = c_m x_m$ $(m = 1, 2, \cdots, M)$. This observation yields the following theorem.

Theorem 3: For a given link-cost function $V(\cdot)$, defining $v(\cdot) := V'(\cdot)$, for any set of user-utilities $\{U_m\}$ we have

$$PoA(\{U_m\}; V) \geq \inf_{c>0} \frac{cv^{-1}(\frac{c}{2}) - V(v^{-1}(\frac{c}{2}))}{cv^{-1}(c) - V(v^{-1}(c))}. \quad (19)$$

B. PoA bound for polynomical link-costs

We apply the above theorem to derive explicit expressions for the bound on the PoA when the link-cost function is polynomial. We start with the simplest case of quadratic link-cost, i.e., $V(x) = bx^2$ where $b > 0$. We then have $v(x) = 2bx$ so that $v^{-1}(y) = \frac{y}{2b}$. Thus, using (19), we obtain

$$PoA(\{U_m\}; V) \geq \inf_{c>0} \frac{c\frac{c}{4b} - V(\frac{c}{4b})}{c\frac{c}{2b} - V(\frac{c}{2b})} = \inf_{c>0} \frac{c\frac{c}{4b} - b(\frac{c}{4b})^2}{c\frac{c}{2b} - b(\frac{c}{2b})^2}$$
$$= \inf_{c>0} \frac{\frac{c^2}{4b}(1 - \frac{1}{4})}{\frac{c^2}{2b}(1 - \frac{1}{2})} = \frac{3}{4}.$$

Thus, when the link-cost is quadratic, the worst-case efficiency loss is no more that 25%.

Similarly, suppose $V(x) = bx^3$ for $x \geq 0$, with $b > 0$. (This is increasing and convex for $x \geq 0$.) Then, using the bound (19) and a similar calculation, we obtain

$$PoA(\{U_m\}; V) \geq \frac{5}{4\sqrt{2}} \geq 0.88.$$

Thus, the worst-case efficiency loss improves to 12% when the link-cost is cubic. In general, suppose the link-cost is

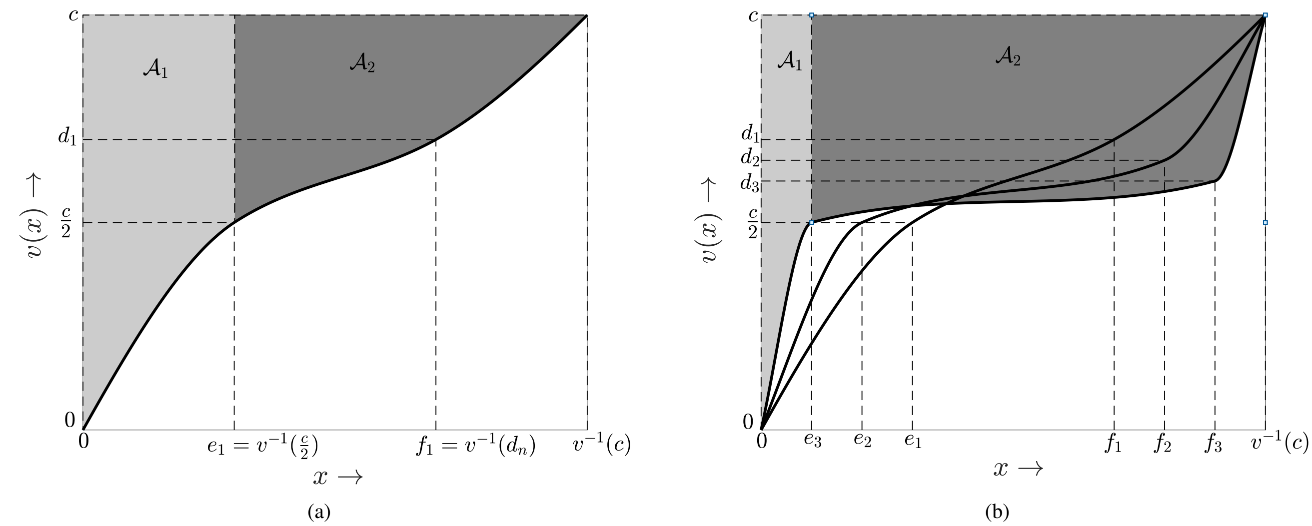

Fig. 1. Geometric interpretation for the PoA bound.

polynomial of degree $n \geq 2$, i.e., $V(x) = bx^n$, $x \geq 0, b > 0$, then the expression for the bound on the PoA is given by

$$PoA(\{U_m\}; V) \quad \geq \quad \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \frac{2n-1}{n-1}. \qquad (20)$$

The above bound as a function of n is increasing and converges to 1 as $n \to \infty$. Thus, the efficiency loss reduces if the link-cost can be modeled using polynomials of higher degrees.

The above observation is in contrast to the result in Section IV where the efficiency loss (for any $\{U_m\}$ and V) is 100% when both the users and the link-supplier are allowed to simultaneously participate in the bidding process.

C. Worst-case bound on the PoA

Although the class of polynomial link-cost functions yield favourable PoA, we now show that there exist link-cost functions V_n, $n \geq 1$, such that the corresponding sequence of PoA-bound converges to 0 as $n \to \infty$. Thus, the worst-case PoA (over all possible $\{U_m\}$ and V) is 0.

To see this, let us first rewrite (19) by expressing V in the integral form $V(x) = \int_0^x v(\tau)d\tau$ to get

$$\begin{aligned} PoA(\{U_m\}; V) \quad &\geq \quad \inf_{c>0} \frac{cv^{-1}(\frac{c}{2}) - \int_0^{v^{-1}(\frac{c}{2})} v(\tau)d\tau}{cv^{-1}(c) - \int_0^{v^{-1}(c)} v(\tau)d\tau} \\ &=: \quad \inf_{c>0} H(c, v). \end{aligned}$$

For a given c and a marginal cost function for the link-supplier $v(\cdot)$, $H(c, v)$ can be geometrically interpreted with the aid of the illustration in Fig.1(a) as follows: the numerator in the PoA formula is the area of the region $\mathcal{A}_1$ (light shaded region) while the denominator is total area of $\mathcal{A}_1$ and $\mathcal{A}_2$ (shaded dark). We then have

$$H(c, v) = \frac{A_1}{A_1 + A_2} = \frac{A_1/A_2}{1 + A_1/A_2}$$

where A_i denotes the area of region $\mathcal{A}_i$ $(i = 1, 2)$. In Fig. 1(a) we have used e_1 to denote $v^{-1}(\frac{c}{2})$; also, $f_1 = v^{-1}(d_1)$ where d_1 is arbitrarily chosen in $\left(\frac{c}{2}, c\right)$. Since V is strictly convex and increasing, it follows that v is strictly increasing.

Now, it is possible to construct a sequence of $v(\cdot)$ functions, say $\{v_n\}$, such that $e_n \downarrow 0$, $d_n \downarrow \frac{c}{2}$ while $f_n \uparrow v^{-1}(c)$; an illustration of such a construction is depicted in Fig. 1(b). Observe that along such a sequence we have $A_1 \downarrow 0$ and $A_2 \uparrow \frac{c}{2}v^{-1}(c) > 0$. As a result we have $H(c, v_n) \to 0$ as $n \to \infty$. Thus, for any given c it is possible to produce pathological link-cost functions whose PoA-bounds are arbitrarily close to 0. Therefore, it is not possible to guarantee a less-than-100% efficiency loss (i.e., a positive PoA) when the class of all possible link-cost functions are considered. Nevertheless, bounding the PoA for a fixed link-cost function is reassuring.

VI. Proof of Main Results

A. Proof of Theorem 2

Proof: We will first show that $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ is a Nash equilibrium. For this, note that once the link-supplier fixes his bids to $\boldsymbol{\beta}^o = \mathbf{0}$, then for any vector of user bids $\mathbf{p} \geq \mathbf{0}$ the system operates in the regime $\sum_i \sqrt{p_i \beta_i^o} \leq C$. Thus, using the first expression in (12), for any $\overline{p}_m > 0$, we have

$$\begin{aligned} Q_m(\overline{p}_m, \mathbf{p^o_{-m}}, \boldsymbol{\beta^o}) \quad &= \quad U_m(0) - \overline{p}_m \\ &< \quad U_m(0) \\ &= \quad Q_m(p_m^o, \mathbf{p^o_{-m}}, \boldsymbol{\beta^o}). \end{aligned}$$

Thus, unilateral deviation from p_m^o is not beneficial for user m $(\forall m)$. Similarly, for any $\overline{\boldsymbol{\beta}}$ such that $\overline{\beta}_m > 0$ for some m, we have

$$Q_L(\overline{\boldsymbol{\beta}}, \mathbf{p^o}) = -V(0) + \sum_m p_m^o = Q_L(\boldsymbol{\beta^o}, \mathbf{p^o})$$

To obtain the above, note that since the users' payments are zero, from (13), the first expression applies. Any other value of β_m does not strictly increase the pay-off of the link-supplier. Thus, $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ is a Nash equilibrium.

We now prove the uniqueness of the Nash equilibrium. Let $(\mathbf{p}^*, \boldsymbol{\beta}^*)$ be a Nash equilibrium. Suppose $p_m^* > 0$ for some m. Then, if $\beta_m^* = 0$ (recalling (12)) the pay-off to user m is

$$\begin{aligned} Q_m(p_m^*, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) &= U_m(0) - p_m^* < U_m(0) \\ &= Q_m(0, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) \end{aligned}$$

which contradicts the assumption that $(\mathbf{p}^*, \boldsymbol{\beta}^*)$ is a Nash equilibrium. On the other hand, if $\beta_m^* > 0$, then the link-supplier can benefit by deviating to the bid $\boldsymbol{\beta}^o$. This is because, the rate-cost incurred by deviating to $\boldsymbol{\beta}^o$ is always strictly lower (since he now provides zero bandwidth). Also, the payment $\sum_m p_m^*$ accrued under $\beta_m^* > 0$ may be better if the system was not already in the regime $\sum_i \sqrt{p_i^* \beta_i^*} \leq C$; if already in that regime the payment remains unchanged. Formally,

$$Q_L(\boldsymbol{\beta}^*, \mathbf{p}^*) < -V(0) + \sum_m p_m^* = Q_L(\boldsymbol{\beta}^o, \mathbf{p}^*)$$

which is again a contradiction. Thus, $p_m^* = 0 \; \forall m$, i.e., $\mathbf{p}^* = \mathbf{0}$.

Now, suppose $\beta_m^* > 0$ for some m. User m can benefit by making a small payment $\overline{p}_m$. Indeed choose a $\overline{p}_m$ satisfying

$$0 < \overline{p}_m \leq \min\left\{C^2/\beta_m^*, q_m\right\} \tag{21}$$

where q_m is the maximizer of the function

$$h(p_m) \;=\; U_m\left(\sqrt{p_m \beta_m^*}\right) - p_m \tag{22}$$

over $p_m \geq 0$. Note that $h(p_m)$ is strictly concave in p_m. Hence, q_m is the unique solution to the optimality condition

$$U_m'\left(\sqrt{q_m \beta_m^*}\right) = 2\sqrt{q_m/\beta_m^*}.$$

Since $U'(\cdot)$ is strictly decreasing with $U'(0) > 0$, we have $q_m > 0$, thus enabling us to choose a $\overline{p}_m$ satisfying (21). The $\min$ term in (21) involving C^2/β_m^* is required to ensure that the first expression of (12) is applicable. Thus, we have

$$\begin{aligned} Q_m(p_m^*, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) &= U_m(0) \text{ (since } p_m^* = 0) \\ &< U_m\left(\sqrt{\overline{p}_m \beta_m^*}\right) - \overline{p}_m \\ &= Q_m(\overline{p}_m, \mathbf{p}_{-\mathbf{m}}^*, \boldsymbol{\beta}^*) \end{aligned}$$

where the inequality is because the function $h(\cdot)$, being strictly concave, is strictly increasing until q_m. The above contradiction implies that $\beta_m^* = 0 \; \forall m$, i.e., $\boldsymbol{\beta}^* = \mathbf{0}$. Hence, $(\mathbf{p}^o, \boldsymbol{\beta}^o)$ is the only Nash equilibrium. ■

B. Proof of Lemma 3

Proof: We first show that the objective function of the problem in (17) is continuous in β_m for each m. We next argue that it is sufficient to consider $\arg\max$ in (17) over a compact set of $\boldsymbol{\beta}$ values. Then the proof is completed by invoking Weierstrass theorem. The details are as follows.

Proof of continuity: To show the continuity of the objective in (17) it suffices to prove that r_{β_m} is continuous in β_m (for all m). For simplicity we omit the subscript m hereafter. Let us first prove right-continuity of r_β at $\beta = 0$. Clearly, since r_β is the solution to $U_m'(r) = 2r/\beta$, we see that $r_\beta \geq 0$ and, moreover, since U_m' is strictly decreasing we have $r_\beta = \frac{U_m'(r_\beta)}{2}\beta \leq \frac{U_m'(0)}{2}\beta$ from which it follows that $r_0 = 0$ so that r_β is continuous at $\beta = 0$. To prove continuity at any $\beta > 0$, we show that r_β is Lipschitz continuous, i.e., $|r_{\beta_1} - r_{\beta_2}| \leq \frac{U_m'(0)}{2}|\beta_1 - \beta_2|$. Without loss of generality assume $\beta_2 > \beta_1$ so that $r_{\beta_2} > r_{\beta_1}$. Then we have (again since $U_m'(\cdot)$ is decreasing)

$$\begin{aligned} r_{\beta_2} - r_{\beta_1} &= U_m'(r_{\beta_2})\beta_2/2 - U_m'(r_{\beta_1})\beta_1/2 \\ &\leq U_m'(r_{\beta_2})\beta_2/2 - U_m'(r_{\beta_2})\beta_1/2 \\ &= (U_m'(r_{\beta_2})/2)(\beta_2 - \beta_1) \\ &\leq (U_m'(0)/2)(\beta_2 - \beta_1). \end{aligned}$$

This establishes Lipschitz continuity.

Proof that it suffices to search for β_m in a bounded set: First note that, using the definition of r_{β_m} and p_m^β, we may write the objective function as

$$\begin{aligned} & -V\left(\sum_m \frac{p_m^{\boldsymbol{\beta}}}{r_{\beta_m}/\beta_m}\right) + \sum_m p_m^{\boldsymbol{\beta}} \\ &\qquad = -V\left(\sum_m \frac{p_m^{\boldsymbol{\beta}}}{U_m'(r_{\beta_m})/2}\right) + \sum_m p_m^{\boldsymbol{\beta}} \\ &\qquad \leq -V\left(\sum_m \frac{p_m^{\boldsymbol{\beta}}}{U_m'(0)/2}\right) + \sum_m p_m^{\boldsymbol{\beta}} \end{aligned}$$

since $V(\cdot)$ is strictly increasing. From the assumption that $V(x)/x \to \infty$ as $x \to \infty$, we see that the last term in the right-hand side above is less than 0 for all $(p_m^\beta, 1 \leq m \leq M)$ with $\sum_m p_m^\beta > P$ for some bounded P. Since each $p_m^\beta \geq 0$, we trivially have that $0 \leq p_m^\beta \leq P$.

From the formula for r_{β_m} and p_m^β, we have $p_m^\beta = r_{\beta_m} U_m'(r_{\beta_m})/2$, and hence $0 \leq r_{\beta_m} U_m'(r_{\beta_m})/2 \leq P$. Under the assumption $rU_m'(r) \to \infty$ as $r \to \infty$, we must then have $0 \leq r_{\beta_m} \leq R$ for some bounded R, and since U_m' is strictly decreasing and strictly positive, we must have $U_m'(r_{\beta_m}) \geq U_m'(R) > 0$. Using this, we then have

$$\beta_m = p_m^\beta/(U_m'(r_{\beta_m})/2)^2 \leq 4P/(U_m'(R))^2 < \infty.$$

This completes the proof. ■

C. Proof of Lemma 4

Proof: Applying Lemma 2 for the game $\mathcal{G}(\{\overline{U}_m\}, V)$, we can express the equilibrium strategy of user-m as $\overline{p}_m^{\boldsymbol{\beta}} = \overline{r}_{\beta_m}^2/\beta_m$ where $\overline{r}_{\beta_m}$ satisfies

$$\overline{r}_{\beta_m} = \frac{\beta_m}{2}\overline{U}_m'(\overline{r}_{\beta_m}) = \frac{\beta_m}{2}a_m = \frac{\beta_m}{2}U'(r_{\beta_m}) = r_{\beta_m}.$$

Thus, we have $\overline{p}_m^{\boldsymbol{\beta}} = p_m^{\boldsymbol{\beta}}$ for all $m = 1, 2, \cdots, M$. ■

D. Proof of Lemma 5

Proof: For simplicity, let us denote

$$\begin{aligned} \Lambda^{\boldsymbol{\beta}^*} &:= \sum_m \left(U_m\left(x_m^{\boldsymbol{\beta}^*}\right) - a_m x_m^{\boldsymbol{\beta}^*}\right) \\ \Lambda^s &:= \sum_m \left(U_m\left(x_m^s\right) - a_m x_m^s\right). \end{aligned}$$

From the concavity of $\{U_m\}$, and since $a_m = U'(r_{\beta^*_m}) = U'(x_m^{\boldsymbol{\beta}^*})$, it follows that $\Lambda^{\boldsymbol{\beta}^*} \geq \Lambda^s$. Thus, we have

$$\begin{aligned}
&PoA(\{U_m\};V)\\
&= \frac{\sum_m U_m\left(x_m^{\boldsymbol{\beta}^*}\right) - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\sum_m U_m\left(x_m^s\right) - V\left(\sum_m x_m^s\right)}\\
&= \frac{\Lambda^{\boldsymbol{\beta}^*} + \sum_m a_m x_m^{\boldsymbol{\beta}^*} - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\Lambda^s + \sum_m a_m x_m^s - V\left(\sum_m x_m^s\right)}\\
&\geq \frac{\Lambda^{\boldsymbol{\beta}^*} + \sum_m a_m x_m^{\boldsymbol{\beta}^*} - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\Lambda^{\boldsymbol{\beta}^*} + \sum_m a_m x_m^s - V\left(\sum_m x_m^s\right)}\\
&\geq \frac{\Lambda^{\boldsymbol{\beta}^*} + \sum_m a_m x_m^{\boldsymbol{\beta}^*} - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\Lambda^{\boldsymbol{\beta}^*} + \max\limits_{\{x_m\}}\Big(\sum_m a_m x_m - V\left(\sum_m x_m\right)\Big)}\\
&\geq \frac{\sum_m a_m x_m^{\boldsymbol{\beta}^*} - V\left(\sum_m x_m^{\boldsymbol{\beta}^*}\right)}{\max\limits_{\{x_m\}}\Big(\sum_m a_m x_m - V\left(\sum_m x_m\right)\Big)}\\
&= PoA(\{\overline{U}_m\};V).
\end{aligned}$$

Note that to obtain the last inequality we have used the identity $\frac{x+y}{x+z} \geq \frac{y}{z}$ whenever $z \geq y$. ■

E. Proof of Theorem 3

Proof: From Lemma 5 it follows that $PoA(\{U_m\};V)$ is lower bounded by the worst case PoA achieved by a system with linear user-utilities. Thus, we can write

$$PoA(\{U_m\};V) \geq \inf\left\{PoA(\{\widetilde{U}_m\};V) : \{\widetilde{U}_m\} \text{ are linear}\right\}.$$

We hence proceed to compute the PoA of a system with user-utilities $\widetilde{U}_m(x_m) = c_m x_m$ where $c_m > 0$ for all $m = 1, 2, \cdots, M$. Without loss of generality, we assume that $c_1 \geq c_2 \geq \cdots \geq c_M$.

First, fix a $\boldsymbol{\beta}$. Then, applying Lemma 2 the equilibrium bid of user-m can be computed as $p_m^{\boldsymbol{\beta}} = r_{\beta_m}^2/\beta_m$, where r_{β_m} is the rate allocated to user-m which is given by

$$r_{\beta_m} = \beta_m \widetilde{U}'_m(r_{\beta_m})/2 = \beta_m c_m/2$$

Now, using (17) the optimal $\boldsymbol{\beta}^*$ can be computed by solving

$$\max\left\{-V\left(\sum_m \frac{\beta_m c_m}{2}\right) + \sum_m \frac{\beta_m c_m^2}{4}\,\middle|\, \beta_m \geq 0\ \forall m\right\}.$$

The solution to the above problem is given by $\beta_m^* = (2/c_1)v^{-1}(c_1/2)$ if $m = 1$; 0 otherwise. Thus, we have

$$\begin{aligned}
\text{Nash Utility} &= \sum_m \widetilde{U}_m(r_{\beta^*_m}) - V\left(\sum_m r_{\beta^*_m}\right)\\
&= c_1 v^{-1}\left(\frac{c_1}{2}\right) - V\left(v^{-1}\left(\frac{c_1}{2}\right)\right). \quad (23)
\end{aligned}$$

Next, the social utility can be computed by solving

$$\max\left\{\sum_m c_m x_m - V\left(\sum_m x_m\right)\,\middle|\, x_m \geq 0\ \forall m\right\}.$$

The solution to the above is given by $x_m^s = v^{-1}(c_1)$ for $m = 1$; 0 otherwise. Thus, we have

$$\text{Social utility} = c_1 v^{-1}(c_1) - V(v^{-1}(c_1)). \quad (24)$$

Finally, using (23) and (24), the PoA can be expressed as

$$PoA(\{\widetilde{U}_m\};V) = \frac{c_1 v^{-1}(\frac{c_1}{2}) - V(v^{-1}(\frac{c_1}{2}))}{c_1 v^{-1}(c_1) - V(v^{-1}(c_1))}.$$

The worst-case PoA is obtained by taking inf over $c_1 > 0$. ■

VII. Conclusion

Data offloading is a good low-cost strategy that leverages existing auxiliary technology for handling the growth of mobile data. Technologies to enable such offloading are now available [1], [2]. Since the auxiliary technology will be owned by third parties, a compensation mechanism should be put in place to encourage them to participate in the offloading process. It is natural that these agents are strategic. This paper demonstrates that mechanisms for offloading should be designed with some care. An earlier work proposed an offloading mechanism (collect bids, allocate offloading amounts, and distribute payments) and designed an iterative procedure to get the system to a competitive equilibrium where all agents benefited, if all agents were price-taking. We showed that if the agents are price-anticipating, this benefit completely disappears. New mechanisms are thus needed when all agents are price-anticipating. We proposed a simple Stackelberg formulation with the supplying agent as a lead player alleviates the problem to some extent. The price of anarchy is then bounded in terms of the true link cost function. The efficiency loss is 25% for quadratic link costs (PoA = 0.75). While there are link cost function for which the efficiency loss, even in the Stackelberg formulation, is close to 100%, these appear to be pathological cases. The proposed mechanism with link suppliers as lead players will have tolerable efficiency loss in most real situations.

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