# Closed-Form Analytical Expression for the Electric Field Distribution in an Ideal Stripline

*Felix Vega, John J. Pantoja, and Chaouki Kasmi*

*Abstract* – In this article, we propose a closed-form explicit solution for the electric field distribution on an ideal stripline. The solution is obtained by applying an exponential conformal transformation to the stripline geometry. The results obtained are in close agreement with numerical simulations. The exact expression presented here can be used to estimate power ratings, voltage breakdown thresholds, characteristic impedance, per-unit-length capacitance, background field values, and field enhancement factors in stripline geometries.

## 1. Introduction

Striplines are widely used as transmission lines in microwave circuit applications and as unit test cells in electromagnetic capability and radio-frequency testing. Several analytical and numerical approaches have been proposed to estimate the electric field distribution in an ideal stripline. Initial approximations using implicit solutions of the multiconductor transmission line theory were proposed by Geer in [1]. In recent years, numerical analysis techniques based on the finite-difference time-domain method [2] and the finite-element method [3] were also proposed; however, the field intensity values are highly dependent on the mesh resolution, particularly near the ends of the central strip.

On the other hand, Kumar and Das [4] have proposed a closed-form expression to calculate the electric field distribution in the line. Their approach consists of applying two conformal transformations to one quadrant of the stripline. However, the expression obtained requires additional numerical integration and normalization to obtain the correct values of the electric field intensity. The result is, therefore, an implicit expression. The same approach is discussed in [5] for the estimation of the per-unit-length capacitance and characteristic impedance of a symmetrical strip transmission line.

Conformal mapping, followed by numerical integration, has also been applied in [6] and [7] to determine an expression of the capacitance and characteristic impedance in a shielded asymmetrical coplanar stripline.

In this article, we derive an exact closed-form expression for the electric field distribution and amplitude on an ideal stripline. We use a preexisting electric field distribution reported in [8]. The obtained equation uses simple algebraic calculations, not requiring additional numerical integration.

The article is organized as follows. In Section 2, the new closed-form solution is obtained. In Section 3, two examples are presented to demonstrate the applicability of the proposed solution. Finally, conclusions are provided in Section 4.

Manuscript received 28 December 2021.

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## 2. Closed-Form Analytical Expression

The stripline line consists of an inner conductor strip of width $d$, centered between two ground planes at a distance $a$ from each other, as shown in Figure 1. For the case of the present analysis, an ideal stripline is considered. This means that the conductor thickness $t$ is negligible or much smaller than both $a$ and $d$. The dielectric space is filled with a homogeneous, nondispersive dielectric material, with permittivity $\varepsilon_0$.

The stripline geometry is defined in the 2D space $Z = x + iy$ by the boundary conditions

$$V(Z) = \begin{cases} 0 \text{ for } Z = x - ai, \ -\infty < x < \infty \\ +V \text{ for } Z = x + \frac{a}{2}i, \ -\frac{d}{2} < x < \frac{d}{2} \\ 0 \text{ for } Z = x, \ -\infty < x < \infty \end{cases} \tag{1}$$

where $-\infty < x < \infty$, $0 \le y \le a$, and $V(Z)$ is the potential in the stripline.

The exponential conformal transformation can be applied to the domain $Z$, obtaining the image domain $W = u + iv$ (see Figure 2),

$$W = u + iv = T(Z) = e^{\frac{Z\pi}{a}} = e^{\frac{x\pi}{a}} e^{i\frac{y\pi}{a}} \tag{2}$$

where $T$ is the exponential conformal transformation.

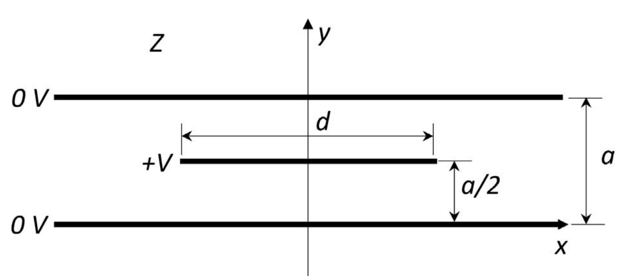


Figure 1. Geometry of an ideal stripline.

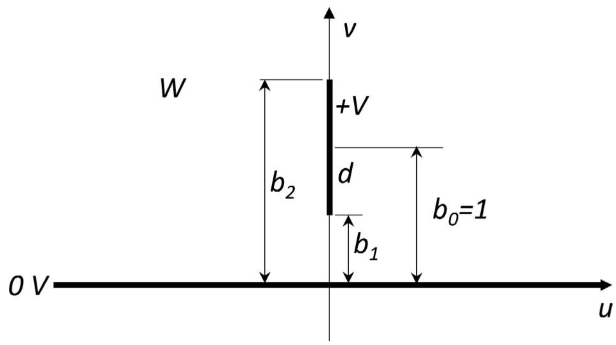


Figure 2. Geometry after conformal transformation.

This can be further expanded as

$$u = e^{\frac{x\pi}{a}}\cos\left(\frac{y\pi}{a}\right)\ v = e^{\frac{x\pi}{a}}\sin\left(\frac{y\pi}{a}\right)$$

The boundary conditions given in (1) are translated to the $W$ domain as

$$V(W) = \begin{cases} 0 \text{ for } W = u,\ -\infty < u < \infty \\ +V \text{ for } W = vi,\ b_1 < v < b_2 \end{cases} \quad (3)$$

where $b_1 = e^{-\frac{d\pi}{2a}}$ and $b_2 = e^{\frac{d\pi}{2a}}$.

This geometry has been solved in [9], where the potential function has been found as

$$V(W) = \frac{V_1}{K(m)}\mathrm{Re}\left(sn^{-1}\left(\frac{-iW}{b_1}, m\right)\right) \quad (4)$$

where $sn^{-1}(\ )$ is the inverse of the Jacobi $sn$ elliptic function [10], $m = b_1^4 = e^{-2d\pi/a}$, and $K(m)$ is the complete elliptic integral of the first kind [10].

The electric field can therefore be obtained as

$$\vec{E}(W) = -\nabla V(W) \quad (5)$$

An explicit solution of this expression was published in [8] as follows:

$$\vec{E}(W) = E_u(W) + iE_v(W) \quad (6)$$

$$E_u(W) = \frac{b_1 V_1}{K(m)}\mathrm{Re}\left(\frac{i}{\sqrt{\left(b_1^2 + W^2\right)\left(b_1^2 + mW^2\right)}}\right) \quad (7)$$

$$E_v(W) = \frac{b_1 V_1}{K(m)}\mathrm{Re}\left(\frac{-1}{\sqrt{\left(b_1^2 + W^2\right)\left(b_1^2 + mW^2\right)}}\right) \quad (8)$$

The electric field obtained using (6) can be translated back to the original stripline ($Z$ space) using

$$\vec{E}(Z) = J \times \vec{E}(W)$$

$$\vec{E}(Z) = \begin{pmatrix} E_x(Z) \\ E_y(Z) \end{pmatrix} = \begin{pmatrix} \frac{du}{dx} & \frac{dv}{dx} \\ \frac{du}{dy} & \frac{dv}{dy} \end{pmatrix} \times \begin{pmatrix} E_u(W) \\ E_v(W) \end{pmatrix} \quad (9)$$

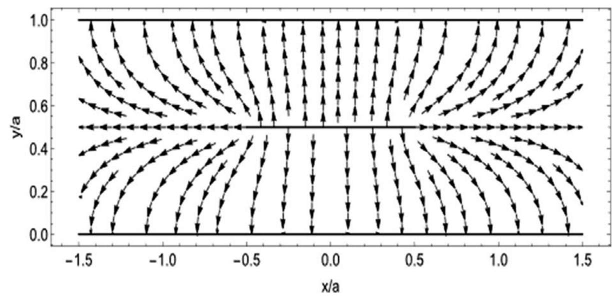


Figure 3. Electric field distribution in a stripline with $a = d$.

where $J$ is the Jacobian matrix. Replacing (2), (7), and (8) in (9), $\vec{E}(Z)$can be expanded as

$$\begin{pmatrix} E_x(Z) \\ E_y(Z) \end{pmatrix} = \frac{\pi e^{\frac{\pi x}{a}} b_1 V_1}{aK(m)} \begin{pmatrix} \cos\left(\frac{\pi y}{a}\right) & \sin\left(\frac{\pi y}{a}\right) \\ -\sin\left(\frac{\pi y}{a}\right) & \cos\left(\frac{\pi y}{a}\right) \end{pmatrix} \times \begin{pmatrix} \mathrm{Re}\left(\frac{i}{\sqrt{\left(b_1^2 + e^{2\pi(x+iy)/a}\right)\left(b_1^2 + me^{2\pi(x+iy)/a}\right)}}\right) \\ \mathrm{Re}\left(\frac{-1}{\sqrt{\left(b_1^2 + e^{2\pi(x+iy)/a}\right)\left(b_1^2 + me^{2\pi(x+iy)/a}\right)}}\right) \end{pmatrix} \quad (10)$$

Equation (10) is valid for the region $x \leq 0$. The fields in the region $x > 0$ can be found using the symmetry of the problem as

$$E(Z)|_{x \geq 0} = -E(-Z^*)^*|_{x \leq 0} \quad (11)$$

where ( )* denotes the complex conjugate operator.

Notice that the determinant of the Jacobian matrix in (9) is 1. This indicates that the transformation preserves area and orientation. The argument of the trigonometric functions in the matrix indicates the rotation angle, which, as stated in (2), is $\theta = \pi y/a$.

Finally, the potential distribution in the stripline can be expressed using (4) and (2) as

$$V(Z) = \frac{V_1}{K(m)}\mathrm{Re}\left(sn^{-1}\left(\frac{-ie^{\frac{(x+iy)\pi}{a}}}{b_1}, m\right)\right) \quad (12)$$

This expression, to the best of the authors' knowledge, cannot be further simplified.

## 3. Numerical Results

In order to validate the proposed expressions, two examples are presented and solved using both the closed-form exact analytical solution proposed in this article and an electrostatic simulation.

### 3.1 Example 1

As an example, Figure 3 presents the field distribution for an ideal stripline of dimensions $a = d$, obtained using (10). The complete elliptical integral $K(m)$ in (10) was calculated using the built-in function

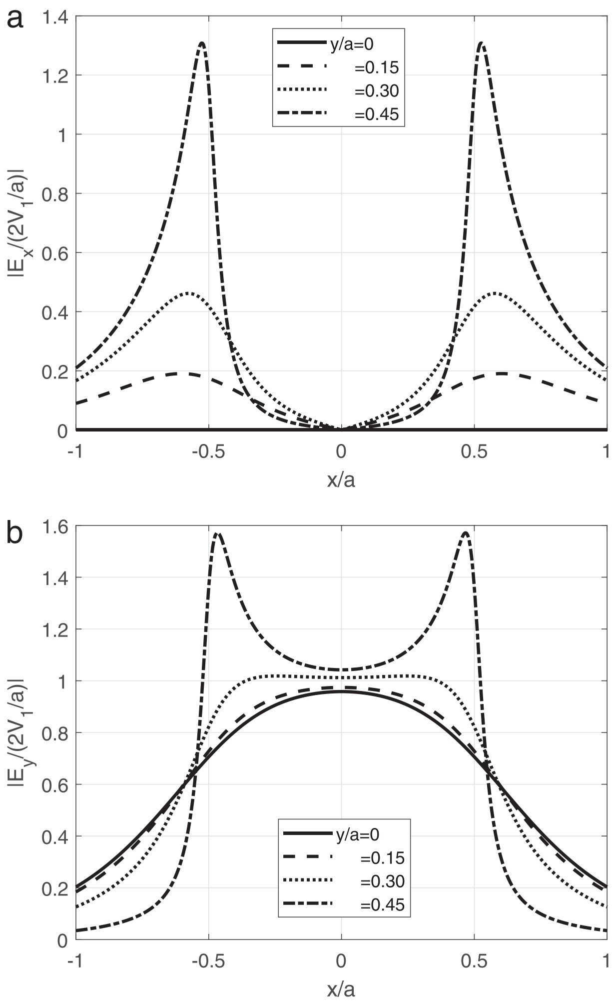


Figure 4. The $x$-component (a) and $y$-component (b) of the total electric field as a function of the $x$-axis for different values of $y$.

of Matlab. Figure 4 shows the magnitude of electric field components $E_x$ and $E_y$, normalized to the uniform electric field value: $E_{norm} = 2V/a$. As can be seen, both components are enhanced at the corners of the strip; this effect is particularly noted for $y/a = 0.45$.

Figure 5 shows a comparison between the electric field components obtained using (10) and the values obtained from an electrostatic simulation using QuickField (finite-element method) [11]. As can be seen, there is excellent agreement between both results.

### 3.2 Example 2

Due to the capacity of operating in transverse electromagnetic mode, the stripline is used in many electromagnetic capability testing applications. In tests, the device is generally placed in the axis $x = 0$, $0 < y < a/2$, where the electric field is uniform and vertical as shown in Figures 3 and 4.

Figure 6 presents the electric field along this axis, normalized to $E_{norm}$, for different $d/a$ ratios. The electric

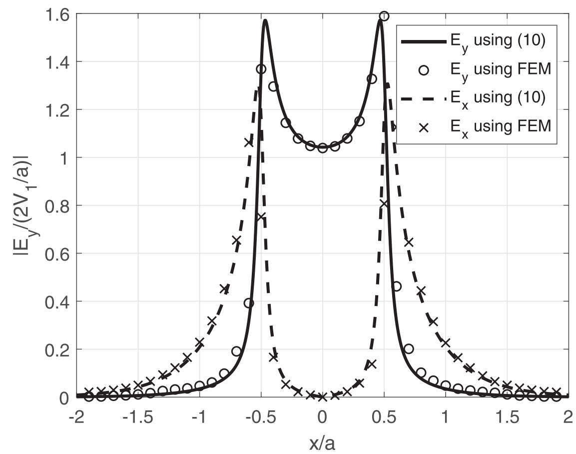


Figure 5. Normalized components of the electric field for $y/a = 0.45$ calculated using (10) compared with values obtained numerically using the finite-element method.

field inhomogeneity increases to a great extent for $d/a$ smaller than 1. Variations higher than 10% from $E_{norm}$ appear for ratios $d/a < 0.75$. For reference, this ratio produces a characteristic impedance of 78.2 ohms in air. Therefore, striplines in air with characteristic impedances higher than 78.2 ohms produce variations higher than 10% in the electric field along the $y$-axis.

Finally, we define the enhancement factor on the central point of the strip ($x = 0$, $y = a/2$) as

$$EF = \frac{Max\left|E_y(ia/2)\right|}{\frac{2V}{a}} \tag{13}$$

Equation (13) is plotted in Figure 7 for different $d/a$ ratios. The figure shows that for $d/a$ ratios smaller than 0.75, the uniformity in the field amplitude is not given.

## 4. Conclusions

A closed-form expression for the electric field in an ideal stripline immersed in a homogeneous dielectric

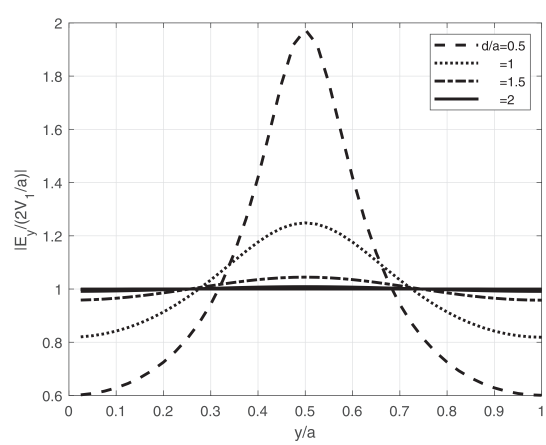


Figure 6. Variation of electric field intensity along the $y$-axis.

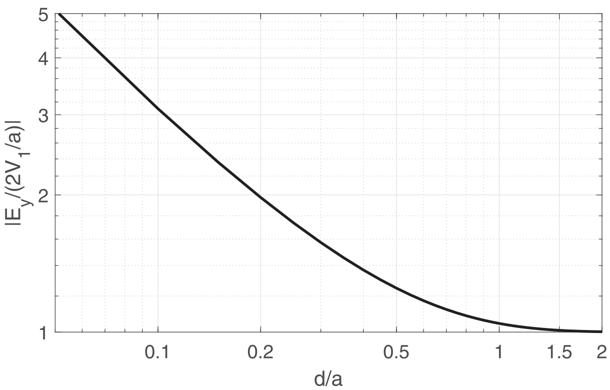


Figure 7. Enhancement factor as a function of the ratio $d/a$.

was presented and discussed. The method, based on an exponential conformal transformation, was described. The distribution and field intensity can be calculated with the presented equation without additional normalization procedures. The closed expression presented here can be used to estimate power ratings, voltage breakdown thresholds, and background field values. Different examples have been proposed demonstrating the applicability of the approach.

## 5. References


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