Characterisations of Nowhere Dense Graphs

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Abstract

Nowhere dense classes of graphs were introduced by Nešetřil and Ossona de Mendez as a model for "sparsity" in graphs. It turns out that nowhere dense classes of graphs can be characterised in many different ways and have been shown to be equivalent to other concepts studied in areas such as (finite) model theory. Therefore, the concept of nowhere density seems to capture a natural property of graph classes generalising for example classes of graphs which exclude a fixed minor, have bounded degree or bounded local tree-width. In this paper we give a self-contained introduction to the concept of nowhere dense classes of graphs focussing on the various ways in which they can be characterised. We also briefly sketch algorithmic applications these characterisations have found in the literature.

1998 ACM Subject Classification G.2.2 Graph Theory

Keywords and phrases Graph Algorithms, Algorithmic Graph Structure Theory, Finite Model Theory, Nowhere Dense Classes of Graphs

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2013.21

Category Invited Talk

1 Introduction

Structural graph theory has proved to be a powerful tool for coping with computational intractability. It provides a wealth of concepts and results that can be used to design efficient algorithms for hard computational problems on specific classes of graphs that occur naturally in applications. Examples include polynomial-time (in fact, fixed-parameter linear) algorithms for problems such as computing dominating sets, independent sets, Hamiltonian cycles, 3-colourings and many other problems on classes of graphs of *bounded tree-width*. See e.g. [6, 8, 7, 5, 4] for surveys on tree-width and the huge number of algorithmic applications. Other examples are *polynomial-time approximation schemes* (PTAS) for problems such as vertex cover, dominating sets or Steiner-forests on planar graphs or, more generally, on graph classes of bounded genus or which exclude a fixed graph as a minor [3, 58, 29, 36, 14].

In their monumental work on graph minors [56], Robertson and Seymour developed a very powerful structure theory for classes of graphs excluding a fixed minor which has found a large number of algorithmic consequences, for instance to constant-factor approximations of colouring problems (see e.g. [19, 20]), to polynomial-time approximation schemes ([36, 19, 14, 21]) or for general parameterized algorithms for problems such as dominating sets and many other in form of bidimensionality theory developed in [17] and subsequent papers. See e.g. [16, 18].

Excluding a fixed graph H as a minor yields classes of graphs for which topological methods can be employed to obtain efficient algorithms for computational problems. A

33rd Int'l Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2013).
Editors: Anil Seth and Nisheeth K. Vishnoi; pp. 21–40
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

different approach is taken in classes of bounded degree, where topology does not play a decisive rôle in the development of algorithms but other approaches succeed in obtaining good algorithms for various problems on input graphs of maximum degree d, for some constant d. Graph classes of bounded degree can be generalised further to classes of *bounded local tree-width* [30, 31], a property where we do not require that the entire graph has small tree-width, but only that every r-neighbourhood of a vertex in the graph has tree-width bounded by some function of its radius r. Again, bounded local tree-width and excluded minors are incomparable concepts.

Whereas many papers provide optimised algorithms for specific, individual problems on certain classes of graphs, another line of research aims at developing general tractability results for a whole and natural class of problems on special classes of inputs. One example is bidimensionality theory as outlined above. Many other examples of such general tractability results, often referred to as *algorithmic meta-theorems*, use definability of computational problems in logical languages to obtain natural classes of problems. The best-known of these results is Courcelle's theorem [12] stating that every algorithmic problem definable in *monadic second-order logic* can be decided in linear time on classes of graphs of bounded tree-width. This includes many common algorithmic problems such as Hamiltonicity, 3-Colourability and many covering problems such as dominating sets. Following Courcelle's result, meta-theorems of various forms have been developed, see e.g. [9, 57, 33, 34, 13, 28] and the surveys [37, 38, 45].

In general, the main goals of this whole line of research described so far, sometimes referred to as *algorithmic graph structure theory*, are the following: we want to understand for natural and important classes of graphs what kind of problems can be solved efficiently on these graphs and to develop the corresponding graph structural and algorithmic techniques; for natural classes of problems we want to understand their general tractability frontier, i.e. the "most general" classes of graphs on which these problems become tractable.

In particular the last aspect has been pursued intensively in research on algorithmic meta-theorems with a quest for finding the largest classes of graphs where problems definable in first-order logic become tractable. First-order definable problems define a natural class of problems including dominating sets, vertex covers, network centres and many others.

As diverse as the examples above of graph classes with a rich algorithmic theory may appear, a feature all these classes have in common is that they are relatively *sparse*, i.e. graphs in these classes have a relatively low number of edges compared to the number of vertices. In fact, classes of graphs excluding a fixed minor can only have a linear number of edges. This suggests that this "sparsity" might be an underlying reason why many problems can be solved efficiently on these classes of graphs, even though they otherwise do not have much in common. This leads to the question how to define a reasonable concept of "sparse classes of graphs".

A first idea to capture the concept of "sparse" classes of graphs is to bound the average degree, i.e. to study classes $\mathcal{C}$ of graphs such that for all $G \in \mathcal{C}$, $\frac{|E(G)|}{|V(G)|} \leq d$ for some constant d. However, given any graph G of order $n := |V(G)|$, we can bound its average degree by "padding", i.e. simply by adding n^2 isolated vertices. While this reduces the average degree below 2, for many problems it does not significantly change the structure of the graph. For instance, adding extra isolated vertices does not really change the problem of evaluating first-order formulas. Hence, this notion is not a satisfactory measure for sparsity in general. Therefore, to prevent padding arguments of this form, we may want to require that sparse graphs remain sparse if we take a subgraph, i.e. that sparse graphs do not have dense subgraphs.

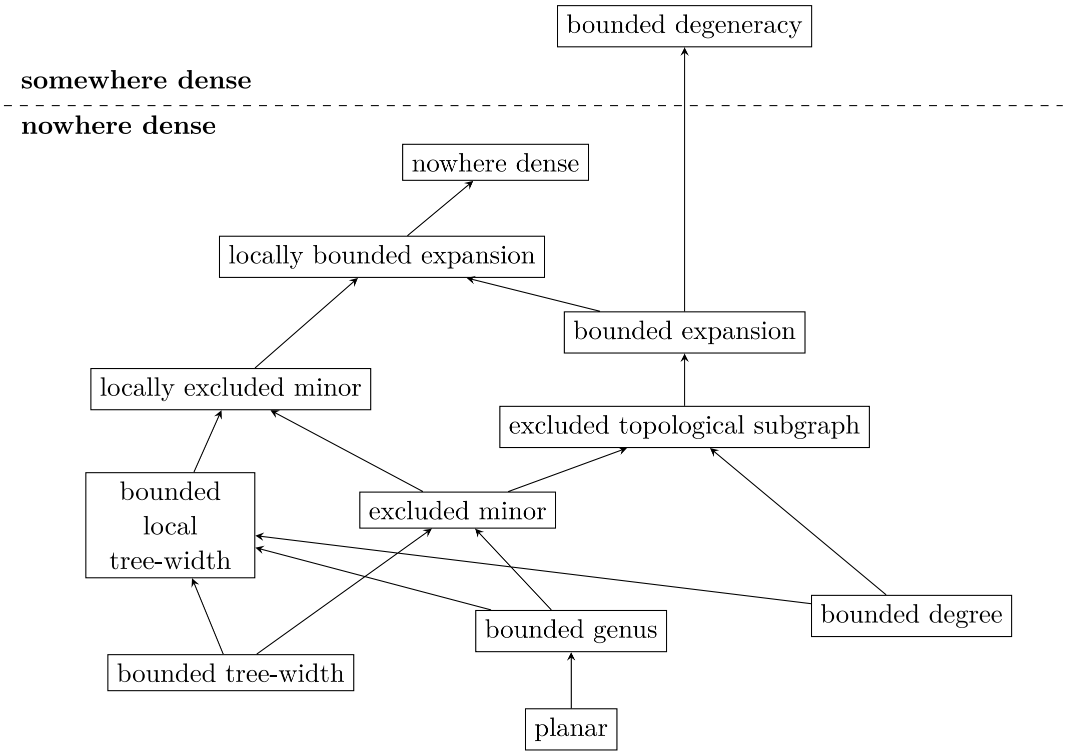

Figure 1 Sparse graph classes.

This leads to the well-studied concept of *degeneracy*. A graph G is d-degenerate if every subgraph of G contains a vertex of degree at most d. In particular, this means (and in fact degeneracy is equivalent to saying) that the average degree of every subgraph is bounded by a constant. But again this concept is not universally satisfactory as we can make every graph 2-degenerate by *subdividing* every edge once. Here, by subdividing an edge we mean the operation of replacing an edge $\{u, v\}$ by a path of length 2. Again, for various problems such as evaluating first-order formulas, this does not change the nature of the graph significantly. Hence, in addition to closure under subgraphs, we may want to require of our notion of sparsity that it should be invariant under such subdivisions or local modifications. The two requirements together exactly yield the concept of *nowhere dense* classes of graphs introduced by Nešetřil and Ossona de Mendez [52]. See [49] for an extensive study of sparse graphs. We defer a formal definition to Section 2, see Definition 2.5.

It turns out that nowhere dense classes of graphs can equivalently be characterised in many different ways which at first sight have very little in common. For instance, nowhere dense classes of graphs can equivalently be characterised by the concept of *uniformly quasi-wideness* (see Section 6), a concept studied in finite model theory; by the existence of *low tree-depth colourings* (see Section 4); by generalised colouring numbers (see Section 3); by a game characterisation (see Section 7.2); or by the model-theoretic concept of *independence*, see [1]. See the individual sections for references. This shows that the concept of nowhere dense classes of graphs is very robust and seems to capture a natural property of graph classes arising independently in many diverse contexts.

By construction, nowhere dense classes of graphs contain all examples of special graph classes mentioned above and thereby form a very general type of graph classes. Figure 1 shows the containment of the various classes mentioned so far. In particular, nowhere dense classes unify the two incomparable concepts of bounded degree or bounded local tree-width on the one hand and excluded minors or excluded topological subgraphs on the other hand. They therefore provide a unifying framework in which to study sparse classes of graphs.

Following their introduction, nowhere dense classes of graphs have been studied also with algorithmic applications in mind. The fact that they can be characterised in many different ways also has very nice algorithmic consequences, as each characterisation yields different algorithmic techniques. For instance, it has been shown that problems such as network centres and dominating sets can be solved by fixed-parameter algorithms on nowhere dense classes of graphs using uniformly quasi-widesness [15] (see Section 6). Using low tree-depth colourings, Nešetřil and Ossona de Mendez [50] showed that the subgraph isomorphism or homomorphiosm problem is fixed-parameter tractable on nowhere dense classes. Furthermore, using the same characterisation, Gajarsky et al. [35] extended the meta-kernelisation framework of [10] to nowhere dense classes of graphs providing polynomial kernels for a large number of algorithmic problems (see Section 4). On an important subclass of nowhere dense graphs, called classes of *bounded expansion* (see Definition 2.8), even more algorithmic applications are known, for instance in database query answering and enumeration [42], which relies on the concept of augmentations (see Section 5), or in approximating dominating sets [25], which is based on generalised colouring numbers (see Section 3).

As mentioned above, in a series of papers starting in the 1990s researchers have tried to investigate the largest classes of graphs on which all first-order definable graph properties can be decided efficiently, or more formally, on which first-order model checking is fixed-parameter tractable. It turns out that for classes of graphs closed under subgraphs, this limit are exactly nowhere dense classes of graphs. For, it was shown in [45] and [27] that on classes of graphs which are not nowhere dense but closed under subgraphs first-order model-checking cannot be fixed-parameter tractable unless FPT = W[1], a consequence widely believed to be false in the parameterized complexity community (see e.g. [24, 32]). On the other hand, very recently it was shown by the authors of this paper that on nowhere dense classes of graphs, first-order model-checking is fixed-parameter tractable [39]. Hence, this is another indication that nowhere dense classes of graphs form a natural limit for certain types of algorithmic problems to be solved efficiently.

As the exposition above already indicates, the appeal of nowhere dense classes and their applications relies on the fact that they are characterised in many different ways. In this paper we provide a self-contained introduction to nowhere dense classes with a strong focus on their different characterisations. In each of the individual sections of this paper we will present some characterisation of nowhere dense classes of graphs, prove its equivalence to other characterisations, state their main properties and sketch some algorithmic consequences.

Notation. We use standard notation in graph theory and refer, e.g., to [23] for details. In particular, we write $d(G)$ for the average degree of a graph G, $\delta(G)$ for the minimum degree and $\Delta(G)$ for its maximum degree. $\Delta^-(\vec{G})$ denotes the maximum in-indegree of a directed graph $\vec{G}$.

2 Sparse graphs

As motivated in the introduction, we want to find the largest number of edges an n-vertex graph can have such that we still find structure that can be used algorithmically. This is closely related to one of the classical questions of extremal graph theory: given a graph H, what is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph? This number is known as the Turán number $ex(n, H)$ of H and Turán determined the exact value of $ex(n, K_t)$ for the complete graph on t vertices, K_t. For algorithmic purposes often a more interesting restriction is to exclude a graph as a topological subgraph. A graph H is a *subdivision* of a graph H' if H can be obtained from H' by replacing edges by vertex disjoint paths. H is a *topological subgraph* or *topological minor* of G, denoted $H \preceq^t G$, if a subdivision of H is isomorphic to a subgraph of G. Mader was one of the first who considered Turán's question for topological subgraphs.

▶ **Theorem 2.1** (Mader [46]). *Given $t \in \mathbb{N}$, there exists a constant c_t depending only on t such that every graph G on n vertices with at least $c_t n$ edges contains a subdivision of the complete graph K_t.*

Bollobás and Thomason [11] and independently Komlós and Szemerédi [44] showed that $c_t \leq ct^2$ for some absolute constant c, hence every graph with average degree at least ct^2 contains a subdivision of K_t. More generally, a graph that contains a subgraph with average degree at least ct^2 contains a subdivision of K_t.

Recent results show that even less structural information suffices to solve many problems efficiently. A graph H' is an *r-subdivision* of a graph H if H' can be obtained from H by replacing edges by vertex disjoint paths of length at most $r + 1$. H is a *topological depth-r subgraph* or *topological depth-r minor* of a graph G, denoted $H \preceq^t_r G$, if an r-subdivision of H is isomorphic to a subgraph of G. In the proofs of the above results, the edges of the complete graphs that are found as topological subgraphs are subdivided by more than a constant number of vertices. It was hence the next step to ask how many edges an n-vertex graph that does not contain an r-subdivision of a complete graph K_t can maximally have. A first result in this direction is implied by a result of Alon, Krivelevich and Sudakov.

▶ **Theorem 2.2** (Alon, Krivelevich, Sudakov [2]). *Let $t \in \mathbb{N}$ and $\epsilon \geq 1/2$. Every graph G on n vertices with at least $\frac{t^2}{2} n^{1+\epsilon}$ edges contains a 2-subdivision of K_t.*

On the other hand, there are well known classes of graphs of girth at least g and $\Omega(n^{1+1/g})$ edges. Any c-subdivision of K_t, where $t \geq 3$, must contain a cycle of length at most $3c$. Hence there are n-vertex graphs with $\Omega(n^{1+1/(3c+1)})$ edges and no c-subdivision of K_t. Dvořák [26] and Jiang [41] independently answered the question for $0 < \epsilon < 1/2$ by showing the following.

▶ **Theorem 2.3** (Dvořák [26], Jiang [41]). *Given $t \in \mathbb{N}$ and $\epsilon > 0$. There exists $n_0 = n_0(t, \epsilon)$ and $c = c(\epsilon)$ such that all graphs G with $n \geq n_0$ vertices and at least $n^{1+\epsilon}$ edges contain a c-subdivision of the complete graph K_t.*

Jiang [41] provides the best bound for the constant $c(\epsilon)$ known today, $c(\epsilon) \leq \lfloor 10/\epsilon \rfloor$. If we apply this result to infinite classes of graphs, we obtain an interesting dichotomy in terms of edge densities. Note that for every graph G with at least one edge and for all $\epsilon \geq 0$,

$$|E(G)| = |V(G)|^\epsilon \Longleftrightarrow \frac{\log |E(G)|}{\log |V(G)|} = \epsilon.$$

▶ **Corollary 2.4.** *Let $\mathcal{C}$ be an infinite class of graphs. Then either for all $r \in \mathbb{N}$*

$$\lim_{n \to \infty} \sup \left\{ \frac{\log |E(H)|}{\log |V(H)|} \;\middle|\; G \in \mathcal{C} \text{ with } |V(G)| \geq n, H \preceq^t_r G \right\} \leq 1 \tag{2.1}$$

or there exists $r \in \mathbb{N}$ with

$$\lim_{n \to \infty} \sup \left\{ \frac{\log |E(H)|}{\log |V(H)|} \;\middle|\; G \in \mathcal{C} \text{ with } |V(G)| \geq n, H \preceq^t_r G \right\} = 2. \tag{2.2}$$

Here we take $\frac{\log |E(H)|}{\log |V(H)|}$ *to be* $-\infty$ *if* $E(H) = \emptyset$.

Proof. Note that the supremum always exists, because $\frac{\log |E(H)|}{\log |V(H)|} \leq 2$ for all H. Furthermore, observe that if an r-subdivision of H_1 is a subgraph of G and an s-subdivision of H_2 is a subgraph of H_1, then an $((r+1)s+1$-subdivision of H_2 is a subgraph of G. Assume that for some $r \in \mathbb{N}$, $\lim_{n\to\infty} \sup \left\{ \frac{\log |E(H)|}{\log |V(H)|} \;\middle|\; G \in \mathcal{C} \text{ with } |V(G)| \geq n, H \preceq_r^t G \right\} = 1 + 2\epsilon$ for some $0 < \epsilon < \frac{1}{2}$. Then there are infinitely many n-vertex graphs H with at least $n^{1+\epsilon}$ edges such that an r-subdivision of H is a subgraph of some $G \in \mathcal{C}$. Let $\mathcal{C}'$ be the class of those graphs. By Theorem 2.3, for all $t \in \mathbb{N}$ there exists $n_0(t, \epsilon)$ and $s := c(\epsilon)$ such that all graphs in $\mathcal{C}'$ contain an s-subdivision of K_t. By our above observation, $\mathcal{C}$ contains $(r+1)s+1$-subdivisions of arbitrary large complete graphs. Then the above limit goes to 2 for $(r+1)s+1$. ◀

The previous corollary was proved by Nešetřil and Ossona de Mendez [52] who showed that the limits defined there form a trichotomy, i.e. that for all classes $\mathcal{C}$ of graphs the lim sup can only take the values $\{0, 1, 2\}$. Those classes for which the limits is ≤ 1 are called nowhere dense.

▶ **Definition 2.5.** Let $\mathcal{C}$ be a class of graphs. $\mathcal{C}$ is *nowhere dense*, if

$$\lim_{n\to\infty} \sup \left\{ \frac{\log |E(H)|}{\log |V(H)|} \;\middle|\; G \in \mathcal{C} \text{ with } |V(G)| \geq n, H \preceq_r^t G \right\} \leq 1.$$

Otherwise $\mathcal{C}$ is called *somewhere dense*.

We can rephrase the definition in the following ways.

▶ **Corollary 2.6.** *A class $\mathcal{C}$ of graphs is nowhere dense if, and only if, for all $r \in \mathbb{N}$ and all $\epsilon > 0$ there is $n_0(r, \epsilon)$ such that all n-vertex graphs $H \preceq_r^t G \in \mathcal{C}$ with $n \geq n_0$ vertices satisfy $|E(H)| \leq n^{1+\epsilon}$.*

▶ **Corollary 2.7.** *A class $\mathcal{C}$ is nowhere dense if and only if there is a function f such that for all $r \in \mathbb{N}$ we have $K_{f(r)} \not\preceq_r^t G$ for all $G \in \mathcal{C}$.*

The original interest of Nešetřil and Ossona de Mendez when studying sparse graph classes was the following subclass of nowhere dense classes [47, 53, 48].

▶ **Definition 2.8.** A class $\mathcal{C}$ has bounded expansion if for every r there exists $n_0(r)$ and $c(r)$ such that all n-vertex graphs $H \preceq_r^t G \in \mathcal{C}$ with $n \geq n_0$ vertices satisfy $|E(H)| \leq c \cdot n$.

▶ Remark. Nowhere dense classes and classes of bounded expansion were originally defined in terms of excluded depth-r minors and not in terms of excluded r-subdivisions. In general, minors and topological subgraphs behave quite different. Surprisingly, densities of bounded depth minors and bounded depth topological subgraphs are strongly related. In our presentation we will always work with topological subgraphs and hence we have defined nowhere dense classes accordingly.

In the rest of this section we present a proof of Theorem 2.3. Our presentation follows [49]. A well-known lemma from graph theory states that any graph with high average degree contains a subgraph of high minimum degree.

▶ **Lemma 2.9.** *Let G be a graph and $1/|V(G)| < \epsilon \leq 1$. Then G has a subgraph H with*

$$\delta(H) \geq (1-\epsilon)\frac{|E(G)|}{|V(G)|} \qquad \textit{and} \qquad |E(H)| \geq \epsilon \cdot |E(G)|.$$

The next lemma will be used to show that in graphs of large minimum degree d we can find a 1-subdivision of H of only slightly smaller minimum degree and such that we can carefully control the order of H. We will use this to show that when we express the minimum degree relative to the order of H, it will in fact grow.

▶ **Lemma 2.10.** *Let A be an n-element set and let $A_1, \dots, A_n \subseteq A$ be a collection of sets of size at least d such that $A = \bigcup_{1\le i\le n} A_i$. Let $d \le s \le n$. Then there exists a set $S \subseteq A$ of size s such that $|A_i \cap S| \ge \lfloor s \cdot d/n \rfloor$ for at least $n/2$ of the A_i.*

Proof. We may assume without loss of generality that every set A_i has exactly d elements. For each i, the number of subsets of A of size s including exactly k elements of A_i is $\binom{n-d}{s-k}\binom{d}{k}$ (choose k elements of A_i and $s-k$ elements of $A \setminus A_i$). Hence the number of subsets of A of size s including at least k elements from A_i is $\sum_{k\le\ell\le s}\binom{n-d}{s-\ell}\binom{d}{\ell}$.

For $k \le d$ consider the bipartite graphs G_k, where one part consists of the s-element subsets of A and the other part consists of the sets A_i. We add an edge (B, A_i) if and only if $|B \cap A_i| \ge k$. The degree of B in G_k corresponds to the number of A_i that B shares at least k elements with. As observed above, every A_i has degree $\sum_{k\le\ell\le s}\binom{n-d}{s-\ell}\binom{d}{\ell}$. Hence

$$|E(G_k)| = n \cdot \sum_{k\le\ell\le s}\binom{n-d}{s-\ell}\binom{d}{\ell}.$$

The average degree of a vertex B in G_k is $\frac{n\cdot\sum_{k\le\ell}\binom{n-d}{s-\ell}\binom{d}{\ell}}{\binom{n}{s}}$ and there must be one vertex S with at least this degree.

Observe that $\frac{\binom{n-d}{s-k}\binom{d}{k}}{\binom{n}{s}}$ is the probability mass function of a hypergeometric distribution with mean $\frac{ds}{n}$. Hence for $k = \lfloor \frac{ds}{n} \rfloor$ the above sum is greater than 1/2 and hence $d(S) \ge n/2$ for this k. We conclude that $|A_i \cap S| = k = \lfloor \frac{ds}{n} \rfloor$ for at least $n/2$ of the A_i. ◀

We now show how to find a subgraph with larger minimum degree with respect to its order.

▶ **Lemma 2.11.** *Let $\rho > 1$. There exists $n_0(\rho)$ such that for all graphs G on $n \ge n_0$ vertices with minimum degree $\delta(G) \ge n^{1/\rho}$ there exists a graph H such that a 1-subdivision of H is a subgraph of G and either*

- *H is a complete graph of order $n^{1/(3\rho^2)}$, or*
- *$\delta(H) \ge |V(H)|^{1/(\rho-1/2)}$ and $|V(H)| \ge \sqrt{n/6}$.*

Proof. Let $\mu := 1/\rho$ and $s := n^{1-\mu+\mu^2/3}$. For each vertex $a_i \in V(G)$ let A_i be the set of neighbours of a_i. Then every set A_i has at least n^μ many vertices, $\bigcup_{1\le i\le n} A_i = A$ and $n^\mu \le s \le n$. By Lemma 2.10 there exists a subset $S \subseteq A$ of size s and a set $T' \subseteq V(G)$ of size at least $n/2$ such that every vertex in T' has at least $n^\mu \cdot n^{1-\mu+\mu^2/3}/n = n^{\mu^2/3}$ neighbours in S. Let $T := T' \setminus S$. T has size at least $t := n/2 - s$. Enumerate the vertices of T as $t_1, \dots, t_t$. We construct a sequence $H_0 \subseteq H_1 \subseteq \dots$ of graphs, where H_0 is the empty graph on vertex set S. Assume that graph H_i has already be constructed. Consider all neighbours of t_{i+1} in S. If they induce a complete graph in H_i define H to be this complete graph of order $n^{\mu^2/3} = n^{1/(3\rho^2)}$. Otherwise, we add the edge e_{i+1} to H_{i+1} between two arbitrary not yet adjacent neighbours u, v of t_{i+1} and associate with this edge the path u, t_{i+1}, v of length 2 and continue the construction. After t steps, we define $H' := H_t$. Then $|E(H')| = t$.

Let $d := n^{\mu-\mu^2/3} > 2$ and $\epsilon := (1-1/d)/(2-1/d)$ (hence $1/n < 1/2 < \epsilon < 1$). Note that $n/d = s$. By Lemma 2.9, H' has a subgraph H with

$$\delta(H) \geq (1-\epsilon)\frac{|E(H')|}{|V(H')|} = \frac{1}{2-1/d} \cdot \frac{n/2 - n/d}{n/d} = \frac{1}{2-1/d} \cdot (2d-1) = d = n^{\mu-\mu^2/3}$$

and

$$\begin{aligned} |E(H)| \geq \epsilon \cdot |E(H')| &= \frac{1-1/d}{2-1/d} \cdot (n/2 - n/d) = (1-1/d) \cdot \frac{d}{2d-1} \cdot \frac{d-2}{2d} \cdot n \\ &= (1-1/d) \cdot \frac{d-2}{4d-2} \cdot n \geq (1-1/d) \cdot \frac{n}{10}. \end{aligned}$$

We conclude that

$$|V(H)| \geq \sqrt{2|E(H)|} \geq \sqrt{(1-1/d)\frac{n}{5}}.$$

For sufficiently large n, $1/d \leq 1/6$ and hence $|V(H)| \geq \sqrt{n/6}$.

On the other hand, as $H \subseteq H'$, we have $|V(H)| \leq n^{1-\mu+\mu^2/3}$, and hence $n \geq |V(H)|^{\frac{1}{1-\mu+\mu^2/3}}$. Then

$$\delta(H) \geq |V(H)|^{\frac{\mu-\mu^2/3}{1-\mu+\mu^2/3}} \geq |V(H)|^{\mu+\frac{\mu^2}{2}}.$$

We conclude the proof by observing that $\left(\frac{1}{\rho} + \frac{1}{2\rho^2}\right)^{-1} = \rho - \frac{1}{2} + \frac{1/2}{2\rho+1} > \rho - \frac{1}{2}$. ◀

We are ready to take the final step.

▶ **Lemma 2.12.** *Let $\rho > 1$. There exists $n_0 = n_0(\rho)$ and $\mu = \mu(\rho) > 0$ such that for all graphs G on $n \geq n_0$ vertices with minimum degree at least $n^{1/\rho}$ we find a complete graph of order n^μ as a 9^ρ subdivision of G.*

Proof. We construct a sequence of graphs $G_0, G_1, \dots, G_k$ such that for each $0 \leq i \leq k$ the graph G_i has order n_i and minimum degree at least $n_i^{1/(\rho-i/2)}$ as follows. Let $G_0 := G$. Iteratively, for each $i \geq 0$, if G_i is not a complete graph we apply Lemma 2.11 to G_i. We get a graph H_i whose 1-subdivision is a subgraph of G_i. If H_i is a 1-subdivision of a complete graph we stop. Otherwise we let $G_{i+1} := H_i$. The process stops after $k \leq 2\rho$ iterations because of the increase of $\delta(G_i)$. We have $n_{i+1} \geq \sqrt{n_i/6}$ and hence $n_{k-1} \geq \frac{1}{6}\left(\frac{n}{6}\right)^{2^{-2\rho}}$. At the next step we find a complete subgraph of size at least $n_{k-1}^{1/(3\rho^2)}$ and we let $\mu > 0$ such that for any $k \leq 2\rho$ we have $n_k \geq n^\mu$. Now every 1-subdivision of a k-subdivision is a $2k+1$ subdivision of the original graph. For simplicity we treat it as a $3k$ subdivision. Hence we find G_k as a $3^{2\rho} = 9^\rho$-subdivision of G. ◀

3 Generalised colouring numbers

As explained in the introduction, degeneracy is another concept to describe sparse graphs. Degeneracy gives rise to an ordering of the vertices of a graph with nice properties in a very natural way. Let d be the degeneracy of G. By induction on the number of vertices we construct an order $v_1 < \ldots < v_n$ such that every vertex v_i has at most d neighbours in $\{v_1, \ldots v_{i-1}\}$. In G, there is a vertex v which has at most d neighbours. $G - v$ is also

d-degenerate and has $n-1$ vertices. By induction, we have an order $v_1 < \ldots < v_{n-1}$ such that every v_i has at most d neighbours in $\{v_1, \ldots, v_{i-1}\}$. Adding v as the largest element to this order gives us an order with the desired properties. This order can for example be used to compute a vertex colouring of $V(G)$ with at most d colours in linear time.

When characterising nowhere dense classes in terms of degeneracy, we have to talk about the degeneracy of topological depth-r subgraphs. For example the class of 1-subdivisions of complete graphs is a class of degeneracy 2 but it is dense at depth 1. The aim of this section is to find a measure which generalises degeneracy and which allows to state that a class is nowhere dense if and only if this measure is bounded by n^ϵ for every sufficiently large n-vertex graph from $\mathcal{C}$. Such generalisations were found by Kierstead and Yang [43] which they called the generalised colouring numbers of a graph. Their theorem is weaker than what they actually proved, as they were not aware of the depth-r minor terminology. Zhu [60] formulated their theorem in terms of topological depth-r subgraphs. The following presentation follows Kierstead and Yang.

For a graph G, let $\Pi(G)$ be the set of all linear orderings of the vertices of G. For $\leq \in \Pi(G)$ and $x, y \in V(G)$, we say that x is *weakly k-reachable*[1] from y if there is a path of length $0 \leq \ell \leq k$ from y to x such that x is the smallest vertex with respect to the ordering. Let $\mathrm{WReach}_k[G, \leq, y]$ be the set of vertices that are weakly k-reachable from y with respect to the ordering. If furthermore, all internal nodes of the path are larger than y in the ordering, then x is called *strongly k-reachable* from y. Let $\mathrm{SReach}_k[G, \leq, y]$ be the set of vertices that are strongly k-reachable from y with respect to the ordering. The *weak k-colouring number* $\mathrm{wcol}_k(G)$ of G is defined as

$$\mathrm{wcol}_k(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} [\mathrm{WReach}_k(G, \leq, v)]$$

and the *k-colouring number* $\mathrm{col}_k(G)$ of G is defined as

$$\mathrm{col}_k(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} [\mathrm{SReach}_k(G, \leq, v)].$$

The aim of this section is to present a proof of the following theorem which follows from Kierstead and Yang's result [43] and Zhu's result [60].

▶ **Theorem 3.1.** *A class $\mathcal{C}$ of graphs is nowhere dense if and only if for every $\epsilon > 0$ there is $n_0 = n_0(r, \epsilon)$ such that $\mathrm{wcol}_r(G') \leq n^\epsilon$ for all n-vertex subgraphs $G' \subseteq G$ of a graph $G \in \mathcal{C}$ with $n \geq n_0$.*

The direction from right to left is easy to see. We show that an $r-1$-subdivision G of a complete graph K_t satisfies $\mathrm{wcol}_r(G) \geq t-1$. To see this, fix any ordering of $V(G)$. Let v be the largest vertex with respect to the ordering that corresponds to a vertex of K_t. For every other vertex w which corresponds to a vertex of K_t, v weakly k-reaches either w or some subdivision vertex on the path of length at most $r-1$ between v and w. Hence if $\mathcal{C}$ is somewhere dense, i.e., it contains arbitrary large complete graphs as $r-1$-subdivisions for some r, then the weak colouring numbers are too large.

To prove the other direction, we first show the following connection between weak-colouring number and colouring number.

▶ **Theorem 3.2** (Kierstead, Yang [43]). *Let G be a graph. Then $\mathrm{col}_k(G) \leq \mathrm{wcol}_k(G) \leq \mathrm{col}_k(G)^k$.*

[1] Note that weak k-reachability is also known as weak k-accessibility and strong k-reachability is known as k-accessibility in the literature.

Proof. The first inequality clearly holds. For the second inequality let $L \in \Pi(G)$. We show by induction on k that

$$\max_{v\in V(G)} |\mathrm{WReach}_k[G,\le,v]| \le (\max_{v\in V(G)} |\mathrm{SReach}_k[G,\le,v]|)^k.$$

For the base step $k=1$, observe that $|\mathrm{WReach}_1[G,\le,v]| = |\mathrm{SReach}_1[G,\le,v]|$. (Note also that the degeneracy of a graph is equal to $\mathrm{wcol}_1(G)+1 = \mathrm{col}_1(G)+1$). Let $k>1$ and $y \in V(G)$. For each $x \in \mathrm{WReach}_k[G,\le,y]$ let P_{xy} be a shortest $x-y$-path such that every vertex $z \in V(P)$ satisfies $x \le_L z$. If $x \ne y$, let w be the first vertex on P_{xy} such that $w <_L y$ and let i be the distance from y to w. Then $w \in \mathrm{SReach}_i[G,\le,y] - \mathrm{SReach}_{i-1}[G,\le,y]$ and $x \in \mathrm{WReach}_{k-i}[G,\le,w]$. It follows that

$$\begin{aligned}
&|\mathrm{WReach}_k[G,\le,y]|\\
&\le 1 + \sum_{i=1}^{k} \big|\mathrm{SReach}_i[G,\le,y] - \mathrm{SReach}_{i-1}[G,\le,y]\big| \cdot \max_{v\in V(G)} \big|\mathrm{WReach}_{k-1}[G,\le,v]\big|\\
&\le \big|\mathrm{SReach}_k[G,\le,y]\big| \cdot \max_{v\in V(G)} \big|\mathrm{WReach}_{k-1}[G,\le,v]\big|.
\end{aligned}$$

Thus by the induction hypothesis

$$\begin{aligned}
|\mathrm{WReach}_k[G,\le,y]| &\le |\mathrm{SReach}_k[G,\le,y]| \cdot \max_{v\in V(G)} |\mathrm{WReach}_{k-1}[G,\le,v]|\\
&\le |\mathrm{SReach}_k[G,\le,y]| \cdot (\max_{v\in V(G)} |\mathrm{SReach}_{k-1}[G,\le,v]|)^{k-1}\\
&\le (\max_{v\in V(G)} |\mathrm{SReach}_k[G,\le,v]|)^k.
\end{aligned}$$

◀

We now show that for a nowhere dense class of graphs, for sufficiently large n-vertex subgraphs G' of graphs from the class, $\mathrm{col}_r(G') \le n^\epsilon$.

▶ **Theorem 3.3.** *There exists a function $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ which is linear in the first argument such that for all $d, r \in \mathbb{N}$ and all classes $\mathcal{C}$ of graphs, if the class $\{H : H \preceq_r^t G, G \in \mathcal{C}\}$ is d-degenerate, then $\mathrm{col}_k(G') \le f(d,r)$ for every subgraph $G' \subseteq G$ of a graph $G \in \mathcal{C}$.*

Proof. Let $G' \subseteq G$ for some $G \in \mathcal{C}$. Define f by

$$f(d,r) = \begin{cases} r+1 & \text{if } d = 1\\ 2d \cdot f(d,r-1)^{2r^2} & \text{else.}\end{cases}$$

If $d=1$, then G' is a forest and it is easy to see that $\mathrm{col}_r(G') \le r+1$.

If $d \ge 2$, we recursively construct an ordering $L = x_1x_2\ldots x_n$ of V as follows. Suppose that we have constructed the final sequence $x_{i+1}\ldots x_n$ of L (if $i=n$ then this sequence is empty). Let $M = \{x_{i+1},\ldots,x_n\}$ be the set of vertices that have already been ordered and let $U = V - M$ be the set of vertices that have not yet been ordered. Notice that even though we have not finished constructing L, we have determined $\mathrm{SReach}_r[G',\le,y]$ for any $y \in M$. However, we have not necessarily determined $\mathrm{WReach}_r[G',\le,y]$. We now have to choose x_i from U. To do this we first define a probability space Ω, where each point in Ω is a graph $H = (U,F)$ such that an r-subdivision of H is a subgraph of G'. For each pair $\{u,v\} \subseteq U$ for which there exists a $u-v$ path of length at most r whose internal vertices are all in M, choose any such path and denote it by P_{uv}. For each vertex $z \in M$ let

$$S_z = \{\{u,v\} \subseteq U : z \in P_{uv}\}.$$

Label each $z \in M$ with a random element chosen from S_z; if $S_z = \emptyset$ then leave z unlabeled. Let F be the set of edges $\{u,v\}$ such that every internal vertex of P_{uv} is labeled with $\{u,v\}$. Then an r-subdivision of H is a subgraph G'. If P_{uv} is defined then the probability that $\{u,v\} \in F$ is

$$\Pr(\{u,v\} \in F) = \prod_{z \in M \cap V(P_{uv})} \frac{1}{|S_z|}.$$

In particular, if $\{u,v\} \in E$ then $\Pr(\{u,v\} \in F) = 1$. Let $E[d_H(u)]$ be the expected value of the degree of u in H. Choose x_i in U such that $E[d_H(x_i)]$ is minimal. We show that $E[d_H(x_i)] \leq 2d$.

Assume towards a contradiction that for all $u \in U$

$$E[d_H(u)] = \frac{\sum_{H' \in \Omega} d_{H'}(u)}{|\Omega|} > 2d.$$

Then

$$|U| \cdot d \geq \frac{\sum_{H' \in \Omega} |E(H')|}{|\Omega|} = \frac{\sum_{H' \in \Omega} \sum_{u \in U} d_{H'}(u)/2}{|\Omega|} > |U| \cdot d.$$

This is a contradiction and completes the construction of L.

We now argue by induction on $s \leq r$ that $|\mathrm{SReach}_s(y)| < f(d,s)$ for all vertices y. The base step $s = 1$ is trivial, so consider the induction step $s = t+1$. Let $y \in V(G')$. Let U and M be the sets at the step before y was added to the order in the above recursion ($\mathrm{SReach}_s[G', \leq, y]$ is determined at this step). For each $z \in M$ and $\{u,v\} \in S_z$ both u and v are in $\mathrm{WReach}_t[G', \leq, z]$. Thus, by induction hypothesis and Theorem 3.2, $|S_z| \leq |\mathrm{WReach}_t[G', \leq, z]|^2 < f^{2t}(d,t)$. It follows that

$$\begin{aligned} 2d \geq E[d_H(y)] &= \sum_{x \in \mathrm{SReach}_s(y)} \Pr(\{x,y\} \in F) \\ &= \sum_{x \in \mathrm{SReach}_s(y)} \prod_{z \in M \cap V(P_{xy})} \frac{1}{|S_z|} > |\mathrm{SReach}_s[G', \leq, y]| \cdot f(d,t)^{-2t^2}. \end{aligned}$$

So $|\mathrm{SReach}_s[G', \leq, y]| < 2d \cdot f(d,t)^{2t^2} = f(d,s)$. ◀

As an algorithmic application, we close this section by demonstrating how generalised colouring numbers can be used in the design of *sparse neighbourhood covers*. *Neighborhood covers* of small radius and small size play a key role in the design of many data structures for distributed systems. See e.g. [54]. In this section we will show that nowhere dense classes of graphs admit sparse neighbourhood covers of small radius and small size.

▶ **Definition 3.4.** For $r \in \mathbb{N}$, an *r-neighbourhood cover* $\mathcal{X}$ of a graph G is a set of connected subgraphs of G called *clusters*, such that for every vertex $v \in V(G)$ there is some $X \in \mathcal{X}$ with $N_r(v) \subseteq X$. The *radius* $\mathrm{rad}(\mathcal{X})$ of a cover $\mathcal{X}$ is the maximum radius of any of its clusters. The *degree* $d^{\mathcal{X}}(v)$ of v in $\mathcal{X}$ is the number of clusters that contain v. The *maximum degree* $\Delta(\mathcal{X})$ of $\mathcal{X}$ is $\Delta(\mathcal{X}) = \max_{v \in V(G)} d^{\mathcal{X}}(v)$. The size of $\mathcal{X}$ is $\|\mathcal{X}\| = \sum_{X \in \mathcal{X}} |X| = \sum_{v \in V(G)} d^{\mathcal{X}}(v)$.

As proved in [39], nowhere dense classes of graphs admit sparse neighbourhood covers. This follows relatively easily from the characterisation of nowhere dense classes by weak colouring numbers (Theorem 3.1).

▶ **Theorem 3.5** (Grohe, Kreutzer, Siebertz [39]). *Let $\mathcal{C}$ be a nowhere dense class of graphs. There is a function f such that for all $r \in \mathbb{N}$ and $\epsilon > 0$ and all graphs $G \in \mathcal{C}$ with $n \geq f(r,\epsilon)$ vertices, there exists an r-neighbourhood cover of radius at most $2r$ and maximum degree at most n^{ϵ} and this cover can be computed in time $f(r,\epsilon) \cdot n^{1+\epsilon}$.*

4 Low tree-depth colourings

Many local problems can be solved by decomposing a graph into smaller pieces on which the problem hopefully becomes easier to solve. In the previous section, we have seen the concept of small radius neighbourhood covers. In this section we will use a graph colouring to define decompositions of graphs. It has been conjectured by Thomas [59] that for every graph K there is an integer k such that if a graph G excludes K as a minor then G has a vertex partition into two graphs with tree-width at most k. DeVos et al. proved the following stronger theorem.

▶ **Theorem 4.1** (DeVos et al. [22]). *For every graph K and every integer $j \geq 1$, there is an integer k, such that every graph with no K-minor has a vertex partition into $j+1$ graphs such that any j parts form a graph with tree width at most k.*

This result was strengthened by Hell and Nešetřil.

▶ **Theorem 4.2** (Hell and Nešetřil [40]). *For every graph K and integer $j \geq 1$, there is an inter $N(K, j)$ such that every graph with no K-minor has a vertex partition into N graphs such that any $j' \leq j$ parts form a graph with tree depth at most j'.*

The aim of this section is to present a characterisation of nowhere dense classes in terms of low tree depth colourings in the sense of the above theorem. Let us first give the formal definitions of tree depth and low tree depth colourings.

An *elimination tree* of a graph G is a rooted tree Y with vertex set $V(G)$ defined recursively as follows. If $V(G) = \{v\}$ then Y is just $\{v\}$. Otherwise, let $w \in V(G)$ be an arbitrary vertex which is chosen as the root of Y. The branches of Y at w are the elimination trees of the connected components of $G - w$ whose roots are the sons of w in Y. The height of a rooted tree Y is the maximum distance of the root to any vertex of the tree. The *tree depth* of a graph G is the minimum height of an elimination tree of G. It follows that we can give the following recursive characterisation.

Let G be a graph. The tree depth $\mathrm{td}(G)$ is defined as

$$\mathrm{td}(G) = \begin{cases} 0 & \text{if } |V(G)| = 1 \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v) & \text{if } G \text{ is connected and } |V(G)| > 1 \\ \max_{1 \leq i \leq k} \mathrm{td}(G_i) & \text{if } G_1, \ldots, G_k \text{ are the connected components of } G. \end{cases}$$

For $r \geq 1$, an r-tree depth colouring of G is a colouring such that every nonempty subgraph $G' \subseteq G$ is coloured by at least $\min\{r, \mathrm{td}(G') + 1\}$ colours. Equivalently, an r-tree depth colouring of G is a colouring such that any $r' \leq r$ colour classes induce a subgraph with tree depth at most $r' + 1$. The minimum number of colours of such a colouring of G is denoted by $\text{td-col}_r(G)$.

The aim of this section is to show the following theorem.

▶ **Theorem 4.3** (Nešetřil and Ossona de Mendez [52]). *A class $\mathcal{C}$ of graphs is nowhere dense if and only if for every $\epsilon > 0$ there is $n_0(r, \epsilon)$ such that $\text{td-col}_r(G') \leq n^{\epsilon}$ for all n-vertex subgraphs $G' \subseteq G$ of a graph $G \in \mathcal{C}$ with $n \geq n_0$.*

As a first step towards the proof of this theorem, we present a result of Zhu [60]. Recall that every nowhere dense class admits an ordering of its vertices such that only few vertices are weakly r-reachable from any other vertex.

▶ **Theorem 4.4** (Zhu [60]). *If G is a graph with $\mathrm{wcol}_{2^{r-2}}(G) \leq m$, then G can be coloured with m colours such that any in connected subgraph $H \subseteq G$ either some colour appears exactly once in H or H gets at least r colours.*

Proof. Let $\leq \in \Pi(G)$ be an ordering of $V(G)$ witnessing $\mathrm{wcol}_{2^{r-2}} \leq m$. Colour the vertices greedily with m colours, using the order L, such that the colour assigned to v is distinct from colours assigned to vertices weakly reachable from v. We claim that this colouring satisfies the desired properties.

Let H be a connected subgraph of G and let v be the minimum vertex of H with respect to L. If the colour $c(v)$ appears exactly once in H then we are done.

Assume $c(v)$ occurs more than once in H. We shall prove that H uses at least r colours. Let $u \neq v$ be a vertex of H with $c(u) = c(v)$ and let $P_0 = v, v_1, \ldots, v_q = u$ be a path in H connecting v and u. We must have $q > 2^{r-2}$, for otherwise v is weakly 2^{r-2}-accessible from u and we should have $c(v) \neq c(u)$. Let $u_0 := v$ and let $P_1 := v_1, \ldots, v_{2^{r-2}}$. Observe that no vertex of P_1 uses colour $c(u_0)$ and P_1 contains 2^{r-2} vertices. Assume $0 \leq j \leq r-2$ and that a vertex u_j of P_j and a subpath P_{j+1} of P_j are chosen such that the following holds. No vertex of P_{j+1} uses the colour of u_j and P_{j+1} contains at least 2^{r-j-2} vertices. We show how to establish this same situation for $j+1$. Let u_{j+1} be the minimum vertex of P_{j+1} with respect to L let P_{j+2} be the largest component of $P_{j+1} - u_{j+1}$. Then u_{j+1} is weakly 2^{r-2}-accessible from each vertex of P_{j+1} and hence no vertex of P_{j+2} uses the colour $c(u_{j+1})$. Moreover, P_{j+2} is a path containing at least 2^{r-j-3} vertices. We repeat this process until $j = r-2$ and obtain vertices $u_0, \ldots, u_{r-1}$ of distinct colours. Hence H uses at least r colours. ◀

The properties of the colouring in the above proof are important enough to give a special name to such colourings.

▶ **Definition 4.5.** An *r-centered colouring* of a graph G is a vertex colouring such that for any connected subgraph $H \subseteq G$, either some colour appears exactly once in H or H gets at least r colours.

It is not difficult to see that indeed such colourings induce low tree depth colourings.

▶ **Lemma 4.6.** *Any r-centered colouring is an r-tree depth colouring.*

Proof. Let c be an r-centered colouring of G. Assume that there is a subgraph $G' \subseteq G$ with $\mathrm{td}(G') = k < r$ which does not get $k+1$ colours. Let G' be minimal with this property. Then G' is connected. As G' does not get $k+1 \leq r$ colours and c is r-centered, there is one colour which occurs exactly once, say this colour is given to vertex v. Then $\mathrm{td}(G' - v) \geq k-1$ and $G' - v$ does not get $k-1$ colours. Hence G' was not minimal. ◀

Let us show how low tree depth colourings bound the edge density of depth-r topological subgraphs.

▶ **Lemma 4.7** (Zhu [60]). *Let G be a graph, $r \in \mathbb{N}$ and $H \preceq_r^t G$. Then*

$$\frac{|E(H)|}{|V(H)|} \leq \binom{\textit{td-col}_{r+2}(G)}{r+2}(r+1).$$

Proof. Consider a vertex colouring c of G with $N = \text{td-col}_{r+2}(G)$ colours such that any $i \leq r+2$ colours induce a subgraph of tree depth at most i. For every set J of $r+2$ colours let $G_J := G[c^{-1}(J)]$ and let Y_J be an elimination tree of height $td(G_J) \leq r+2$ of G_J (which is in fact a rooted forest).

Suppose that $V(H) = [k]$. As $H \preceq_r^t G$, there are vertices $h_1, \ldots, h_k \in V(G)$ and mutually internally disjoint paths $P_{ij} \subseteq G$ of length at most $(r+1)$ from h_i to h_j for all edges $ij \in E(H)$. Let J_{ij} be a set of $r+2$ colours such that $c(V(P_{ij})) \subseteq J_{ij}$. Then $P_{ij} \subseteq G_{J_{ij}}$, and there is a vertex v_{ij} on P_{ij} that is a common ancestor of h_i and h_j in the elimination tree $Y_{ij} = Y_{J_{ij}}$. Possibly, $v_{ij} = h_i$ or $v_{ij} = h_j$. We orient the edge $e = ij$ from i to j if $h_i = v_{ij}$ and from j to i if $v_{ij} = h_j$. If neither $h_i = v_{ij}$ nor $h_j = v_{ij}$ then we orient the edge ij arbitrarily. Let $\vec{H}$ be the resulting oriented graph.

To bound the number of edges of H, we bound the maximum in-degree $\Delta^{-1}(\vec{H})$. Let $j \in V(H)$. For every edge $ij \in E(\vec{H})$, the vertex v_{ij} is a proper ancestor of h_i in the elimination tree Y_{ij}, and there are at most $r+1$ such ancestors. Moreover, for distinct edges $ij, i'j$ the vertices v_{ij} and $v_{i'j}$ are distinct, because the paths P_{ij} are internally disjoint. Thus

$$\Delta^-(\vec{H}) \leq \sum_J (r+1) = \binom{N}{r+2}(r+1),$$

where the sum ranges over all sets J of at most $r+2$ colours. It follows that

$$|E(H)| \leq |V(H)| \cdot \Delta^-(\vec{H}) \leq |V(H)| \binom{N}{r+2}(r+1).$$

◀

Note that in order to conclude the proof with Corollary 2.6, we have to express $\frac{|E(H)|}{|V(H)|}$ with respect to $|V(H)|$ and not with respect to $|V(G)|$. For r-subdivisions this is no problem though. Take a minimal subgraph G' of G such that an r-subdivision is a subgraph of G'. Then G' has at most $|V(H)|^2 \cdot r$ edges.

We demonstrate the algorithmic applications of low tree depth colourings by showing that the *subgraph isomophism problem* can be solved in time $\mathcal{O}(n^{1+\epsilon})$ on nowhere dense classes of graphs for any fixed template H. Recall that the subgraph isomorphism problem is the problem, given two graphs H and G as input, to decide whether G contains a subgraph isomorphic to H. For a fixed template H and $\epsilon > 0$, the problem can be solved on any nowhere dense class $\mathcal{C}$ of graphs as follows. Let $h := |V(H)|$ and set $\epsilon' := \frac{\epsilon}{h}$. Let $G \in \mathcal{C}$. To decide whether G contains a subgraph G' isomorphic to H, we first apply Theorem 4.3 to obtain a colouring γ of $V(G)$ with at most $n^{\epsilon'}$ colours such that any h colour classes together induce a subgraph of G of tree depth at most h. Note that while we have only given a proof of the existence of such colouring, such a colouring γ can be computed in time $\mathcal{O}(n^{1+\epsilon'})$ using the concept of augmentations introduced in the next section (see e.g. [53, 47]). Furthermore, it is well-known (and follows, e.g. from Courcelle's theorem mentioned in the introduction) that on any class of graphs of bounded tree depth there is a linear time algorithm for solving the subgraph isomorphism problem for a fixed template H. Hence, to verify whether G contains a subgraph isomorphic to H we can compute all $(n^{\epsilon'})^h = n^{h \cdot \epsilon'} = n^{\epsilon}$ subgraphs $G_{c_1,\ldots,c_h}$ induced by exactly h colour classes $c_1, \ldots, c_h$ with respect to γ and for each test in linear time whether $G_{c_1,\ldots,c_h}$ contains H as an isomorphic subgraph. Together this yields the required running time.

5 Augmentations

For many algorithms it is essential to compute an ordering that witnesses $\mathrm{wcol}_r(G) \leq n^{\epsilon}$ or a td-col$_r$-colouring with at most n^{ϵ} colours. Not surprisingly, the problem of computing an optimal such ordering is NP-complete in general (it is easy to modify the proof of Pothen [55], showing that computing the tree depth of a graph is NP-complete). The question whether the problem is fixed-parameter tractable (with parameter $\mathrm{wcol}_r(G)$) is an interesting open question, yet even a positive answer would not help in the context of nowhere dense classes of graphs, as the parameter is only bounded by n^ϵ for such classes. Yet there are good approximation algorithms, see e.g. Dvořák [25]. We are going to present another way of approximating $\mathrm{wcol}_r(G)$ in order to present another important method for nowhere dense classes of graphs. The idea is as follows. Assume that an order witnessing $\mathrm{wcol}_r(G) = k$ has been found. We define a directed graph $\vec{H}_r$ on the same vertex set as G by adding an edge from u to v if and only if u is weakly r-accessible from v. $\vec{H}_r$ has the following properties. For all pairs u, v of vertices such that $\mathrm{dist}_G(u, v) \leq r$ one of the following holds. Either there is an edge (u, v) or (v, u) in $\vec{H}_r$ or there is a vertex w such that (w, u) and (w, v) are edges of $\vec{H}_r$ (if the first two cases do not hold, consider the smallest vertex w on the path from u to v). Furthermore, every vertex of $\vec{H}_r$ has indegree at most k.

▶ **Lemma 5.1.** *Let G be a graph and let $r > 0$. Let $\vec{H}$ be a directed graph such that for all pairs u, v of vertices with $\mathrm{dist}_G(u, v) \leq r$ one of the following holds. Either there is an edge (u, v) or (v, u) in $\vec{H}$ or there is a vertex w such that (w, u) and (w, v) are edges of $\vec{H}$ and such that every vertex of $\vec{H}$ has indegree at most d. Then $\mathrm{wcol}_r(G) \leq 2(d + 1)^2$.*

Proof. As $\Delta^-(\vec{H}_r) \leq d$, the underlying undirected graph H is $2d$-degenerate and we can order the vertices of H such that each vertex has at most $2d$ smaller neighbours. Denote this order by $\leq$. For each vertex $v \in V(G)$ we count the number of endvertices of paths of length at most r from v such that the endvertex is the smallest vertex of the path. This number bounds $|\mathrm{WReach}_r[G, \leq, v)]|$.

By assumption, for each such path with endvertex w, we either have an edge (v, w) or an edge (w, v) or there is u on the path and we have edges $(u, v), (u, w)$ in $\vec{H}$. By construction of the order there are at most $2d$ edges (v, w) or (w, v) such that $w < v$. Furthermore, we have at most d edges (u, v), as v has indegree at most d and for each such u there are at most $2d$ edges (u, w) such that $w < u$ by construction of the order. These are exactly the pairs of edges we have to consider, as no vertex on the path from v to w may be smaller than w. Hence in total we have $|\mathrm{WReach}_r[G, \leq, v]| \leq 2d + 2d^2 + 1 \leq 2(d + 1)^2$. (Note that we have to add 1 because $\mathrm{WReach}_r[G, \leq, v]$ contains v which is not reachable by any edges in the above way). ◀

It was shown by Nešetřil and Ossona de Mendez that we can iteratively compute a good approximation to the above on nowhere dense classes of graphs.

▶ **Definition 5.2.** Let $\vec{G}$ be a directed graph. A *tight* 1*-transitive fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with the same vertex set, including all the arcs of $\vec{G}$ and such that for all distinct vertices u, v, w

- if $(u, w), (w, v) \in E(\vec{G})$ then $(u, v) \in E(\vec{H})$,
- if $(u, w), (v, w) \in E(\vec{G})$ then (u, v) or (v, u) are arcs of $\vec{H}$ and
- for all $(u, v) \in E(\vec{H})$, either $(u, v) \in E(\vec{G})$ or there is some w such that $(u, w), (w, v) \in E(\vec{G})$ or $(u, w), (v, w) \in E(\vec{G})$.

Let us show that edge densities of depth-r topological subgraphs are relatively stable under tight 1-transitive fraternal augmentations.

▶ **Definition 5.3.** Let H, G be graphs. The *lexicographic product* of G and H is defined as the graph $G \bullet H$ with vertex set and edge set respectively:

$$V(G \bullet H) = V(G) \times V(H)$$

$$E(G \bullet H) = \big\{\{(x, y), (x', y')\} : \{x, x'\} \in E(G) \text{ or } (x = x' \text{ and } \{y, y'\} \in E(H))\big\}.$$

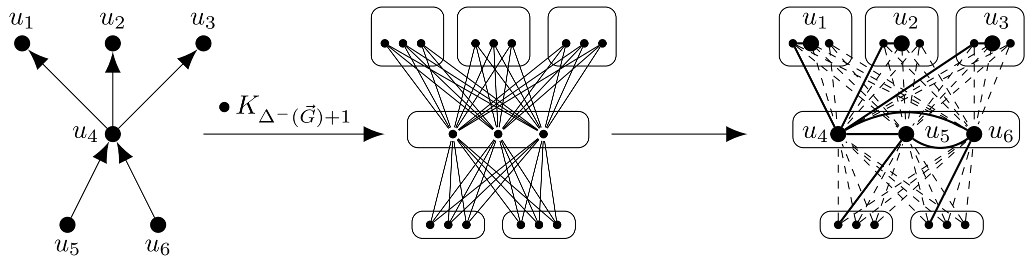

Figure 2 Proof sketch of Lemma 5.5.

The following is not hard to see.

▶ **Lemma 5.4.** *Let G be a graph and let $r, t, m \in \mathbb{N}$. If G has an r-centered colouring with m colours then $G \bullet K_t$ has an r centered colouring with $t \cdot m$ colours.*

The following was shown by Nešetřil and Ossona de Mendez [49, Lemma 7.2].

▶ **Lemma 5.5.** *Let $\vec{G}$ be a directed graph and let $\vec{H}$ be a tight 1-transitive fraternal augmentation of $\vec{G}$. Let $t := \Delta^-(\vec{G}) + 1$ and let G, H be the undirected graphs underlying $\vec{H}$ and $\vec{G}$, respectively. Then a 1-subdivision of H is a subgraph of $G \bullet K_t$.*

We only sketch the proof of the lemma. The main construction is illustrated in Figure 2. We want to show that the tight 1-transitive fraternal augmentation of the graph $\vec{G}$ on the left hand side of the figure is a 1-subdivision of a subgraph of $G \bullet K_3$. $G \bullet K_3$ is illustrated in the middle section of the figure, where rounded rectangles correspond to copies of K_3. The right hand side of the figure shows how the tight 1-transitive fraternal augmentation of $\vec{G}$ can be found as a 1-subdivision. Here, the thick black edges show the edges that are subdivided once. To improve readability we have omitted the edges from u_5 and u_6 to u_1, u_2, u_3 as these are not subdivided. Note that the only purpose of the two edges going to the two rectangles on the bottom is that in this way we can extend the construction to larger graphs $\vec{G}$. See [49, Lemma 7.2] for details.

By Lemma 4.7 we can conclude that H has small degeneracy if G had an r-centered colouring with few colours and if the indegree of $\vec{G}$ was not too large. Both is the case for nowhere dense classes of graphs. We can hence reorient the degenerate graph H to obtain again a small indegree orientation. Note furthermore that $(G \bullet K_t) \bullet K_s \cong G \bullet K_{t \cdot s}$. We can hence iterate the augmentation procedure for r times and obtain as a result a directed graph with the properties required by Lemma 5.1. It was shown by Nešetřil and Ossona de Mendez that this procedure can be implemented in time $\mathcal{O}(n^{1+\epsilon})$ on nowhere dense classes of graphs.

Augmentations are a key algorithmic tool for nowhere dense classes of graphs. Due to space restriction we refrain from giving explicit algorithmic applications here and refer, e.g., to [42] instead, where augmentations are used for query answer enumeration in nowhere dense classes of databases.

6 Quasi-wideness

A set $A \subseteq V(G)$ is called r-scattered if $N_r(u) \cap N_r(v) = \emptyset$ for all distinct $u, v \in A$.

▶ **Definition 6.1.** A class $\mathcal{C}$ of graphs is *uniformly quasi-wide* with margin $s : \mathbb{N} \to \mathbb{N}$ and $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ if for all $r, k \in \mathbb{N}$, if $G \in \mathcal{C}$ and $W \subseteq V(G)$ with $|W| > N(r, k)$, then there is a set $S \subseteq W$ with $|S| < s(r)$, such that W contains an r-scattered set of size at least k in $G - S$.

The aim of this section is to show the following theorem.

▶ **Theorem 6.2.** *A class $\mathcal{C}$ is nowhere dense if and only if $\mathcal{C}$ is uniformly quasi-wide.*

Obviously, if $\mathcal{C}$ is somewhere dense then it is not uniformly quasi-wide. We will hence show the other direction.

The first step of showing that nowhere density implies uniformly quasi-wideness is to find large independent sets. We will make use of the following Ramsey Theorem.

▶ **Theorem 6.3.** *For $k \geq 1$, let $n_1, \ldots, n_k \in \mathbb{N}$. There exists a number $R = R(n_1, \ldots, n_k)$, called Ramsey number, which is minimum with the following property. For every complete graph G on at least R vertices with edges coloured by colours $\{1, \ldots, k\}$, there exists some $1 \leq i \leq k$ such that G contains an induced subgraph of size at least n_i such that every edge has colour i.*

The theorem implies that every sufficiently large graph contains either a large independent set or a large complete graph. As nowhere dense classes of graphs do not contain large complete graphs, we conclude that in large graphs from the class we find large independent sets.

To go from 1-independent sets to 2-independent sets, we will use the following lemma which was proved by Nešetřil and Ossona de Mendez. The proof makes use of Theorem 6.3 in it's general form.

▶ **Lemma 6.4** (Nešetřil and Ossona de Mendez [51])**.** *There is a function Θ such that for all $s, k, a, b \in \mathbb{N}$ and all bipartite graphs $G = (A \cup B, E)$ with $|A| \geq \Theta(k, a, b, s)$ at least one of the following holds.*

- *There exists in B a subset of size at most s whose removal leaves in A a 2-independent set of size k.*
- *A includes the branch vertices of a 1-subdivision of the complete graph K_a.*
- *B includes $s+1$ vertices that form one side of a complete bipartite subgraph $K_{s+1,b}$ in G.*

Given a graph G and an independent set $A \subseteq V(G)$ of size at least $\Theta(k, a, b, s)$ we can just construct the bipartite graph with part $B = N(A)$ and establish the situation of the above lemma. For a nowhere dense class of graphs and sufficiently large a, b, s, we again know that we are in the first case.

We now apply the above arguments iteratively for r times. Given a large $2k$-independent set in a graph, we contract the k-neighbourhoods of the elements of the independent set and find large a $2k+1$ independent set. Given a large $2r+1$-independent set, we contract the $2k+1$-neighbourhoods of the elements of the set, and after the deletion of few elements, we find a $2k+2$-independent set. Note that we do not delete contracted vertices but vertices of $V(G)$ and hence we really delete only few vertices.

Algorithmically, the concept of uniformly quasi-wideness is very useful in designing bounded depth search trees for parameterized algorithms for solving problems such as dominating sets and network centres. We refer to [15] for examples demonstrating this technique.

7 Game-theoretic characterisation

As a final characterisation we show that nowhere dense classes of graphs can also be defined in terms of a game, which we call the splitter game introduced in [39].

▶ **Definition 7.1** (Splitter game). Let G be a graph and let $\ell, m, r > 0$. The (ℓ, m, r)-*splitter game* on G is played by two players, "Connector" and "Splitter", as follows. We let $G_0 := G$. In round $i+1$ of the game, Connector chooses a vertex $v_{i+1} \in V(G_i)$. Then Splitter picks a subset $W_{i+1} \subseteq N_r^{G_i}(v_{i+1})$ of size at most m. We let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus W_{i+1}]$. Splitter wins if $G_{i+1} = \emptyset$. Otherwise the game continues at G_{i+1}. If Splitter has not won after ℓ rounds, then Connector wins.

A *strategy* for Splitter is a function f, which associates to every partial play $(v_1, W_1, \dots, v_s, W_s)$ with associated sequence $G_0, \dots, G_s$ of graphs and move $v_{s+1} \in V(G_s)$ by Connector a set $W_{s+1} \subseteq N_r^{G_s}(v_{s+1})$ of size at most m. A strategy f is a *winning strategy* for Splitter in the (ℓ, m, r)-splitter game on G if Splitter wins every play in which he follows the strategy f. If Splitter has a winning strategy, we say that he *wins* the (ℓ, m, r)-splitter game on G.

▶ **Theorem 7.2** (Grohe, Kreutzer, Siebertz [39]). *Let $\mathcal{C}$ be a nowhere dense class of graphs.*

1. *For every $r > 0$ there are $\ell, m > 0$, such that for every $G \in \mathcal{C}$, Splitter wins the (ℓ, m, r)-splitter game on G.*
2. *Conversely, if for every $r > 0$ there are $\ell, m > 0$ such that for every graph $G \in \mathcal{C}$, Splitter wins the (ℓ, m, r)-splitter game, then $\mathcal{C}$ is nowhere dense.*

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