Towards Concept Analysis in Categories: Limit Inferior as Algebra, Limit Superior as Coalgebra*

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Abstract

While computer programs and logical theories begin by declaring the concepts of interest, be it as data types or as predicates, network computation does not allow such global declarations, and requires *concept mining* and *concept analysis* to extract shared semantics for different network nodes. Powerful semantic analysis systems have been the drivers of nearly all paradigm shifts on the web. In categorical terms, most of them can be described as bicompletions of enriched matrices, generalizing the Dedekind-MacNeille-style completions from posets to suitably enriched categories. Yet it has been well known for more than 40 years that ordinary categories themselves in general do not permit such completions. Armed with this new semantical view of Dedekind-MacNeille completions, and of matrix bicompletions, we take another look at this ancient mystery. It turns out that simple categorical versions of the *limit superior* and *limit inferior* operations characterize a general notion of Dedekind-MacNeille completion, that seems to be appropriate for ordinary categories, and boils down to the more familiar enriched versions when the limits inferior and superior coincide. This explains away the apparent gap among the completions of ordinary categories, and broadens the path towards categorical concept mining and analysis, opened in previous work.

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It is an open problem whether there exists a sup- and inf-complete category $\mathbb{A}''''$ with a sup- and inf-dense embedding $\mathbb{A} \to \mathbb{A}''''$ in analogy to the Dedekind completion of an ordered set.

Joachim Lambek [15, Introduction]

No Lambek extension of the one-object category $\mathbb{Z}_4$ has finite limits.

John Isbell [10, Thm. 3.1]

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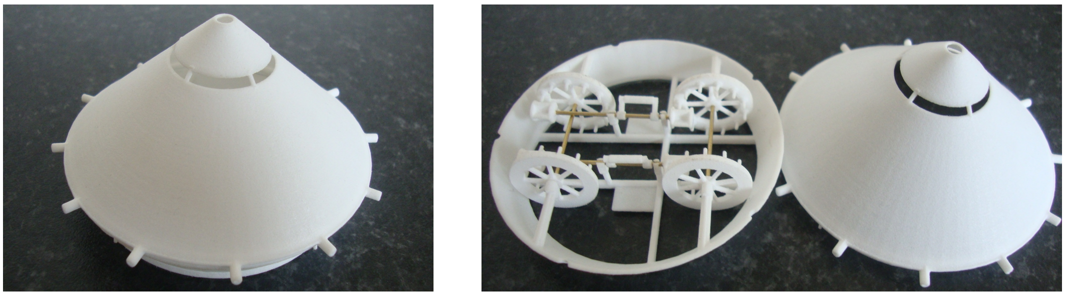

Figure 1 Unidentified object: The external and the internal view.

1 Introduction

1.1 Problem of concept mining and analysis

Suppose you come across upon the object depicted in Fig. 1. The conic top is easily removed to uncover the mechanism on the right. What is this thing?

You would surely approach the problem from both directions at once: on one hand, you would look how the parts fit together and try to discern the *structural components* of the device; on the other hand, you would twiddle with some parts and watch what moves together, trying to figure out the *functional modules.* The parts that move together may not be next to each other, but they probably belong to the same functional module. The parts that are related structurally are more likely to be related functionally. If you manage to discern some distinct components corresponding to distinct functionalities, then each such component-function pair will presumably correspond to a *concept* conceived by the designer of the device. By analyzing the device you will extract the designer's idea.

Similar analyses are formalized under different names in different research communities: some speak of *concept analysis*, some of *knowledge acquisition*, *semantic indexing*, or *data mining* [4, 8, 22]. The application domains and the formalisms vary very widely, from mathematical taxonomy [11], through text analysis [32] and pattern recognition [5], to web search and recommender systems [31]. The importance of formalizing and implementing concept analysis grew rapidly with the advent of the web, as almost anything found on the web requires some sort of concept mining and analysis, not only because there are no global semantical declarations, and the meaning has to be extracted from the network structure [23], but also to establish trust [25]. Diverse toy examples of such concept analysis tasks, motivating the modeling approach extended in this paper, can be found in [24, 25, 27, 28].

The analytic process that a formal concept analyst may initiate upon an encounter with the unidentified object from Fig. 1 is thus not all that different from what a curious child would do: they would both start by recording the observed components on one hand, and the observed functionalities on the other, and they would note which components are related to which functionalities. With the "yes-no" relations, the formal version of this process leads to the simple and influential method that goes under the name *Formal Concept Analysis (FCA)* [7, 6]. If the relations between the components and the functionalities are quantified by real numbers and stored in *pattern matrices*, then the analysis usually proceeds by the methods of statistics and linear algebra, and goes under the name *Principal Component Analysis (PCA)* [13], or *Latent Semantic Analysis (LSA)* [17], etc. It performs the singular value decomposition of the pattern matrix, and thus mines the concepts as the eigenspaces of the induced linear operators.

Interestingly, if you wanted to record that the unidentified device has 4 identical wheels, and that each wheel has 12 identical cogs, and that two of the wheels are related to two different functionalities, driving and steering, you would be led beyond the familiar concept mining approaches. While the experts in these approaches would surely figure out multiple tricks to record what is needed (e.g. by using multi-level pattern matrices), the straightforward approach leads beyond the FCA matrices of 0s and 1s, and beyond the LSA matrices of real numbers, to matrices of sets between components on one hand, matrices of sets between functionalities on the other hand, and matrices of sets between components and functionalities in-between. You would construct a category of components, a category of functionalities, and a profunctor/distributor between them. If the cog is recognized as a part, then a coproduct of 12 cogs would be embedded in each wheel. If the cogs are attached with rivets, then their morphisms may not be monic, since the distinctions of some of their parts may be obliterated through deformations. So why have such categorical models not been used in concept analysis?

Many of the concept mining approaches derived from LSA are instances of *spectral decomposition* [1]. Formalized in terms of *enriched category theory* [14], the problem of concept mining turns out to be an instance of a general spectral decomposition problem [25, 27], which can also be viewed as a problem of *minimal bicompletion* of a suitably enriched matrix [28]. Even the standard linear algebra of LSA seems to be an instance of such bicompletion, over a suitable category[1] of real numbers. The problem of minimal bicompletions of enriched categories, which subsume the Dedekind-MacNeille completions of posets, is the special case, arising when a category itself is viewed as a matrix. Instantiated to categories enriched over sets, also known as *"ordinary"* categories, this turned out to be a strange problem, as suggested by the quotations at the very beginning of the paper. Maybe this is the reason for the notable absence of ordinary categories in the extensive concept mining toolkits? We sketch the problem of bicompletions of ordinary categories in the next section.

1.2 Problem of minimal bicompletions of matrices and categories

Throughout the paper, we assume familiarity with the basic concepts of category theory, e.g. at the level of [20]. To understand the general approach to concept mining through minimal bicompletions, explained in this section, the reader may need some ideas about enriched categories as well, e.g. as presented in [14]. Beyond this section, the rest of the paper will be about ordinary categories.

Suppose that we have thus proceeded as in the preceding section, and built a category of components $\mathbb{A}$ and a category of functionalities $\mathbb{B}$. If we have recorded just the inclusion relations, then each of these categories is a poset, i.e. enriched over the ordered monoid $(\{0,1\},\wedge,1)$. If we have recorded the distances among the components on one hand, and among the functionalities on the other, then our categories are metric spaces [18], viewed as categories enriched over the monoidal poset $([0,\infty],+,0)$. If we capture the components and the functionalities as ordinary categories, then $\mathbb{A}$ and $\mathbb{B}$ are enriched over the monoidal category $(\mathsf{Set},\times,1)$.

[1] not poset!

$$
\cfrac{\cfrac{\Phi\colon \mathbb{A}^o \times \mathbb{B} \to \mathcal{V}}{\Phi^{\#}\colon \mathbb{A} \to \left(\mathcal{V}^{\mathbb{B}}\right)^o \qquad\qquad \Phi_{\#}\colon \mathbb{B} \to \mathcal{V}^{\mathbb{A}^o}}}{\cfrac{\Phi^{*}\colon \mathcal{V}^{\mathbb{A}^o} \to \left(\mathcal{V}^{\mathbb{B}}\right)^o \qquad\qquad \Phi_{*}\colon \left(\mathcal{V}^{\mathbb{B}}\right)^o \to \mathcal{V}^{\mathbb{A}^o}}{\overleftarrow{\Phi} = \Phi_*\Phi^*\colon \mathcal{V}^{\mathbb{A}^o} \to \mathcal{V}^{\mathbb{A}^o} \qquad\qquad \overrightarrow{\Phi} = \Phi^*\Phi_*\colon \left(\mathcal{V}^{\mathbb{B}}\right)^o \to \left(\mathcal{V}^{\mathbb{B}}\right)^o}}
$$

Figure 2 Deriving the two extensions and the two kernels of a matrix Φ.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, tail, "\nabla"] \arrow[d, "\Phi"', "\circ" marking] \arrow[rd, dashed, "\Phi^{\#}" near start] \arrow[rr, bend left=50, "\nabla_\Phi"] & \Downarrow\!\mathbb{A} \arrow[loop above, "\overleftarrow{\Phi}"] \arrow[r, bend left=10, "\overline{\Phi}"] \arrow[r, phantom, "\perp"] \arrow[d, bend right=15, "\Phi^*"'] \arrow[d, phantom, "\dashv"] & \left(\Downarrow\!\mathbb{A}\right)^{\overleftarrow{\Phi}} \arrow[l, bend left=10, "U"] \arrow[d, "\overleftrightarrow{\Phi}", "\circ" marking] \\
\mathbb{B} \arrow[r, tail, "\Delta"'] \arrow[ru, dashed, "\Phi_{\#}"' near start] \arrow[rr, bend right=50, "\Delta_\Phi"'] & \Uparrow\!\mathbb{B} \arrow[loop below, "\overrightarrow{\Phi}"] \arrow[u, bend right=15, "\Phi_*"'] \arrow[r, bend right=10, "\underline{\Phi}"'] \arrow[r, phantom, "\perp"] & \left(\Uparrow\!\mathbb{B}\right)^{\overrightarrow{\Phi}} \arrow[l, bend right=10, "V"']
\end{tikzcd}
$$

Figure 3 Minimal bicompletion of a matrix Φ.

1.2.1 The setting of minimal bicompletion

The relationships between the components and the functionalities will be expressed as a $\mathcal{V}$-enriched functor $\Phi\colon \mathbb{A}^o \times \mathbb{B} \to \mathcal{V}$, where $\mathcal{V}$ is the enriching category, such as $\{0,1\}, [0,\infty]$ or Set above. We call such $\mathcal{V}$-enriched functor a *matrix*. In particular, given a $\mathcal{V}$-matrix $\Phi\colon \mathbb{A}^o \times \mathbb{B} \to \mathcal{V}$ we derive its extensions as in Fig. 2.

The functors $\Phi^{\#}$ and $\Phi_{\#}$ are the transpositions of Φ. The presheaves in the form $\Phi_{\#}b$ and the postsheaves in the form $\Phi^{\#}a$ are called Φ-*representable*. The functors Φ^* and Φ_* are the Kan extensions [14, Ch. 4] of $\Phi^{\#}$ and $\Phi_{\#}$. Since they form an adjunction, their composite $\overleftarrow{\Phi}$ is a monad and $\overrightarrow{\Phi}$ is a comonad.

When the enrichment is clear from the context, it is convenient to abbreviate the matrix $\mathbb{A}^o \times \mathbb{B} \to \mathcal{V}$ to $\mathbb{A} \nrightarrow \mathbb{B}$ and the completions $\mathcal{V}^{\mathbb{A}^o}$ and $\left(\mathcal{V}^{\mathbb{B}}\right)^o$ to $\Downarrow\!\mathbb{A}$ and $\Uparrow\!\mathbb{B}$ respectively, so that the derivations in Fig. 2 give the diagram in Fig. 3 where ∇ and Δ are the Yoneda embeddings [14, Sec. 2.4]. The monad $\overleftarrow{\Phi} = \Phi_*\Phi^*$, induced by the Kan extensions $\Phi^* \dashv \Phi_*\colon \Uparrow\!\mathbb{B} \to \Downarrow\!\mathbb{A}$, induces the category of (Eilenberg-Moore) algebras $\left(\Downarrow\!\mathbb{A}\right)^{\overleftarrow{\Phi}}$, whereas the comonad $\overrightarrow{\Phi} = \Phi^*\Phi_*$ induces $\left(\Uparrow\!\mathbb{B}\right)^{\overrightarrow{\Phi}}$. The functor $\nabla_\Phi = \overline{\Phi} \circ \nabla$ maps $\mathbb{A}$ to free $\overleftarrow{\Phi}$-algebras generated by the representable presheaves, whereas the functor $\Delta_\Phi = \underline{\Phi} \circ \Delta$ maps $\mathbb{B}$ to cofree $\overrightarrow{\Phi}$-coalgebras cogenerated by the representable postsheaves.

1.2.2 Familiar cases

When $\mathcal{V} = \{0,1\}$, the $\mathcal{V}$-enriched categories $\mathbb{A}$ and $\mathbb{B}$ are posets. Then $\Downarrow\!\mathbb{A}$ consists of antitone maps $\overleftarrow{L}: \mathbb{A}^o \to \{0,1\}$, or equivalently of the lower-closed sets in $\mathbb{A}$, whereas $\Uparrow\!\mathbb{B}$ consists of the monotone maps $\overrightarrow{U}: \mathbb{B} \to \{0,1\}$, or equivalently of the upper-closed sets in $\mathbb{B}$. The Yoneda embedding $\nabla\colon \mathbb{A} \to \Downarrow\!\mathbb{A}$ is then the supremum (or join) completion, and the $\Delta\colon \mathbb{B} \to \Uparrow\!\mathbb{B}$ is the infimum (or meet) completion. A matrix $\Phi\colon \mathbb{A}^o \times \mathbb{B} \to \{0,1\}$ corresponds to a subset of the product poset which is lower closed in $\mathbb{A}$ and upper closed in $\mathbb{B}$. Its extensions are then

$$\Phi^* \overleftarrow{L} \quad = \quad \left\{ u \in \mathbb{B} \;\middle|\; \forall x.\, \overleftarrow{L}(x) \Rightarrow \Phi(x,u) \right\} \tag{1}$$

$$\Phi_* \overrightarrow{U} \quad = \quad \left\{ \ell \in \mathbb{A} \;\middle|\; \forall y.\, \overrightarrow{U}(y) \Rightarrow \Phi(\ell,y) \right\} \tag{2}$$

Intuitively, $\Phi^* \overleftarrow{L}$ can be construed as the set of upper bounds in $\mathbb{B}$ of the Φ-image of the lower set $\overleftarrow{L}$, whereas $\Phi_* \overrightarrow{U}$ can be construed as the set of Φ-lower bounds of the upper set $\overrightarrow{U}$. The operator $\overleftarrow{\Phi} = \Phi_* \Phi^*$ thus maps each lower set $\overleftarrow{L}$ to the set of the Φ-lower bounds of the set of its Φ-upper bounds; whereas the operator $\overrightarrow{\Phi} = \Phi^* \Phi_*$ maps each upper set $\overrightarrow{U}$ to the set of the Φ-upper bounds of its Φ-lower bounds. Both operators are thus *closure operators*. Their lattices of closed sets $(\Downarrow\!\mathbb{A})^{\overleftarrow{\Phi}}$ and $(\Uparrow\!\mathbb{B})^{\overrightarrow{\Phi}}$ turn out to be isomorphic, and form the *nucleus* of Φ [27]. A $\overleftarrow{\Phi}$-closed set in $\mathbb{A}$ and the corresponding $\overrightarrow{\Phi}$-closed set in $\mathbb{B}$, of course, completely determine each other, but the most informative presentation carries both, as Dedekind-style cuts. When $\mathbb{A} = \mathbb{B}$ and $\Phi \subseteq \mathbb{A}^o \times \mathbb{A}$ is the partial ordering

$$\Phi(x,y) \quad \iff \quad x \le y \tag{3}$$

then the nucleus is just the Dedekind-MacNeille completion $\Updownarrow\!\mathbb{A}$ of the poset $\mathbb{A}$ [21]. This is the *minimal* bicompletion, in the sense that the embedding $\mathbb{A} \to \Updownarrow\!\mathbb{A}$ preserves any suprema and infima that $\mathbb{A}$ may already have, and only adds those that do not yet exist [21, 2, 12, III.3.11]. The consequence of this minimality is that every element of the completion $\Updownarrow\!\mathbb{A}$ is *both* a supremum and an infimum of the elements of $\mathbb{A}$. The nucleus of a $\{0,1\}$-matrix is a minimal bicompletion in a similar sense, as are the nuclei of $[0,1]$-matrices, and of $[0,\infty]$-matrices[2]: the nuclei give the *semantic bicompletions* of matrices, uncovering their concepts [27, 28].

1.2.3 The trouble with ordinary categories

Our main concern in the present paper are the minimal bicompletions of matrices and categories enriched over $(\mathsf{Set}, \times, 1)$. Categories enriched in Set are usually called *ordinary* categories. Set-matrices are variably called *profunctors* or *distributors*. We increase the wealth of terminology by calling them *matrices*. The functors $\overleftarrow{\alpha} \in \Downarrow\!\mathbb{A} = \mathsf{Set}^{\mathbb{A}^o}$ are called *presheaves*. The functors $\overrightarrow{\beta} \in \Uparrow\!\mathbb{B} = \left(\mathsf{Set}^{\mathbb{B}}\right)^o$ are usually called *covariant functors to* Set, but we call them *postsheaves*. We use without further explanation the well known fact [9, 19] that presheaves are equivalent to discrete fibrations, and that postsheaves are equivalent with discrete opfibrations.

We also call the categorical limits the *infima*, and the categorical colimits the *suprema*, following Lambek's 1966 Lectures on Completions of Categories [15], quoted at the beginning

[2] Since the monoidal posets $([0,\infty], +, 0)$ and $([0,1], \times, 1)$ are isomorphic as monoidal categories, all statements about categories enriched over them transfer trivially. However, isomorphisms are not always trivial phenomena. E.g., the Laplace transform is an isomorphism, which maps differential operations into algebraic operations, and thus allows solving differential equations as algebraic equations, and mapping back the solutions [30]. In a similar way, it often happens that a distance space presentation of a data pattern, enriched over $([0,\infty], +, 0)$, displays some geometric content, whereas an isomorphic proximity lattice presentation of the same data pattern, enriched over $([0,1], \times, 1)$, displays some generalized order structure, not apparent in the first interpretation.

of this paper. The Yoneda embeddings $\nabla\colon \mathbb{A} \to \Downarrow\mathbb{A}$ and $\Delta\colon \mathbb{B} \to \Uparrow\mathbb{B}$ are then again, respectively, the supremum and the infimum completion, this time of the categories $\mathbb{A}$ and $\mathbb{B}$. The transposes $\Phi^{\#}$ and $\Phi_{\#}$ now extend to the adjunction $\Phi^* \dashv \Phi_*\colon \Uparrow\mathbb{B} \to \Downarrow\mathbb{A}$, which are defined similarly to (1–2). More precisely, the mappings between the $\mathbb{A}$-presheaves and $\mathbb{B}$-postsheaves

$$\frac{\overleftarrow{\alpha} : \mathbb{A}^o \to \mathsf{Set}}{\Phi^*\overleftarrow{\alpha} : \mathbb{B} \to \mathsf{Set}} \qquad\qquad \frac{\overrightarrow{\beta} : \mathbb{B} \to \mathsf{Set}}{\Phi_*\overrightarrow{\beta} : \mathbb{A}^o \to \mathsf{Set}}$$

are defined as follows

$$\Phi^*\overleftarrow{\alpha}(u) \;=\; \varprojlim_{x\in\mathbb{A}}\Big(\overleftarrow{\alpha}(x) \Rightarrow \Phi(x,u)\Big) \;=\; \Downarrow\mathbb{A}\Big(\overleftarrow{\alpha}, \Phi_{\#}u\Big) \tag{4}$$

$$\Phi_*\overrightarrow{\beta}(\ell) \;=\; \varprojlim_{y\in\mathbb{B}}\Big(\overrightarrow{\beta}(y) \Rightarrow \Phi(\ell,y)\Big) \;=\; \Uparrow\mathbb{B}\Big(\Phi^{\#}\ell, \overrightarrow{\beta}\Big) \tag{5}$$

Here we write $X \Rightarrow Y$ for the set exponents Y^X not only because the multiple exponents tend to "fly away" in the latter notation, but also to emphasize the parallel with (1–2). When $\mathbb{A} = \mathbb{B}$ is the same category, and $\Phi = H\colon \mathbb{A}^o \times \mathbb{A} \to \mathsf{Set}$ is the hom-set matrix, then $H^*\overleftarrow{\alpha}(u)$ is the set of (right) cones from the presheaf $\overleftarrow{\alpha}$, viewed as a diagram, to the object u as the tip of the cone. Dually, $H_*\overrightarrow{\beta}(\ell)$ is the set of (left) cones from the tip ℓ to the diagram $\overrightarrow{\beta}$. For a general matrix $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$, thinking of the elements of each set $\Phi(a,b)$ as "arrows" from $a \in \mathbb{A}$ to $b \in \mathbb{B}$ also allows thinking of $\overrightarrow{\varrho} \in \Phi^*\overleftarrow{\alpha}(u)$ as a (right) "cone" from the diagram $\overleftarrow{\alpha}$ in $\mathbb{A}$ to the tip $u \in \mathbb{B}$, and of $\overleftarrow{\lambda} \in \Phi_*\overleftarrow{\beta}(\ell)$ as a (left) "cone" from the tip $\ell \in \mathbb{A}$ to a diagram $\overleftarrow{\beta}$ in $\mathbb{B}$. The presheaves and postsheaves of (4) and (5) thus generalize the lower and the upper sets of (1) and (2).

At the very beginning of his lectures, Lambek raised the question of the Dedekind-MacNeille completion of a category, and left it open. He did not raise the general question of semantic completions of matrices (profunctors, or distributors) only because the semantical impact was not clear at the time; but the general situation from Fig. 3 was well known. Lambek's open question of the Dedekind-MacNeille completion of a category was closed by Isbell a couple of years later, who showed in [10, Sec. 3] that already the group $\mathbb{Z}_4$, viewed as a category with a single object, cannot have a completion generated both by the suprema and by the infima.

However, taking a broader semantical view, and seeking semantic completions of matrices, shows that the story does not really end with Isbell's counterexample. A semantic completion of a matrix, relating, say, the parts and the moves observed within a device like the one on Fig. 1, should uncover the concepts underlying the design of the device. These concepts are expressed through the structural component of the device, and through its functional units. When the matrix is enriched over a monoidal poset, then there is a one-to-one correspondence between the structural components and the functional modules, and they form the nucleus of the matrix [27, 28]. In reality, though, a single structural component may play a role in several functional modules, and vice versa. While the posetal enrichment cannot capture this, the enrichment in sets, or in a proper category of real numbers, can record how many copies of a given a part are used for a certain function. Modeled in this way, the spaces of structural components and of functional modules will not be isomorphic. The concepts will not be uncovered as a single category of component-function pairs, like in the posetal case, but as a nontrivial matrix relating some component-concepts approximated by their functionalities with some function-concepts approximated by the components that perform them.

Contributions

To spell this out, we consider the following technical questions:

(a) What kind of completions of a given matrix $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$ are provided by the categories $(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ and $(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$? (The idea is that the former captures the component-concepts, the latter the function-concepts.)

(b) What kind of matrix $\overleftrightarrow{\Phi}\colon (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}} \looparrowright (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ is the minimal bicompletion of $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$? (Capturing the relations between the component-concepts and the function-concepts.)

Our approach to these questions is based on a new family of limits and colimits, introduced in the next section. It seems intuitive and appropriate to call them *limit inferior*, and *limit superior*. For consistency, we also revert, albeit just for the duration of this paper[3], from *limits* and *colimits* to *infima* and *suprema*, following Lambek [15]. The reader is reminded that in posets

- the limit inferior is the *supremum* of the lower bounds of a set, whereas
- the limit superior is the *infimum* of the upper bounds.

Mutatis mutandis, the categorical concepts will behave similarly.

Overview of the paper

In Sec. 2, we propose the answers to the above question. Sec. 2.1 spells out the preliminaries. Sec. 2.2 defines categorical limits inferior and superior and characterizes their completions. Sec. 2.3 proposes an answer to question (a) above. Sec. 2.4 proposes an answer to question (b) above. In Sec. 3 we study some simple examples, illustrating and validating the introduced concepts. Sec. 3.1 describes a monadicity workflow useful for analyzing the examples. Sec. 3.2 characterizes completions of constant matrices. Sections 3.3 and 3.4 characterize completions of the matrices representing groups or posets, respectively. Sec. 3.5 characterizes completions of a vector in the (cyclic) group $\mathbb{Z}_p$ of prime order p. Sec. 4 closes the paper, to some extent.

Due to the space constraints of this conference paper and the scope of the presented material, all proofs and many lemmas had to be moved into the appendices. Full details will require a significantly longer paper.

2 Categorical limit inferior and limit superior

2.1 Preliminaries

Although suprema and infima are very basic concepts, familiar to most readers, and easily found in [20, Sec. III.3–4], we spell them out here not only to introduce the notation and practice using the words *infimum* and *supremum* instead of limit and colimit, but also to align these familiar definitions with the variations needed to define the *limit superior* and the *limit inferior*.

Let $\mathbb{C}$ and $\mathbb{J}$ be categories and $\mathbb{C}^{\mathbb{J}}$ the category of functors between them, with natural transformations as morphisms. Let $\square\colon \mathbb{C} \to \mathbb{C}^{\mathbb{J}}$ be the functor taking each object of x of $\mathbb{C}$ to the constant functor $\square x\colon \mathbb{J} \to \mathbb{C}$, which maps all objects of $\mathbb{J}$ to $x \in \mathbb{C}$ and all morphisms of $\mathbb{J}$ to id_x.

[3] We hope that our terminological contributions, advancing from "profunctors" and "distributors" to "matrices" and from "covariant functors to Set" to "postsheaves", as well as retreating from "limits" to "infima" and from "colimits" to "suprema", will not end up being the central features of the paper.

The suprema and the infima in $\mathbb{C}$ can be defined as, respectively, the left and the right adjoint of the constant functor, i.e.

$$\varinjlim \dashv \Box \dashv \varprojlim \quad : \quad \mathbb{C}^{\mathbb{J}} \to \mathbb{C}$$

These adjunctions can be viewed as the natural bijections

$$\mathbb{C}^{\mathbb{J}}(F, \Box x) \quad \cong \quad \mathbb{C}\big(\varinjlim F, x\big) \tag{6}$$

$$\mathbb{C}^{\mathbb{J}}(\Box x, F) \quad \cong \quad \mathbb{C}\big(x, \varprojlim F\big) \tag{7}$$

It is well known that the Yoneda embeddings realize the $\varinjlim$ and $\varprojlim$-completions [20, Sec. X.6]:

- $\nabla \colon \mathbb{C} \to \Downarrow\mathbb{C}$ is the $\varinjlim$-completion of $\mathbb{C}$, whereas
- $\Delta \colon \mathbb{C} \to \Uparrow\mathbb{C}$ is the $\varprojlim$-completion of $\mathbb{C}$

where

- $\Downarrow\mathbb{C}$ denotes the category $\mathsf{Set}^{\mathbb{C}^o}$ of $\mathbb{C}$-*presheaves*, or equivalently[4] the category of discrete fibrations over $\mathbb{C}$,
- $\Uparrow\mathbb{C}$ denotes the category $\left(\mathsf{Set}^{\mathbb{C}}\right)^o$ of $\mathbb{C}$-*postsheaves*, or equivalently the opposite category of discrete opfibrations over $\mathbb{C}$.

For completeness, we note the following well known and routinely checkable fact.

▶ **Lemma 2.1.** *Given a functor $F \colon \mathbb{J} \to \mathbb{C}$, consider the presheaf and the postsheaf*

$$\left(\overleftarrow{F} : \mathbb{C}/\!\!/F \to \mathbb{C}\right) \in \Downarrow\mathbb{C} \qquad\qquad \left(\overrightarrow{F} : F/\!\!/\mathbb{C} \to \mathbb{C}\right) \in \Uparrow\mathbb{C} \tag{8}$$

where $\mathbb{C}/\!\!/F$ is the category of connected components[5] *of the comma category $\mathbb{C}/F$ from $\mathrm{Id}_{\mathbb{C}}$ to F, whereas $F/\!\!/\mathbb{C}$ is the category of connected components of the comma category $F/\mathbb{C}$ the other way around [20, Sections II.6 and IX.3]. Then*

$$\varinjlim F \;=\; \varinjlim \overleftarrow{F} \qquad\qquad \varprojlim F \;=\; \varprojlim \overrightarrow{F} \tag{9}$$

Notations have been introduced in Sec. 1.2, especially in Figures 2 and 3. The next section studies the special case $\Phi = H \colon \mathbb{C} \looparrowright \mathbb{C}$ of the matrix of hom-sets of a category.

2.2 Limit inferior and limit superior over a category

▶ **Definition 2.2.** For arbitrary categories $\mathbb{C}$ and $\mathbb{J}$ we define

- the category of *left saturated diagrams* $\mathbb{C}^{\mathbb{J}}_{\Downarrow}$ to consist of
  - objects $|\mathbb{C}^{\mathbb{J}}_{\Downarrow}| = |\mathbb{C}^{\mathbb{J}}|$
  - morphisms $\mathbb{C}^{\mathbb{J}}_{\Downarrow}(F, G) = \Downarrow\mathbb{C}\left(H_*\overrightarrow{F}, H_*\overrightarrow{G}\right)$
- the category of *right saturated diagrams* $\mathbb{C}^{\mathbb{J}}_{\Uparrow}$ to consist of
  - objects $|\mathbb{C}^{\mathbb{J}}_{\Uparrow}| = |\mathbb{C}^{\mathbb{J}}|$
  - morphisms $\mathbb{C}^{\mathbb{J}}_{\Uparrow}(F, G) = \Uparrow\mathbb{C}\left(H^*\overleftarrow{F}, H^*\overleftarrow{G}\right)$

[4] The equivalence between the "indexed" and "fibered" versions of sheaves lies at the heart of Grothendieck's descent theory [9, VI], but also generalizes to substantially different purposes [29, 26].

[5] To be precise, each fiber category $(\mathbb{C}/\!\!/F)_x$ at $x \in \mathbb{C}$ is defined to be the category of connected components of the fiber category $(\mathbb{C}/F)_x = \mathbb{C}/Fx$.

▶ **Definition 2.3.** In a category $\mathbb{C}$ we define

- the *limit inferior* operation $\overrightarrow{\lim}$ over left diagrams from $\mathbb{J}$ by the adjunction

  $$\overrightarrow{\lim} \dashv \square \ : \ \mathbb{C} \to \mathbb{C}^{\mathbb{J}}_{\Downarrow}$$

  which can be viewed as the natural bijection

  $$\mathbb{C}^{\mathbb{J}}_{\Downarrow}(F, \square x) \ \cong \ \mathbb{C}\left(\overrightarrow{\lim} F, x\right)$$

- the *limit superior* operation $\overleftarrow{\lim}$ over right diagrams from $\mathbb{J}$ by the adjunction

  $$\square \dashv \overleftarrow{\lim} \ : \ \mathbb{C}^{\mathbb{J}}_{\Uparrow} \to \mathbb{C}$$

  which can be viewed as the natural bijection

  $$\mathbb{C}^{\mathbb{J}}_{\Uparrow}(\square x, F) \ \cong \ \mathbb{C}\left(x, \overleftarrow{\lim} F\right)$$

▶ Remark. Note that the operations $\overrightarrow{\lim}$ and $\overleftarrow{\lim}$ are defined over *arbitrary* diagrams. Indeed, the objects of the categories of *saturated* diagrams are arbitrary diagrams; the saturation is imposed on them in the definitions of the morphisms in these categories.

The operations $\varinjlim$ and $\varprojlim$ are also defined over arbitrary diagrams, but differently: the supremum of a diagram is equal to the supremum of the induced presheaf; and the infimum of a diagram is equal to the infimum of the induced postsheaf, as stated in Lemma 2.1. This is analogous to lattices, where a supremum of a set is equal to the supremum of its lower closure, whereas the infimum of a set is the infimum of the upper closure. However, the limit inferior of a diagram is the supremum of the presheaf induced by the postsheaf induced by the diagram; and the limit superior is the infimum of the postsheaf induced by the presheaf induced by the diagram. In a partially ordered set, the limit inferior of a set is the join of the lower bounds of all of its upper bounds; whereas the limit superior of a set is the meet of the upper bounds of all of its lower bounds.

▶ **Lemma 2.4.** *Every representable presheaf ∇x is a free algebra in $\Downarrow\mathbb{C}^{\overleftarrow{H}}$, with $\nabla x \overset{\eta}{\cong} \overleftarrow{H}\nabla x$. Every representable postsheaf Δx is a cofree coalgebra in $\Uparrow\mathbb{C}^{\overrightarrow{H}}$, with $\overrightarrow{H}\Delta x \overset{\varepsilon}{\cong} \Delta x$.*

▶ **Proposition 2.5.** *Every $\overleftarrow{H}$-algebra is a limit inferior in $\Downarrow\mathbb{C}^{\overleftarrow{H}}$ of representable presheaves, viewed as $\overleftarrow{H}$-algebras. Every $\overrightarrow{H}$-coalgebra is a limit superior in $\Uparrow\mathbb{C}^{\overrightarrow{H}}$ of representable postsheaves, viewed as $\overrightarrow{H}$-coalgebras.*

▶ **Corollary 2.6.** *$\Downarrow\mathbb{C}^{\overleftarrow{H}}$ is $\overrightarrow{\lim}$-complete. $\Uparrow\mathbb{C}^{\overrightarrow{H}}$ is $\overleftarrow{\lim}$-complete.*

▶ **Theorem 2.7.** *The extended Yoneda embeddings realize the limit inferior and limit superior completions:*

- *$\nabla_H : \mathbb{C} \xrightarrow{\nabla} \Downarrow\mathbb{C} \xrightarrow{\overline{H}} \Downarrow\mathbb{C}^{\overleftarrow{H}}$ is the $\overrightarrow{\lim}$-completion of $\mathbb{C}$, whereas*
- *$\Delta_H : \mathbb{C} \xrightarrow{\Delta} \Uparrow\mathbb{C} \xrightarrow{\underline{H}} \Uparrow\mathbb{C}^{\overrightarrow{H}}$ is the $\overleftarrow{\lim}$-completion of $\mathbb{C}$.*

2.3 Limit inferior and limit superior over a matrix

Given a category $\mathbb{C}$, Lem. 2.1 implies that the suprema and the infima, defined by (6) and (7) respectively, can be viewed as the left and the right adjoint of the corresponding Yoneda embeddings:

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "\nabla"'] \arrow[r, phantom, "\perp"] & \Downarrow\mathbb{C} \arrow[l, bend right, "\varinjlim"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{C} \arrow[r, bend right, "\Delta"'] \arrow[r, phantom, "\top"] & \Uparrow\mathbb{C} \arrow[l, bend right, "\varprojlim"'] \end{tikzcd}$$

Given a matrix $\Phi \colon \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}$, the suprema and the infima *weighted* by its transposes $\Phi^\# \colon \mathbb{A} \to \Uparrow\mathbb{B}$ and $\Phi_\# \colon \mathbb{B} \to \Downarrow\mathbb{A}$ can similarly be viewed as adjoints:

$$\begin{tikzcd} \mathbb{B} \arrow[r, bend right, "\Phi_\#"'] \arrow[r, phantom, "\perp"] & \Downarrow\mathbb{A} \arrow[l, bend right, "\varinjlim_\Phi"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{A} \arrow[r, bend right, "\Phi^\#"'] \arrow[r, phantom, "\top"] & \Uparrow\mathbb{B} \arrow[l, bend right, "\varprojlim_\Phi"'] \end{tikzcd}$$

It is, of course, well known and easy to see that the weighted limits can in ordinary categories be reduced to the ordinary limits. The situation is slightly more subtle with the weighted inferior and superior limits. To align the two situations, note that the adjunctions

$$\mathbb{B}\left(\varinjlim_\Phi \overleftarrow{\alpha}, b\right) \cong \Downarrow\mathbb{A}\left(\overleftarrow{\alpha}, \Phi_\# b\right) \qquad\qquad \mathbb{A}\left(a, \varprojlim_\Phi \overrightarrow{\beta}\right) \cong \Uparrow\mathbb{B}\left(\Phi^\# a, \overrightarrow{\beta}\right)$$

will now become

$$\mathbb{A}\left(\overrightarrow{\lim}_\Phi \overrightarrow{\beta}, a\right) \cong (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\left(\Phi_* \overrightarrow{\beta}, \nabla_\Phi a\right) \qquad \mathbb{B}\left(b, \overleftarrow{\lim}_\Phi \overleftarrow{\alpha}\right) \cong (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}\left(\Delta_\Phi b, \Phi^* \overleftarrow{\alpha}\right)$$

▶ **Definition 2.8.** Given a matrix $\Phi \colon \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}$, with the induced extensions as in Fig. 3, we define the operations Φ-*limit inferior* $\overrightarrow{\lim}_\Phi$ and $\overleftarrow{\lim}_\Phi$ by the following adjunctions

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend right, "\nabla_\Phi"'] \arrow[r, phantom, "\perp"] & (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}} \arrow[l, bend right, "\overrightarrow{\lim}_\Phi"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{B} \arrow[r, bend right, "\Delta_\Phi"'] \arrow[r, phantom, "\top"] & (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}} \arrow[l, bend right, "\overleftarrow{\lim}_\Phi"'] \end{tikzcd}$$

where ∇_Φ and Δ_Φ are as defined in Fig. 3.

2.3.1 Two pairs of "Yoneda embeddings"

In this section we spell out the basic properties of the two kinds of "Yoneda embeddings" induced by a matrix $\Phi \colon \mathbb{A} \looparrowright \mathbb{B}$:

- $\overleftarrow{\Phi}$-algebra representables and $\overrightarrow{\Phi}$-coalgebra representables

$$\nabla_\Phi \colon \mathbb{A} \to (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}} \qquad\qquad \Delta_\Phi \colon \mathbb{B} \to (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$$

- Φ-representable presheaves and postsheaves

$$\Phi^\# \colon \mathbb{A} \to (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}} \qquad\qquad \Phi_\# \colon \mathbb{B} \to (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$$

The underlying functors are as in Fig. 3. The structures are as follows.

▶ **Lemma 2.9.** *Every presheaf* $\overleftarrow{\alpha} \in \Downarrow\mathbb{A}$ *induces the* $\overrightarrow{\Phi}$*-coalgebra* $\Phi^*\overleftarrow{\alpha} \xrightarrow{\Phi^*\eta} \Phi^*\Phi_*\Phi^*\overleftarrow{\alpha}$. *Every* Φ*-representable postsheaf* $\Phi^\# a$ *is thus canonically a* $\overrightarrow{\Phi}$*-coalgebra, since* $\Phi^\# a = \Phi^*\nabla a$.

Any postsheaf $\overrightarrow{\beta} \in \Uparrow\mathbb{B}$ *induces the* $\overleftarrow{\Phi}$*-algebra* $\Phi_*\overrightarrow{\beta} \xleftarrow{\Phi_*\varepsilon} \Phi_*\Phi^*\Phi_*\overrightarrow{\beta}$. *Every* Φ*-representable presheaf* $\Phi_\# b$ *is thus canonically a* $\overleftarrow{\Phi}$*-algebra, since* $\Phi_\# b = \Phi_*\Delta b$.

▶ **Lemma 2.10** (Matrix Yoneda Lemma)**.** *For every* $a \in \mathbb{A}$ *and every* $\overrightarrow{\beta} \in \Uparrow\mathbb{B}$ *there is a natural bijection*

$$(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\left(\nabla_\Phi a, \Phi_*\overrightarrow{\beta}\right) \cong \Phi_*\overrightarrow{\beta}(a) \tag{10}$$

For every $b \in \mathbb{B}$ *and every* $\overleftarrow{\alpha} \in \Downarrow\mathbb{A}$ *there is a natural bijection*

$$(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}\left(\Phi^*\overleftarrow{\alpha}, \Delta_\Phi b\right) \cong \Phi^*\overleftarrow{\alpha}(b) \tag{11}$$

▶ **Corollary 2.11** (Matrix Yoneda embedding)**.**

$$(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\left(\nabla_\Phi a, \Phi_\# b\right) \cong \Phi(a,b) \cong (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}\left(\Phi^\# a, \Delta_\Phi b\right) \tag{12}$$

2.3.2 Completeness and generation

▶ **Corollary 2.12.** *$(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ is $\overrightarrow{\lim}_\Phi$-complete. $(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ is $\overleftarrow{\lim}_\Phi$-complete.*

▶ **Proposition 2.13.** *Every $\overleftarrow{\Phi}$-algebra is a limit inferior in $(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ of $\overleftarrow{\Phi}$-algebra representables. Every $\overrightarrow{\Phi}$-coalgebra is a limit superior in $(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ of $\overrightarrow{\Phi}$-coalgebra representables.*

▶ **Theorem 2.14.** *The Φ-extended Yoneda embeddings realize the $\overrightarrow{\lim}_\Phi$-completion and $\overleftarrow{\lim}_\Phi$-completion:*

- $\nabla_\Phi : \mathbb{A} \xrightarrow{\nabla} \Downarrow\mathbb{A} \xrightarrow{\overleftarrow{\Phi}} (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ *is the $\overrightarrow{\lim}$-completion of $\mathbb{A}$, whereas*
- $\Delta_\Phi : \mathbb{B} \xrightarrow{\Delta} \Uparrow\mathbb{B} \xrightarrow{\overrightarrow{\Phi}} (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ *is the $\overleftarrow{\lim}$-completion of $\mathbb{B}$.*

2.4 Minimal bicompletion of a matrix

2.4.1 Loose extensions

In general, a matrix $\Phi\colon \mathbb{A} \nrightarrow \mathbb{B}$ always induces a *loose* extension $\Updownarrow\Phi\colon (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}} \nrightarrow (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$, defined

$$\Updownarrow\Phi(a,b) \quad = \quad \left\{ f \in \Downarrow\mathbb{A}\Big(\overleftarrow{\alpha}, \Phi_*\overrightarrow{\beta}\Big) \;\middle|\; \begin{tikzcd} \Phi_*\Phi^*\overleftarrow{\alpha} \arrow[r, "\Phi_*\Phi^*f"] \arrow[d, "a"'] & \Phi_*\Phi^*\Phi_*\overrightarrow{\beta} \\ \overleftarrow{\alpha} \arrow[r, "f"'] & \Phi_*\overrightarrow{\beta} \arrow[u, "\Phi_*b"'] \end{tikzcd} \right\} \tag{13}$$

▶ **Proposition 2.15.** *Each of the following squares commutes if and only if the other one commutes.*

$$\begin{tikzcd} \Phi_*\Phi^*\overleftarrow{\alpha} \arrow[r, "\Phi_*\Phi^*f"] \arrow[d, "a"'] & \Phi_*\Phi^*\Phi_*\overrightarrow{\beta} \\ \overleftarrow{\alpha} \arrow[r, "f"'] & \Phi_*\overrightarrow{\beta} \arrow[u, "\Phi_*b"'] \end{tikzcd} \qquad \Longleftrightarrow \qquad \begin{tikzcd} \Phi^*\Phi_*\Phi^*\overleftarrow{\alpha} \arrow[r, "\Phi^*\Phi_*f'"] \arrow[d, "\Phi^*a"'] & \Phi^*\Phi_*\overrightarrow{\beta} \\ \Phi^*\overleftarrow{\alpha} \arrow[r, "f'"'] & \beta \arrow[u, "b"'] \end{tikzcd}$$

The commutativity of the preceding squares implies the commutativity of the following squares, which are each other's transposes.

$$\overleftarrow{\alpha} \xrightarrow{f} \Phi_*\overrightarrow{\beta} \underset{\eta}{\overset{\Phi_*b}{\rightrightarrows}} \Phi_*\Phi^*\Phi_*\overrightarrow{\beta} \qquad \Longleftrightarrow \qquad \begin{tikzcd} \Phi^*\Phi_*\Phi^*\overleftarrow{\alpha} \arrow[r, "\Phi^*\Phi_*f'"] & \Phi^*\Phi_*\overrightarrow{\beta} \\ \Phi^*\overleftarrow{\alpha} \arrow[u, "\Phi_*\eta"] \arrow[r, "f'"'] & \overrightarrow{\beta} \arrow[u, "b"'] \end{tikzcd}$$

$$\Phi^*\Phi_*\Phi^*\overleftarrow{\alpha} \underset{\varepsilon}{\overset{\Phi^*a}{\rightrightarrows}} \Phi^*\overleftarrow{\alpha} \xrightarrow{f'} \overrightarrow{\beta} \qquad \Longleftrightarrow \qquad \begin{tikzcd} \Phi_*\Phi^*\overleftarrow{\alpha} \arrow[r, "\Phi_*\Phi^*f'"] \arrow[d, "a"'] & \Phi_*\Phi^*\Phi_*\overrightarrow{\beta} \arrow[d, "\Phi_*\varepsilon"] \\ \overleftarrow{\alpha} \arrow[r, "f'"'] & \Phi_*\overrightarrow{\beta} \end{tikzcd}$$

▶ **Conjecture 2.16.** $\Updownarrow\Phi$ *isomorphic with the matrix*

$$\Updownarrow\Phi(a,b) \quad = \quad \left(\Downarrow(\mathbb{A}\times\mathbb{B}^o)\right)^{\overleftarrow{\Phi}\times\overrightarrow{\Phi}}\left(\overleftarrow{\alpha}\times\overrightarrow{\beta},\Phi\right)$$

which is equivalent to the matrix of the adjunction $\Phi^{\circledast}\dashv\Phi_{\circledast}\colon(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}\to(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$*, defined*

$$\Phi^{\circledast}\overleftarrow{\alpha}(u) \quad = \quad (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\left(\overleftarrow{\alpha},\Phi^{\#}u\right) \tag{14}$$

$$\Phi_{\circledast}\overrightarrow{\beta}(\ell) \quad = \quad (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}\left(\Phi_{\#}\ell,\overrightarrow{\beta}\right) \tag{15}$$

with the structure maps induced by composition with the structure maps $a\colon\Phi_*\Phi^*\overleftarrow{\alpha}\to\overleftarrow{\alpha}$ *and* $b\colon\overrightarrow{\beta}\to\Phi^*\Phi_*\overrightarrow{\beta}$*.*

2.4.2 Tight extensions

But this loose extension is of little semantical value. E.g., when Φ is a partial ordering like in (3), $\Updownarrow\Phi$ picks all pairs of a saturated lower set and a saturated upper set which are contained in each other's sets of bounds, but do not necessarily contain *all* such bounds. So it does not capture the Dedekind cuts.

The tight extension $\overleftrightarrow{\Phi}$ brings us closer to the Dedekind cuts:

$$\overleftrightarrow{\Phi}(a,b) \quad = \quad \{f\in\Updownarrow\Phi(a,b)\mid f \text{ is mono, and } f' \text{ is epi}\} \tag{16}$$

Since $(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ and $(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ are regular categories, $\overleftrightarrow{\Phi}$ can be extracted from $\Updownarrow\Phi$ by two closure operators: first extracting the mono factors, and then the epis of their transposes, or equivalently the other way around. After the factorizations, in the first case the transpose of the resulting epi will be mono; in the second the transpose of the resulting mono will be epi. Either way, the process will stop.

The resulting matrix $\overleftrightarrow{\Phi}$ will be a reflective submatrix of $\Updownarrow\Phi$. The completeness and the generation will be inherited, but tight. We need to prove that the inferior limits that existed in $\mathbb{A}$ and the superior limits that existed in $\mathbb{B}$ are preserved.

▶ **Conjecture 2.17.** *For every matrix* $\Phi\colon\mathbb{A}\to\mathbb{B}$*, the tight extension* $\overleftrightarrow{\Phi}\colon(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\looparrowright(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ *is the minimal bicompletion.*

3 When does limit inferior boil down to limit?

By the couniversal property of the (Eilenberg-Moore) categories of algebras for a monad [16, Part 0.6], there are always the comparison adjunctions between $\Downarrow\mathbb{A}$ and $(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$, and between $\Uparrow\mathbb{B}$ and $(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$, as displayed in the leftmost diagram of Fig. 4, since the monad $\overleftarrow{\Phi}$ and the comonad $\overrightarrow{\Phi}$ are induced by the adjunction $\Phi^*\dashv\Phi_*\colon\Uparrow\mathbb{B}\to\Downarrow\mathbb{A}$. When these comparisons are equivalences, then this adjunction transfers to the two Eilenberg-Moore categories, as indicated in the rightmost diagram of Fig. 4. Moreover, the inferior Φ-limits $\overrightarrow{\lim}_{\Phi}$ in $\mathbb{B}$ then boil down to the suprema $\underrightarrow{\lim}$ in $\mathbb{A}$, whereas the superior Φ-limits $\overleftarrow{\lim}_{\Phi}$ in $\mathbb{A}$ boil down to the infima $\underleftarrow{\lim}$ in $\mathbb{B}$. In terms of the concept mining example from the Introduction, the structural components represented in $(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ can be computed as infima functions in $\Uparrow\mathbb{B}$, whereas the functional modules represented in $(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ can be computed as suprema of parts in $\Downarrow\mathbb{A}$. Connecting the extensions $\Updownarrow\Phi$ and $\overleftrightarrow{\Phi}$ along the equivalences $\Downarrow\mathbb{A}\simeq(\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ and $\Uparrow\mathbb{B}\simeq(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ shows that all loose extensions are already tight.

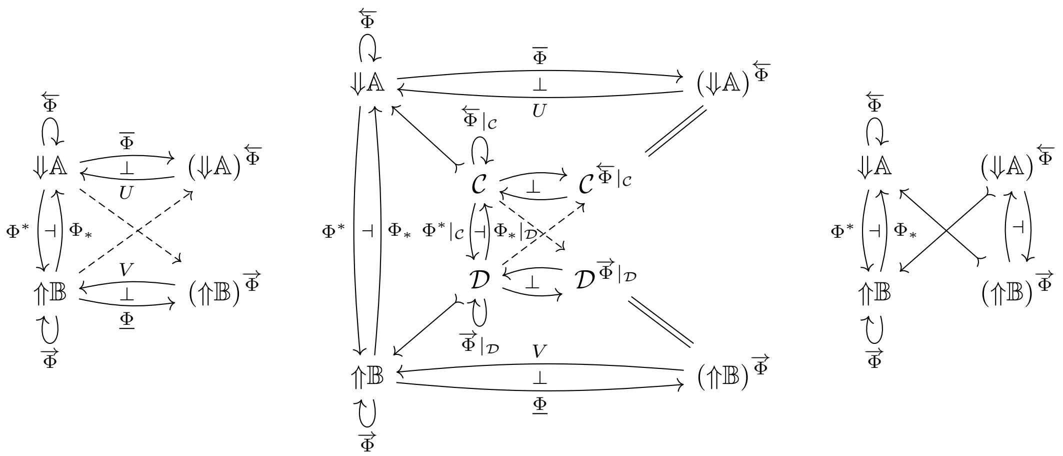

Figure 4 Comparisons between the $\underleftarrow{\lim}$- and $\overrightarrow{\lim}_{\Phi}$-completions, and between the $\underrightarrow{\lim}$- and $\overleftarrow{\lim}_{\Phi}$-completions.

▶ **Proposition 3.1.** *For any matrix $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$, the extensions $\Phi^* \dashv \Phi_*\colon \Uparrow\mathbb{B} \to \Downarrow\mathbb{A}$ are both monadic if and only if the loose and the tight extensions coincide, i.e. $\Updownarrow\Phi \simeq \overleftrightarrow{\Phi}$.*

The notion of *monadicity* [20, Sec. VI.7] here precisely captures the equivalences of interest, as $\Uparrow\mathbb{B} \simeq (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}$ means that Φ_* is monadic and $\Downarrow\mathbb{A} \simeq (\Uparrow\mathbb{B})^{\overrightarrow{\Phi}}$ means that Φ^* is (co)monadic. In this section, we study the monadicity of the extensions Φ_* and Φ^* in order to gain insight into the situations when the inferior and superior limits boil down to the ordinary limits, and the situations when they genuinely provide new information.

3.1 Monadicity workflow

As a reminder, we quote the Precise Monadicity Theorem in Appendix B. Intuitively, its impact on the concrete instances of our situation is that it allows constructing the inferior limits, which are in principle the suprema of lower bounds, as specific maximal cones into the infima.

We begin describing a convenient setting of subcategories, as displayed in the middle in Fig. 4. When $\Phi_*\colon \Uparrow\mathbb{B} \to \Downarrow\mathbb{A}$ restricts to a monadic functor $\mathcal{D} \to \mathcal{C}$, so that $\mathcal{D} \simeq \mathcal{C}^{\overleftarrow{\Phi}|_{\mathcal{C}}}$, then we have an embedding $(\Downarrow\mathbb{A})^{\overleftarrow{\Phi}} \rightarrowtail \Uparrow\mathbb{B}$ as indicated in the rightmost diagram in Fig. 4.

In the general framework of an adjunction as in Appendix B, items (a–b) of the Monadicity Theorem say that the induced Eilenberg-Moore category $\mathcal{C}^T$ is coreflective within the category $\mathcal{D}$ whenever $\mathcal{D}$ has and U preserves reflexive U-split coequalizers. However, its converse does not hold (see Example 3.6 and Prop. 3.8). The task is thus to spell out the full subcategories $\mathcal{C} \subseteq \Downarrow\mathbb{A}$, $\mathcal{D} \subseteq \Uparrow\mathbb{B}$ explicitly, even if we cannot apply the Monadicity Theorem to the setting $\mathcal{C} = \Downarrow\mathbb{A}$, $\mathcal{D} = \Uparrow\mathbb{B}$. Towards this goal, and to simplify calculations with the algebras, we propose the following lemma.

▶ **Definition 3.2.** An object B is said to be a retract of an object A if there exist morphisms $B \to A \to B$ whose composite is id_B. For a full subcategory $\mathcal{F} \subseteq \mathcal{E}$, we denote by $\mathrm{Retr}_{\mathcal{E}}(\mathcal{F}) \subseteq \mathcal{E}$ the full subcategory of all retracts in $\mathcal{E}$ of objects in $\mathcal{F}$.

Notational conventions. For a functor G, we denote its full image by $\mathrm{Im}\, G$. For a category $\mathcal{E}$ and its full subcategories $\mathcal{F}, \mathcal{F}'$, we loosely use $\mathcal{F} \subseteq \mathcal{F}'$ to denote that any object in $\mathcal{F}$ is isomorphic in $\mathcal{E}$ to some object in $\mathcal{F}'$.

▶ **Lemma 3.3.** *Let $F \dashv U\colon \mathcal{B} \to \mathcal{A}$ be an adjunction and T be its monad on $\mathcal{A}$.*

$$
\begin{tikzcd}
\mathcal{A} \arrow[loop above, "T"] \arrow[rr, bend right=10] \arrow[rr, phantom, "\top"] \arrow[dd, bend right, "F"'] \arrow[dd, phantom, "\dashv"] & & \mathcal{A}^T \arrow[ll, bend right=10] \\
& \mathcal{C} \arrow[loop above, "T|_{\mathcal{C}}"] \arrow[ul, tail] \arrow[r, bend right=10] \arrow[r, phantom, "\top"] \arrow[d, bend right, "F|_{\mathcal{C}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{C}^{T|_{\mathcal{C}}} \arrow[l, bend right=10] \arrow[uu, tail] \\
\mathcal{B} \arrow[uu, bend right, "U"'] \arrow[r, equal] & \mathcal{B} \arrow[u, bend right, "U"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[loop above, "T"] \arrow[r, bend right=10] \arrow[r, phantom, "\top"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \mathcal{A}^T \arrow[l, bend right=10] \arrow[dl, bend left, "L"] \arrow[dl, phantom, "\top"] & \\
\mathcal{B} \arrow[u, bend right, "U"'] \arrow[ur, "K"] & & \mathcal{D} \arrow[ul, bend right, tail] \arrow[ll, bend left, tail]
\end{tikzcd}
$$

1. *Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory such that $\operatorname{Im} U \subseteq \mathcal{C}$. If $\operatorname{Retr}_{\mathcal{A}}(\operatorname{Im} U) \subseteq \mathcal{C}$, then the canonical inclusion is an equivalence of categories $\mathcal{C}^{T|_{\mathcal{C}}} \simeq \mathcal{A}^T$.*
2. *Let $\mathcal{D}$ be the full subcategory that depicts the equivalence induced by the comparison functor $K\colon \mathcal{B} \to \mathcal{A}^T$ and its partial left adjoint $L\colon \mathcal{A}^T \rightharpoonup \mathcal{B}$ (see e.g. [14, Sec. 1.11]). Then, $\operatorname{Retr}_{\mathcal{B}}(\operatorname{Im} F) \subseteq \mathcal{D}$. In particular, $\operatorname{Retr}_{\mathcal{B}}(\operatorname{Im} F) \subseteq \mathcal{A}^T$ if the category $\mathcal{A}^T$ is a (coreflective) subcategory of $\mathcal{B}$ by $L\colon \mathcal{A}^T \rightarrowtail \mathcal{B}$.*

The above lemma intuitively means

- we need at most retracts of images under U in $\mathcal{A}$, and that
- we need at least retracts of images under F in $\mathcal{B}$,

in order to obtain a monadic functor of the form $\mathcal{A}^T \simeq \mathcal{D} \xrightarrow{U|_{\mathcal{D}}} \mathcal{C}$. In the later discussion, we restrict an adjunction as the diagram below and calculate the category of T-algebras by $\mathcal{A}^T = (\operatorname{Retr}(\operatorname{Im} U))^{T'} \supseteq (\operatorname{Retr}(\operatorname{Im} F))$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[loop above, "T"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \operatorname{Retr}(\operatorname{Im} U) \arrow[loop above, "T'"] \arrow[l, tail] \arrow[r, bend right=10] \arrow[r, phantom, "\top"] \arrow[d, bend right, "F'"'] \arrow[d, phantom, "\dashv"] & \mathcal{A}^T \arrow[l, bend right=10] \\
\mathcal{B} \arrow[u, bend right, "U"'] & \operatorname{Retr}(\operatorname{Im} F) \arrow[l, tail] \arrow[u, bend right, "U'"'] \arrow[ur, bend right, tail, "K'"'] &
\end{tikzcd}
\quad \text{where} \quad
\begin{aligned}
T' &= U'F' = T|_{\operatorname{Retr}(\operatorname{Im} U)} \ , \\
F' &= F|_{\operatorname{Retr}(\operatorname{Im} U)} \ , \\
U' &= U|_{\operatorname{Retr}(\operatorname{Im} F)} \ , \\
K' &= K|_{\operatorname{Retr}(\operatorname{Im} F)}
\end{aligned}
$$

3.2 Completing constant matrices

Any set R can be viewed as a constant matrix $\widetilde{R}\colon 1 \looparrowright 1$ by setting $\widetilde{R}(0,0) = R$, where $1 = \{0\}$. We abuse notation and write $\widetilde{R}$ as R. The extensions $R^* \dashv R_*\colon \mathsf{Set}^o \to \mathsf{Set}$ are thus $R^*X = R_*X = R^X$, and they induce the continuation monad $\overleftarrow{R}X = R^{R^X}$ on Set, and the same comonad $\overrightarrow{R}$ on Set^o.

Lem. B.2 in the Appendix B helps characterizing the monadicity of R^* and R_*.

▶ **Proposition 3.4.** *For a set R with at least 2 elements, the functor $R_*\colon \mathsf{Set}^o \to \mathsf{Set}$ is monadic. When R is a singleton, then the monad $\overleftarrow{R}$ on Set has a single algebra, and the comonad $\overrightarrow{R}$ on Set^o has a single coalgebra. When R is empty, then they have two algebras and coalgebras respectively.*

▶ **Corollary 3.5.** *The loose extension of the constant matrix R is always in the form $\Updownarrow R : \mathsf{Set} \looparrowright \mathsf{Set}$ with $\Updownarrow R(X,Y) = \mathsf{Set}(X,Y)$. The tight extension is*

- $\overleftrightarrow{R} = \Updownarrow R : \mathsf{Set} \looparrowright \mathsf{Set}$ *when R has at least 2 elements*
- $\overleftrightarrow{1} : 1 \looparrowright 1$ *with $\overleftrightarrow{1}(0,0) = 1$, where $1 = \{0\}$*
- $\overleftrightarrow{0} : 2 \looparrowright 2$ *with $\overleftrightarrow{0}(x,y) = 1$ if and only if $x \leq y$ within $2 = \{0 < 1\}$.*

3.3 Completing groups

Let $\mathbb{C}$ be a group G, viewed as a one-object category with invertible morphisms. The category $\Downarrow G$ of presheaves is the category of right G-sets, or the category $G^o\text{-}\mathsf{Set}$ of (left) G^o-sets. Indeed as a discrete fibration over the one-object category, the total category of the presheaf is a set X with an action $X \times G \to X$. The adjunction $H^* \dashv H_*$ is given explicitly as follows. We think of G as a (left G, right G)-set by the multiplication. For a right G-set X, the (left) G-set H^*X is the set $G^o\text{-}\mathsf{Set}(X, G)$ with the action $(g \cdot f)(i) = g\big(f(i)\big)$. Similarly, $H_*Y = G\text{-}\mathsf{Set}(Y, G)$ with the right action $(f \cdot g)(i) = \big(f(i)\big)g$ for a left G-set Y.

We assume later that the group G is nontrivial (i.e. G has at least two elements).

▶ **Example 3.6.** The diagram $0 \to 1 \rightrightarrows 1 + 1$ displays an equalizer of a reflexive H_*-split pair of left G-maps (i.e. a coequalizer in $\Uparrow G$). However, the image of this diagram under H_* is not a coequalizer: $1 \leftarrow 0 \leftleftarrows 0$.

▶ **Proposition 3.7.** *We have* $\operatorname{Im} H^* \simeq \{0\} \cup \{G^I \mid I \in \mathsf{Set}\}$ *and* $\operatorname{Retr}(\operatorname{Im} H^*) \simeq \{1\} \cup \{G \times I \mid I \in \mathsf{Set}\}$*, where* G^I *is the exponential in* Set *with the pointwise multiplication* $(g \cdot f)(i) = g\big(f(i)\big)$ *and* $G \times I$ *is the free* G*-set generated by the set* I *(i.e.* $g \cdot \langle h, i\rangle = \langle gh, i\rangle$*).*

We denote by $G\text{-}\mathsf{Set}_{1,\text{free}}$ the full subcategory $\{1\} \cup \{G \times I \mid I \in \mathsf{Set}\} \subseteq G\text{-}\mathsf{Set}$ of a singleton and free G-sets.

▶ **Proposition 3.8.** *The functor* $H_* \colon (G\text{-}\mathsf{Set}_{1,\text{free}})^o \to G^o\text{-}\mathsf{Set}_{1,\text{free}}$ *is monadic. In particular, the category* $(G^o\text{-}\mathsf{Set})^{\overleftarrow{H}}$ *of* $\overleftarrow{H}$*-algebras is equivalent to* $(G\text{-}\mathsf{Set}_{1,\text{free}})^o$*.*

▶ **Corollary 3.9.** *The loose extension of a nontrivial group is the canonical connection of its left and right actions. The tight extension is the canonical extension of its free actions.*

3.4 Completing posets

Let $\mathbb{C}$ be a poset $(P, \leq)$. We denote the poset of lower sets of P by $\downarrow P$, and the poset of upper sets of P by $\uparrow P$.[6] They are respectively the join and the meet completions. While P's categorical supremum completion $\Downarrow P = \mathsf{Set}^{P^o}$ and its infimum completion $\Uparrow P = \big(\mathsf{Set}^P\big)^o$ are proper categories, its limit inferior completion $\Downarrow P^{\overleftarrow{H}}$, and its limit superior completion $\Uparrow P^{\overrightarrow{H}}$, although still constructed over Set — turn out to be both equivalent to a lattice, and in particular to P's Dedekind-MacNeille completion $\updownarrow P$.

▶ **Lemma 3.10.** *The lattice of subobjects of the terminal object in* $\Downarrow P$ *is isomorphic to* $\downarrow P$*. The lattice of quotient objects of the initial object of* $\Uparrow P$ *is isomorphic to* $\uparrow P$*. The full subcategories* $\downarrow P \subseteq \Downarrow P$ *and* $\uparrow P \subseteq \Uparrow P$ *contain all the representables.*

▶ **Lemma 3.11.** *The adjunction* $H^* \dashv H_* \colon \Uparrow P \to \Downarrow P$ *restricts to an adjunction between posets* $\downarrow P, \uparrow P$ *(i.e. a Galois connection), which coincides with the* $\{0, 1\}$*-enriched construction. Moreover,* $\operatorname{Im} H^* = \operatorname{Im}(H^*|_{\downarrow P})$ *and* $\operatorname{Im} H_* = \operatorname{Im}(H_*|_{\uparrow P})$*.*

▶ **Corollary 3.12.** *It holds* $\operatorname{Retr}_{\Uparrow P}(\operatorname{Im} H^*) = \operatorname{Im}(H^*|_{\downarrow P})$*.*

Therefore, the category $(\Downarrow P)^{\overleftarrow{H}}$ is nothing more than the category of algebras for the adjunction $\uparrow P \leftrightarrows \downarrow P$.

▶ **Proposition 3.13.** *There exist equivalences of categories* $(\Downarrow P)^{\overleftarrow{H}} \simeq \updownarrow P \simeq (\Uparrow P)^{\overrightarrow{H}}$*.*

▶ **Corollary 3.14.** *The tight extension* $\overleftrightarrow{P}$ *of a poset* P *coincides with its Dedekind-MacNeille completion* $\updownarrow P$*.*

[6] The poset $\downarrow P$ is ordered by $L \leq L' \iff L \subseteq L'$, whereas $\uparrow P$ is ordered by $U \leq U' \iff U \supseteq U'$.

$\Phi = \underline{1}\Phi_1 + \underline{p}\Phi_p$	$\Phi_p = 0$	$\Phi_p \geq 1$
$\Phi_1 = 0$	$\{0,1\}$ $\dashv$ $\{0,\underline{1}\}^o$	Set $\dashv$ $(\{\underline{1}\} \cup \{\underline{p}U_p \mid U_p \in \mathsf{Set}\})^o$
$\Phi_1 = 1$	$\{1\}$ $\dashv$ $\{\underline{1}\}^o$	Set $\dashv$ $\{\underline{1} + \underline{p}U_p \mid U_p \in \mathsf{Set}\}^o$
$\Phi_1 \geq 2$	Set $\dashv$ $\{\underline{1}U_1 \mid U_1 \in \mathsf{Set}\}^o$	Set $\dashv$ $(\mathbb{Z}_p\text{-}\mathsf{Set})^o$

Figure 5 The $\overleftarrow{\lim}_\Phi$-completion and the $\overrightarrow{\lim}_\Phi$-completion of a $\mathbb{Z}_p$-vector $\Phi\colon 1 \looparrowright \mathbb{Z}_p$.

3.5 Completing a $\mathbb{Z}_p$-vector

A *vector* is a matrix in the form $\Phi\colon 1 \looparrowright \mathbb{B}$. We consider the vectors in $\mathbb{B} = \mathbb{Z}_p$, viewed as an additive cyclic group of prime order p. Every $\mathbb{Z}_p$-set X has an orbit-decomposition $X \cong 1 \times X_1 + \mathbb{Z}_p \times X_p$ where the action on 1 is trivial and the action on $\mathbb{Z}_p$ is defined by the addition.. We abbreviate the decomposition $1 \times X_1 + \mathbb{Z}_p \times X_p$ as $\underline{1}X_1 + \underline{p}X_p$.

▶ **Lemma 3.15.** $\mathbb{Z}_p\text{-}\mathsf{Set}(\underline{1}X_1 + \underline{p}X_p, \underline{1}Y_1 + \underline{p}Y_p) \;\cong\; Y_1^{X_1}(Y_1 + pY_p)^{X_p}$.

Hence for a vector $\Phi = \underline{1}\Phi_1 + \underline{p}\Phi_p$, the adjunction $\Phi^* \dashv \Phi_*\colon (\mathbb{Z}_p\text{-}\mathsf{Set})^o \to \mathsf{Set}$ is explicitly

$$\begin{aligned} \Phi^* L &\cong (\underline{1}\Phi_1 + \underline{p}\Phi_p)^L && (L \in \mathsf{Set}), \\ \Phi_* U &\cong \Phi_1^{U_1}(\Phi_1 + p\Phi_p)^{U_p} && (U = \underline{1}U_1 + \underline{p}U_p \in \mathbb{Z}_p\text{-}\mathsf{Set}). \end{aligned}$$

▶ **Lemma 3.16.** *Let $f, g\colon U \rightrightarrows U'$ be a reflexive pair in $\mathbb{Z}_p$-Set. The $\mathbb{Z}_p$-sets U, U' have suitable isomorphisms to their orbit-decompositions such that the right square of the diagram*

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, dashed, "\cong"] & U \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\cong"] & U' \arrow[d, "\cong"] \\ \underline{1}E_1 + \underline{p}E_p \arrow[r, "\underline{1}e_1 + \underline{p}e_p"] & \underline{1}U_1 + \underline{p}U_p \arrow[r, shift left, "\underline{1}f_1 + \underline{p}f_p"] \arrow[r, shift right, "\underline{1}g_1 + \underline{p}g_p"'] & \underline{1}U_1' + \underline{p}U_p' \end{tikzcd}$$

serially commutes for some maps $f_1, g_1\colon U_1 \rightrightarrows U_1'$, $f_p, g_p\colon U_p \rightrightarrows U_p'$. Moreover, equalizers $E_1 \xrightarrow{e_1} U_1$, $E_p \xrightarrow{e_p} U_p$ of the pairs (f_1, g_1), (f_p, g_p), which induce an equalizer $E \xrightarrow{e} U$ of the pair (f, g) in $\mathbb{Z}_p$-Set, satisfy the condition of Lem. B.2.2.

Let us find full subcategories $\mathrm{Retr}(\mathrm{Im}\,\Phi_*) \subseteq \mathcal{C} \subseteq \mathsf{Set}$ and $\mathrm{Retr}(\mathrm{Im}\,\Phi^*) \subseteq \mathcal{D} \subseteq (\mathbb{Z}_p\text{-}\mathsf{Set})^o$ to fit the scheme of Fig. 4.

▶ **Proposition 3.17.** *Fig. 5 depicts a restriction of the adjunction $\Phi^* \dashv \Phi_*\colon (\mathbb{Z}_p\text{-}\mathsf{Set})^o \to \mathsf{Set}$ that makes both Φ^* and Φ_* monadic, without changing the categories of algebras:*

$$\begin{tikzcd} \mathsf{Set} \arrow[d, bend right, "\Phi^*"'] \arrow[d, phantom, "\dashv"] & \mathcal{C} \arrow[l, hook'] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] \arrow[dr, "\simeq"] & \mathcal{C}^{\overleftarrow{\Phi}|_{\mathcal{C}}} \arrow[r, equal] & \mathsf{Set}^{\overleftarrow{\Phi}} \\ (\mathbb{Z}_p\text{-}\mathsf{Set})^o \arrow[u, bend right, "\Phi_*"'] & \mathcal{D} \arrow[l, hook'] \arrow[u, bend right] \arrow[ur, "\simeq"'] & \mathcal{D}^{\overrightarrow{\Phi}|_{\mathcal{D}}} \arrow[r, equal] & (\mathbb{Z}_p\text{-}\mathsf{Set}^o)^{\overrightarrow{\Phi}} \end{tikzcd} .$$

Thus, the subcategories $\mathcal{C}$ and $\mathcal{D}$ are equivalent to the $\overleftarrow{\lim}_\Phi$- and $\overrightarrow{\lim}_\Phi$-completions, respectively.

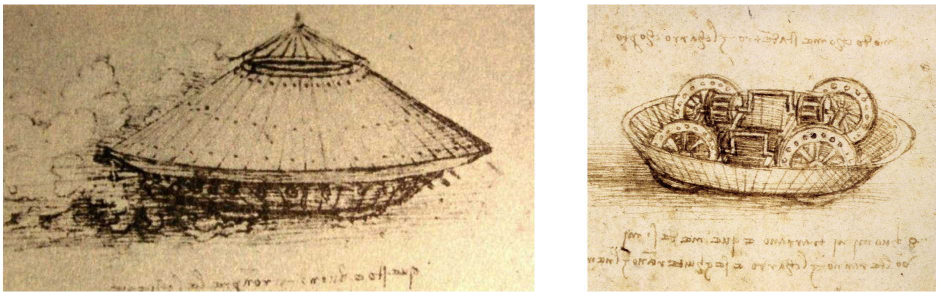

Figure 6 Identified object: The external and the internal view

4 Conclusion

Deploying the categorical concept analysis of the unidentified object from Fig. 1 according to the technical recipes proposed in this paper, our diligent reader has surely uncovered that the mysterious device consists of two main structural components: the internal mechanism of wheels and gears, and the external protection shell. On the other hand, the detailed categorical analysis has surely displayed three main functional modules: moving, defending from the outside attacks, and attacking from inside. As desired, the tight matrix then clearly shows that the object must be a model of a man-powered armored combat vehicle from XV century. It was conceived by Leonardo da Vinci, whose drawings are reproduced on Fig. 6. The advances of category theory will undoubtedly permit us to better understand Leonardo's conceptualizations of warfare.

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A Appendix: Proofs

Proof of Thm. 2.7. Suppose that $\mathbb{X}$ is a category with all limits inferior, and that $G\colon \mathbb{C} \to \mathbb{X}$ is an arbitrary functor. We show that G has a unique extension $G'\colon (\Downarrow\mathbb{C})^{\overleftarrow{H}} \to \mathbb{X}$, such that

$$G \quad = \quad G' \circ \overline{H} \circ \nabla \tag{17}$$

where $\overline{H}\colon \Downarrow\mathbb{C} \to (\Downarrow\mathbb{C})^{\overleftarrow{H}}$, as defined on Fig. 3 instantiated to $\Phi = H$, maps $\mathbb{C}$-presheaves to free $\overleftarrow{H}$-algebras, i.e. it is the left adjoint of the forgetful functor $U\colon (\Downarrow\mathbb{C})^{\overleftarrow{H}} \to \Downarrow\mathbb{C}$. The construction is illustrated on the following diagram.

$$\begin{tikzcd} & & (\Downarrow\mathbb{C})^{\overleftarrow{H}} \arrow[dd, dashed, "G'"] \\ & \Downarrow\mathbb{C} \arrow[ur, "\overline{H}"] & \\ \mathbb{C} \arrow[ur, "\nabla"] \arrow[rr, "G"'] & & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \overleftarrow{\gamma} & \overleftarrow{H}\overleftarrow{\gamma} \arrow[l, "a"'] \arrow[d, mapsto, "G'"] \\ & \overrightarrow{\lim}\ G \circ \overrightarrow{\varphi} \end{tikzcd}$$

Given an arbitrary $\overleftarrow{H}$-algebra $\overleftarrow{\gamma} \xleftarrow{\ a\ } \overleftarrow{H}\overleftarrow{\gamma}$ in $\Downarrow\mathbb{C}$, we construct the equalizer of postsheaves

$$\overrightarrow{\varphi} \rightarrowtail H_*\overleftarrow{\gamma} \underset{\eta_{H_*\overleftarrow{\gamma}}}{\overset{H_*a}{\rightrightarrows}} H_*\overleftarrow{H}\overleftarrow{\gamma}$$

which is a coequalizer in $\Uparrow\mathbb{C}$. Note that the $\overleftarrow{H}$-algebra a displays the presheaf $\overleftarrow{\gamma}$ as the coequalizer of the H^*-image of the pair $\langle H_*a, \eta_{H_*\overleftarrow{\gamma}}\rangle$. The $\overleftarrow{H}$-algebra a itself is the coequalizer of the free $\overleftarrow{H}$-algebras over this image, *and* as the limit inferior as decomposed in Proposition 2.5. We set the G'-image of the $\overleftarrow{H}$-algebra a to be the limit inferior of the functor $\mathbb{F} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{G} \mathbb{X}$. Equation (17) follows from Lem. 2.4. The fact that G' preserves inferior limits follows from the fact that every inferior limit cone $H^*\overrightarrow{F} \xrightarrow{\ \lambda\ } \overleftarrow{\gamma}$ factors through any structure map $\overleftarrow{H}\overleftarrow{\gamma} \xrightarrow{\ a\ } \overleftarrow{\gamma}$: the factorization is the composite $H^*\overrightarrow{F} \xrightarrow{\ \lambda\ } \overleftarrow{\gamma} \xrightarrow{\ \eta\ } H^*\overleftarrow{\gamma}$, which obviously boils down to λ when further postcomposed with a. The uniqueness follows from Proposition 2.5. ◀

Proof of Lem. 2.10. Consider a natural transformation $\psi \in (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\left(\Phi_*\Phi^\# a, \Phi_*\overrightarrow{\beta}\right)$ on the left-hand side of (10). By (5) and by the naturality of ψ, for $f \in \mathbb{A}(x,a)$ the left-hand square diagram in Fig. 7 must commute.

$$\begin{tikzcd}[column sep=small] \Phi_*\Phi^\# a(a) = \Uparrow\mathbb{B}\left(\Phi^\# a, \Phi^\# a\right) \arrow[r, "\psi_a"] \arrow[d, "\Phi_*\Phi^\# a(f)\ \ (-)\circ f"] & \Uparrow\mathbb{B}\left(\Phi^\# a, \overrightarrow{\beta}\right) = \Phi_*\overrightarrow{\beta}(a) \arrow[d, "(-)\circ f\ \ \Phi_*\overrightarrow{\beta}(f)"] \\ \Phi_*\Phi^\# a(x) = \Uparrow\mathbb{B}\left(\Phi^\# x, \Phi^\# a\right) \arrow[r, "\psi_x"] & \Uparrow\mathbb{B}\left(\Phi^\# x, \overrightarrow{\beta}\right) = \Phi_*\overrightarrow{\beta}(x) \end{tikzcd} \qquad \begin{tikzcd} \left[\nabla_\Phi a\right](\nabla_\Phi a) \arrow[r, "{[\psi]}"] \arrow[d, "{[\nabla_\Phi a]\widehat{f}}"] & \left[\Phi_*\overrightarrow{\beta}\right](\nabla_\Phi a) \arrow[d, "{[\Phi_*\overrightarrow{\beta}]\widehat{f}}"] \\ \left[\nabla_\Phi a\right](\nabla_\Phi x) \arrow[r, "{[\psi]}"'] & \left[\Phi_*\overrightarrow{\beta}\right](\nabla_\Phi x) \end{tikzcd}$$

Figure 7 Matrix Yoneda squares

Recall that $\overleftarrow{\Phi}$-algebras like $\nabla_\Phi a, \Phi_*\overrightarrow{\beta}\colon \mathbb{A}^o \to \mathsf{Set}$ always canonically extend to functors $\left[\nabla_\Phi a\right] \to \left[\Phi_*\overrightarrow{\beta}\right] : (\Downarrow\mathbb{A})^o_{\overleftarrow{\Phi}} \to \mathsf{Set}$, and that $\overleftarrow{\Phi}$-algebra homomorphism $\psi\colon \nabla_\Phi a \to \Phi_*\overrightarrow{\beta}$ extend to $[\psi]\colon \left[\nabla_\Phi a\right] \to \left[\Phi_*\overrightarrow{\beta}\right]$. It follows that a $\overleftarrow{\Phi}$-algebra homomorphism ψ must be natural with respect to all homomorphisms between free $\overleftarrow{\Phi}$-algebras, and not just with respect to those arising from $\mathbb{A}$.

In particular, consider the natural isomorphism

$$\Uparrow\mathbb{B}\left(\Phi^{\#}x, \Phi^{\#}a\right) \overset{(a)}{=} \Phi_*\Phi^{\#}a(x) \overset{(b)}{\cong} \Downarrow\mathbb{A}\left(\nabla x, \Phi_*\Phi^{\#}a\right) \overset{(c)}{=} (\Downarrow\mathbb{A})^{o}_{\overleftarrow{\Phi}}\left(\nabla_\Phi a, \nabla_\Phi x\right) \quad (18)$$

where (a) is based on (5), (b) on the usual Yoneda lemma, and (c) on the definition of the Kleisli category $(\Downarrow\mathbb{A})_{\overleftarrow{\Phi}}$. Every $f \in \Uparrow\mathbb{B}\left(\Phi^{\#}x, \Phi^{\#}a\right)$ thus induces a unique homomorphism $\widehat{f} \in (\Downarrow\mathbb{A})^{o}_{\overleftarrow{\Phi}}\left(\nabla_\Phi a, \nabla_\Phi x\right)$, and vice versa. The naturality condition on $\left[\psi\right]$ now implies that the right-hand square on Fig. 7 must commute, which implies

$$\left[\psi\right]_{\nabla_\Phi x}\left(\widehat{f}\right) = \left[\psi\right]_{\nabla_\Phi x}\circ\left[\nabla_\Phi a\right]\widehat{f}\left(\mathrm{id}_{\nabla_\Phi a}\right) = \left[\Phi_*\overrightarrow{\beta}\right]\widehat{f}\circ\left[\psi\right]_{\nabla_\Phi a}\left(\mathrm{id}_{\nabla_\Phi a}\right) = \left[\Phi_*\overrightarrow{\beta}\right]\widehat{f}\left(\left[\Psi\right]\right) \quad (19)$$

where $\left[\Psi\right] = \left[\psi\right]_{\nabla_\Phi a}\left(\mathrm{id}_{\nabla_\Phi a}\right)$. Hence the bijection between the natural transformations $\left[\psi\right] \colon \left[\nabla_\Phi a\right] \to \left[\Phi_*\overrightarrow{\beta}\right]$ and the elements $\left[\Psi\right]$ of $\left[\Phi_*\overrightarrow{\beta}\right]\left(\nabla_\Phi a\right)$. The restriction to $\psi \colon \nabla_\Phi a \to \Phi_*\overrightarrow{\beta}$ of $\left[\psi\right]$ must be coherent with respect to the natural bijection (18), which means that ψ must be natural with respect to $f \in \Uparrow\mathbb{B}\left(\Phi^{\#}x, \Phi^{\#}a\right)$ just like $\left[\psi\right]$ was with respect to $\widehat{f} \in (\Downarrow\mathbb{A})^{o}_{\overleftarrow{\Phi}}\left(\nabla_\Phi a, \nabla_\Phi x\right)$. The naturality of the left-hand square in Fig. 7 now gives

$$\psi_x\left(f\right) = \psi_x \circ \nabla_\Phi a f\left(\mathrm{id}_{\Phi^{\#}a}\right) = \Phi_*\overrightarrow{\beta}(f)\circ\psi_a\left(\mathrm{id}_{\Phi^{\#}a}\right) = \Phi_*\overrightarrow{\beta}(f)\Psi \quad (20)$$

where $\Psi = \psi_a\left(\mathrm{id}_{\Phi^{\#}a}\right)$. Hence the bijection between the $\overleftarrow{\Phi}$-algebra homomorphisms $\psi \in (\Downarrow\mathbb{A})^{\overleftarrow{\Phi}}\left(\nabla_\Phi a, \Phi_*\overrightarrow{\beta}\right)$ and the elements Ψ of $\Phi_*\overrightarrow{\beta}\,(a)$, as claimed in (10). Claim (11) is proven dually. ◀

Proof of Lem. 3.3.

1. Let $A \xleftarrow{h} UFA$ be a T-algebra. By the unit law of Eilenberg-Moore algebras, we have a retract $A \overset{\eta_A}{\rightarrowtail} UFA \overset{h}{\twoheadrightarrow} A$. In particular, the underlying object A of the algebra is a retract of an image under U.
2. Firstly, we shall show that

$$\mathcal{D} = \left\{B \in \mathcal{B} \;\middle|\; B \xleftarrow{\varepsilon_B} FUB \underset{\varepsilon_{FUB}}{\overset{FU\varepsilon_B}{\leftleftarrows}} FUFUB \text{ is a coequalizer}\right\} \quad (21)$$

as a full subcategory of $\mathcal{B}$. Recall that $KB = (UB \xleftarrow{U\varepsilon_B} UFUB)$. For a T-algebra $A \xleftarrow{h} UFA$, its image under $L \colon \mathcal{A}^T \rightharpoonup \mathcal{B}$ is defined by the representability:

$$\begin{aligned}\mathcal{B}\big(L(A \xleftarrow{h} UFA), B\big) &\cong \mathcal{A}^T\big((A \xleftarrow{h} UFA), (UB \xleftarrow{U\varepsilon_B} UFUB)\big)\\ &\cong \left\{f \in \mathcal{B}(FA, B) \;\middle|\; B \xleftarrow{f} FA \underset{\varepsilon_{FA}}{\overset{Fh}{\leftleftarrows}} FUFA \text{ commutes}\right\},\end{aligned}$$

where the latter isomorphism is essentially shown at the item (a) of the precise monadicity theorem. In particular, a counit $LKB \to B$ of the partial adjunction $L \dashv K$ (exists and) is an isomorphism if and only if the diagram in (21) is a equalizer in $\mathcal{B}$, because the above bijection maps $\mathrm{id}_{KB} \in \mathcal{A}^T(KB, KB)$ to $\varepsilon_B \in \mathcal{B}(FUB, B)$.

Secondly, we claim that $\mathrm{Im}\, F \subseteq \mathcal{D}$ as full subcategories of $\mathcal{B}$. It is obvious as the following is a split coequalizer diagram:

$$\begin{tikzcd} FA \arrow[r, bend right, "F\eta_A"'] & FUFA \arrow[l, "\varepsilon_{FA}"'] \arrow[r, bend right, "FUF\eta_A"'] & FUFUFA \arrow[l, shift right, "FU\varepsilon_{FA}"'] \arrow[l, shift left, "\varepsilon_{FUFA}"] \end{tikzcd} .$$

Finally, we shall prove that the subcategory $\mathcal{D} \subseteq \mathcal{B}$ is closed under taking retracts. Let D be an object of $\mathcal{D}$, and $B \overset{s}{\rightarrowtail} D \overset{r}{\twoheadrightarrow} B$ be a retract of D in $\mathcal{B}$. We have shown that the upper row of the following diagram is a coequalizer in $\mathcal{B}$.

$$\begin{tikzcd}
D \arrow[d, shift right, "r"'] & FUD \arrow[l, "\varepsilon_D"'] \arrow[d, shift right, "FUr"'] & FUFUD \arrow[l, shift right, "FU\varepsilon_D"'] \arrow[l, shift left, "\varepsilon_{FUD}"] \arrow[d, shift right, "FUFUr"'] \\
B \arrow[u, shift right, "s"'] & FUB \arrow[l, "\varepsilon_B"'] \arrow[u, shift right, "FUs"'] & FUFUB \arrow[l, shift right, "FU\varepsilon_B"'] \arrow[l, shift left, "\varepsilon_{FUB}"] \arrow[u, shift right, "FUFUs"']
\end{tikzcd}$$

The squares commutes serially, and all the columns are retracts. It is a straightforward consequence that the lower row of the diagram is also a coequalizer. ◀

▶ **Lemma A.1.** *Let X be a G-set and J be a set. Any G-map $f\colon X \to G \times J$ to the free G-set generated by J is a composite $X \cong G \times I \xrightarrow{\mathrm{id}_G \times k} G \times J$ for some $k\colon I \to J$ in* Set.

Proof of Lem. A.1. Let $I = f^{-1}(\{e\} \times J)$. The action of X induces an isomorphism $X \cong G \times I$ of G-sets. ◀

▶ **Lemma A.2.** *A retract of a singleton in G-Set is a singleton. A retract of a free G-set is free.*

Proof of Lem. A.2. The first claim is obvious. The latter claim is by Lem. A.1. ◀

Proof of Prop. 3.7. Let X be a right G-set. If there exists a right G-map $X \to G$, there exists an isomorphism $X \cong I \times G$ for some set I by Lem. A.1. Then, we have a bijection

$$H^*X = G^o\text{-}\mathsf{Set}(X, G) \cong G^o\text{-}\mathsf{Set}(I \times G, G) \cong \mathsf{Set}(I, G) = G^I \ ,$$

which is moreover an isomorphism of left G-sets. If X does not have a right G-map $X \to G$, then we have $H^*X = 0$. For instance, letting $X = 1$ gives $H^*1 = 0$ since $|G| \geq 2$. ◀

Proof of Prop. 3.8. By the Monadicity Theorem, this proposition reduces to the following two lemmas. ◀

▶ **Lemma A.3.** *The following hold.*
1. *The category G-$\mathsf{Set}_{1,\mathrm{free}}$ has reflexive equalizers.*
2. *The functor $H_*\colon (G\text{-}\mathsf{Set}_{1,\mathrm{free}})^o \to G^o\text{-}\mathsf{Set}_{1,\mathrm{free}}$ preserves reflexive coequalizers.*

Proof. By Lem. A.2, a reflexive pair in G-$\mathsf{Set}_{1,\mathrm{free}}$ is either $1 \rightrightarrows 1$ or $G \times I \rightrightarrows G \times J$. The pair $1 \rightrightarrows 1$ trivially has an equalizer that is preserved by any functor.

Let $r\colon G\times J \to G\times I$ be a common retraction in G-$\mathsf{Set}_{1,\mathrm{free}}$ of the pair $(f, h)\colon G\times I \rightrightarrows G\times J$. We may assume $r = \mathrm{id}_G \times r'$ for some map $r'\colon J \to I$ by Lem. A.1. Define a map $f'\colon I \to J$ by $\langle g_i, f'(i)\rangle = f(\langle e, i\rangle)$ for each $i \in I$ where it turns out to hold $g_i = e$ since

$$\langle e, i\rangle = r\big(f(\langle e, i\rangle)\big) = r\big(\langle g_i, f'(i)\rangle\big) = \big\langle g_i, r'\big(f'(i)\big)\big\rangle \ .$$

Moreover for any $g \in G$, it holds

$$f(\langle g, i\rangle) = f(g \cdot \langle e, i\rangle) = g \cdot f(\langle e, i\rangle) = g \cdot \langle e, f'(i)\rangle = \langle g, f'(i)\rangle \ .$$

Therefore, there exist maps $f', h'\colon I \rightrightarrows J$ such that $f = \mathrm{id}_G \times f'$, $h = \mathrm{id}_G \times h'$, and r' is a common retraction of the pair (f', h') in Set.

Using an equalizer $E \to I \rightrightarrows J$ in Set, we have an equalizer $G \times E \to G \times I \rightrightarrows G \times J$ in $G\text{-}\mathsf{Set}_{1,\mathrm{free}}$. We shall show that this (co)equalizer is preserved by $H_* \colon (G\text{-}\mathsf{Set}_{1,\mathrm{free}})^o \to G^o\text{-}\mathsf{Set}_{1,\mathrm{free}}$, i.e. the diagram

$$G^E \leftarrow G^I \leftleftarrows G^J$$

is a coequalizer of right G-sets. Their underlying sets form a coequalizer diagram in Set because the functor $|G|^{(-)} \colon \mathsf{Set}^o \to \mathsf{Set}$ preserves reflexive coequalizers for $|G| \geq 2$ by Lem. B.2. Hence, the diagram is also a coequalizer in $G^o\text{-}\mathsf{Set}$. ◀

▶ **Lemma A.4.** *The functor $H_* \colon (G\text{-}\mathsf{Set}_{1,\mathrm{free}})^o \to G^o\text{-}\mathsf{Set}_{1,\mathrm{free}}$ reflects isomorphisms.*

Proof. Let $f \colon X \to Y$ be a morphism in $G\text{-}\mathsf{Set}_{1,\mathrm{free}}$ such that $H_* f \colon H_* Y \to H_* X$ is an isomorphism. There are three cases of the G-map f:

$$1 \to 1 \ , \qquad\qquad G \times I \to 1 \ , \qquad\qquad G \times I \to G \times J \ .$$

The first two cases are trivial. For the last case, we may assume that $f = \mathrm{id}_G \times k$ for some map $k \colon I \to J$ by Lem. A.1. Then, the right G-bijection $H_* f \colon H_*(G \times J) \to H_*(G \times I)$ can be written as $G^k \colon G^J \to G^I$. By $|G| \geq 2$, the map k is a bijection, which shows that the G-map $f = \mathrm{id}_G \times k$ is an isomorphism. ◀

Proof of Lem. 3.10. The terminal object in $\Downarrow P$ is a constant presheaf 1. A presheaf $\overleftarrow{\alpha} \in \Downarrow P$ is a subobject of the presheaf 1 if and only if $\overleftarrow{\alpha} \rightarrowtail 1$ in Set for any $x \in P$.

A representable presheaf ∇x is a subobject of 1, since $(\nabla x)(y) = P(y, x) \rightarrowtail 1$. ◀

Proof of Lem. 3.11. Firstly, we shall show that $\operatorname{Im} H^* \subseteq \uparrow P$. Let $\overleftarrow{\alpha} \in \Downarrow P$ be a presheaf. We shall show $H^* \overleftarrow{\alpha} \in \uparrow P$. The set $(H^* \overleftarrow{\alpha})(x) = \Downarrow P(\overleftarrow{\alpha}, \nabla x)$ has at most one element for any $x \in P$, since ∇x is a subobject of a terminal object $1 \in \Downarrow P$. In particular, the postsheaf $H^* \overleftarrow{\alpha} \in \Uparrow P$ is an upper set of P.

Dually, we have $\operatorname{Im} H_* \subseteq \downarrow P$. Then, the adjunction $H^* \dashv H_*$ restricts as

$$\begin{tikzcd} \downarrow P \arrow[r, rightarrowtail] \arrow[d, bend right, "H^*|_{\downarrow P}"'] \arrow[d, phantom, "\dashv"] & \Downarrow P \arrow[d, bend right, "H^*"'] \arrow[d, phantom, "\dashv"] \\ \uparrow P \arrow[r, rightarrowtail] \arrow[u, bend right, "H_*|_{\uparrow P}"'] & \Uparrow P \arrow[u, bend right, "H_*"'] \end{tikzcd} \ .$$

It is obvious by definition that the restricted adjunction coincides with the $\{0,1\}$-enriched construction.

The claim $\operatorname{Im} H^* = \operatorname{Im}(H^*|_{\downarrow P})$ follows from that the embedding $\downarrow P \rightarrowtail \Downarrow P$ has a left adjoint $\Downarrow P \to \downarrow P$, which maps a presheaf $\overleftarrow{\alpha}$ to the image of $\overleftarrow{\alpha} \to 1$. Just for reference, we describe the following elementary proof, which boils down to the above argument.

A presheaf $\overleftarrow{\alpha}$ can be written as a canonical colimit $\overleftarrow{\alpha} = \varinjlim_{i \in \mathbb{I}} \nabla a_i$. Let $L \subseteq P$ be the image of $\overleftarrow{\alpha} \to 1$, i.e.

$$L = \left\{ x \in P \;\middle|\; \overleftarrow{\alpha}(x) \text{ is nonempty} \right\} = \bigcup_{i \in |\mathbb{I}|} \{ x \in P \mid x \leq a_i \} = \bigcup_{i \in |\mathbb{I}|} \nabla a_i \ .$$

We shall show that this lower set $L \in \downarrow P$ satisfies $H^* L = H^* \overleftarrow{\alpha}$. We have

$$H^* L = H^* \bigcup_{i \in |\mathbb{I}|} \nabla a_i = \bigcap_{i \in |\mathbb{I}|} H^* \nabla a_i = \bigcap_{i \in |\mathbb{I}|} \Delta a_i \qquad \in \uparrow P \ ,$$

$$H^* \overleftarrow{\alpha} = H^* \varinjlim_{i \in \mathbb{I}} \nabla a_i = \varinjlim_{i \in \mathbb{I}} H^* \nabla a_i = \varinjlim_{i \in \mathbb{I}} \Delta a_i \qquad \in \Uparrow P$$

by the adjunctions $H^*|_{\downarrow P} \dashv H_*|_{\uparrow P}$ and $H^* \dashv H_*$, respectively. The colimit $\varinjlim_{i\in\mathbb{I}} \Delta a_i$ in $\Uparrow P$ is a limit in Set^P, and it is just a product in Set^P because the objects Δa_i are subobjects of 1 in Set^P. Therefore, $H^*L = H^*\overleftarrow{\alpha}$. ◀

Proof of Cor. 3.12. A retract of an upper set $U \subseteq P$ is also an upper set, because the retract is always U itself. Thus,

$$\begin{aligned} \mathrm{Retr}_{\Uparrow P}(\mathrm{Im}\, H^*) &= \mathrm{Retr}_{\Uparrow P}(\mathrm{Im}(H^*|_{\downarrow P})) && \text{by Lem. 3.11} \\ &= \mathrm{Im}(H^*|_{\downarrow P}) && \text{by } \mathrm{Im}(H^*|_{\downarrow P}) \subseteq \uparrow P. \end{aligned}$$ ◀

Proof of Prop. 3.13. By Lem. 3.11 and Cor. 3.12. ◀

Proof of Lem. 3.15. For a $\mathbb{Z}_p$-set $Y = \underline{1}Y_1 + \underline{p}Y_p$, an element $y \in Y$ forms a $\mathbb{Z}_p$-map $y\colon \underline{1} \to Y$ if and only if $y \in \underline{1}Y_1$. For the free $\mathbb{Z}_p$-set $\underline{p}$, an element $y \in Y$ bijectively corresponds to a $\mathbb{Z}_p$-map $\underline{p} \to Y$. Since an orbit-decomposition is a coproduct,

$$\begin{aligned} \mathbb{Z}_p\text{-}\mathsf{Set}(\underline{1}X_1 + \underline{p}X_p, \underline{1}Y_1 + \underline{p}Y_p) &\cong \big(\mathbb{Z}_p\text{-}\mathsf{Set}(\underline{1}, \underline{1}Y_1 + \underline{p}Y_p)\big)^{X_1} \big(\mathbb{Z}_p\text{-}\mathsf{Set}(\underline{p}, \underline{1}Y_1 + \underline{p}Y_p)\big)^{X_p} \\ &\cong Y_1^{X_1}(Y_1 + pY_p)^{X_p} \ . \end{aligned}$$ ◀

Proof of Lem. 3.16. Let r be a common retraction of the pair (f, g), and $I = \mathrm{Im}\, f \cup \mathrm{Im}\, g \cong \underline{1}I_1 + \underline{p}I_p$. By the existence of retraction, we have $f = f_1' + f_p'\colon \underline{1}U_1 + \underline{p}U_p \to \underline{1}U_1' + \underline{p}U_p'$ for some $\mathbb{Z}_p$-maps f_1', f_p', and similar for g. Hence, there exists $r|_I = r_1' + r_p'\colon \underline{1}I_1 + \underline{p}I_p \to \underline{1}U_1 + \underline{p}U_p$. We may assume $r_1' + r_p' = \underline{1}r_1 + \underline{p}r_p$ by modifying the coercing isomorphism $U' \cong \underline{1}U_1' + \underline{p}U_p'$ on $I \subseteq U'$. Under the assumption, we obtain $f_1' + f_p' = \underline{1}f_1 + \underline{p}f_p$ and $g_1' + g_p' = \underline{1}g_1 + \underline{p}g_p$.

The reflexive equalizer in $\mathbb{Z}_p$-Set is also a reflexive equalizer in Set, which induces the following pullback of injections in Set by Lem. B.2.1.

$$\begin{tikzcd} E_1 + pE_p \arrow[r, tail, "e_1+pe_p"] \arrow[d, tail, "e_1+pe_p"'] & U_1 + pU_p \arrow[d, tail, "f_1+pf_p"] \\ U_1 + pU_p \arrow[r, tail, "g_1+pg_p"'] & U_1' + pU_p' \end{tikzcd} .$$

Changing the base by maps $U_1' \to U_1' + pU_p'$, $U_p' \to U_1' + pU_p'$ concludes the proof. ◀

Proof of Prop. 3.17. It is easy to check the full subcategories $\mathcal{C}, \mathcal{D}$ contain all retracts of images. Then, by Lem. 3.3.1, we have only to show that the restrictions $\Phi_*|_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{C}$, $\Phi^*|_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{D}$ are monadic functors. By the Monadicity Theorem, it follows from the following two lemmas that the functor $\Phi_*|_{\mathcal{D}}$ is monadic. The comonadicity of $\Phi^*|_{\mathcal{C}}$ is shown similarly to the comonadicity of the restriction of $R^*\colon \mathsf{Set} \to \mathsf{Set}^o$ (Prop. 3.4) for $R = \Phi_1 + p\Phi_p$. ◀

▶ **Lemma A.5.** *For the adjunction $\Phi^*|_{\mathcal{C}} \dashv \Phi_*|_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{C}$ in Fig. 5, the category $\mathcal{D}$ has and the functor $\Phi_*|_{\mathcal{D}}$ preserves reflexive coequalizers.*

Proof. By Lem. 3.16, an equalizer in $\mathbb{Z}_p$-Set of a reflexive pair in $\mathcal{D}^o \subseteq \mathbb{Z}_p$-$\mathsf{Set}$ can be taken as

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, equal] & U \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, equal] & U' \arrow[d, equal] \\ \underline{1}E_1 + \underline{p}E_p \arrow[r, "\underline{1}e_1+\underline{p}e_p"] & \underline{1}U_1 + \underline{p}U_p \arrow[r, shift left, "\underline{1}f_1+\underline{p}f_p"] \arrow[r, shift right, "\underline{1}g_1+\underline{p}g_p"'] & \underline{1}U_1' + \underline{p}U_p' \end{tikzcd}$$

for some equalizers

$$E_1 \xrightarrow{e_1} U_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} U'_1 \ , \qquad\qquad E_p \xrightarrow{e_p} U_p \underset{g_p}{\overset{f_p}{\rightrightarrows}} U'_p \ .$$

It is easy to show that $\underline{1}E_1 + \underline{p}E_p \in \mathcal{D}$. For example, if $\Phi = \underline{1} + \underline{p}\Phi_p$ then $U_1 = U'_1 = 1$, which implies $E_1 = 1$.

By Lem. B.2.2, the diagrams

$$\Phi_1^{E_1} \xleftarrow{\Phi_1^{e_1}} \Phi_1^{U_1} \underset{\Phi_1^{g_1}}{\overset{\Phi_1^{f_1}}{\leftleftarrows}} \Phi_1^{U'_1} \ , \quad (\Phi_1 + p\Phi_p)^{E_p} \xleftarrow{(\Phi_1+p\Phi_p)^{e_p}} (\Phi_1 + p\Phi_p)^{U_p} \underset{(\Phi_1+p\Phi_p)^{g_p}}{\overset{(\Phi_1+p\Phi_p)^{f_p}}{\leftleftarrows}} (\Phi_1 + p\Phi_p)^{U'_p}$$

are split coequalizers. Hence, their pointwise product

$$\Phi_*(\underline{1}E_1 + \underline{p}E_p) \xleftarrow{\Phi_*(\underline{1}e_1+\underline{p}e_p)} \Phi_*(\underline{1}U_1 + \underline{p}U_p) \underset{\Phi_*(\underline{1}g_1+\underline{p}g_p)}{\overset{\Phi_*(\underline{1}f_1+\underline{p}f_p)}{\leftleftarrows}} \Phi_*(\underline{1}U'_1 + \underline{p}U'_p)$$

is a (split) coequalizer. ◀

▶ **Lemma A.6.** *The right adjoint functor $\Phi_*|_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{C}$ in Fig. 5 reflects isomorphisms.*

Proof. Let $f \colon U \to U'$ be a $\mathbb{Z}_p$-map. In general, f is of the form

$$\underline{1}U_1 + \underline{p}(U_{p,1} + U_{p,p}) \ \to \ \underline{1}U'_1 + \underline{p}U'_p$$

induced by maps $f_1 \colon U_1 \to U'_1$, $g \colon U_{p,1} \to U'_1$, $h \colon U_{p,p} \to pU'_p$ up to isomorphisms, by Lem. 3.15. Modify the embedding $\underline{p}U_{p,p} \rightarrowtail U$, and we may assume further that the map h is induced by a map $f_p \colon U_{p,p} \to U'_p$. Then, $f = [\mathrm{id}_{\underline{1}} \times f_1, ! \times g, \mathrm{id}_{\underline{p}} \times f_p]$ where $! \colon \underline{p} \to \underline{1}$ is a unique $\mathbb{Z}_p$-map.

Assume that $U, U' \in \mathcal{D}$ and that $\Phi_* f \colon \Phi_* U' \to \Phi_* U$ is a bijection. We have only to prove the bijectivity of the maps f_1, f_p and $U_{p,1} = 0$. The map $\Phi_* f$ factors through an injection as the following diagram:

$$\begin{tikzcd}
 & \Phi_1^{U_1}\Phi_1^{U_{p,1}}(\Phi_1 + p\Phi_p)^{U_{p,p}} \arrow[d, tail] \\
\Phi_1^{U'_1}(\Phi_1 + p\Phi_p)^{U'_p} \arrow[ur, "\langle \Phi_1^{f_1}, \Phi_1^{g}\rangle \times (\Phi_1+p\Phi_p)^{f_p}"] \arrow[r, "\Phi_* f"'] & \Phi_1^{U_1}(\Phi_1 + p\Phi_p)^{U_{p,1}}(\Phi_1 + p\Phi_p)^{U_{p,p}}
\end{tikzcd} .$$

Since the injection must be a bijection, we have

$$\Phi_1^{U_1}(\Phi_1 + p\Phi_p)^{U_{p,p}} = 0 \quad \text{or} \quad \Phi_p = 0 \quad \text{or} \quad U_{p,1} = 0 \ .$$

The rest is straightforward for each $\Phi \colon 1 \nrightarrow \mathbb{Z}_p$.

For example, let $\Phi = \underline{p}\Phi_p$. The full subcategory $\mathcal{D} \subseteq (\mathbb{Z}_p\text{-}\mathsf{Set})^o$ contains only $\mathbb{Z}_p$-sets with trivial actions, i.e. $\mathcal{D} \subseteq \mathsf{Set}^o \subseteq (\mathbb{Z}_p\text{-}\mathsf{Set})^o$. In particular, we have $U_{p,1} = U_{p,p} = 0$, and the claim reduces to the monadicity of a restriction of the functor $\Phi_1^{(-)} \colon \mathsf{Set}^o \to \mathsf{Set}$. ◀

B Appendix: General propositions

▶ **Proposition B.1** (Precise monadicity theorem). *Let $U\colon \mathcal{D} \to \mathcal{C}$ be a functor that has a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$, and $T = U \circ F$ be the induced monad.*

$$\begin{tikzcd}
\mathcal{C} \arrow[loop above, "T"] \arrow[r, bend right, "F^T"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \mathcal{C}^T \arrow[l, bend right, "U^T"'] \arrow[dl, dashed, bend left, "L"] \\
\mathcal{D} \arrow[u, bend right, "U"'] \arrow[ur, "K"] \arrow[ur, phantom, "\top"]
\end{tikzcd}
\qquad
\begin{array}{l}
KD \;=\; (UD \xleftarrow{U\varepsilon_D} UFUD)\ , \\
L(C \xleftarrow{h} UFC) \leftarrow FC \underset{\varepsilon_{FC}}{\overset{Fh}{\leftleftarrows}} FUFC \quad \text{is a coequalizer.}
\end{array}$$

(a) *The comparison functor $K\colon \mathcal{D} \to \mathcal{C}^T$ has a left adjoint $L\colon \mathcal{C}^T \to \mathcal{D}$ if the category $\mathcal{D}$ has reflexive U-split coequalizers.*
(b) *The functor L is full and faithful if $\mathcal{D}$ has and U preserves reflexive U-split coequalizers.*
(c) *The comparison functor K is full and faithful if U reflects isomorphisms [3, Sec. 3.3].*
In particular, the right adjoint functor U is monadic if U creates reflexive U-split coequalizers.

Conversely, for a monad T, the forgetful functor $U^T\colon \mathcal{C}^T \to \mathcal{C}$ creates U^T-split coequalizers.

The statements and the proof of the following lemma is inspired by the proof of the monadicity of powerset functors $\Omega^{(-)}$ (see e.g. [3, Sec. 5.1]).

▶ **Lemma B.2.** *Let X, Y be sets, and $f, g\colon X \rightrightarrows Y$ be maps.*

1. *If the pair (f, g) is reflexive, then the maps f, g are injections and the diagram*

$$\begin{tikzcd}
Z \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[d, "f"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd} \tag{22}$$

is a pullback for an equalizer $Z \xrightarrow{e} X$ of the pair (f, g).

2. *Let R be a nonempty set and $e\colon Z \to X$ be a map such that the above diagram (22) is a pullback. If the map f is an injection, then $R^Z \xleftarrow{R^e} R^X \underset{R^g}{\overset{R^f}{\leftleftarrows}} R^Y$ is a split coequalizer. In particular, the functor $R^{(-)}\colon \mathsf{Set}^o \to \mathsf{Set}$ preserves such coequalizers.*

Proof of Lem. B.2.

1. Obvious.

2. Firstly, the map e is an injection, because f is. Fix an element $r \in R$. Since the maps e, f are injective, we may and shall define maps $R^Z \xrightarrow{e_r} R^X \xrightarrow{f_r} R^Y$ by

$$e_r(k)(x) = \begin{cases} k(z) & \text{if } x = e(z), \\ r & \text{otherwise} \end{cases} \qquad\qquad f_r(h)(y) = \begin{cases} h(x) & \text{if } y = f(x), \\ r & \text{otherwise} \end{cases}$$

where $h\colon X \to R$ and $k\colon Z \to R$. The maps give a splitting

$$\begin{tikzcd}
R^Z \arrow[r, bend right, "e_r"'] & R^X \arrow[l, "R^e"'] \arrow[r, bend right=50, "f_r"'] & R^Y \arrow[l, shift right, "R^f"'] \arrow[l, shift left, "R^g"]
\end{tikzcd}\ ,$$

i.e. the diagrams

$$
\begin{tikzcd}
 & R^Z \\
R^Z \arrow[ur, "\mathrm{id}"] \arrow[r, "e_r"'] & R^X \arrow[u, "R^e"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R^X \\
R^X \arrow[ur, "\mathrm{id}"] \arrow[r, "f_r"'] & R^Y \arrow[u, "R^f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R^Z \arrow[r, "e_r"] & R^X \\
R^X \arrow[u, "R^e"] \arrow[r, "f_r"'] & R^Y \arrow[u, "R^f"']
\end{tikzcd}
$$

commute. ◀