Codensity Liftings of Monads

Shin-ya Katsumata and Tetsuya Sato

Research Institute for Mathematical Sciences, Kyoto University
Kitashirakawaoiwakecho, Sakyoku, Kyoto, 606-8502, Japan
`{sinya,satoutet}@kurims.kyoto-u.ac.jp`

Abstract

We introduce a method to lift monads on the base category of a fibration to its total category using *codensity monads*. This method, called *codensity lifting*, is applicable to various fibrations which were not supported by the categorical $\top\top$-lifting. After introducing the codensity lifting, we illustrate some examples of codensity liftings of monads along the fibrations from the category of preorders, topological spaces and extended psuedometric spaces to the category of sets, and also the fibration from the category of binary relations between measurable spaces. We next study the liftings of *algebraic operations* to the codensity-lifted monads. We also give a characterisation of the class of liftings (along posetal fibrations with fibred small limits) as a limit of a certain large diagram.

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1 Introduction

Inspired by Lindley and Stark's work on extending the concept of reducibility candidates to monadic types [9, 10], the first author previously introduced its semantic analogue called *categorical* $\top\top$*-lifting* in [6]. It constructs a lifting of a strong monad $\mathcal{T}$ on the base category of a closed-structure preserving fibration $p : \mathbb{E} \to \mathbb{B}$ to its total category. The construction takes the inverse image of the continuation monad on the total category along the canonical monad morphism $b : \mathcal{T} \to (- \Rightarrow TR) \Rightarrow TR$ in the base category, which exists for any strong monad $\mathcal{T}$:

$$\begin{tikzcd}
\mathcal{T}^{\top\top} \arrow[r, dotted] & (- \Rightarrow S) \Rightarrow S \\
\mathcal{T} \arrow[r, "b"'] & (- \Rightarrow TR) \Rightarrow TR
\end{tikzcd}$$

The objects R and S (such that $TR = pS$) are presupposed parameters of this $\top\top$-lifting, and by varying them we can derive various liftings of $\mathcal{T}$. The categorical $\top\top$-lifting has been used to construct logical relations for monads [7] and to analyse the concept of preorders on monads [8].

One key assumption for the $\top\top$-lifting to work is that the fibration p preserves the *closed structure*, so that the continuation monad $(- \Rightarrow S) \Rightarrow S$ on the total category becomes a lifting of the continuation monad $(- \Rightarrow TR) \Rightarrow TR$ on the base category. Although many such fibrations are seen in the categorical formulations of logical relations [12, 3, 7], requiring fibrations to preserve closed structures on their total categories imposes a technical limitation to the applicability of the categorical $\top\top$-lifting. Indeed, outside the categorical semantics of type theories, it is common to work with the categories that have no closed structure. In the

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study of coalgebras, *predicate / relational liftings* of functors and monads are fundamental structures to formulate modal operators and (bi)simulation relations, and the underlying categories of them are not necessarily closed. For instance, the category **Meas** of measurable spaces, which is unlikely to be cartesian closed, is used to host labelled Markov processes. The categorical $\top\top$-lifting does not work in such situations.

To overcome this technical limitation, in this paper we introduce an alternative lifting method called *codensity lifting*. The idea is to replace the continuation monad $(- \Rightarrow S) \Rightarrow S$ with the *codensity monad* $\mathbf{Ran}_S S$ given by a right Kan extension. We then ask fibrations to preserve the right Kan extension, which is often fulfilled by the preservation of limits. We demonstrate that the codensity lifting is applicable to lift monads on the base categories of the following fibrations:

$$
\begin{tikzcd}
\mathbf{Pre} \arrow[d] \\
\mathbf{Set}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Top} \arrow[d] \\
\mathbf{Set}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{ERel}(\mathbf{Meas}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{BRel}(\mathbf{Meas}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Pred} \arrow[d] \\
\mathbf{Meas} \arrow[r, "\Delta"'] & \mathbf{Meas}^2 \arrow[r, "(\times)\circ U^2"'] & \mathbf{Set}
\end{tikzcd}
\quad
\begin{tikzcd}
U^*\mathbf{EPMet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{EPMet} \arrow[d] \\
\mathbf{Meas} \arrow[r, "U"'] & \mathbf{Set}
\end{tikzcd}
$$

Another issue when we have a lifting $\dot{\mathcal{T}}$ of a monad $\mathcal{T}$ is the liftability of *algebraic operations* for $\mathcal{T}$ to the lifting $\dot{\mathcal{T}}$. For instance, let $\dot{\mathcal{T}}$ be a lifting of the powerset monad $\mathcal{T}_p$ on **Set** along the canonical forgetful functor $p : \mathbf{Top} \to \mathbf{Set}$, which is a fibration. A typical algebraic operation for $\mathcal{T}_p$ is the union of A-indexed families of sets: $\mathrm{union}^A_X(f) = \bigcup_{a \in A} f(a)$. Then the question is whether we can "lift" the ordinary function $\mathrm{union}^A_X : A \pitchfork T_p X \to T_p X$ to a continuous function of type $A \pitchfork \dot{T}(X, \mathcal{O}_X) \to \dot{T}(X, \mathcal{O}_X)$ for every topological space $(X, \mathcal{O}_X)$. We show that the liftability of algebraic operations to codensity liftings has a good characterisation in terms of the parameters supplied to the codensity liftings.

We are also interested in the categorical property of the *collection* of liftings of a monad $\mathcal{T}$ (along a limited class of fibrations). We show a characterisation of the class of liftings of $\mathcal{T}$ as a limit of a large diagram of partial orders. This is yet an abstract categorical result, we believe that this will be helpful to construct and enumerate the possible liftings of a given monad $\mathcal{T}$.

1.1 Preliminaries

We use white bold letters $\mathbb{B}, \mathbb{C}, \mathbb{E}, \cdots$ to range over locally small categories. We sometimes identify an object in a category $\mathbb{C}$ and a functor of type $1 \to \mathbb{C}$.

We do a lot of 2-categorical calculations in **CAT**. To reduce the notational burden, we omit writing the composition operator $\circ$ between functors, or a functor and a natural transformation. For instance, for functors G, F, P, Q and a natural transformation $\alpha : P \to Q$, by $G\alpha F$ we mean the natural transformation $G(\alpha_{FI}) : G \circ P \circ F(I) \to G \circ Q \circ F(I)$. We use $\bullet$ and $*$ for the vertical and horizontal compositions of natural transformations, respectively.

Let A be a set and $X \in \mathbb{C}$. An *A-fold cotensor of* X is a pair of an object $A \pitchfork X$ and an A-indexed family of projection morphisms $\{\pi_a : A \pitchfork X \to X\}_{a \in A}$. They satisfy the following universal property: for any A-indexed family of morphisms $\{f_a : B \to X\}_{a \in A}$, there exists a unique morphism $m : B \to A \pitchfork X$ such that $\pi_a \circ m = f_a$ holds for all $a \in A$. Here are some examples of cotensors:

1. When $\mathbb{C} = \mathbf{Set}$, the function space $A \Rightarrow X$ and the evaluation function $\pi_a(f) = f(a)$ give an A-fold cotensor of X.
2. When $\mathbb{C}$ has small products, the product of A-fold copies of X and the associated projections give an A-fold cotensor of X.

3. When $\mathbb{C}$ has A-fold cotensors, any functor category $[\mathbb{D}, \mathbb{C}]$ also has A-fold cotensors, which can be given pointwisely: $(A \pitchfork F)X = A \pitchfork (FX)$.

A *right Kan extension* of $F : \mathbb{A} \to \mathbb{C}$ along $G : \mathbb{A} \to \mathbb{D}$ is a pair of a functor $\mathbf{Ran}_G F : \mathbb{D} \to \mathbb{C}$ and a natural transformation $c : \mathbf{Ran}_G F \circ G \to F$ making the following mapping ϕ_H:

$$\phi_H(\alpha) = c \bullet (\alpha G) : [\mathbb{D}, \mathbb{C}](H, \mathbf{Ran}_G F) \to [\mathbb{A}, \mathbb{C}](H \circ G, F)$$

bijective and natural on $H \in [\mathbb{D}, \mathbb{C}]$. A functor $p : \mathbb{C} \to \mathbb{C}'$ *preserves* a right Kan extension $(\mathbf{Ran}_G F, c)$ if $(p(\mathbf{Ran}_G F), pc)$ is a right Kan extension of pF along G. Thus for any right Kan extension $(\mathbf{Ran}_G(pF), c')$ of pF along G, we have $p(\mathbf{Ran}_G F) \simeq \mathbf{Ran}_G(pF)$ by the universal property.

Let $\mathcal{T}$ be a monad on a category $\mathbb{C}$. Its components are denoted by (T, η, μ). The Kleisli lifting of a morphism $f : I \to TJ$ is $\mu_J \circ Tf : TI \to TJ$, denoted by $f^\#$. We write $J : \mathbb{C} \to \mathbb{C}_\mathcal{T}$ and $K : \mathbb{C}_\mathcal{T} \to \mathbb{C}$ for the Kleisli adjunction of $\mathcal{T}$, and $\epsilon : JK \to \mathrm{Id}_\mathbb{C}$ for the counit of this adjunction. When $\mathcal{T}$ is decorated with an extra symbol, like $\dot{\mathcal{T}}$, the same decoration is applied to the notation of adjunction, like $\dot{\eta}, \dot{J}, \dot{\epsilon}$, etc.

For the definition of fibrations and related concepts, see [4].

▶ **Proposition 1** ([4, Exercise 9.2.4]). *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, and assume that $\mathbb{B}$ has small limits. If p has fibred small limits, then $\mathbb{E}$ has small limits and p preserves them.*

2 Codensity Lifting of Monads

Fix a fibration $p : \mathbb{E} \to \mathbb{B}$ and a monad $\mathcal{T}$ on $\mathbb{B}$. We first introduce the main subject of this study, *liftings* of $\mathcal{T}$.

▶ **Definition 2.** A *lifting* of $\mathcal{T}$ (along p) is a monad $\dot{\mathcal{T}}$ on $\mathbb{E}$ such that $p\dot{T} = Tp, p\dot{\eta} = \eta p$ and $p\dot{\mu} = \mu p$.

We do not require *fibredness* on $\dot{\mathcal{T}}$. The codensity lifting is a method to construct a lifting of $\mathcal{T}$ from the following data called *lifting parameter*.

▶ **Definition 3.** A *lifting parameter* (for $\mathcal{T}$) is a span $\mathbb{B}_\mathcal{T} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ of functors such that $KR = pS$. We say that it is *single* if $\mathbb{A} = 1$.

A single lifting parameter is thus a pair (R, S) of objects $R \in \mathbb{B}$ and $S \in \mathbb{E}_{TR}$. This is the same data used in the original (single-result) categorical $\top\top$-lifting in [6].

In this section we first introduce the *codensity lifting* under the situation where the fibration and the lifting parameter satisfy the following *codensity condition*.

▶ **Definition 4.** We say that a fibration $p : \mathbb{E} \to \mathbb{B}$ and a functor $S : \mathbb{A} \to \mathbb{E}$ satisfy the *codensity condition* if

1. a right Kan extension of S along S exists, and
2. $p : \mathbb{E} \to \mathbb{B}$ preserves this right Kan extension.

Later in Section 6, we give the codensity lifting without relying on the codensity condition. Although it is applicable to wider situations, the codensity lifting using the right Kan extension given below has a conceptually simpler description.

The codensity condition relates the size of $\mathbb{A}$ and the completeness of $\mathbb{E}$.

▶ **Proposition 5.** *Let p be a fibration and $\mathbb{A}$ be a category. If one of the following conditions holds:*

1. *$\mathbb{E}$ has, and p preserves cotensors, and $\mathbb{A} = 1$*
2. *$\mathbb{E}$ has, and p preserves small products, and $\mathbb{A}$ is small discrete*
3. *$\mathbb{E}$ has, and p preserves small limits, and $\mathbb{A}$ is small*
then for any functor $S : \mathbb{A} \to \mathbb{E}$ from a category $\mathbb{A}$ satisfying the condition, the pair p, S satisfies the codensity condition.

▶ **Proposition 6.** *For any fibration p and right adjoint functor $S : \mathbb{A} \to \mathbb{E}$, p, S satisfies the codensity condition.*

Proof. Let P be a left adjoint of S. Then the assignment $F \mapsto FP$ extends to a right Kan extension of F along S. This Kan extension is *absolute* [11, Proposition X.7.3]. ◀

Fix a lifting parameter $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ and assume that the fixed p, S satisfies the codensity condition. We take a right Kan extension $(\mathbf{Ran}_S S, c_S : (\mathbf{Ran}_S S)S \to S)$. As p preserves this right Kan extension, $(p(\mathbf{Ran}_S S), pc_S)$ is a right Kan extension of pS along S. Thus the following mapping:

$$\underline{(-)} = pc_S \bullet -S : [\mathbb{E}, \mathbb{B}](H, p(\mathbf{Ran}_S S)) \to [\mathbb{A}, \mathbb{B}](HS, pS).$$

is bijective and natural on $H : \mathbb{E} \to \mathbb{B}$. We write $\overline{(-)}$ for its inverse.

The right Kan extension $\mathbf{Ran}_S S$ is the functor part of the *codensity monad* [11, Exercise X.7.3]. Its unit $u_S : \mathrm{Id} \to \mathbf{Ran}_S S$ and multiplication $m_S : (\mathbf{Ran}_S S)\mathbf{Ran}_S S \to \mathbf{Ran}_S S$ are respectively given by the unique natural transformations such that $c_S \bullet u_S S = \mathrm{id}_S$ and $c_S \bullet m_S S = c_S \bullet (\mathbf{Ran}_S S)c_S$.

The codensity lifting constructs a lifting $\mathcal{T}^{\top\top} = (T^{\top\top}, \eta^{\top\top}, \mu^{\top\top})$ of $\mathcal{T}$ along p as follows.

We first lift the endofunctor T. We send $K\epsilon R : KJpS = KJKR \to KR = pS$ to $\overline{K\epsilon R} : Tp \to p(\mathbf{Ran}_S S)$, then take its cartesian lifting with respect to $\mathbf{Ran}_S S$; This is possible because $[\mathbb{E}, p] : [\mathbb{E}, \mathbb{E}] \to [\mathbb{E}, \mathbb{B}]$ is a fibration. We name the cartesian lifting σ. We then define $T^{\top\top}$ to be the codomain of σ.

$$\begin{tikzcd}
T^{\top\top} \arrow[r, dotted, "\sigma"] & \mathbf{Ran}_S S & {[\mathbb{E}, \mathbb{E}]} \arrow[d, "{[\mathbb{E}, p]}"] \\
Tp \arrow[r, "\overline{K\epsilon R}"'] & p(\mathbf{Ran}_S S) & {[\mathbb{E}, \mathbb{B}]}
\end{tikzcd}$$

We next lift the unit η. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Id}_{\mathbb{E}} \arrow[dr, dotted, "\eta^{\top\top}"'] \arrow[drr, bend left, "u_S"] & & & \\
& T^{\top\top} \arrow[r, "\sigma"'] & \mathbf{Ran}_S S & {[\mathbb{E}, \mathbb{E}]} \arrow[dd, "{[\mathbb{E}, p]}"] \\
p \arrow[dr, "\eta p"'] \arrow[drr, bend left, "pu_S"] & & & \\
& Tp \arrow[r, "\overline{K\epsilon R}"'] & p(\mathbf{Ran}_S S) & {[\mathbb{E}, \mathbb{B}]}
\end{tikzcd}$$

The triangle in the base category commutes by:

$$\overline{K\epsilon R} \bullet \eta p = \overline{K\epsilon R \bullet \eta p S} = \overline{K\epsilon R \bullet \eta KR} = \overline{\mathrm{id}_{KR}} = \overline{\mathrm{id}_{pS}} = pu_S.$$

Therefore from the universal property of σ, we obtain the unique natural transformation $\eta^{\top\top}$ above ηp making the triangle in the total category commute.

We finally lift the multiplication μ. Consider the following diagram.

$$
\begin{tikzcd}
T^{\top\top}T^{\top\top} \arrow[r, "T^{\top\top}\sigma"] \arrow[dr, dotted, "\mu^{\top\top}"'] & T^{\top\top}\mathbf{Ran}_S S \arrow[r, "\sigma\mathbf{Ran}_S S"] & (\mathbf{Ran}_S S)\mathbf{Ran}_S S \arrow[dr, "m_S"] & & \\
& T^{\top\top} \arrow[rr, "\sigma"] & & \mathbf{Ran}_S S & {[\mathbb{E},\mathbb{E}]} \arrow[dd, "{[\mathbb{E},p]}"] \\
TTp \arrow[r, "T\overline{K\epsilon R}"] \arrow[dr, "\mu p"'] & Tp(\mathbf{Ran}_S S) \arrow[r, "\overline{K\epsilon R}\mathbf{Ran}_S S"] & p(\mathbf{Ran}_S S)\mathbf{Ran}_S S \arrow[dr, "pm_S"] & & \\
& Tp \arrow[rr, "\overline{K\epsilon R}"'] & & p(\mathbf{Ran}_S S) & {[\mathbb{E},\mathbb{B}]}
\end{tikzcd}
$$

The pentagon in the base category commutes by:

$$
\begin{aligned}
\underline{pm_S \bullet \overline{K\epsilon R}\mathbf{Ran}_S S \bullet T\overline{K\epsilon R}} &= pc_S \bullet p(\mathbf{Ran}_S S)c_S \bullet \overline{K\epsilon R}(\mathbf{Ran}_S S)S \bullet T\overline{K\epsilon R}S \\
\text{(interchange law)} &= pc_S \bullet \overline{K\epsilon R}S \bullet Tpc_S \bullet T\overline{K\epsilon R}S = K\epsilon R \bullet KJK\epsilon R \\
&= K\epsilon R \bullet \mu KR = K\epsilon R \bullet \mu pS = \underline{\overline{K\epsilon R} \bullet \mu p}.
\end{aligned}
$$

Therefore from the universal property of σ, we obtain the unique morphism $\mu^{\top\top}$ above μp making the pentagon in the total category commute. We take $\mu^{\top\top}$ as the lifting of μ.

▶ **Theorem 7.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, $\mathcal{T}$ be a monad on $\mathbb{B}$, $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter for $\mathcal{T}$, and assume that p, S satisfies the codensity condition. The tuple $\mathcal{T}^{\top\top} = (T^{\top\top}, \eta^{\top\top}, \mu^{\top\top})$ constructed as above is a lifting of $\mathcal{T}$ along p.*

▶ **Corollary 8.** *The cartesian morphism $\sigma : T^{\top\top} \to \mathbf{Ran}_S S$ is a monad morphism.*

Any lifting of $\mathcal{T}$ along p can be obtained by the codensity lifting, although the choice of the lifting parameter is rather canonical.

▶ **Theorem 9.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, $\mathcal{T}$ be a monad on $\mathbb{B}$ and $\dot{\mathcal{T}}$ be a lifting of $\mathcal{T}$. Then there exists a lifting parameter R, S such that p, S satisfies the codensity condition and $\dot{\mathcal{T}} \simeq \mathcal{T}^{\top\top}$.*

Proof. We write $p_k : \mathbb{E}_{\dot{\mathcal{T}}} \to \mathbb{B}_{\mathcal{T}}$ for the canonical functor extending $p : \mathbb{E} \to \mathbb{B}$ to Kleisli categories. Then the span $\mathbb{B}_{\mathcal{T}} \xleftarrow{p_k} \mathbb{E}_{\dot{\mathcal{T}}} \xrightarrow{\dot{K}} \mathbb{E}$ is a lifting parameter that satisfies the codensity condition by Proposition 6. We can even choose $\mathbf{Ran}_{\dot{K}}\dot{K}$ so that it equals $\dot{\mathcal{T}}$. Then the morphism $\overline{K\epsilon p_k} : Tp \to p(\mathbf{Ran}_{\dot{K}}\dot{K}) = Tp$ becomes the identity morphism. Hence $\dot{\mathcal{T}}$ is isomorphic to $\mathcal{T}$. ◀

3 Examples of Codensity Liftings with Single Lifting Parameters

We illustrate some examples of the codensity liftings of monads. The fibration $p : \mathbb{E} \to \mathbb{B}$ appearing in each example has fibred small limits, and its base category $\mathbb{B}$ has small limits. Hence $\mathbb{E}$ also has small limits that are preserved by p (Proposition 1). We focus on the codensity liftings of monads with single lifting parameters. We give a general scheme to calculate them.

▶ **Proposition 10.** *Let $p : \mathbb{E} \to \mathbb{B}$ a fibration such that p has fibred small limits and $\mathbb{B}$ has small limits, $\mathcal{T}$ be a monad on $\mathbb{B}$, and $R \in \mathbb{B}, S \in \mathbb{E}_{TR}$ be a single lifting parameter. Then the functor part of $\mathcal{T}^{\top\top}$ satisfies*

$$T^{\top\top} X \simeq \bigwedge_{f \in \mathbb{E}(X,S)} ((pf)^{\#})^{-1}(S) \tag{1}$$

where $\bigwedge$ stands for the fibred product in $\mathbb{E}_{T(pX)}$.

3.1 Lifting Set-Monads to the Category of Preorders

The canonical forgetful functor $p : \mathbf{Pre} \to \mathbf{Set}$ from the category $\mathbf{Pre}$ of preorders and monotone functions is a fibration with fibred small limits: the inverse image of a preorder $(J, \leq_J)$ along a function $f : I \to J$ is the preorder $(I, \leq_I)$ given by $i \leq_I i' \iff f(i) \leq_J f(i')$. The fibred small limits are given by the set-theoretic intersections of preorders on the same set. We note that p does not preserve exponentials, hence the $\top\top$-lifting in [6] is not applicable to p.

We consider the codensity lifting of a monad $\mathcal{T}$ over $\mathbf{Set}$ along $p : \mathbf{Pre} \to \mathbf{Set}$ with a single lifting parameter: a pair of $R \in \mathbf{Set}$ and $S = (TR, \leq) \in \mathbf{Pre}$. By instantiating (1), for every $(X, \leq_X) \in \mathbf{Pre}$ (X for short), the preorder $T^{\top\top}X$ is of the form $(TX, \leq_X^{\top\top})$ where the preorder $\leq_X^{\top\top}$ is given by

$$x \leq_X^{\top\top} y \iff \forall f \in \mathbf{Pre}(X, S) \;.\; (pf)^{\#}(x) \leq (pf)^{\#}(y). \tag{2}$$

We further instantiate this by letting $\mathcal{T}$ be the powerset monad $\mathcal{T}_p$, $R = 1$ and $\leq$ be the following partial orders on $T_p 1 = \{\emptyset, 1\}$:

1. Case $\leq = \{(\emptyset, \emptyset), (\emptyset, 1), (1, 1)\}$. The homset $\mathbf{Pre}(X, S)$ is isomorphic to the set $\mathbf{Up}(X)$ of upward closed subsets of X, and (2) is rewritten to:
$$\begin{aligned} x \leq_X^{\top\top} y \quad &\iff \quad (\forall F \in \mathbf{Up}(X) \;.\; x \cap F \neq \emptyset \implies y \cap F \neq \emptyset) \\ &\iff \quad \forall i \in x \;.\; \exists j \in y \;.\; i \leq_X j, \end{aligned}$$
that is, $\leq_X^{\top\top}$ is the lower preorder.
2. Case $\leq = \{(\emptyset, \emptyset), (1, \emptyset), (1, 1)\}$. By the similar argument, $\leq^{\top\top}$ is the upper preorder:
$$x \leq_X^{\top\top} y \quad \iff \quad \forall j \in y \;.\; \exists i \in x \;.\; i \leq_X j.$$

In order to make $\leq^{\top\top}$ the *convex preorder* on $\mathcal{T}_p$:

$$x \leq_X^{\top\top} y \iff (\forall i \in x \;.\; \exists j \in y \;.\; i \leq_X j) \wedge (\forall j \in y \;.\; \exists i \in x \;.\; i \leq_X j),$$

we supply the cotupling $\mathbf{Set}_{\mathcal{T}_p} \leftarrow 1+1 \to \mathbf{Pre}$ of the above lifting parameters to the codensity lifting.

3.2 Lifting Set-Monads to the Category of Topological Spaces

The canonical forgetful functor $p : \mathbf{Top} \to \mathbf{Set}$ from the category $\mathbf{Top}$ of topological spaces and continuous functions is a fibration with fibred small limits. For a topological space $(X, \mathcal{O}_X)$ and a function $f : Y \to X$, the inverse image topological space $f^*(X, \mathcal{O}_X)$ is given by $(Y, \{f^{-1}(U) \mid U \in \mathcal{O}_X\})$. We note that each fibre category $\mathbf{Top}_X$ is the poset of topological spaces on a set X ordered in the opposite direction, that is, $(X, \mathcal{O}_1) \leq (X, \mathcal{O}_2)$ holds if and only if $\mathcal{O}_2 \subseteq \mathcal{O}_1$.

We consider the codensity lifting of a monad $\mathcal{T}$ over $\mathbf{Set}$ along $p : \mathbf{Top} \to \mathbf{Set}$ with a single lifting parameter: a pair of $R \in \mathbf{Set}$ and $S = (TR, \mathcal{O}_S) \in \mathbf{Top}$. By instantiating (1), for every $(X, \mathcal{O}_X) \in \mathbf{Top}$ (X for short), $T^{\top\top}X$ is the topological space $(TX, T^{\top\top}\mathcal{O}_X)$ whose topology $T^{\top\top}\mathcal{O}_X$ is the coarsest one making every set $((pf)^{\#})^{-1}(U)$ open, where f and U range over $\mathbf{Top}(X, S)$ and $\mathcal{O}_S$, respectively.

We further instantiate this by letting $\mathcal{T} = \mathcal{T}_p$, $R = 1$, and $\mathcal{O}_S$ be the following topologies on $T_p 1$. The topologies given to powersets by the following liftings are similar to *lower* and *upper Vietoris topology.*

1. Case $\mathcal{O}_S = \{\emptyset, \{1\}, \{\emptyset, 1\}\}$. The topology $T_p^{\top\top}\mathcal{O}_X$ is the coarsest one making every set $\{V \subseteq pX \mid V \cap U \neq \emptyset\}$ open, where U ranges over $\mathcal{O}_X$. We call this *lower Vietoris lifting.*
2. Case $\mathcal{O}_S = \{\emptyset, \{\emptyset\}, \{\emptyset, 1\}\}$. The topology $T_p^{\top\top}\mathcal{O}_X$ is the coarsest one making every set $\{V \subseteq pX \mid V \subseteq U\}$ open, where U ranges over $\mathcal{O}_X$. We call this *upper Vietoris lifting.*

3.3 Simulations on Labelled Markov Processes by Codensity Lifting

We next move on to the category **Meas** of measurable spaces and measurable functions between them. Recall that **Meas** has small limits (as the canonical forgetful functor $U : \mathbf{Meas} \to \mathbf{Set}$ is topological). We introduce some notations: For $X \in \mathbf{Meas}$, by $\mathcal{M}_X$ we mean the σ-algebra of X. For $X \in \mathbf{Top}$, by $\mathcal{B}X \in \mathbf{Meas}$ we mean the Borel (measurable) space of X.

We consider the following two fibrations q, r obtained by the change-of-base of the subobject fibration of **Set**:

$$\begin{tikzcd}
\mathbf{ERel}(\mathbf{Meas}) \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{BRel}(\mathbf{Meas}) \arrow[rr] \arrow[d, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathbf{Pred} \arrow[d, "p"] \\
\mathbf{Meas} \arrow[r, "\Delta"'] & \mathbf{Meas}^2 \arrow[r, "U^2"'] & \mathbf{Set}^2 \arrow[r, "Prod"'] & \mathbf{Set}
\end{tikzcd}$$

Here, Δ is the diagonal functor and and $Prod$ is the product functor. The legs q and r of the change-of-base are fibrations with fibred small limits. [1] The explicit description of $\mathbf{BRel}(\mathbf{Meas})$ is:

- An object X is a triple, whose components are denoted by X_0, X_1, X_2, such that X_1, X_2 are measurable spaces and $X_0 \subseteq UX_0 \times UX_1$.
- A morphism $(f_1, f_2) : X \to Y$ is a pair of measurable functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ such that $(Uf_1 \times Uf_2)(X_0) \subseteq X_1$.

The explicit description of $\mathbf{ERel}(\mathbf{Meas})$ is:

- An object is a pair, whose components are denoted by X_0, X_1, such that X_1 is a measurable space and $X_0 \subseteq UX_1 \times UX_1$.
- A morphism $f : X \to Y$ is a measurable function $f : X_1 \to Y_1$ such that $(Uf \times Uf)(X_0) \subseteq Y_0$.

For a binary relation $R \subseteq X \times Y$ and $A \subseteq X$, the image of A by R is defined to be the set $\{y \in Y \mid \exists x \in A \,.\, (x, y) \in R\}$, and is denoted by $R[A]$.

For $X \in \mathbf{Meas}$, by $\mathbf{SPMsr}(X)$ we mean the set of sub-probability measures on X. We equip it with the σ-algebra generated from the sets of the following form:

$$\{\mu \in \mathbf{SPMsr}(X) \mid \mu(U) \in V\} \quad (U \in \mathcal{M}_X, V \in \mathcal{M}_{\mathcal{B}[0,1]}),$$

and denote this measurable space by GX. The assignment $X \mapsto GX$ can be extended to a monad $\mathcal{G}$ on **Meas**, called *Giry monad* [2]. Notice that $G1 = \mathcal{B}[0, 1]$.

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[1] **BRel** and **ERel** stands for binary relations and endo-relations, respectively.

We consider the codensity lifting of $\mathcal{G}$ along $r : \mathbf{ERel}(\mathbf{Meas}) \to \mathbf{Meas}$ with a single lifting parameter $R = 1$ (the one-point measurable space) and $S = (\leq, G1)$; here $\leq$ is the usual order on $[0,1] = U(G1)$. By instantiating (1), we obtain

$$(v_1, v_2) \in (G^{\top\top}X)_0 \iff \forall f \in \mathbf{ERel}(\mathbf{Meas})(X, S) \ . \int_{X_1} f \ dv_1 \leq \int_{X_1} f \ dv_2.$$

▶ **Theorem 11.** *The relation part $(G^{\top\top}X)_0$ satisfies:*

$$(v_1, v_2) \in (G^{\top\top}X)_0 \iff (\forall U \in \mathcal{M}_{X_1} \ . \ X_0[U] \subseteq U \implies v_1(U) \leq v_2(U)).$$

Proof.

($\subseteq$) Suppose $(v_1, v_2) \in G^{\top\top}X_0$. Let $U \in \mathcal{M}_{X_1}$ be a measurable set satisfying $X_0[U] \subseteq U$. The indicator function χ_U is a morphism in $\mathbf{ERel}(\mathbf{Meas})$ from X to S. Hence,

$$v_1(U) = \int_{X_1} \chi_U \ dv_1 \leq \int_{X_1} \chi_U \ dv_2 = v_2(U).$$

($\supseteq$) Suppose that $X_0[U] \subseteq U \implies v_1(U) \leq v_2(U)$ holds for all $U \in \mathcal{M}_{X_1}$. Let $f \in \mathbf{ERel}(\mathbf{Meas})(X, S)$ be a morphism and $\sum_{i=0}^{n} \alpha_i \chi_{A_i} \leq f$ be a positive measurable simple function. Without loss of generality, we may assume $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n$ and $\sum_{i=0}^{n} \alpha_i \leq 1$. Let C_i be $f^{-1}\left([\sum_{k=0}^{i} \alpha_k, 1]\right)$, the inverse image of the closed interval $[\sum_{k=0}^{i} \alpha_k, 1]$ along f. We have $\sum_{i=0}^{n} \alpha_i \chi_{A_i} \leq \sum_{i=0}^{n} \alpha_i \chi_{C_i} \leq f$, and we obtain $C_i \in \mathcal{M}_{X_1}$ and $X_0[C_i] \subseteq C_i$ because $f \in \mathbf{ERel}(\mathbf{Meas})(X, S)$. Hence,

$$\sum_{i=0}^{n} \alpha_i v_1(A_i) \leq \sum_{i=0}^{n} \alpha_i v_1(C_i) \leq \sum_{i=0}^{n} \alpha_i v_2(C_i) \leq \int_{X_1} f \ dv_2.$$

This implies

$$\int_{X_1} f \ dv_1 = \sup \left\{ \sum_{i=0}^{n} \alpha_i v_1(A_i) \ \middle| \ \sum_{i=0}^{n} \alpha_i \chi_{A_i} \leq f \right\} \leq \int_{X_1} f \ dv_2.$$ ◀

This lifting is related to the concept of *simulation relation* between two states on the same *labelled Markov process (LMP)* in [15]. Let *Act* be a set (of actions). An LMP over $X_1 \in \mathbf{Meas}$ is a measurable function $x : X_1 \to \mathit{Act} \pitchfork GX_1$. Then a reflexive relation $X_0 \subseteq UX_1 \times UX_1$ is a *simulation* in the sense of [15, Definition 3] if and only if x is a morphism of type $(X_0, X_1) \to \mathit{Act} \pitchfork G^{\top\top}(X_0, X_1)$ in $\mathbf{ERel}(\mathbf{Meas})$.

We next consider the codensity lifting of the product Giry monad $\mathcal{G}^2$ on $\mathbf{Meas}^2$ along $q : \mathbf{BRel}(\mathbf{Meas}) \to \mathbf{Meas}^2$ with a single lifting parameter $R = (1,1)$ and $S = (\leq, G1, G1)$. By instantiating (1), we obtain

$$(v_1, v_2) \in (G^{\top\top}X)_0 \iff \forall (f_1, f_2) \in \mathbf{ERel}(\mathbf{Meas})(X, S) \ . \int_{X_1} f_1 \ dv_1 \leq \int_{X_2} f_2 \ dv_2.$$

▶ **Theorem 12.** *The relation part $(G^{\top\top}X)_0$ satisfies:*

$$(v_1, v_2) \in (G^{\top\top}X)_0 \iff (\forall U \in \mathcal{M}_{X_1}, V \in \mathcal{M}_{X_2} \ . \ X_0[U] \subseteq V \implies v_1(U) \leq v_2(V)).$$

Employing this lifting, we naturally obtain the concept of simulation relation between two states in different LMPs. Let $X \in \mathbf{BRel}(\mathbf{Meas})$ and $x_i : X_i \to \mathit{Act} \pitchfork GX_i$ be LMPs

$(i = 1, 2)$. We say that X is a simulation from x_1 to x_2 if (x_1, x_2) is a morphism of type $X \to Act \pitchfork G^{\top\top} X$ in $\mathbf{BRel}(\mathbf{Meas})$. This is equivalent to:

$$\forall (s_1, s_2) \in X_0 \ . \ \forall U \in \mathcal{M}_{X_1}, V \in \mathcal{M}_{X_2} \ . \ X_0[U] \subseteq V \implies x_1(s_1)(U) \le x_2(s_2)(V).$$

One natural property we expect on simulation relations between LMPs is the *composability*. However, $\mathcal{G}^{\top\top}$ fails to satisfy the *lax compositionality* $(G^{\top\top} X)_0; (G^{\top\top} Y)_0 \subseteq (G^{\top\top}(X; Y))_0$ for general X, Y; here ";" is the left-first relation composition. Therefore the above definition of simulation relation is not closed under the relation composition. One way to solve this problem is to require each simulation relation X to preserve measurability in the following sense: $\forall U \in \mathcal{M}_{X_1} \ . \ X_0[U] \in \mathcal{M}_{X_2}$.

3.4 Kantorovich Metric by Codensity Lifting

An *extended pseudometric space* (we drop "extended" hereafter) is a pair (X, d) of a set X and a pseudometric $d : X \times X \to [0, \infty]$ giving distances (including ∞) between elements in X. The axioms for pseudometrics are

$$d(x, x) = 0, \quad d(x, y) = d(y, x), \quad d(x, y) + d(y, z) \ge d(x, z).$$

For pseudometric spaces $(X, d), (Y, e)$, a function $f : X \to Y$ is *non-expansive* if for any $x, x' \in X$, $d(x, x') \ge e(f(x), f(x'))$ holds. We define $\mathbf{EPMet}$ to be the category of extended pseudometric spaces and non-expansive functions. The canonical forgetful functor $p : \mathbf{EPMet} \to \mathbf{Set}$ is a fibration with fibred small limits. The inverse image of a pseudometric (Y, d) along a function $f : X \to Y$ is given by $f^*(Y, d) = (X, d \circ (f \times f))$. The fibred small limit of pseudometric spaces $\{(X, d_i)\}_{i \in I}$ above the same set X is given by the pointwise sup of pseudometrics: $\bigwedge_{i \in I}(X, d_i) = (X, \sup_{i \in I} d_i)$.

We first consider the codensity lifting of a monad $\mathcal{T}$ on $\mathbf{Set}$ along $p : \mathbf{EPMet} \to \mathbf{Set}$ with a single lifting parameter: a pair of $R \in \mathbf{Set}$ and $S = (TR, s) \in \mathbf{EPMet}$. By instantiating (1), for every $(X, d) \in \mathbf{EPMet}$ (X for short), the pseudometric space $T^{\top\top} X$ is of the form $(TX, T^{\top\top} d)$ where the pseudometric $T^{\top\top} d$ is given by

$$T^{\top\top} d(c, c') = \sup_{f \in \mathbf{EPMet}(X, S)} s(f^{\#}(c), f^{\#}(c')).$$

The following example is inspired by Ogawa's work deriving Kantorovich metric on subprobability distributions [13]. We perform the following change-of-base of the fibration

$$\begin{tikzcd} U^*(\mathbf{EPMet}) \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{EPMet} \arrow[d, "p"] \\ \mathbf{Meas} \arrow[r, "U"'] & \mathbf{Set} \end{tikzcd}$$

We obtain a new fibration q with fibred small limits. An object in $U^*(\mathbf{EPMet})$ is a pair of a measurable space $(X, \mathcal{M}_X)$ and a pseudometric d on X. A morphism from $((X, \mathcal{M}_X), d)$ to $((Y, \mathcal{M}_Y), e)$ in $U^*(\mathbf{EPMet})$ is a measurable function $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ that is also non-expansive with respect to pseudometrics d and e.

We consider the codensity lifting of $\mathcal{G}$ along $q : U^*\mathbf{EPMet} \to \mathbf{Meas}$ with the following single lifting parameter: a pair of $R = 1$ and $S = (G1, s) = (\mathcal{B}[0, 1], s)$ where $s(x, y) = |x - y|$. For every $(X, d) \in \mathbf{EPMet}$ (X for short), $G^{\top\top} X$ is the pair of the measurable space GX and the following pseudometric $G^{\top\top} d$ on the set $\mathbf{SPMsr}(X)$ of subprobability measures on X:

$$G^{\top\top} d(v_1, v_2) = \sup_f \left| \int_X f dv_1 - \int_X f dv_2 \right|;$$

in the above sup, f ranges over $U^*\mathbf{EPMet}(X, S)$, the set of measurable functions of type $X \to \mathcal{B}[0,1]$ that are also non-expansive, that is, $\forall x, y \in UX \,.\, d(x,y) \geq |f(x) - f(y)|$. The pseudometric $G^{\top\top}d$ between subprobability measures is called *Kantorovich metric* [5].

We briefly mention two works related to this lifting.

- In a recent work [1], Baldan et al. introduces *Kantorovich lifting* of **Set**-functors. Although they consider lifting of general **Set**-functors rather than **Set**-monads, their lifting scheme is very close to the codensity lifting of **Set**-monads along $p : \mathbf{EPMet} \to \mathbf{Set}$.
- Ogawa reported that the Kantorovich metric on finite subprobability distributions can be derived using the technique of *observational algebra* [13].

4 Lifting Algebraic Operations to Codensity-Lifted Monads

We adopt the concept of *algebraic operation* [14] for general monads, and discuss their liftings to codensity-lifted monads. The following definition is a modification of [14, Proposition 2] for non-strong monads, and coincides with the original one when $\mathbb{C} = \mathbf{Set}$.

▶ **Definition 13.** Let $\mathbb{C}$ be a category, A be a set and assume that $\mathbb{C}$ has A-fold cotensors. An *A-ary algebraic operation* for a monad $\mathcal{T}$ on $\mathbb{C}$ is a natural transformation $\alpha : A \pitchfork K \to K$ (see Section 1.1 for K). We write $\mathbf{Alg}(\mathcal{T}, A)$ for the class of A-ary algebraic operations for $\mathcal{T}$.

▶ **Example 14.** For each set A, the powerset monad $\mathcal{T}_p$ has the algebraic operation of *A-ary set-union* $\mathrm{union}^A_X : A \pitchfork T_pX \to T_pX$ given by $\mathrm{union}^A_X(f) = \bigcup_{x \in A} f(x)$.

Fix a fibration $p : \mathbb{E} \to \mathbb{B}$, a monad $\mathcal{T}$ on $\mathbb{B}$, a set A and assume that $\mathbb{E}$ has and p preserves A-fold cotensors.

▶ **Definition 15.** Let $\dot{\mathcal{T}}$ be a lifting of $\mathcal{T}$ along p. A *lifting* of an algebraic operation $\alpha \in \mathbf{Alg}(\mathcal{T}, A)$ to $\dot{\mathcal{T}}$ is an algebraic operation $\dot\alpha \in \mathbf{Alg}(\dot{\mathcal{T}}, A)$ such that $p\dot\alpha = \alpha p_k$; here $p_k : \mathbb{E}_{\dot{\mathcal{T}}} \to \mathbb{B}_{\mathcal{T}}$ is the canonical extension of p to Kleisli categories. We write $\mathbf{Alg}_\alpha(\dot{\mathcal{T}}, A)$ for the class $\{\dot\alpha \in \mathbf{Alg}(\dot{\mathcal{T}}, A) \mid p\dot\alpha = \alpha p_k\}$ of liftings of α to $\dot{\mathcal{T}}$.

▶ **Example 16.** (Continued from Example 14) Let $\dot{\mathcal{T}}$ be a lifting of $\mathcal{T}_p$ along $p : \mathbf{Top} \to \mathbf{Set}$. Since p is faithful, there is at most one lifting of union^A to $\dot{\mathcal{T}}$. It exists if and only if for every $(X, \mathcal{O}_X) \in \mathbf{Top}$, union^A_X is a continuous function of type $A \pitchfork \dot{T}(X, \mathcal{O}_X) \to \dot{T}(X, \mathcal{O}_X)$.

We give a characterisation of the liftings of algebraic operations to codensity-lifted monads. Fix a lifting parameter $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ and assume that p, S satisfies the codensity condition. Note that the canonical extension $p_k : \mathbb{E}_{\mathcal{T}^{\top\top}} \to \mathbb{B}_{\mathcal{T}}$ of p satisfies

$$p_k J^{\top\top} = Jp, \quad pK^{\top\top} = Kp_k, \quad p\eta^{\top\top} = \eta p, \quad p_k\epsilon^{\top\top} = \epsilon p_k.$$

Starting from a natural transformation $\alpha_0 : A \pitchfork S \to S$ such that $p\alpha_0 = \alpha R$, we construct a lifting $\phi(\alpha_0) \in \mathbf{Alg}_\alpha(\mathcal{T}^{\top\top}, A)$ of α as follows.

From $A \pitchfork S = (A \pitchfork \mathrm{Id}_{\mathbb{E}})S$, the natural trasnformation α_0 induces the mate $\overline{\alpha_0} : A \pitchfork \mathrm{Id}_{\mathbb{E}} \to \mathbf{Ran}_S S$. We then obtain the following situation:

$$\begin{tikzcd}
A \pitchfork \mathrm{Id}_{\mathbb{E}} \arrow[dr, dotted, "\beta"'] \arrow[drr, bend left, "\overline{\alpha_0}"] & & & \\
& T^{\top\top} \arrow[r, "\sigma"'] & \mathbf{Ran}_S S & {[\mathbb{E},\mathbb{E}]} \arrow[ddd, "{[\mathbb{E},p]}"] \\
A \pitchfork p \arrow[dr, "\alpha Jp \bullet A \pitchfork \eta p"'] \arrow[drr, bend left, "\overline{\alpha R}"] & & & \\
& Tp \arrow[r, "\overline{K\epsilon R}"'] & p(\mathbf{Ran}_S S) & {[\mathbb{E},\mathbb{B}]}
\end{tikzcd}$$

The triangle in the base category commutes by:

$$\begin{aligned}
\overline{K\epsilon R} \bullet \alpha Jp \bullet A \pitchfork \eta p &= \overline{K\epsilon R \bullet \alpha JpS \bullet A \pitchfork \eta pS} = \overline{K\epsilon R \bullet \alpha JKR \bullet A \pitchfork \eta KR} \\
&= \overline{(K\epsilon \bullet \alpha JK \bullet A \pitchfork \eta K)R} = \overline{(\alpha \bullet A \pitchfork K\epsilon \bullet A \pitchfork \eta K)R} = p\overline{\alpha_0}.
\end{aligned}$$

We thus obtain the unique morphism β above $\alpha Jp \bullet A \pitchfork \eta p$ making the triangle in the total category commute. Using this β, we define $\phi(\alpha_0) : A \pitchfork K^{\top\top} \to K^{\top\top}$ by

$$\phi(\alpha_0) = K^{\top\top}\epsilon^{\top\top} \bullet \beta K^{\top\top} : A \pitchfork K^{\top\top} \to K^{\top\top}.$$

This algebraic operation is a lifting of α to $T^{\top\top}$:

$$p\phi(\alpha_0) = p(K^{\top\top}\epsilon^{\top\top} \bullet \beta K^{\top\top}) = (K\epsilon \bullet \alpha JK \bullet A \pitchfork \eta K)p_k = (\alpha \bullet A \pitchfork K\epsilon \bullet A \pitchfork \eta K)p_k = \alpha p_k.$$

The following theorem shows that ϕ characterises the class of liftings of α to the codensity-lifted monads. It is an analogue of Theorem 11 in [7], which is stated for the categorical $\top\top$-lifting.

▶ **Theorem 17.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, $\mathcal{T}$ be a monad on $\mathbb{B}$, and $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter, and A be a set. Suppose that $\mathbb{B}, \mathbb{E}$ has, and p preserves A-fold cotensor. Then for any $\alpha \in \mathbf{Alg}(\mathcal{T}, A)$, the mapping ϕ constructed as above has the following type and is bijective:*

$$\phi : [\mathbb{A}, \mathbb{E}]_{\alpha R}(A \pitchfork S, S) \to \mathbf{Alg}_\alpha(\mathcal{T}^{\top\top}, A).$$

▶ **Example 18.** (Continued from Example 16) We look at liftings of $\mathrm{union}^A \in \mathbf{Alg}(\mathcal{T}_p, A)$ to the codensity liftings of $\mathcal{T}_p$ along $p : \mathbf{Top} \to \mathbf{Set}$ with some single lifting parameters.

Let $R \in \mathbf{Set}$ and $S = (T_pR, \mathcal{O}_S) \in \mathbf{Top}$ be a single lifting parameter. Theorem 17 is instantiated to the following statement: a lifting of union^A to $\mathcal{T}_p^{\top\top}$ exists if and only if $\mathrm{union}_R^A : A \pitchfork T_pR \to T_pR$ is a continuous function of type $A \pitchfork S \to S$. Here, $A \pitchfork S$ is the product of A-fold copies of S, and its topology $\mathcal{O}_{A \pitchfork S}$ is generated from all the sets of the form $\pi_a^{-1}(U)$, where a and U range over A and $\mathcal{O}_S$, respectively. We further instantiate the single lifting parameter as follows (see Section 3.2):

1. Case $R = 1, \mathcal{O}_S = \{\emptyset, \{1\}, \{\emptyset, 1\}\}$. For any set A, union_1^A is a continuous function of type $A \pitchfork S \to S$ because $(\mathrm{union}_1^A)^{-1}(\{1\}) = \bigcup_{a \in A} \pi_a^{-1}(\{1\}) \in \mathcal{O}_{A \pitchfork S}$. From Theorem 17, for any set A, union^A lifts to the lower Vietoris lifting $\mathcal{T}_p^{\top\top}$.
2. Case $R = 1, \mathcal{O}_S = \{\emptyset, \{\emptyset\}, \{\emptyset, 1\}\}$. For any finite set A, union_1^A is a continuous function of type $A \pitchfork S \to S$ because $(\mathrm{union}_1^A)^{-1}(\{\emptyset\}) = \bigcap_{a \in A} \pi_a^{-1}(\{\emptyset\}) \overset{*}{\in} \mathcal{O}_{A \pitchfork S}$. On the other hand, the membership $\overset{*}{\in}$ does not hold when A is infinite. From Theorem 17, for any set A, union^A lifts to the upper Vietoris lifting $\mathcal{T}_p^{\top\top}$ if and only if A is finite.

5 Pointwise Codensity Lifting

Fix a fibration $p : \mathbb{E} \to \mathbb{B}$, a monad $\mathcal{T}$ on $\mathbb{B}$ and a lifting parameter $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$. When $\mathbb{A}$ is a large category, or $\mathbb{B}, \mathbb{E}$ are not very complete, the right Kan extension $\mathbf{Ran}_S S$ may not exist, hence the codensity lifting in Section 2 is not applicable to lift $\mathcal{T}$. In this section we introduce an alternative method (called *pointwise codensity lifting*) that relies on fibred limits of p. The point of this method is to swap the order of computation. Instead of taking the inverse image after computing $\mathbf{Ran}_S S$, we first take the inverse image of the components of $\mathbf{Ran}_S S$, bringing everything inside a fibre, then compute the right Kan extension as a fibred limit.

We assume that $\mathbb{A}$ is small (resp. large) and p has fibred small (resp. large) limits. The pointwise codensity lifting lifts $\mathcal{T}$ as follows.

We first lift T to an object mapping $\dot{T} : |\mathbb{E}| \to |\mathbb{E}|$. Let $X \in \mathbb{E}$. Consider the following diagram:

$$\begin{tikzcd}
X \downarrow S \arrow[r, "\pi_X"] \arrow[d, "!_{X\downarrow S}"'] \arrow[dr, phantom, "\Rightarrow\gamma_X"] & \mathbb{A} \arrow[r, "R"] \arrow[d, "S"] & \mathbb{B}_{\mathcal{T}} \arrow[r, equal] \arrow[d, "K"'] \arrow[r, phantom, shift right=3, "\Rightarrow\epsilon"] & \mathbb{B}_{\mathcal{T}} \arrow[d, "K"] \\
1 \arrow[r, "X"'] & \mathbb{E} \arrow[r, "p"'] & \mathbb{B} \arrow[r, "T"'] \arrow[ur, "J"'] & \mathbb{B}
\end{tikzcd}$$

where $(X \downarrow S, \pi_X, !_{X\downarrow S}, \gamma_X)$ is the *comma category*. The middle square commutes as R, S is a lifting parameter. We let $\delta_X = K\epsilon R\pi_X \bullet Tp\gamma_X$ be the composite natural transformation, and take the inverse image of $S\pi_X$ along δ_X:

$$\begin{tikzcd}
\delta_X^{-1}(S\pi_X) \arrow[r, dotted, "\overline{\delta_X}(S\pi_X)"] & S\pi_X & {[X \downarrow S, \mathbb{E}]} \arrow[d, "{[X\downarrow S, p]}"] \\
TpX!_{X\downarrow S} \arrow[r, "\delta_X"'] & KR\pi_X & {[X \downarrow S, \mathbb{B}]}
\end{tikzcd}$$

We obtain a functor $\delta_X^{-1}(S\pi_X) : X \downarrow S \to \mathbb{E}$ such that $p\delta_X^{-1}(S\pi_X) = TpX!_{X\downarrow S}$. We then define $T^{\top\top}X$ by $T^{\top\top}X = \lim(\delta_X^{-1}(S\pi_X))$, where right hand side is the fibred limit. In the following calculations we will use the vertical projection and the tupling operation of this fibred limit, denoted by

$$P_X : (T^{\top\top}X)!_{X\downarrow S} \to \delta_X^{-1}(S\pi_X),$$
$$\langle - \rangle : [X \downarrow S, \mathbb{E}]_{f!_{X\downarrow S}}(Y!_{X\downarrow S}, \delta_X^{-1}(S\pi_X)) \to \mathbb{E}_f(Y, T^{\top\top}X) \quad (f \in \mathbb{E}(Y, TpX)).$$

We next lift η. Consider the following diagram:

$$\begin{tikzcd}
X!_{X\downarrow S} \arrow[drr, bend left, "\gamma_X"] \arrow[dr, dotted, "\eta'_X"'] & & & \\
& \delta_X^{-1}(S\pi_X) \arrow[r, "\overline{\delta_X}(S\pi_X)"'] & S\pi_X & {[X \downarrow S, \mathbb{E}]} \arrow[ddd, "{[X\downarrow S, p]}"] \\
pX!_{X\downarrow S} \arrow[drr, bend left, "p\gamma_X"] \arrow[dr, "\eta pX!_{X\downarrow S}"'] & & & \\
& TpX!_{X\downarrow S} \arrow[r, "\delta_X"'] & KR\pi_X & {[X \downarrow S, \mathbb{B}]}
\end{tikzcd}$$

where the lower triangle commute by:

$$\delta_X \bullet \eta pX!_{X\downarrow S} = K\epsilon R\pi_X \bullet \eta pS\pi_X \bullet p\gamma_X = K\epsilon R\pi_X \bullet \eta KR\pi_X \bullet p\gamma_X = p\gamma_X.$$

Therefore there exists the unique natural transformation η'_X above $\eta pX!_{X\downarrow S}$ making the upper triangle commute. We define $\eta^{\top\!\top}_X = \langle \eta'_X \rangle$, which is above ηpX.

We finally lift the Kleisli lifting $(-)^{\#}$ of $\mathcal{T}$. Let $g : X \to T^{\top\!\top} Y$ be a morphism in $\mathbb{E}$, and $f = P_Y \bullet g!_{Y\downarrow S} : X!_{Y\downarrow S} \to \delta_Y^{-1}(S\pi_Y)$ be a morphism, which is above $pg!_{Y\downarrow S}$ and satisfies $g = \langle f \rangle$. We obtain the composite natural transformation $\overline{\delta_Y}(S\pi_Y) \bullet f : X!_{Y\downarrow S} \to \delta_Y^{-1}(S\pi_Y) \to S\pi_Y$. From the universal property of the comma category, we obtain the unique functor $M_f : Y \downarrow S \to X \downarrow S$ such that $\pi_X M_f = \pi_Y$ and $\gamma_X M_f = \overline{\delta_Y}(S\pi_Y) \bullet f$. We next consider the following diagram:

$$\begin{tikzcd}
\delta_X^{-1}(S\pi_X)M_f \arrow[dr, dotted, "f^\flat"'] \arrow[drr, bend left, "\overline{\delta_X}(S\pi_X)M_f"] & & & \\
& \delta_Y^{-1}(S\pi_Y) \arrow[r, "\overline{\delta_Y}(S\pi_Y)"'] & S\pi_Y & {[Y\downarrow S, \mathbb{E}]} \arrow[ddd, "{[Y\downarrow S, p]}"] \\
TpX!_{Y\downarrow S} \arrow[dr, "\mu pY!_{Y\downarrow S} \bullet Tpf"'] \arrow[drr, bend left, "\delta_X M_f"] & & & \\
& TpY!_{Y\downarrow S} \arrow[r, "\delta_Y"'] & KR\pi_Y & {[Y\downarrow S, \mathbb{B}]}
\end{tikzcd}$$

where the lower triangle commutes. Therefore there exists the unique natural transformation $f^\flat$ above $\mu pY!_{Y\downarrow S} \bullet Tpf = \mu pY!_{Y\downarrow S} \bullet Tpg!_{Y\downarrow S} = (pg)^{\#}!_{Y\downarrow S}$ making the upper triangle commute. Then we define $g^{\#^{\top\!\top}} = \langle f^\flat \bullet P_X M_f \rangle$, which is above $(pg)^{\#}$.

▶ **Theorem 19.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration with fibred small (resp. large) limits, $\mathcal{T}$ be a monad on $\mathbb{B}$, $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter for $\mathcal{T}$ and assume that $\mathbb{A}$ is small (resp. large). The tuple $(T^{\top\!\top}, \eta^{\top\!\top}, (-)^{\#^{\top\!\top}})$ constructed as above is a Kleisli triple on $\mathbb{E}$, and the corresponding monad is a lifting of $\mathcal{T}$.*

The pointwise codensity lifting coincides with the codensity lifting in Section 2, provided that $\mathbf{Ran}_S S$ and $p(\mathbf{Ran}_S S)$ are both pointwise.

▶ **Theorem 20.** *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, $\mathcal{T}$ be a monad on $\mathbb{B}$ and $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter. Assume that p, S satisfies the codensity condition, and moreover $\mathbf{Ran}_S S$ and $p(\mathbf{Ran}_S S)$ are both pointwise. Then $((\overline{K\epsilon R})^{-1}(\mathbf{Ran}_S S))X \simeq \lim(\delta_X^{-1}(S\pi_X))$.*

6 Characterising $\mathbf{lift}(\mathcal{T})$ as a Limit

We give a characterisation of the class of liftings of $\mathcal{T}$ as a limit of a large diagram. This is shown for *posetal* fibrations $p : \mathbb{E} \to \mathbb{B}$ with fibred small limits, which bijectively correspond to functors of type $\mathbb{B}^{op} \to \mathbf{Lat}_\wedge$; here $\mathbf{Lat}_\wedge$ is the category of complete lattices and meet-preserving functions. Notice that each fibre actually admits *large* limits computed by meets.

Fix such a fibration $p : \mathbb{E} \to \mathbb{B}$ and a monad $\mathcal{T}$ on $\mathbb{B}$. Since p is posetal, p is faithful. Thus we regard each homset $\mathbb{E}(X, Y)$ as a subset of $\mathbb{B}(pX, pY)$, and make p implicit.

▶ **Definition 21.** We define $\mathbf{lift}(\mathcal{T})$ to be the class of liftings of $\mathcal{T}$ along p. We introduce a partial order $\preceq$ on them by $\dot{T} \preceq \dot{T}' \iff \forall X \in \mathbb{E} \,.\, \dot{T}X \leqslant \dot{T}'X \quad (\text{in } \mathbb{E}_{T(pX)})$.

The partially ordered class $\mathbf{lift}(\mathcal{T})$ admits arbitrary large meets given by the pointwise meet.

We introduce a specific notation for the codensity liftings of $\mathcal{T}$ with a single lifting parameter R, S. By $[S]^R$ we mean the codensity lifting $\mathcal{T}^{\top\top}$ with R, S. Using Proposition 10, it is given as: $[S]^R X = \bigwedge_{f \in \mathbb{E}(X,S)} (f^\#)^{-1}(S)$.

▶ **Definition 22.** Let $X \in \mathbb{E}$. An object $S \in \mathbb{E}_{T(pX)}$ is *closed* with respect to X if 1) $\eta_{pX} \in \mathbb{E}(X, S)$ and 2) for all $f \in \mathbb{E}(X, S)$, we have $f^\# \in \mathbb{E}(S, S)$.

▶ **Proposition 23.** *Let $X \in \mathbb{E}$. Then $S \in \mathbb{E}_{T(pX)}$ is closed with respect to X if and only if $S = [S]^{pX} X$.*

▶ **Definition 24.** We define $\mathbf{Cls}(\mathcal{T}, X)$ to be the subposet $(\{S \mid S = [S]^{pX} X\}, \leq)$ of $\mathbb{E}_{T(pX)}$ consisting of closed objects with respect to X. We also define the following mappings:

$$\mathbf{Cls}(\mathcal{T}, X) \underset{q_X}{\overset{[-]^{pX}}{\rightleftarrows}} \mathbf{lift}(\mathcal{T}), \quad [S]^{pX} = T^{\top\top(pX,S)}, \quad q_X(\dot{T}) = \dot{T}X.$$

The mapping q_X is monotone, while $[-]^{pX}$ is *not*, because its argument is used both in positive and negative way. Still, we have the following adjoint-like relationship:

▶ **Theorem 25.** *For each $X \in \mathbb{E}$, we have $q_X \circ [-]^{pX} = \mathrm{id}_{\mathbf{Cls}(\mathcal{T},X)}$ and $\mathrm{id}_{\mathbf{lift}(\mathcal{T})} \preceq [-]^{pX} \circ q_X$.*

We define a function $\phi_{X,Y} : \mathbf{Cls}(\mathcal{T}, X) \to \mathbf{Cls}(\mathcal{T}, Y)$ by $\phi_{X,Y}(S) = q_Y([S]^{pX}) = [S]^{pX} Y$. This is not monotone. Theorem 25 asserts that $\phi_{X,X} = \mathrm{id}_{\mathbf{Cls}(\mathcal{T},X)}$. Using the second inequality of Theorem 25, for each $X, Y \in \mathbb{E}$, we also have

$$[S]^{pX} \preceq [[S]^{pX} Y]^{pY} = [\phi_{X,Y}(S)]^{pY}. \tag{3}$$

From Theorem 25, $\dot{T}$ is a lower bound of the class $\{[q_X(\dot{T})]^{pX} \mid X \in \mathbb{E}\}$. In fact, $\dot{T}$ is the *greatest* lower bound:

▶ **Theorem 26.** *For any lifting $\dot{\mathcal{T}}$ of $\mathcal{T}$, we have $\dot{T} = \bigwedge_{X \in \mathbb{E}} [q_X(\dot{T})]^{pX}$.*

▶ **Definition 27.** We say that $X \in \mathbb{E}$ is a *split subobject* of $Y \in \mathbb{E}$, (denoted by $X \lhd Y$) if there is a split monomorphism $m : X \to Y$.

▶ **Lemma 28.** *Let $X \lhd Y$ in $\mathbb{E}$. The following holds: 1) $\phi_{Y,X} \circ q_Y = q_X$. 2) For any $Z \in \mathbb{E}$, $\phi_{Y,X} \circ \phi_{Z,Y} = \phi_{Z,X}$. 3) $[q_Y(\dot{T})]^{pY} \preceq [q_X(\dot{T})]^{pX}$.*

Let us write $\mathbf{Split}(\mathbb{E})$ for the large preorder of $\mathbb{E}$-objects ordered by $\lhd$. We extend $\mathbf{Cls}(\mathcal{T}, -)$ to a functor of type $\mathbf{Split}(\mathbb{E})^{op} \to \mathbf{Pre}$ by $\mathbf{Cls}(\mathcal{T}, X \lhd Y) = \phi_{Y,X} : \mathbf{Cls}(\mathcal{T}, Y) \to \mathbf{Cls}(\mathcal{T}, X)$. This is indeed a functor thanks to Theorem 25 (for identity) and Lemma 28-2 (for composition). Moreover, $q_X : \mathbf{lift}(\mathcal{T}) \to \mathbf{Cls}(\mathcal{T}, X)$ is a large cone over the diagram $\mathbf{Cls}(\mathcal{T}, -)$ by Lemma 28-1. When $\mathbf{Split}(\mathbb{E})$ is directed, q is a limiting cone.

▶ **Theorem 29.** *Suppose that $\mathbf{Split}(\mathbb{E})$ is directed. Then the cone $q_X : \mathbf{lift}(\mathcal{T}) \to \mathbf{Cls}(\mathcal{T}, X)$ over the large diagram $\mathbf{Cls}(\mathcal{T}, -)$ is limiting.*

7 Conclusion and Future Work

We introduced the codensity lifting of monads along the fibrations that preserve the right Kan extensions giving codensity monads (this codensity condition was relaxed later in Section 5). The codensity lifting allows us to lift various monads on non-closed base / total categories, which was not possible by the previous $\top\top$-lifting [6].

Theorem 29 is an analogue of the characterisation of the collection of preorders on a **Set**-monad as a limit of $\mathbf{Card}^{op}$-chain in [8]. There we exploited this characterisation to enumerate the collection of preorders on some monads. We are wondering whether Theorem 29 is also useful to identify all the liftings of a given monad $\mathcal{T}$.

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