Canonical Coalgebraic Linear Time Logics

Corina Cîrstea

University of Southampton, UK
cc2@ecs.soton.ac.uk

Abstract

We extend earlier work on linear time fixpoint logics for coalgebras with branching, by showing how propositional operators arising from the choice of branching monad can be canonically added to these logics. We then consider two semantics for the uniform modal fragments of such logics: the previously-proposed, step-wise semantics and a new semantics akin to those of path-based logics. We prove that the two semantics are equivalent, and show that the canonical choice made for resolving branching in these logics is crucial for this property. We also state conditions under which similar, non-canonical logics enjoy the same property – this applies both to the choice of a branching modality and to the choice of linear time modalities. Our logics allow reasoning about linear time behaviour in systems with non-deterministic, probabilistic or weighted branching. In all these cases, the logics enhanced with propositional operators gain in expressiveness. Another contribution of our work is a reformulation of fixpoint semantics, which applies to any coalgebraic modal logic whose semantics arises from a one-step semantics.

1998 ACM Subject Classification F.1.1 Models of Computation, F.4.1 Modal Logic

Keywords and phrases coalgebra, linear time logic, fixpoint logic

Digital Object Identifier 10.4230/LIPIcs.CALCO.2015.66

1 Introduction

Several recent works focus on the study of trace semantics for coalgebras with branching, and of associated trace logics. The majority of these works concerns *finite* traces, for which a coalgebraic account exists that uses using finality in either the Kleisli [7] or the Eilenberg-Moore category [10] of the monad defining the branching type. Logics for finite traces were studied in [11], with the approach involving a dual adjunction between the category of Eilenberg-Moore algebras of the branching monad and itself.

A canonical, modular approach to defining *maximal* (including infinite) traces for coalgebras with branching was proposed in [1], and linear time coalgebraic fixpoint logics that match this notion of linear time behaviour were studied in [2]. The logics in loc. cit. are interpreted over coalgebras of type $\mathsf{T}F$, with $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ a branching monad and $F : \mathsf{Set} \to \mathsf{Set}$ a (typically polynomial) endofunctor. They use modalities arising from the endofunctor F to formalise properties of linear time behaviours, and a hidden modality arising canonically from the branching monad T to abstract away branching at each step. The semantics of these logics is quantitative, and uses $\mathsf{T}1$, with 1 a one-element set, as the domain of truth values.

$$\begin{tikzcd} & & x_2 \arrow[r] & * \\ x_0 \arrow[r, "a"] & x_1 \arrow[ur, "b"] \arrow[dr, "c"'] & & \\ & & x_3 \arrow[r] & * \end{tikzcd}$$

Figure 1

The logics have no propositional operators, and attempting to incorporate them in the usual way at the level of linear time formulas fails, as such operators do not always interact as expected with the branching modality. For example, taking T to be the powerset monad and $F = 1 + A \times \mathsf{Id}$, the associated (two-valued) linear time logics use the standard diamond modality implicitly in their semantics to abstract away branching at each step, and unary modalities

6th International Conference on Algebra and Coalgebra in Computer Science (CALCO'15).
Editors: Lawrence S. Moss and Paweł Sobociński; pp. 66–85
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

$[a]$ (with the formula $[a]\varphi$ stating that an a-transition is observed next, and subsequently φ holds) together with a nullary modality $*$ (expressing successful termination) to formalise properties of F-behaviours. The intention is that a linear time formula (containing modalities $*$ and $[a]$ with $a \in A$) should hold in a state of a $\mathcal{P}F$-coalgebra if there exists a maximal trace from that state satisfying the given formula. Unfortunately, the step-wise semantics makes the addition of a conjunction operator to the logic tricky: the obvious semantics (stipulating that $\varphi \wedge \psi$ holds in a state if both φ and ψ hold in that state) wrongly results in the pointed $\mathcal{P}(1 + A \times \mathsf{Id})$-coalgebra in Figure 1 satisfying the linear time formula $[a][b] * \wedge [a][c]*$, as there is no maximal trace starting in x_0 and satisfying both $[a][b]*$ and $[a][c]*$. The underlying problem is that, by abstracting away branching in a step-wise manner, information about which traces from a given state (x_1 in this case) satisfy a given formula (e.g. $[b]*$ or $[c]*$) is lost. In [2], this issue is dealt with by incorporating restricted versions of propositional operators into (additional) modalities, thereby enhancing the logic for F-behaviours.

Here we take a different approach, by showing how to incorporate propositional operators that arise canonically from the branching monad T into these logics. Our approach resembles that of [11] and involves lifting the logics to the Eilenberg-Moore category of T. This guarantees a smooth interaction between propositional operators and the branching modality, thereby avoiding the previously-mentioned problem. For concrete Ts, the resulting propositional operators add expressiveness to the logics: arbitrary disjunctions for non-deterministic branching, sub-convex combinations for probabilistic branching and linear combinations for weighted branching.

To justify the canonical choice for the branching modality employed by our logics, we provide an alternative, equivalent path-based semantics for their *uniform modal fragments*. Roughly speaking, these fragments only contain formulas whose modal depth is uniformly n for some $n \in \omega$; however, for formulas without variables, the uniformity condition is slightly less restrictive (see Section 5 for details). The definition of the path-based semantics involves the use of a canonical distributive law of T over F to flatten finite-step $\mathsf{T}F$-behaviours into T-branches of finite-step F-behaviours. For example, if $\mathsf{T} = \mathcal{P}$, the alternative semantics exactly captures the idea that a linear time formula holds in a state of a $\mathcal{P}F$-coalgebra if there exists a maximal trace from that state satisfying the given formula. The equivalence result crucially depends on the choice of branching modality. Its proof assumes canonical choices also for the linear time modalities, but a generalisation to logics where these modalities are not canonical is subsequently stated. In particular, this generalisation applies to modalities incorporating restricted versions of propositional operators, as used in [2].

Our technical approach relies on rephrasing the logics of [2] in the, by now standard, dual adjunction framework (originating with [16, 15] and later generalised by several other authors). In addition, we show how fixpoint logics can be accounted for in this framework, by exploiting the existence of a *coalgebraic structure* on their modal fragments.

Our results hold for coalgebras of arbitrary compositions of polynomial endofunctors with (possibly several occurrences of) a single branching monad on the category Set. However, for simplicity of presentation, this paper restricts to $\mathsf{T}F$-coalgebras.

A key assumption of the paper is that the branching monad T is commutative and *partially additive* (see Section 2). If, in addition to being partially additive, T is also assumed to be finitary, then one can show (see Remark 2.4) that T is isomorphic to a *weighted monad*, that is $\mathsf{T}X \simeq S^X$ with $(S, +, 0, \bullet, 1)$ a partial commutative semiring. Our examples of branching monads include finitary ones (modelling weighted branching that arises from partial commutative semirings) as well as infinitary ones (in particular, the full powerset monad and the sub-probability distribution monad). While the latter (infinitary) examples have a similar flavour to weighted branching, they do not strictly fall in this category.

Our contributions are:

1. We rephrase the logics of [2] in the dual adjunction framework (Section 3).
2. We extend these logics with propositional operators arising canonically from the monad T, by moving to the Eilenberg-Moore category of T (Section 4). The enhanced logics gain in expressiveness (see Example 4.4).
3. We show how fixpoint semantics fits into the dual adjunction framework (Sections 3 and 4). Our approach applies to any coalgebraic modal logic defined in this framework.
4. We show that the uniform modal fragments of the logics in [2] and of their canonical extensions with propositional operators admit an equivalent, path-based semantics (Theorems 5.13 and 5.24).
5. We show that this equivalence result depends crucially on the canonical choice for the branching modality. We also state conditions under which other choices for this modality, as used e.g. in [6, 12], enjoy the same property (Theorem 5.16). As an example, this allows the standard box modality to be used in linear time logics for coalgebras with non-deterministic, *non-empty* branching (see Example 5.15).
6. We generalise the equivalence result to other choices of linear time modalities, as used e.g. in [2] (Theorem 5.17).

Given the last two points, this paper is not only about canonical linear time logics, but also about equally well-behaved non-canonical ones.

Related Work. Several quantitative logics for probabilistic systems have been studied in the literature. Among these, the closer in spirit to our logics are perhaps those of [8], which have a linear time flavour with a semantics for modal operators that involves weighted averages. However, unlike the logics considered here, the logics of [8] employ conjunction and disjunction operators with several possible fuzzy interpretations (e.g. minimum or multiplication for conjunctions), none of which is canonical.

Following [2], a similar approach to defining *finite* trace logics has been taken in [12]. The logics in loc. cit. are parametric in the choice of both branching and linear time modalities, but contain neither propositional operators nor fixpoints. Moreover, the consequences of the generality resulting from parametricity are not sufficiently explored. As we show later (Example 5.14), the equivalence of the step-wise semantics with a more standard path-based semantics does not hold in general.

2 Preliminaries

2.1 Partially Additive Monads

We use commutative monads (T, η, μ) on Set (where $\eta : \mathsf{Id} \Rightarrow \mathsf{T}$ and $\mu : \mathsf{T} \circ \mathsf{T} \Rightarrow \mathsf{T}$ are the *unit* and *multiplication* of T) to capture branching in coalgebraic types. We write $\mathsf{st}_{X,Y} : X \times \mathsf{T}Y \to \mathsf{T}(X \times Y)$ and $\mathsf{dst}_{X,Y} : \mathsf{T}X \times \mathsf{T}Y \to \mathsf{T}(X \times Y)$ for the *strength* and respectively *double strength* maps of such a monad. The *swapped strength* map $\mathsf{st}'_{X,Y} : \mathsf{T}X \times Y \to \mathsf{T}(X \times Y)$ is defined using the *twist map* $\mathsf{tw}_{X,Y} : X \times Y \to Y \times X$ (taking $(x, y) \in X \times Y$ to (y, x)) by

$$\mathsf{T}X \times Y \xrightarrow{\mathsf{tw}_{\mathsf{T}X,Y}} Y \times \mathsf{T}X \xrightarrow{\mathsf{st}_{Y,X}} \mathsf{T}(Y \times X) \xrightarrow{\mathsf{T}\mathsf{tw}_{Y,X}} \mathsf{T}(X \times Y)$$

▶ **Example 2.1.** As examples of commutative monads, we consider:

1. The *powerset monad* $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, modelling nondeterministic computations: $\mathcal{P}(X) = \{U \mid U \subseteq X\}$, with unit given by singletons, multiplication given by unions, strength given by $\mathsf{st}_{X,Y}(x, V) = \{x\} \times V$ and double strength given by $\mathsf{dst}_{X,Y}(U, V) = U \times V$.

2. The *semiring monad* $\mathsf{T}_S : \mathsf{Set} \to \mathsf{Set}$, with $(S, +, 0, \bullet, 1)$ a commutative semiring, modelling weighted computations: $\mathsf{T}_S(X) = \{\varphi : X \to S \mid \mathsf{supp}(\varphi) \text{ is finite}\}$, where $\mathsf{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ is the *support* of φ. Its unit and multiplication are given by $\eta_X(x)(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$ and $\mu_X(\Phi) = \sum_{\varphi \in \mathsf{supp}(\Phi)} \sum_{x \in \mathsf{supp}(\varphi)} \Phi(\varphi) \bullet \varphi(x)$, while its strength and double strength are given by $\mathsf{st}_{X,Y}(x, \psi)(z, y) = \begin{cases} \psi(y) & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$ and $\mathsf{dst}_{X,Y}(\varphi, \psi)(z, y) = \varphi(z) \bullet \psi(y)$. As an example we consider the *tropical semiring* $W = (\mathbb{N}^\infty, \min, \infty, +, 0)$, with the weights being thought of as costs.
3. The *sub-probability distribution monad* $\mathcal{S} : \mathsf{Set} \to \mathsf{Set}$, modelling probabilistic computations: $\mathcal{S}(X) = \{\varphi : X \to [0, 1] \mid \sum_{x \in \mathsf{supp}(\varphi)} \varphi(x) \leq 1\}$[1]. Its unit, multiplication, strength and double strength are defined similarly to those of the semiring monad.

It was shown in [13, 4] that any commutative monad $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ induces a commutative monoid structure on the set $\mathsf{T}1$, with $1 = \{*\}$ a one-element set. The monoid multiplication $\bullet : \mathsf{T}1 \times \mathsf{T}1 \to \mathsf{T}1$ is given by the composition

$$\mathsf{T}1 \times \mathsf{T}1 \xrightarrow{\mathsf{dst}_{1,1}} \mathsf{T}(1 \times 1) \xrightarrow{\mathsf{T}\pi_2} \mathsf{T}1$$

whereas the unit is given by $\eta_1(*) \in \mathsf{T}1$.

In addition to being commutative, all the monads in Example 2.1 are *partially additive* [1]. Commutative, partially additive monads $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ were shown in loc. cit. to induce a partial commutative semiring structure on the set $\mathsf{T}1$. The resulting partial semirings serve as the domains of truth values for the logics in [2]. To interpret fixpoint formulas, these logics make use of a partial order on the set $\mathsf{T}1$, canonically induced by the partial addition operation on $\mathsf{T}1$.

In order to recall the definition of partially additive monads, we note that any monad $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ with $\mathsf{T}\emptyset = 1$ is such that, for any X, $\mathsf{T}X$ has a *zero element* $0 \in \mathsf{T}X$, obtained as $(\mathsf{T}!_X)(*)$. This yields a *zero map* $0 : Y \to \mathsf{T}X$ for any X, Y, given by

$$Y \xrightarrow{!_Y} \mathsf{T}\emptyset \xrightarrow{T!_X} \mathsf{T}X$$

with the maps $!_Y : Y \to \mathsf{T}\emptyset$ and $!_X : \emptyset \to X$ arising by finality and initiality, respectively. Partial additivity is then defined using the following map:

$$T(X + Y) \xrightarrow{\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle} \mathsf{T}X \times \mathsf{T}Y \tag{1}$$

where $p_1 = [\eta_X, 0] : X + Y \to \mathsf{T}X$ and $p_2 = [0, \eta_Y] : X + Y \to \mathsf{T}Y$.

▶ **Definition 2.2** ([1]). A monad $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ is called *partially additive* if $\mathsf{T}\emptyset = 1$ and the map in (1) is a monomorphism.

If the map in (1) is an isomorphism, then T is called *additive*. Additive monads were studied in [13, 4].

[1] This definition allows for sub-probability distributions with *countable* support.

A (partially) additive monad T induces a (partial) addition operation $+$ on the set $\mathsf{T}X$, given by $\mathsf{T}[1_X, 1_X] \circ q_{X,X}$:

$$\begin{tikzcd}
\mathsf{T}X & \mathsf{T}(X+X) \arrow[l, "\mathsf{T}[1_X,1_X]"'] \arrow[r, shift left, "\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle"] & \mathsf{T}X \times \mathsf{T}X \arrow[l, shift left, dashed, "q_{X,X}"] \arrow[ll, bend left, "+"]
\end{tikzcd}$$

where $q_{X,X} : \mathsf{T}X \times \mathsf{T}X \to \mathsf{T}(X+X)$ is the (partial) left inverse of the map $\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle$. That is, $a + b$ is defined if and only if $(a, b) \in \mathsf{Im}(\langle \mu_X \circ \mathsf{T}p_1, \mu_Y \circ \mathsf{T}p_2 \rangle)$. Hence, when T is additive, $+$ is a total operation.

The next result relates commutative, partially additive monads to *partial commutative semirings*. The latter are given by a set S carrying a partial commutative monoid structure $(S, +, 0)$ as well as a commutative monoid structure $(S, \bullet, 1)$, with $\bullet$ distributing over $+$. Specifically, for all $s, t, u \in S$, $s \bullet 0 = 0$, and whenever $t + u$ is defined, so is $s \bullet t + s \bullet u$ and moreover $s \bullet (t + u) = s \bullet t + s \bullet u$. A similar result in [4] relates additive monads and commutative semirings.

▶ **Proposition 2.3** ([1]). *Let T be a commutative, (partially) additive monad. Then $(\mathsf{T}1, 0, +, \bullet, \eta_1(*))$ is a (partial) commutative semiring.*

▶ Remark 2.4. If, in addition to being partially additive, T is also finitary, then one can show that T is isomorphic to the *partial semiring monad* $\mathsf{T}_S : \mathsf{Set} \to \mathsf{Set}$ induced by the partial commutative semiring $S = (\mathsf{T}1, 0, +, \bullet, \eta_1(*))$. This monad is defined similarly to the semiring monad T_S of Example 2.1, except that this time only those finitely-supported $\varphi : X \to S$ for which the sum $\sum\limits_{x \in \mathsf{supp}(\varphi)} \varphi(x)$ is defined are considered in $\mathsf{T}_S X$. That this yields a monad follows as for the sub-probability distribution monad. The previous observation then follows from $\mathsf{T}\emptyset = 1 \simeq (\mathsf{T}1)^{\emptyset}$, together with the existence of (natural) isomorphisms $\mathsf{T}X \simeq \mathsf{T}(\coprod_{x \in X} 1) \simeq \mathsf{T}_S X$ for X finite and non-empty, with the latter isomorphism being a consequence of the definition of $+$ on $\mathsf{T}1$ for T partially additive – $\mathsf{T}_S X$ is the subset of $\prod_{x \in X}(\mathsf{T}1) \simeq (\mathsf{T}1)^X$ reached by a map $\mathsf{T}(\coprod_{x \in X} 1) \to \prod_{x \in X}(\mathsf{T}1)$ defined similarly to the map in (1). While not all the monads in Example 2.1 are finitary ($\mathcal{P}$ and $\mathcal{S}$ are not), their finitary versions can be phrased as partial semiring monads.

For a partially additive monad T, the partial monoid $(\mathsf{T}1, +, 0)$ can be used to define a preorder relation on $\mathsf{T}1$:

$$x \sqsubseteq y \quad \text{if and only if} \quad \text{there exists } z \in S \text{ such that } x + z = y$$

It is shown in [1] that $\sqsubseteq$ has $0 \in S$ as bottom element and is preserved by $\bullet$ in each argument.

▶ **Example 2.5.** For the partially additive monads in Example 2.1, one obtains the commutative semirings $(\{\bot, \top\}, \vee, \bot, \wedge, \top)$ when $\mathsf{T} = \mathcal{P}$, $W = (\mathbb{N}^{\infty}, \min, \infty, +, 0)$ when $\mathsf{T} = \mathsf{T}_W$ and the *partial* commutative semiring $([0, 1], +, 0, *, 1)$ when $\mathsf{T} = \mathcal{S}$ (with $a + b$ defined if and only if $a + b \leq 1$). The preorders associated to these (partial) semirings are all partial orders: $\leq$ on $\{\bot, \top\}$ for $\mathsf{T} = \mathcal{P}$, $\geq$ on $\mathbb{N}^{\infty}$ for $\mathsf{T} = \mathsf{T}_W$, and $\leq$ on $[0, 1]$ for $\mathsf{T} = \mathcal{S}$.

From this point onwards, T denotes a commutative, partially additive monad with associated partial commutative semiring $(\mathsf{T}1, 0, +, \bullet, \eta_1(*))$ and associated preorder $\sqsubseteq$. We further assume that the unit of $\bullet$ is a top element for $\sqsubseteq$, and that $\sqsubseteq$ is both an *ω-chain complete* and an *ω^{op}-chain complete* partial order, that is, any increasing (decreasing) chain has a least upper bound (greatest lower bound). These assumptions hold for all the preorders in Example 2.5.

2.2 Coalgebraic Linear Time Logics

We now recall briefly a variant of the logics proposed in [2]. The difference w.r.t. loc. cit. is the lack of the propositional constant $\top$. The presence of $\top$ in the syntax of the logics would allow one to also express properties of *partial* traces. The logics below allow the formulation of properties of *completed*, i.e. maximal traces only, as defined in [1].

The syntax of the logics is given by

$$\varphi ::= x \mid [\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)}) \mid \mu x.\varphi \mid \nu x.\varphi, \qquad x \in \mathcal{V},\ \lambda \in \Lambda$$

with $\mathcal{V}$ a set of variables and Λ a set of modal operators with associated *generalised predicate liftings* $[\![\lambda]\!] : (\mathsf{T}1)^{-} \times \ldots \times (\mathsf{T}1)^{-} \Rightarrow (\mathsf{T}1)^{F-}$. Then, for a $\mathsf{T}F$-coalgebra (X, γ) and a valuation $V : \mathcal{V} \to (\mathsf{T}1)^X$ (interpreting the variables in $\mathcal{V}$ as generalised predicates over X), a formula φ is itself interpreted as a generalised predicate $[\![\varphi]\!]_\gamma^V \in (\mathsf{T}1)^X$, defined inductively on the structure of φ by

- $[\![x]\!]_\gamma^V = V(x)$,
- $[\![[\lambda](\varphi_1, \ldots, \varphi_{\mathsf{ar}(\lambda)})]\!]_\gamma^V = \gamma^*(\mathsf{ext}_{FX}([\![\lambda]\!]_X([\![\varphi_1]\!]_\gamma^V, \ldots, [\![\varphi_{\mathsf{ar}(\lambda)}]\!]_\gamma^V)))$, where the generalised predicate lifting $\mathsf{ext} : (\mathsf{T}1)^{-} \Rightarrow (\mathsf{T}1)^{\mathsf{T}-}$, called *extension lifting* in [2], takes a generalised predicate $p : X \to \mathsf{T}1$ to the generalised predicate $\mu_1 \circ \mathsf{T}p : \mathsf{T}X \to \mathsf{T}1$ (with $\mu : \mathsf{T}^2 \Rightarrow \mathsf{T}$ the monad multiplication), and where $\gamma^* : (\mathsf{T}1)^{\mathsf{T}FX} \to (\mathsf{T}1)^X$ is given by pre-composition with $\gamma : X \to \mathsf{T}FX$.
- $[\![\mu x.\varphi]\!]_\gamma^{V\setminus\{x\}}$ ($[\![\nu x.\varphi]\!]_\gamma^{V\setminus\{x\}}$) is the least (respectively greatest) fixpoint of the operator on $(\mathsf{T}1)^X$ defined by $p \longmapsto [\![\varphi]\!]_\gamma^{V[p/x]}$, where the valuation $V[p/x] : \mathcal{V} \to (\mathsf{T}1)^X$ is given by $V[p/x](x) = p$ and $V[p/x](y) = V(y)$ for $y \in \mathcal{V} \setminus \{x\}$.

The use of the extension lifting in the definition of the semantics allows the branching present in the coalgebra γ to be abstracted away in a step-wise manner. For the operator in the last clause to be order-preserving, monotonicity of both ext and the generalised predicate liftings $[\![\lambda]\!]$, with $\lambda \in \Lambda$, is required. The fact that ext is monotone follows by an argument similar to that of [1, Proposition 5.3], with the proof making use of the definition of the order $\sqsubseteq$ on $\mathsf{T}1$ in terms of the partial addition operation on $\mathsf{T}1$. Monotonicity in each argument of the generalised predicate liftings $[\![\lambda]\!]$, with $\lambda \in \Lambda$, was shown in [2] under the assumptions that F is a polynomial functor and that the $[\![\lambda]\!]$s are canonically derived from a presentation of F as a coproduct of finite products of identity functors. (Given such a presentation, each coproduct component Id^n yields a modality of arity n. The existence of least, respectively greatest fixpoints as required by the last clause then follows by [5, Theorem 8.22]. We note that this result only requires an order-preserving operator on a complete partial order. If, in addition, $\mathsf{T}1$ is a complete lattice (which is the case in all our examples), then the Knaster-Tarski fixpoint theorem (see e.g. [5, Theorem 2.35]) also applies, and provides a characterisation of least (greatest) fixpoints as least pre-fixpoints (respectively greatest post-fixpoints).

▶ **Example 2.6.** For $F = 1 + A \times \mathsf{Id} \simeq 1 + \coprod_{a\in A} \mathsf{Id}$, the modal operators arising from the structure of F are a nullary modality $*$ together with unary modalities $[a]$ with $a \in A$. The associated predicate liftings, canonically derived from the structure of F, are given by $[\![*]\!]_X \in (\mathsf{T}1)^{FX}$, $[\![*]\!]_X(\iota_1(*)) = 1$ and $[\![*]\!]_X(\iota_a(x)) = 0$ for $x \in X$, and respectively $[\![a]\!]_X : (\mathsf{T}1)^X \to (\mathsf{T}1)^{FX}$, $[\![a]\!]_X(p)(\iota_1(*)) = 0$ and $[\![a]\!]_X(p)(\iota_{a'}(x)) = \begin{cases} p(x), & \text{if } a' = a \\ 0, & \text{otherwise} \end{cases}$, for $p \in (\mathsf{T}1)^X$ and $x \in X$. Similarly, for $F = A \times \mathsf{Id} \times \mathsf{Id} \simeq \coprod_{a\in A}(\mathsf{Id} \times \mathsf{Id})$, the induced modal operators are binary modalities $[a]$ with $a \in A$, with associated predicate liftings given by $[\![a]\!]_X(p_1, p_2)(\iota_{a'}(x, y)) = \begin{cases} p_1(x) \bullet p_2(y), & \text{if } a' = a \\ 0, & \text{otherwise} \end{cases}$, for $p_1, p_2 \in (\mathsf{T}1)^X$ and $x, y \in X$.

(Similar use of the monoid multiplication $\bullet$ is made for any generalised predicate lifting of arity ≥ 2 derived canonically from a polynomial endofunctor F.) Irrespective of the choice of F, when $\mathsf{T} = \mathcal{P}$, a formula φ of the resulting logic holds in a state of a $\mathsf{T}F$-coalgebra if that state admits a maximal trace (element of the final F-coalgebra) satisfying φ. Also, when $\mathsf{T} = \mathcal{S}$ ($T = \mathsf{T}_W$), $[\![\varphi]\!]_\gamma : X \to \mathsf{T}1$ measures the likelihood (respectively minimal cost) with which a maximal trace satisfying φ is exhibited by states of a $\mathsf{T}F$-coalgebra (X, γ).

3 Coalgebraic Linear Time Logics via Dual Adjunctions

This section rephrases the generalised predicate lifting approach to defining the semantics of coalgebraic linear time logics in terms of dual adjunctions.

For an endofunctor $F : \mathsf{Set} \to \mathsf{Set}$, the dual adjunction approach to defining a logic for F-coalgebras involves a contravariant adjunction $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathsf{op}} \arrow[l, bend left, "P"] \end{tikzcd}$, a functor $L : \mathcal{A} \to \mathcal{A}$ and a natural transformation $\delta : LP \Rightarrow PF$. These yield a logic for F-coalgebras with syntax given by the initial L-algebra $(\mathcal{L}, \alpha)$ and with semantics $[\![\_]\!]_\gamma : \mathcal{L} \to PX$, for an F-coalgebra (X, γ), defined as the unique L-algebra homomorphism from α to $P\gamma \circ \delta_X$:

$$\begin{tikzcd} L(\mathcal{L}) \arrow[r, "L[\![\_]\!]_\gamma"] \arrow[dd, "\alpha"'] & LPX \arrow[d, "\delta_X"] \\ & PFX \arrow[d, "P\gamma"] \\ \mathcal{L} \arrow[r, "[\![\_]\!]_\gamma"] & PX \end{tikzcd}$$

To match the syntax and semantics of the logics in [2], we consider the dual adjunction $\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathsf{op}} \arrow[l, bend left, "P"] \end{tikzcd}$ with $S = P = (\mathsf{T}1)^{-}$. Following previous work on the modular construction of coalgebraic logics [3] (see also [12] for a similar approach to defining *forgetful logics*), we take a modular approach to defining a natural transformation $\delta : LP \Rightarrow P\mathsf{T}F$ that captures the above use of the extension lifting ext and of the generalised predicate liftings $[\![\lambda]\!]$ derived from the structure of F. The ingredients required for this are:

- an endofunctor $L : \mathsf{Set} \to \mathsf{Set}$ specifying the syntax of a logic for F-coalgebras, together with a natural transformation $\delta : LP \Rightarrow PF$, providing a one-step semantics for this logic,
- a natural transformation $\sigma : \mathsf{Id}P \Rightarrow P\mathsf{T}$, providing a one-step semantics for a logic for T-coalgebras.

The use of the identity functor to define a syntax for T reflects the fact that, in the logics of [2], the branching modality is hidden from the syntax. Then, to capture the use of the extension predicate lifting ext in the definition of the semantics, the components of $\sigma : P \Rightarrow P\mathsf{T}$ must be given by

$$X \xrightarrow{p} \mathsf{T}1 \quad \overset{\sigma}{\longmapsto} \quad \mathsf{T}X \xrightarrow{\mathsf{T}p} \mathsf{T}^2 1 \xrightarrow{\mu_1} \mathsf{T}1 \tag{2}$$

Following [2], other choices for a modality that abstracts away branching have been considered: both [6] and [12] propose using an arbitrary T-algebra structure $\tau : \mathsf{T}^2 1 \to \mathsf{T}1$ instead of μ_1 in the definition of σ. While most of the results in this paper concern the canonical choice of σ, we also explore the more general σs arising from a choice of τ as above. For this, we need the following lemma, where we write $\mathsf{Alg}(\mathsf{T})$ for the category of Eilenberg-Moore algebras of the monad T.

▶ **Lemma 3.1.** *For any* $(\mathsf{T}1,\tau)$ *in* $\mathsf{Alg}(\mathsf{T})$*, with induced* $\sigma : P \Rightarrow P\mathsf{T}$*, we have* $\sigma_{\mathsf{T}} \circ \sigma = P\mu \circ \sigma$*.*

Proof. $\sigma_{\mathsf{T}X} \circ \sigma_X$ maps a predicate $p : X \to P1$ to the predicate $\tau \circ \mathsf{T}\tau \circ \mathsf{T}^2 p$, whereas $P\mu_X \circ \sigma_X$ maps p to $\tau \circ \mathsf{T}p \circ \mu_X$. The conclusion now follows from the commutativity of

$$\begin{tikzcd}
\mathsf{T}^2 X \arrow[r, "\mathsf{T}^2 p"] \arrow[d, "\mu_X"'] & T^3 1 \arrow[r, "\mathsf{T}\tau"] \arrow[d, "\mu_{\mathsf{T}1}"'] & \mathsf{T}^2 1 \arrow[d, "\tau"] \\
\mathsf{T}X \arrow[r, "\mathsf{T}p"] & \mathsf{T}^2 1 \arrow[r, "\tau"] & \mathsf{T}1
\end{tikzcd}$$

where the left and right squares follow by naturality of μ and from $\tau \in \mathsf{Alg}(\mathsf{T})$, respectively. ◀

We now return to the endofunctor F and discuss the canonical choice for the corresponding L and $\delta : LP \Rightarrow PF$. As in [2], we assume that F is a polynomial endofunctor, and hence naturally isomorphic to a coproduct of finite (including empty) products of identity functors. Presenting F in this way canonically determines a set of modal operators (as already sketched in Example 2.6).

▶ **Definition 3.2.** Let $L ::= F = \coprod_{\lambda \in \Lambda} X^{\mathsf{ar}(\lambda)}$, with Λ a set of modal operators with specified arities. Also, let $\delta : LP \Rightarrow PF$ be given by

$$\begin{tikzcd}
(PX)^{\mathsf{ar}(\lambda)} \arrow[r, "\bullet_X \circ (P\pi_1 \times \ldots \times P\pi_{\mathsf{ar}(\lambda)})"] \arrow[d, "\iota_\lambda"'] & P(X^{\mathsf{ar}(\lambda)}) \arrow[r, "e_\lambda"] & P(\coprod_{\lambda \in \Lambda} X^{\mathsf{ar}(\lambda)}) \\
\coprod_{\lambda \in \Lambda} (PX)^{\mathsf{ar}(\lambda)} \arrow[urr, dashed, "\delta_X"'] & &
\end{tikzcd}$$

where in the above $\bullet_Y : (PY)^n \to PY$ is given by the transpose of the map

$$((\mathsf{T}1)^Y \times \ldots \times (\mathsf{T}1)^Y) \times Y \longrightarrow \mathsf{T}1 \;, \qquad (p_1, \ldots, p_n, y) \mapsto p_1(y) \bullet \ldots \bullet p_n(y)$$

with $\bullet : \mathsf{T}1 \times \mathsf{T}1 \to \mathsf{T}1$ the multiplication operation on $\mathsf{T}1$ (extended to an n-ary operation), and with $e_\lambda : P(X^{\mathsf{ar}(\lambda)}) \to P(\coprod_{\lambda \in \Lambda} X^{\mathsf{ar}(\lambda)})$ being given by

$$X^{\mathsf{ar}(\lambda)} \xrightarrow{p} \mathsf{T}1 \qquad \stackrel{e_\lambda}{\longmapsto} \qquad \coprod_{\lambda \in \Lambda} X^{\mathsf{ar}(\lambda)} \xrightarrow{[0,\ldots,p,\ldots,0]} \mathsf{T}1$$

The particular choice of L and δ in Definition 3.2 corresponds to a syntax with modal operators $\lambda \in \Lambda$, with associated generalised predicate liftings given by $e_\lambda \circ \bullet_X \circ (P\pi_1 \times \ldots \times P\pi_{\mathsf{ar}(\lambda)})$. That is, for $\lambda \in \Lambda$, the associated predicate lifting takes $(p_1, \ldots, p_{\mathsf{ar}(\lambda)})$ with $p_i : X \to \mathsf{T}1$ to the generalised predicate taking $x \in X$ to $e_\lambda(p_1(x) \bullet \ldots \bullet p_{\mathsf{ar}(\lambda)}(x)) \in \mathsf{T}1$. In particular, the generalised predicate liftings described in Example 2.6 are of this form. Moreover, as explained in Section 2.2, these generalised predicate liftings are monotone.

Having fixed $\delta : LP \Rightarrow PF$ and $\sigma : P \Rightarrow P\mathsf{T}$, a logic $\mathcal{L}$ for $\mathsf{T}F$-coalgebras arises from the one-step semantics specified by the natural transformation $\sigma_F \circ \delta$:

$$LP \overset{\delta}{\Longrightarrow} PF \overset{\sigma_F}{\Longrightarrow} P\mathsf{T}F$$

That is, for a $\mathsf{T}F$-coalgebra (X, γ), the map $[\![\_]\!]_\gamma : \mathcal{L} \to PX$ arises as the unique L-algebra homomorphism from the initial L-algebra $(\mathcal{L}, \alpha)$ to $(PX, P\gamma \circ \sigma_{FX} \circ \delta_X)$. More generally, for a valuation $V : \mathcal{V} \to PX$, $[\![\_]\!]_\gamma^V : \mathcal{L}^{\mathcal{V}} \to PX$ is defined as the unique L-algebra homomorphism from the free L-algebra $(\mathcal{L}^{\mathcal{V}}, \alpha^{\mathcal{V}})$ over $\mathcal{V}$ to $(PX, P\gamma \circ \sigma_{FX} \circ \delta_X)$ which extends V:

$$\begin{tikzcd}
L(\mathcal{L}^{\mathcal{V}}) \arrow[r, "L({[\![_]\!]}_\gamma^V)"] \arrow[dd, "\alpha^{\mathcal{V}}"'] & LPX \arrow[d, "\sigma_{FX} \circ \delta_X"] \\
 & P\mathsf{T}FX \arrow[d, "P\gamma"] \\
\mathcal{L}^{\mathcal{V}} \arrow[r, "{[\![_]\!]}_\gamma^V"] & PX
\end{tikzcd}$$

Extending the logic $\mathcal{L}$ with fixpoint formulas can now be done as before. We write $\mathcal{L}_\mu$ for the resulting logic, and conclude the section by providing an alternative definition of the semantics of fixpoint formulas. This exploits the existence of a *coalgebraic structure* on the modal fragment of the logic, and will later smoothly generalise to the case where the logics carry $\mathsf{Alg}(\mathsf{T})$-structure. To this end, we let $\varphi \in \mathcal{L}^{\{x\}+\mathcal{V}}$ and consider the $\mathcal{V} + L(\_)$-coalgebra $(\mathcal{L}^{\{x\}+\mathcal{V}}, \beta_\varphi)$, with $\beta_\varphi : \mathcal{L}^{\{x\}+\mathcal{V}} \to \mathcal{V} + L\mathcal{L}^{\{x\}+\mathcal{V}}$ the unique L-algebra homomorphism satisfying $\beta_\varphi(v) = v$ for $v \in \mathcal{V}$ and $\beta_\varphi(x) = \varphi$. (Note that the set $\mathcal{V} + L\mathcal{L}^{\{x\}+\mathcal{V}}$ inherits L-algebra structure from $(\mathcal{L}^{\{x\}+\mathcal{V}}, \alpha^{\{x\}+\mathcal{V}})$, as $\mathcal{L}^{\{x\}+\mathcal{V}} \simeq \{x\} + \mathcal{V} + L\mathcal{L}^{\{x\}+\mathcal{V}}$.)

▶ **Lemma 3.3.** *Let (X, γ) be a $\mathsf{T}F$-coalgebra, let $V : \mathcal{V} \to PX$ be a valuation, and let $\varphi \in \mathcal{L}^{\{x\}+\mathcal{V}}$. Consider the operator $\mathsf{O} : [\mathcal{L}^{\{x\}+\mathcal{V}}, PX] \to [\mathcal{L}^{\{x\}+\mathcal{V}}, PX]$ defined by $f \mapsto [V, P\gamma] \circ (\mathsf{id} + (\sigma_{FX} \circ \delta_X)) \circ (\mathsf{id} + Lf) \circ \beta_\varphi$:*

$$\begin{tikzcd}
\mathcal{V}+L\mathcal{L}^{\{x\}+\mathcal{V}} \arrow[r, "\mathsf{id}+Lf"] & \mathcal{V}+LPX \arrow[d, "\mathsf{id}+(\sigma_{FX}\circ\delta_X)"] \\
 & \mathcal{V}+P\mathsf{T}FX \arrow[d, "{[V,P\gamma]}"] \\
\mathcal{L}^{\{x\}+\mathcal{V}} \arrow[uu, "\beta_\varphi"] \arrow[r, "f"] & PX
\end{tikzcd}$$

Then $[\![\mu x.\varphi]\!]^V_\gamma$ ($[\![\nu x.\varphi]\!]^V_\gamma$) is given by $f_0(x)$, with $f_0 : \mathcal{L}^{\{x\}+\mathcal{V}} \to PX$ the least (resp. greatest) fixpoint of O.

Each application of the operator O above computes a new approximation of the semantics of formulas in $\mathcal{L}^{\{x\}+\mathcal{V}}$, obtained by replacing occurrences of the variable x by φ, and using the previous approximation for the semantics of φ. We note that, by definition, $\mathsf{O}(f)$ extends $V : \mathcal{V} \to PX$, and therefore so does f_0.

▶ Remark 3.4. In practice, one only needs the set of subformulas of $\varphi[\varphi/x]$, not the entire $\mathcal{L}^{\{x\}+\mathcal{V}}$, to define $[\![\mu x.\varphi]\!]^V_\gamma$ and $[\![\nu x.\varphi]\!]^V_\gamma$. This set inherits a $\mathcal{V} + L(\_)$-coalgebra structure from β_φ.

▶ Remark 3.5. Transporting the previous diagram via the dual adjunction to the category of spaces, we obtain an operator on $[X, S\mathcal{L}^{\{x\}+\mathcal{V}}]$:

$$\begin{tikzcd}
S\mathcal{V}\times\mathsf{T}FS\mathcal{L}^{\{x\}+\mathcal{V}} \arrow[d, "\mathsf{id}+(\sigma^\flat_{L\mathcal{L}^{\{x\}+\mathcal{V}}}\circ\mathsf{T}\delta^\flat_{\mathcal{L}^{\{x\}+\mathcal{V}}})"'] & S\mathcal{V}\times\mathsf{T}FX \arrow[l, "\mathsf{id}\times\mathsf{T}Ff^\flat"'] \\
S\mathcal{V}\times SL\mathcal{L}^{\{x\}+\mathcal{V}} \arrow[d, "(\beta_\varphi)^\flat"'] & \\
S\mathcal{L}^{\{x\}+\mathcal{V}} & X \arrow[l, "f^\flat"'] \arrow[uu, "{[V^\flat,\gamma]}"']
\end{tikzcd}$$

where $\delta^\flat : FS \Rightarrow SL$ and $\sigma^\flat : \mathsf{T}S \Rightarrow S$ are the *mates* of δ and σ, respectively (see e.g. [12] for a definition). Since taking least/greatest fixpoints in both $[\mathcal{L}^{\{x\}+\mathcal{V}}, PX]$ and $[X, S\mathcal{L}^{\{x\}+\mathcal{V}}]$ amounts to taking least/greatest fixpoints of operators on *generalised relations* on $X \times \mathcal{L}^{\{x\}+\mathcal{V}}$ (see [1, 2] for a treatment of generalised relations induced by T), the semantics of $\mathcal{L}_\mu$ can alternatively be defined in the category of spaces.

4 Enhanced Coalgebraic Linear Time Logics

$\mathcal{L}^{\mathcal{V}}$ and $\mathcal{L}_\mu$ only contain modal operators, not also propositional ones. We now show how to canonically add propositional operators to $\mathcal{L}^{\mathcal{V}}$ and $\mathcal{L}_\mu$, by lifting these logics to $\mathsf{Alg}(\mathsf{T})$.

It follows e.g. from [9, Exercise 5.4.11] that for T a strong monad and $(A, \alpha) \in \mathsf{Alg}(\mathsf{T})$, the dual adjunction $\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathsf{op}} \arrow[l, bend left, "P"] \end{tikzcd}$ with $S = P = A^{-}$ lifts to $\begin{tikzcd} \mathsf{Alg}(\mathsf{T}) \arrow[r, bend left, "\tilde{S}"] \arrow[r, phantom, "\perp"] & \mathsf{Set}^{\mathsf{op}} \arrow[l, bend left, "\tilde{P}"] \end{tikzcd}$, with $\tilde{S} = (A, \alpha)^{-}$ and $\tilde{P} = A^{-}$, where the T-algebra required in the definition of $\tilde{P}X$ is the transpose of $\mathsf{T}(A^X) \times X \xrightarrow{\mathsf{st}'_{A^X, X}} \mathsf{T}(A^X \times X) \xrightarrow{\mathsf{Teval}} \mathsf{T}A \xrightarrow{\alpha} A$. We then have $S = \tilde{S}\mathsf{Free}$ and $P = U\tilde{P}$, where $\mathsf{Free} : \mathsf{Set} \to \mathsf{Alg}(\mathsf{T})$ takes X to $(\mathsf{T}X, \mu_X)$ and $U : \mathsf{Alg}(\mathsf{T}) \to \mathsf{Set}$ takes (B, β) to B:

$$\begin{tikzcd}
\mathsf{Alg}(\mathsf{T}) \arrow[d, shift left, "U"] \arrow[dr, bend left, "\tilde{S}"] & \\
\mathsf{Set} \arrow[u, shift left, "\mathsf{Free}"] \arrow[r, bend left, "S"] & \mathsf{Set}^{\mathsf{op}} \arrow[l, bend left, "P"] \arrow[ul, bend left, "\tilde{P}"]
\end{tikzcd}$$

As before, our choice of (A, α) will be either $(\mathsf{T}1, \mu_1)$ or an arbitrary $(\mathsf{T}1, \tau) \in \mathsf{Alg}(\mathsf{T})$. Irrespective of this, we can lift the functor $L : \mathsf{Set} \to \mathsf{Set}$ from Section 3 to $\tilde{L} : \mathsf{Alg}(\mathsf{T}) \to \mathsf{Alg}(\mathsf{T})$ by taking $\tilde{L} = \mathsf{Free}LU$. Then, the one-step semantics $\delta : LU\tilde{P} = LP \Rightarrow PF = U\tilde{P}F$ lifts to $\tilde{\delta} ::= \delta^\sharp : \tilde{L}\tilde{P} = \mathsf{Free}LU\tilde{P} \Rightarrow \tilde{P}F$.

There is no need for a similar lifting of the identity functor on Set with associated one-step semantics σ to $\mathsf{Alg}(\mathsf{T})$, since the components of σ are already $\mathsf{Alg}(\mathsf{T})$-homomorphisms – this follows from an equivalent definition of $\sigma_X : (\mathsf{T}1)^X \to (\mathsf{T}1)^{\mathsf{T}X}$ as the transpose of the unique extension of $\mathsf{eval} : (\mathsf{T}1)^X \times X \to \mathsf{T}1$ to a 2-*linear map*[2] $(\mathsf{T}1)^X \times \mathsf{T}X \to \mathsf{T}1$, as shown in [14, Proposition 4.1]. We therefore simply write $\tilde{\sigma} : \tilde{P} \Rightarrow \tilde{P}\mathsf{T}$ for the natural transformation whose components are given by those of $\sigma : P \Rightarrow PT$.

This yields new logics $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$ carrying T-algebra structure, and associated semantics $[\![\_]\!]_\gamma : \tilde{\mathcal{L}} \to \tilde{P}X$ and $[\![\_]\!]_\gamma^{V^\sharp} : \tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})} \to \tilde{P}X$, for each $\mathsf{T}F$-coalgebra (X, γ) and valuation $V : \mathcal{V} \to PX$ (extending to a T-algebra homomorphism $V^\sharp : \mathsf{Free}(\mathcal{V}) \to \tilde{P}X$). The syntax of these logics contains propositional operators arising from the T-algebra structure (see Example 4.4 at the end of this section for operators induced by specific monads) and modal operators $\lambda \in \Lambda$. To add fixpoints to these logics, we can now proceed as in Lemma 3.3.

▶ **Definition 4.1.** Let (X, γ) be a $\mathsf{T}F$-coalgebra, let $V : \mathcal{V} \to PX$ be a valuation, and let $\varphi \in \tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})}$. Consider the operator $\tilde{\mathsf{O}} : [\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})}, \tilde{P}X] \to [\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})}, \tilde{P}X]$ defined by $\tilde{f} \mapsto [V^\sharp, P\gamma] \circ (\mathsf{id} + (\tilde{\sigma}_{FX} \circ \tilde{\delta}_X)) \circ (\mathsf{id} + \tilde{L}\tilde{f}) \circ \tilde{\beta}_\varphi$:

$$\begin{tikzcd}
\mathsf{Free}(\mathcal{V}) + \tilde{L}\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V}} \arrow[r, "\mathsf{id}+\tilde{L}\tilde{f}"] & \mathsf{Free}(\mathcal{V}) + \tilde{L}\tilde{P}X \arrow[d, "\mathsf{id}+(\tilde{\sigma}_{FX}\circ\tilde{\delta}_X)"] \\
 & \mathsf{Free}(\mathcal{V}) + \tilde{P}\mathsf{T}FX \arrow[d, "{[V^\sharp, P\gamma]}"] \\
\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})} \arrow[uu, "\tilde{\beta}_\varphi"] \arrow[r, "\tilde{f}"] & \tilde{P}X
\end{tikzcd}$$

Then $[\![\mu x.\varphi]\!]_\gamma^{V^\sharp}$ (respectively $[\![\nu x.\varphi]\!]_\gamma^{V^\sharp}$) is defined as $\tilde{f}_0(x)$, where $\tilde{f}_0 : \tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})} \to \tilde{P}X$ is the least (respectively greatest) fixpoint of $\tilde{\mathsf{O}}$. We write $\tilde{\mathcal{L}}_\mu$ for the resulting fixpoint logic.

[2] A 2-linear map is required to preserve the T-algebra structure in the second argument, where the assumed T-algebra structures on $\mathsf{T}X$ and $\mathsf{T}1$ are the free ones (μ_X and μ_1 respectively).

Now observe that for a formula $\varphi \in \mathcal{L}^{\mathcal{V}}$, one can consider the semantics of its translation to $\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$, in addition to the semantics $[\![\varphi]\!]^{\mathcal{V}}_{\gamma}$. As expected, the two agree:

▶ **Proposition 4.2.** *Let $!_{\mathcal{V}} : (\mathcal{L}^{\mathcal{V}}, \alpha^{\mathcal{V}}) \to (U\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}, U\tilde{\alpha}^{\mathsf{Free}(\mathcal{V})} \circ \eta_{LU\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}})$ be the unique L-algebra morphism arising by freeness of $(\mathcal{L}^{\mathcal{V}}, \alpha^{\mathcal{V}})$. Then $[\![\varphi]\!]^{V}_{\gamma} = U[\![!_{\mathcal{V}}(\varphi)]\!]^{V^\sharp}_{\gamma}$ for $\varphi \in \mathcal{L}^{\mathcal{V}}$.*

Proof (sketch). The conclusion follows by freeness of $(\mathcal{L}^{\mathcal{V}}, \alpha^{\mathcal{V}})$ from the commutativity of

$$\begin{tikzcd}
L\mathcal{L}^{\mathcal{V}} \arrow[r, "L!_{\mathcal{V}}"] \arrow[dddd, "\alpha^{\mathcal{V}}"'] & LU\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})} \arrow[r, "LU[\![_]\!]^{V^\sharp}_{\gamma}"] \arrow[d, "\eta_{LU\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}}"'] & LU\tilde{P}X \arrow[r, equal] \arrow[d, "\eta_{LU\tilde{P}X}"] & LPX \arrow[dd, "\delta_X"] \\
& U\mathsf{Free}LU\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})} \arrow[r, "U\mathsf{Free}LU[\![_]\!]^{V^\sharp}_{\gamma}"] \arrow[ddd, "U\tilde{\alpha}^{\mathsf{Free}(\mathcal{V})}"'] & U\mathsf{Free}LU\tilde{P}X \arrow[d, "U\delta^\sharp_X"] & \\
& & U\tilde{P}FX \arrow[r, equal] \arrow[d, "U\tilde{\sigma}_{FX}"] & PFX \arrow[d, "\sigma_{FX}"] \\
& & U\tilde{P}\mathsf{T}FX \arrow[r, equal] \arrow[d, "U\tilde{P}\gamma"] & P\mathsf{T}FX \arrow[d, "P\gamma"] \\
\mathcal{L}^{\mathcal{V}} \arrow[r, "!_{\mathcal{V}}"] & U\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})} \arrow[r, "U[\![_]\!]^{V^\sharp}_{\gamma}"] & U\tilde{P}X \arrow[r, equal] & PX
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\eta_{\mathcal{V}}"] \arrow[rr, bend right, "V"'] & U\mathsf{Free}(\mathcal{V}) \arrow[r, "UV^\sharp"] & PX
\end{tikzcd}$$

◀

Finally, Proposition 4.2 extends to formulas in $\mathcal{L}_{\mu}$.

▶ **Proposition 4.3.** *Let $V : \mathcal{V} \to PX$, let $f_0 : \mathcal{L}^{\{x\}+\mathcal{V}} \to PX$ be the least (greatest) fixpoint of the operator O in Lemma 3.3, and let $\tilde{f}_0 : \tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})} \to \tilde{P}X$ be the least (resp. greatest) fixpoint of the operator $\tilde{\mathsf{O}}$ in Definition 4.1. Then, $\tilde{f}_0 \circ !_{\{x\}+\mathcal{V}} = f_0$.*

Proof (sketch). The conclusion follows from the fact that if $\tilde{f}$ is a least (greatest) fixpoint of $\tilde{\mathsf{O}}$, then $U\tilde{f} \circ !_{\{x\}+\mathcal{V}}$ is a least (respectively greatest) fixpoint of O. This, in turn, follows from the commutativity of the left, top and right trapezoids in the following diagram

$$\begin{tikzcd}
\mathcal{V} + L\mathcal{L}^{\{x\}+\mathcal{V}} \arrow[rrrr, "\mathsf{id}+Lf"] \arrow[dr, "{[U\iota_1,U\iota_2]\circ(\iota+L!_{\{x\}+\mathcal{V}})}"] & & & & \mathcal{V} + LPX \arrow[dl, "{[U\iota_1,U\iota_2]\circ(\iota+\eta_{LPX})}"'] \arrow[dd, "\mathsf{id}+(\sigma_{FX}\circ\delta_X)"] \\
& U(\mathsf{Free}(\mathcal{V}) + \tilde{L}\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})}) \arrow[rr, "U(\mathsf{id}+\tilde{L}\tilde{f})"] & & U(\mathsf{Free}(\mathcal{V}) + \tilde{L}\tilde{P}X) \arrow[d, "U(\mathsf{id}+(\tilde{\sigma}_{FX}\circ\tilde{\delta}_X))"] & \\
& & & U(\mathsf{Free}(\mathcal{V}) + \tilde{P}\mathsf{T}FX) \arrow[d, "{U[V^\sharp,P\gamma]}"] & \mathcal{V} + P\mathsf{T}FX \arrow[dd, "{[V,P\gamma]}"] \\
& U\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})} \arrow[rr, "U\tilde{f}"] \arrow[uu, "U\tilde{\beta}_{\varphi}"] & & U\tilde{P}X \arrow[dr, equal] & \\
\mathcal{L}^{\{x\}+\mathcal{V}} \arrow[uuuu, "\beta_{\varphi}"] \arrow[ur, "!_{\{x\}+\mathcal{V}}"] \arrow[rrrr, "f"] & & & & PX
\end{tikzcd}$$

which is equivalent to the statement that for $\varphi \in \mathcal{L}^{\{x\}+\mathcal{V}}$, the additional structure in $\tilde{\mathcal{L}}^{\mathsf{Free}(\{x\}+\mathcal{V})}$ is not needed when defining $[\![\mu X.!_{\{x\}+\mathcal{V}}(\varphi)]\!]^{V^\sharp}_{\gamma}$ and $[\![\nu X.!_{\{x\}+\mathcal{V}}(\varphi)]\!]^{V^\sharp}_{\gamma}$. ◀

▶ **Example 4.4.**

1. For $\mathsf{T} = \mathcal{P}$, $\mathsf{Alg}(\mathsf{T})$ is isomorphic to the category of join semi-lattices, and the enhanced logic contains arbitrary disjunctions. With this, one can encode a "next" modality by letting $\bigcirc\varphi ::= \bigvee_{\lambda\in\Lambda}[\lambda](\varphi,\ldots,\varphi)$. This modality turns out to be very useful, for example, the formula $\nu x.\bigcirc x$ is true in a state of a $\mathcal{P}\circ F$-coalgebra if there exists a maximal trace from that state. For $F = A\times \mathsf{Id}$ and $\bigcirc$ as above, the formula $[a](\nu x.\bigcirc x)$ is true in a state if there exists a maximal (hence infinite) trace from that state that starts with an a. (Recall that our logics do not contain a propositional constant $\top$, and therefore partial traces cannot be formalised without using fixpoint operators.) For $F = A\times\mathsf{Id}$ and $[\overline{a}]\varphi ::= \bigvee_{b\in A\setminus\{a\}}[b]\varphi$ for $a\in A$, the formula $\nu x.\mu y.([a]x\vee[\overline{a}]y)$ is true in a state if there exists a maximal trace from that state containing an infinite number of as. Finally, for $F = 1 + A\times\mathsf{Id}$ and $[A]\varphi ::= \bigvee_{a\in A}[a]\varphi$, the formula $\mu x.(*\vee[A]x)$ holds in a state if there exists a *finite* maximal trace from that state.
2. For $\mathsf{T} = \mathcal{S}$, $\mathsf{Alg}(\mathsf{T})$ is isomorphic to the category of positive convex algebras, and the enhanced logic contains sub-convex combinations of formulas. With this, one can encode properties where preference is given to one observable linear time behaviour over another. For $F = 1 + A\times\mathsf{Id}$ and $[A]$ as above, the formula $\mu x.(\frac{1}{2} * + \frac{1}{4}[A]x)$ measures the likelihood of termination, in such a way that the smaller the number of steps required for termination, the higher the value associated to the formula by the semantics.
3. For $\mathsf{T} = \mathsf{T}_S$ with $S = (S,+,0,\bullet,1)$ a commutative semiring, $\mathsf{Alg}(\mathsf{T})$ is isomorphic to the category of modules over S, and the enhanced logic contains finite linear combinations of formulas. As in the previous case, the resulting logic supports weighted choices.

5 Path-based Semantics for Coalgebraic Linear Time Logics

This section provides alternative path-based semantics for what we call the *uniform fragments* of the logics $\mathcal{L}^{\mathcal{V}}$ and $\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$, and proves their equivalence to the already-defined step-wise semantics. The main results (Theorems 5.13 and 5.24) assume canonical choices for both the branching and the linear time modalities, but generalisations to non-canonical choices (subject to additional requirements) are also discussed.

▶ **Definition 5.1.** The *uniform fragment* $u\mathcal{L}^{\mathcal{V}}$ of the logic $\mathcal{L}^{\mathcal{V}}$ is given by $\bigcup\limits_{n\in\omega}\mathcal{L}^{\mathcal{V}}_n$, with $\mathcal{L}^{\mathcal{V}}_n = L^n\mathcal{V}$ consisting of formulas of rank n, for $n\in\omega$.

The uniform fragment $u\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$ of $\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$ is defined similarly, namely by $u\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})} ::= \bigcup\limits_{n\in\omega}\tilde{L}^n\mathsf{Free}(\mathcal{V})$.

A more concrete description of the set $\mathcal{L}^{\mathcal{V}}_n$ is as the set of formulas with nesting depth of modal operators at most n, and with each occurrence of a variable being in the scope of exactly n modal operators.

▶ **Example 5.2.** For $L:\mathsf{Set}\to\mathsf{Set}$ of the form $LX = \coprod_{\lambda\in\Lambda}X^{\mathsf{ar}(\lambda)}$, and for $\lambda_i\in\Lambda$ a modality of arity i, with $i\in\{0,1,2\}$, $[\lambda_2]([\lambda_1]X,[\lambda_0])$, $[\lambda_1]X\vee[\lambda_1]Y\vee[\lambda_0]$ and $[\lambda_1][\lambda_1]X\vee[\lambda_0]$ are uniform modal formulas, whereas $[\lambda_2]([\lambda_1]X,[\lambda_1][\lambda_0])$ and $[\lambda_1]X\vee[\lambda_1][\lambda_1]Y$ are not (where we assume $\mathsf{T}=\mathcal{P}$, and therefore $\vee$ is a propositional operator of the enhanced logic).

▶ Remark 5.3. When $\mathcal{V}=\emptyset$, $u\mathcal{L}^{\mathcal{V}}$ coincides with the full logic $\mathcal{L}^{\mathcal{V}}$. However, when $\mathcal{V}\neq\emptyset$, the inclusion $u\mathcal{L}^{\mathcal{V}}\subseteq\mathcal{L}^{\mathcal{V}}$ is strict. All the example formulas in this paper (e.g. all the modal formulas used to define the fixpoint formulas in Example 4.4) are uniform ones. Moreover, most modal formulas used in practice to define fixpoint formulas appear to belong to the uniform fragment.

5.1 Path-based Semantics for $u\mathcal{L}^{\mathcal{V}}$

For each polynomial endofunctor F and commutative monad T, one can define a canonical distributive law of T over F as shown below. This can be used to give a path-based semantics for the uniform fragment of the logic $\mathcal{L}_{\mathcal{V}}$, by delaying the use of the natural transformation σ when defining the interpretation of formulas in $u\mathcal{L}^{\mathcal{V}}$ for as long as possible.

▶ **Definition 5.4.** For $F = \coprod_{\lambda\in\Lambda} X^{\mathsf{ar}(\lambda)}$ the *canonical distributive law* $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ is given by

$$\begin{tikzcd}
(\mathsf{T}X)^{\mathsf{ar}(\lambda)} \arrow[r, "\mathsf{dst}_{\mathsf{ar}(\lambda)}"] \arrow[d, "\iota_\lambda"'] & \mathsf{T}(X^{\mathsf{ar}(\lambda)}) \arrow[r, "\mathsf{T}\iota_\lambda"] & \mathsf{T}(\coprod_{\lambda\in\Lambda} X^{\mathsf{ar}(\lambda)}) = \mathsf{T}FX \\
F\mathsf{T}X = \coprod_{\lambda\in\Lambda}(\mathsf{T}X)^{\mathsf{ar}(\lambda)} \arrow[urr, dashed, "\lambda_X"'] & &
\end{tikzcd}$$

where $\mathsf{dst}_n : (\mathsf{T}X)^n \to \mathsf{T}(X^n)$ is either $\eta_1 : 1 \to \mathsf{T}1$ (if $n = 0$), the identity map (if $n = 1$), or defined in the obvious way from the double strength of the monad T (if $n \geq 2$).

Given a $\mathsf{T}F$-coalgebra (X, γ), unfolding the coalgebra map $n \geq 1$ times yields a map $(\mathsf{T}F)^{n-1}\gamma \circ \ldots \circ \gamma : X \to (\mathsf{T}F)^n X$. Alternatively, one can use the distributive law λ to flatten, at each unfolding step, the branching arising from the presence of T in the coalgebra type:

▶ **Definition 5.5.** For a $\mathsf{T}F$-coalgebra (X, γ) and $n \geq 1$, let $\gamma_n : X \to \mathsf{T}F^n X$ be given by

- $\gamma_1 = \gamma$
- $\gamma_{n+1} = \mu_{F^{n+1}X} \circ \mathsf{T}\lambda_{F^n X} \circ \mathsf{T}F\gamma_n \circ \gamma$:
$$X \xrightarrow{\gamma} \mathsf{T}FX \xrightarrow{\mathsf{T}F\gamma_n} \mathsf{T}F\mathsf{T}F^n X \xrightarrow{\mathsf{T}\lambda_{F^n X}} \mathsf{T}^2F^{n+1}X \xrightarrow{\mu_{F^{n+1}X}} \mathsf{T}F^{n+1}X$$

In order to relate the maps $(\mathsf{T}F)^{n-1}\gamma \circ \ldots \circ \gamma$ and γ_n, with $n \geq 1$, we note that any distributive law $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ yields natural transformations $(\mathsf{T}F)^n \Rightarrow \mathsf{T}F^n$:

▶ **Definition 5.6.** Let $\lambda_n : (\mathsf{T}F)^n \Rightarrow \mathsf{T}F^n$ for $n \geq 1$ be defined inductively by:

- $\lambda_1 = \mathsf{id}$
- $\lambda_{n+1} = \mu_{F^{n+1}} \circ \mathsf{T}\lambda_{F^n} \circ \mathsf{T}F\lambda_n$ for $n \geq 1$:
$$\mathsf{T}F(\mathsf{T}F)^n \xRightarrow{\mathsf{T}F\lambda_n} \mathsf{T}F\mathsf{T}F^n \xRightarrow{\mathsf{T}\lambda_{F^n}} \mathsf{T}^2F^{n+1} \xRightarrow{\mu_{F^{n+1}}} \mathsf{T}F^{n+1}$$

We can now state the following:

▶ **Lemma 5.7.** *For $n \geq 1$, we have $\gamma_n = \lambda_n \circ (\mathsf{T}F)^{n-1}\gamma \circ \ldots \circ \gamma$.*

Proof. Induction on n. The base case is trivial. The inductive step follows from the commutativity of

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[d, "\gamma"] & \mathsf{T}FX \arrow[r, "\mathsf{T}F\gamma_n"] & \mathsf{T}F\mathsf{T}F^n X \arrow[r, "\mathsf{T}\lambda_{F^n X}"] & \mathsf{T}^2F^{n+1}X \arrow[r, "\mu_{F^{n+1}}"] & \mathsf{T}F^{n+1}X \\
\mathsf{T}FX \arrow[rr, "\mathsf{T}F((\mathsf{T}F)^{n-1}\gamma\circ\ldots\circ\gamma)"] & & (\mathsf{T}F)^{n+1}X \arrow[u, "\mathsf{T}F\lambda_n"] \arrow[urr, "\lambda_{n+1}"'] & &
\end{tikzcd}$$

which, in turn, follows by the induction hypothesis and the definition of λ_{n+1}. ◀

We are finally ready to define an alternative semantics for $u\mathcal{L}^{\mathcal{V}}$. For this, recall that $u\mathcal{L}^{\mathcal{V}} = \bigcup_{n\in\omega} \mathcal{L}^{\mathcal{V}}_n$ with $\mathcal{L}^{\mathcal{V}}_n = L^n\mathcal{V}$ for $n \in \omega$.

▶ **Definition 5.8** (Path-based Semantics for $u\mathcal{L}^{\mathcal{V}}$)**.** Let $\varphi \in \mathcal{L}_n^{\mathcal{V}}$, let (X, γ) be a $\mathsf{T}F$-coalgebra, and let $V : \mathcal{V} \to PX$ be a valuation. Define $(\!|\varphi|\!)_\gamma^V$ as the image of φ under the composition

$$L^n\mathcal{V} \xrightarrow{L^nV} L^nPX \xrightarrow{(\delta_n)_X} PF^nX \xrightarrow{\sigma_{F^nX}} P\mathsf{T}F^nX \xrightarrow{P\gamma_n} PX$$

where $\delta_n : L^nP \Rightarrow PF^n$ performs n successive applications of δ:

- $\delta_1 = \delta$
- $\delta_{n+1} = \delta_{F^n} \circ L\delta_n$ for $n \geq 1$:
  $$L^{n+1}P \xRightarrow{L\delta_n} LPF^n \xRightarrow{\delta_{F^n}} PF^{n+1}$$

Thus, in the path-based semantics, in order to interpret a formula of rank n, the n-step behaviour of a state in a $\mathsf{T}F$-coalgebra is flattened into branches of n-step F-behaviours (using γ_n), and this results in the natural transformation σ (or equivalently, the extension lifting ext) only being used once, rather than at each unfolding of the coalgebra map.

Next, we show that the step-wise and path-based semantics for $u\mathcal{L}^{\mathcal{V}}$ are equivalent. For this, we need the following inductive formulation of the step-wise semantics for the uniform fragment $u\mathcal{L}^{\mathcal{V}}$:

▶ **Definition 5.9.** Let $V : \mathcal{V} \to PX$ be a valuation. For $n \geq 1$, let $\xi^n : L^n\mathcal{V} \to P(\mathsf{T}F)^nX$ be defined by:

- $\xi_1 = \sigma_{FX} \circ \delta_X \circ LV$:
  $$L\mathcal{V} \xrightarrow{LV} LPX \xrightarrow{\delta_X} PFX \xrightarrow{\sigma_{FX}} P\mathsf{T}FX$$
- $\xi_{n+1} = \sigma_{F(\mathsf{T}F)^nX} \circ \delta_{(\mathsf{T}F)^nX} \circ L\xi_n$ for $n \geq 1$:
  $$L^{n+1}\mathcal{V} \xrightarrow{L\xi_n} LP(\mathsf{T}F)^nX \xrightarrow{\delta_{(\mathsf{T}F)^nX}} PF(\mathsf{T}F)^nX \xrightarrow{\sigma_{F(\mathsf{T}F)^nX}} P(\mathsf{T}F)^{n+1}X$$

▶ **Lemma 5.10.** *For formulas in $\mathcal{L}_n^{\mathcal{V}}$, the step-wise semantics is obtained by post-composing ξ_n with $P\gamma \circ \ldots \circ P(\mathsf{T}F)^{n-1}\gamma : P(\mathsf{T}F)^nX \to PX$.*

Proof. Immediate from $\mathcal{L}_n^{\mathcal{V}} = L^n\mathcal{V}$. ◀

The last ingredient required for the proof of equivalence of the two semantics is the following key lemma, which allows us to move from alternating the use of the natural transformations σ and δ (as is done in the step-wise semantics) to only using the natural transformation σ once (as is done in the path-based semantics).

Since later in the paper we discuss other choices for σ, obtained by replacing $(\mathsf{T}1, \mu_1)$ with an arbitrary T-algebra $(\mathsf{T}1, \tau)$, most of the proof of the lemma uses τ instead of μ_1.

▶ **Lemma 5.11.** *Let $\delta : LP \Rightarrow PF$ and $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ be as in Definitions 3.2 and 5.4, respectively, and let $\sigma : P \Rightarrow P\mathsf{T}$ be the natural transformation induced by $\tau := \mu_1 : \mathsf{T}^21 \to \mathsf{T}1$, given by (2). Then the following diagram commutes:*

$$\begin{tikzcd} LP \arrow[r, Rightarrow, "L\sigma"] \arrow[d, Rightarrow, "\delta"'] & LP\mathsf{T} \arrow[r, Rightarrow, "\delta_{\mathsf{T}}"] & PF\mathsf{T} \\ PF \arrow[rr, Rightarrow, "\sigma_F"] & & P\mathsf{T}F \arrow[u, Rightarrow, "P\lambda"'] \end{tikzcd} \tag{3}$$

Proof. The statement follows by expanding the corresponding definitions of δ and σ:

Given $(p_i) \in \iota_\lambda(PX)^n$ (with $n = \mathsf{ar}(\lambda) \geq 2$), we have:

$$\begin{pmatrix} X \\ \downarrow p_i \\ \mathsf{T}1 \end{pmatrix} \overset{L\sigma_X}{\longmapsto} \begin{pmatrix} \mathsf{T}X \\ \downarrow \mathsf{T}p_i \\ \mathsf{T}^21 \\ \downarrow \tau \\ \mathsf{T}1 \end{pmatrix} \overset{\delta_{\mathsf{T}X}}{\longmapsto} \begin{matrix} \coprod_{\lambda\in\Lambda}(\mathsf{T}X)^{\mathsf{ar}(\lambda)} \\ \downarrow [0,\ldots,\bullet(\tau\circ\mathsf{T}p_i),\ldots,0] \\ \mathsf{T}1 \end{matrix}$$

and

$$\left(\begin{array}{c} X \\ \big\downarrow{\scriptstyle p_i} \\ \mathsf{T}1 \end{array}\right) \xmapsto{\delta_X} \begin{array}{c} \coprod_{\lambda\in\Lambda} X^{\mathsf{ar}(\lambda)} \\ \big\downarrow{\scriptstyle [0,\ldots,\bullet p_i,\ldots,0]} \\ \mathsf{T}1 \end{array} \xmapsto{\sigma_{FX}} \begin{array}{c} \mathsf{T}\coprod_{\lambda\in\Lambda} X^{\mathsf{ar}(\lambda)} \\ \big\downarrow{\scriptstyle \mathsf{T}[0,\ldots,\bullet p_i,\ldots,0]} \\ \mathsf{T}^2 1 \\ \big\downarrow{\scriptstyle \tau} \\ \mathsf{T}1 \end{array} \xmapsto{P\lambda} \begin{array}{c} \coprod_{\lambda\in\Lambda}(\mathsf{T}X^{\mathsf{ar}(\lambda)}) \\ \big\downarrow{\scriptstyle +_{\lambda\in\Lambda}\mathsf{dst}_{\mathsf{ar}(\lambda)}} \\ \coprod_{\lambda\in\Lambda}\mathsf{T}(X^{\mathsf{ar}(\lambda)}) \\ \big\downarrow{\scriptstyle [\mathsf{T}\iota_1,\ldots,\mathsf{T}\iota_n]} \\ \mathsf{T}\coprod_{\lambda\in\Lambda} X^{\mathsf{ar}(\lambda)} \\ \big\downarrow{\scriptstyle \mathsf{T}[0,\ldots,\bullet p_i,\ldots,0]} \\ \mathsf{T}^2 1 \\ \big\downarrow{\scriptstyle \tau} \\ \mathsf{T}1 \end{array}$$

where for $q_i : X \to \mathsf{T}1$ with $i \in \{1,\ldots,n\}$, $\bullet(p_i) : X^n \to \mathsf{T}1$ takes $(x_1,\ldots,x_n)$ to $p_1(x_1)\bullet \ldots \bullet p_n(x_n)$. Thus, the commutativity of (3) amounts to the commutativity of

$$\begin{tikzcd} \mathsf{T}X \times \mathsf{T}X \arrow[r, "\mathsf{T}p_1\times\mathsf{T}p_2"] \arrow[d, "\mathsf{dst}_{X,X}"'] & \mathsf{T}^2 1 \times \mathsf{T}^2 1 \arrow[rr, "\tau\times\tau"] \arrow[d, "\mathsf{dst}_{\mathsf{T}1,\mathsf{T}1}"'] & & \mathsf{T}1 \times \mathsf{T}1 \arrow[d, "\bullet"] \\ \mathsf{T}(X\times X) \arrow[r, "\mathsf{T}(p_1\times p_2)"] & \mathsf{T}(\mathsf{T}1\times\mathsf{T}1) \arrow[r, "\mathsf{T}\bullet"] & \mathsf{T}^2 1 \arrow[r, "\tau"] & \mathsf{T}1 \end{tikzcd} \tag{4}$$

where for simplicity we assume $n = 2$. For $\tau = \mu_1$, the latter follows easily by naturality of dst (left rectangle) and exploiting the equivalent definition of $\bullet$ as $\mathsf{T}\pi_1 \circ \mathsf{dst}_{\mathsf{T}1,\mathsf{T}1}$, as given in [1] (right rectangle). The proof in the case when $\mathsf{ar}(\lambda) = 1$ is trivial, whereas the proof in the case when $\mathsf{ar}(\lambda) = 0$ uses the fact that $(\mathsf{T}1, \tau)$ is a T-algebra (and hence $\tau \circ \mathsf{T}\eta_1 = \mathsf{id}$). ◀

▶ Remark 5.12. The commutativity of (4) in the proof of Lemma 5.11 relies on the well-behavedness of μ_1 w.r.t. the double strength map. Replacing $\mu_1 : \mathsf{T}^2 1 \to \mathsf{T}1$ by an arbitrary T-algebra structure $\tau : \mathsf{T}^2 1 \to \mathsf{T}1$ will not, in general, make this diagram commute. For $\mathsf{T} = \mathcal{P}$, an example is the $\Box$-modality $\tau_\Box : \mathsf{T}^2 1 \to \mathsf{T}1$, defined from the $\Diamond$-modality μ_1 via the swap map $\mathsf{swap} : \mathcal{P}1 \to \mathcal{P}1$: $\tau = \mathsf{swap} \circ \mu_1 \circ \mathsf{T}\mathsf{swap}$; an easy calculation shows that commutativity of the previously mentioned diagram fails in this case.

Using Lemma 5.11, we can now state and prove the announced equivalence result.

▶ **Theorem 5.13.** *Let $\delta : LP \Rightarrow PF$, $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ and $\sigma : P \Rightarrow P\mathsf{T}$ be as in Lemma 5.11. Also, let (X,γ) be a $\mathsf{T}F$-coalgebra and let $V : \mathcal{V} \to PX$ be a valuation. For $\varphi \in u\mathcal{L}^{\mathcal{V}}$, $[\![\varphi]\!]^V_\gamma = (\!|\varphi|\!)^V_\gamma$.*

Proof. Since $u\mathcal{L}^{\mathcal{V}} = \bigcup_{n\in\omega} L^n\mathcal{V}$, the claim will follow from the commutativity of:

$$\begin{tikzcd} L^n\mathcal{V} \arrow[rr, "\xi_n"] \arrow[d, "L^nV"'] & & P(\mathsf{T}F)^nX \arrow[r, "P(\mathsf{T}F)^{n-1}\gamma\circ\ldots\circ\gamma"] & PX \\ L^nPX \arrow[r, "(\delta_n)_X"'] & PF^nX \arrow[r, "\sigma_{F^nX}"'] & P\mathsf{T}F^nX \arrow[u, "P(\lambda_n)_X"'] \arrow[ur, "P\gamma_n"'] & \end{tikzcd}$$

where the right triangle commutes by Lemma 5.7, and the commutativity of the left rectangle is proved below by induction on n.

The case $n = 1$ is trivial. The inductive step follows from the commutativity of

$$\begin{tikzcd}
L^{n+1}\mathcal{V} \arrow[rr, "L\xi_n"] \arrow[d, "L^{n+1}V"'] & & LP(\mathsf{T}F)^n X \arrow[r, "\delta_{(\mathsf{T}F)^n X}"] & PF(\mathsf{T}F)^n X \arrow[r, "\sigma_{F(\mathsf{T}F)^n X}"] & P(\mathsf{T}F)^{n+1} X \\
L^{n+1}PX \arrow[r, "L(\delta_n)_X"] \arrow[dr, "(\delta_{n+1})_X"'] & LPF^n X \arrow[r, "L\sigma_{F^n X}"] \arrow[d, "\delta_{F^n X}"] & LP\mathsf{T}F^n X \arrow[r, "\delta_{\mathsf{T}F^n X}"] \arrow[u, "LP(\lambda_n)_X"'] & PF\mathsf{T}F^n X \arrow[r, "\sigma_{F\mathsf{T}F^n}"] \arrow[u, "PF(\lambda_n)_X"'] & P\mathsf{T}F\mathsf{T}F^n X \arrow[u, "P\mathsf{T}F(\lambda_n)_X"'] \\
 & PF^{n+1} X \arrow[rr, "\sigma_{F^{n+1} X}"'] & & P\mathsf{T}F^{n+1} X \arrow[r, "\sigma_{\mathsf{T}F^{n+1} X}"'] \arrow[u, "P\lambda_{F^n X}"'] & P\mathsf{T}^2 F^{n+1} X \arrow[u, "P\mathsf{T}\lambda_{F^n X}"']
\end{tikzcd}$$

where the top arrow is ξ_{n+1}, the top-left rectangle commutes by the induction hypothesis, the top-middle rectangle commutes by naturality of δ, the bottom-left triangle is the definition of δ_{n+1}, the bottom-middle rectangle commutes by Lemma 5.11, the top-right and bottom-right rectangles commute by naturality of σ, and finally the long arrow from $L^{n+1}\mathcal{V}$ to $P(\mathsf{T}F)^{n+1}X$ is $P\lambda_{n+1} \circ \sigma_{F^{n+1}X} \circ (\delta_{n+1})_X \circ L^{n+1}V$ as required – the latter follows from:

$$\begin{aligned}
P\mathsf{T}F\lambda_n \circ P\mathsf{T}\lambda_{F^n} \circ \sigma_{\mathsf{T}F^{n+1}} \circ \sigma_{F^{n+1}} \quad &= \quad \text{(Lemma 3.1)} \\
P\mathsf{T}F\lambda_n \circ P\mathsf{T}\lambda_{F^n} \circ P\mu_{F^{n+1}} \circ \sigma_{F^{n+1}} \quad &= \quad \text{(Definition 5.6)} \\
P\lambda_{n+1} \circ \sigma_{F^{n+1}} &
\end{aligned}$$

This concludes the proof. ◀

The next example confirms that by using $\tau_\Box$ instead of μ_1 to resolve branching for $\mathsf{T} = \mathcal{P}$, Theorem 5.13 does not hold for functors F with associated linear time modalities of arity 2 or greater.

▶ **Example 5.14.** Assume that $\tau_\Box$ is used to resolve branching, and consider the following $\mathcal{P}(1 + A \times \mathsf{Id} \times \mathsf{Id})$-coalgebra (X, γ):

$$\begin{tikzcd}
 & & x_1 \arrow[r, squiggly] & b \arrow[r] \arrow[dr] & x_3 \arrow[r, squiggly] & * \\
x_0 \arrow[r, squiggly] & a \arrow[r] \arrow[ur] & x_2 & & x_4 \arrow[r, squiggly] & *
\end{tikzcd}$$

where $\rightsquigarrow$ is used for nondeterministic transitions (and thus x_2 is a deadlock state). Under the step-wise semantics, the formula $[a](*, *)$ does not hold in x_0, as although x_2 satisfies $*$ (since it has no outgoing transitions), x_1 does not: according to the definition, for a state to satisfy $*$, all transitions from that state (if any) must be terminating ones. However, the map $\gamma_2 : X \to \mathcal{P}(1 + A \times (1 + A \times X) \times (1 + A \times X))$ maps x_0 to the empty set: again, this is because x_2 has no transitions and therefore the flattening performed by γ_2 results in an empty set of linear time behaviours of depth 2; as a result, under the path-based semantics, the formula holds.

In spite of the above, a generalisation of Theorem 5.13 to the case when $\sigma : P \Rightarrow P\mathsf{T}$ arises from an arbitrary T-algebra $(\mathsf{T}1, \tau)$ can be formulated, as suggested by the next example.

▶ **Example 5.15.** The case $\mathsf{T}' = \mathcal{P}^+$ with $\mathcal{P}_+ : \mathsf{Set} \to \mathsf{Set}$ cannot be directly covered by our approach, since in this case $\mathsf{T}'\emptyset \neq 1$. However, any $\mathsf{T}'F$-coalgebra can be viewed as a $\mathsf{T}F$-coalgebra with $\mathsf{T} = \mathcal{P}$, and for $\mathsf{T}F$-coalgebras arising in this way, the proof of Theorem 5.13 does generalise, as it only requires the following to commute (instead of (3)):

$$\begin{tikzcd}
LP \arrow[r, Rightarrow, "L\sigma"] \arrow[d, Rightarrow, "\delta"'] & LP\mathsf{T}' \arrow[r, Rightarrow, "\delta_{\mathsf{T}'}"] & PF\mathsf{T}' \\
PF \arrow[rr, Rightarrow, "\sigma_F"] & & P\mathsf{T}'F \arrow[u, Rightarrow, "P\lambda"']
\end{tikzcd}$$

Using the same reasoning as in Lemma 5.11, the above follows from the commutativity of

$$\begin{tikzcd}
\mathsf{T}'X \times \mathsf{T}'X \arrow[r, "\iota_X \times \iota_X"] \arrow[d, "\mathsf{dst}'_{X,X}"'] & \mathsf{T}X \times \mathsf{T}X \arrow[r, "\mathsf{T}p_1 \times \mathsf{T}p_2"] \arrow[d, "\mathsf{dst}_{X,X}"'] & \mathsf{T}^2 1 \times \mathsf{T}^2 1 \arrow[rr, "\tau \times \tau"] \arrow[d, "\mathsf{dst}_{\mathsf{T}1,\mathsf{T}1}"'] & & \mathsf{T}1 \times \mathsf{T}1 \arrow[d, "\bullet"] \\
\mathsf{T}'(X \times X) \arrow[r, "\iota_{X \times X}"] & \mathsf{T}(X \times X) \arrow[r, "\mathsf{T}(p_1 \times p_2)"] & \mathsf{T}(\mathsf{T}1 \times \mathsf{T}1) \arrow[r, "\mathsf{T}\bullet"] & \mathsf{T}^2 1 \arrow[r, "\tau"] & \mathsf{T}1
\end{tikzcd}$$

where $\iota : \mathsf{T}' \Rightarrow \mathsf{T}$ is the inclusion. Thus, commutativity of (4) in the proof of Lemma 5.11 can be replaced by the commutativity of the outer diagram below:

$$\begin{tikzcd}
\mathsf{T}'\mathsf{T}1 \times \mathsf{T}'\mathsf{T}1 \arrow[r, "\iota_{\mathsf{T}1} \times \iota_{\mathsf{T}1}"] \arrow[d, "\mathsf{dst}'_{\mathsf{T}1,\mathsf{T}1}"'] & \mathsf{T}^2 1 \times \mathsf{T}^2 1 \arrow[rr, "\tau \times \tau"] \arrow[d, "\mathsf{dst}_{\mathsf{T}1,\mathsf{T}1}"'] & & \mathsf{T}1 \times \mathsf{T}1 \arrow[d, "\bullet"] \\
\mathsf{T}'(\mathsf{T}1 \times \mathsf{T}1) \arrow[r, "\iota_{\mathsf{T}1 \times \mathsf{T}1}"'] & \mathsf{T}(\mathsf{T}1 \times \mathsf{T}1) \arrow[r, "\mathsf{T}\bullet"] & \mathsf{T}^2 1 \arrow[r, "\tau"] & \mathsf{T}1
\end{tikzcd} \tag{5}$$

which states that the right rectangle in (4) commutes on the sub-domain $\mathsf{T}'\mathsf{T}1 \times \mathsf{T}'\mathsf{T}1$ of $\mathsf{T}^2 1 \times \mathsf{T}^2 1$. An easy calculation shows that this holds for $\tau_\Box$.

The argument in the previous example can be captured in a more general result on the equivalence between the path-based and the step-wise semantics for $u\mathcal{L}^{\mathcal{V}}$.

▶ **Theorem 5.16.** *Let T' be a commutative sub-monad of the monad T, let $\delta : LP \Rightarrow PF$ and $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ be as in Definitions 3.2 and 5.4, respectively, and let $\sigma : P \Rightarrow P\mathsf{T}$ be induced by a choice of $\tau : \mathsf{T}^2 1 \to \mathsf{T}1$ that makes the outer diagram in (5) commute. Then for a $\mathsf{T}'F$-coalgebra (X, γ) (viewed as a $\mathsf{T}F$-coalgebra), a valuation $V : \mathcal{V} \to PX$ and a formula $\varphi \in u\mathcal{L}^{\mathcal{V}}$, $[\![\varphi]\!]^V_\gamma = (\!|\varphi|\!)^V_\gamma$.*

In particular, Theorem 5.16 applies when $\mathsf{T}' = \mathsf{T}$ and $\tau : \mathsf{T}^2 1 \to \mathsf{T}1$ is such that the right rectangle in (5) commutes.

Finally, we note that the proof of Theorem 5.13 only makes use of the specific (canonical) choice of linear time modalities when it comes to applying Lemma 5.11. As a result, a generalisation of Theorem 5.13 to an arbitrary choice of linear time modalities can also be stated.

▶ **Theorem 5.17.** *Let $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ be as in Definition 5.4, and let $L : \mathsf{Set} \to \mathsf{Set}$, $\delta : LP \Rightarrow PF$ and $\sigma : P \to P\mathsf{T}$ (induced by $\tau : \mathsf{T}^2 1 \to \mathsf{T}1$) be such that Lemma 5.11 holds. Then the path-based and the step-wise semantics of $u\mathcal{L}^{\mathcal{V}}$ coincide.*

▶ **Example 5.18.** Modalities incorporating restricted disjunctions, as used e.g. in [2], can easily be added. For example, when $F = A \times \mathsf{Id} \simeq \coprod_{a \in A} \mathsf{Id}$, one can consider additional (binary) modalities of the form $[a]\_ \sqcup [b]\_$ with $a \neq b \in A$, with the obvious one-step interpretation:

$$\delta_X([a]p \sqcup [b]q)(\iota_c(x)) = \begin{cases} p(x), & \text{if } c = a \\ q(x), & \text{if } c = b \\ 0, & \text{otherwise} \end{cases}$$

This generalises to any polynomial functor F and similar disjunction-like modalities.

5.2 Path-based Semantics for $u\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$

Giving a path-based semantics for $u\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$ can be done in much the same way as for $u\mathcal{L}^{\mathcal{V}}$, since the logic functor used to deal with branching is still the identity functor (now on $\mathsf{Alg}(\mathsf{T})$). For completeness, this section sketches the main definitions and results, all very similar to their counterparts in Section 5.1.

▶ **Definition 5.19.** Let $\tilde{\delta}_n : \tilde{L}^n\tilde{P} \Rightarrow \tilde{P}F^n$ be given by:

- $\tilde{\delta}_1 = \tilde{\delta}$
- $\tilde{\delta}_{n+1} = \tilde{\delta}_{F^n} \circ \tilde{L}\tilde{\delta}_n$ for $n \geq 1$.

▶ **Definition 5.20** (Path-based Semantics for $u\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$)**.** Let $\varphi \in \tilde{\mathcal{L}}_n^{\mathsf{Free}(\mathcal{V})}$, let (X,γ) be a $\mathsf{T}F$-coalgebra, and let $V : \mathcal{V} \to PX$ be a valuation. Define $(\!|\varphi|\!)_\gamma^{V^\sharp}$ as the image of φ under the composition

$$\tilde{L}^n\mathsf{Free}(\mathcal{V}) \xrightarrow{\tilde{L}^n V^\sharp} \tilde{L}^n\tilde{P}X \xrightarrow{(\tilde{\delta}_n)_X} \tilde{P}F^nX \xrightarrow{\tilde{\sigma}_{F^nX}} \tilde{P}\mathsf{T}F^nX \xrightarrow{\tilde{P}\gamma_n} \tilde{P}X$$

▶ **Definition 5.21.** Let $V : \mathcal{V} \to PX$ be a valuation. For $n \geq 1$, let $\tilde{\xi}^n : \tilde{L}^n\mathsf{Free}(\mathcal{V}) \to \tilde{P}(\mathsf{T}F)^nX$ be defined by:

- $\tilde{\xi}_1 = \tilde{\sigma}_{FX} \circ \tilde{\delta}_X \circ \tilde{L}V^\sharp$,
- $\tilde{\xi}_{n+1} = \tilde{\sigma}_{F(\mathsf{T}F)^nX} \circ \tilde{\delta}_{(\mathsf{T}F)^nX} \circ \tilde{L}\tilde{\xi}_n$ for $n \geq 1$.

▶ **Lemma 5.22.** *For formulas in $\tilde{\mathcal{L}}_n^{\mathsf{Free}(\mathcal{V})}$, the step-wise semantics is obtained by post-composing $\tilde{\xi}_n$ with $\tilde{P}\gamma \circ \ldots \circ \tilde{P}(\mathsf{T}F)^{n-1}\gamma : \tilde{P}(\mathsf{T}F)^nX \to PX$.*

The next lemma allows Theorem 5.13 to be lifted to the logics $\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$.

▶ **Lemma 5.23.** *Let $\delta : LP \Rightarrow PF$, $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ and $\sigma : P \Rightarrow P\mathsf{T}$ be as in Lemma 5.11, and let $\tilde{\delta} : \tilde{L}\tilde{P} \Rightarrow \tilde{P}F$ and $\tilde{\sigma} : \tilde{P} \Rightarrow \tilde{P}\mathsf{T}$ arise from δ and σ as before. Then the following diagram commutes:*

$$\begin{tikzcd}
\tilde{L}\tilde{P} \arrow[r, Rightarrow, "\tilde{L}\tilde{\sigma}"] \arrow[d, Rightarrow, "\tilde{\delta}"'] & \tilde{L}\tilde{P}\mathsf{T} \arrow[r, Rightarrow, "\tilde{\delta}_{\mathsf{T}}"] & \tilde{P}F\mathsf{T} \\
\tilde{P}F \arrow[rr, Rightarrow, "\tilde{\sigma}_F"'] & & \tilde{P}\mathsf{T}F \arrow[u, Rightarrow, "\tilde{P}\lambda"']
\end{tikzcd} \tag{6}$$

Proof. By freeness of $\tilde{L}\tilde{P}$, it suffices to show that pre-composing the image under U of the above diagram with $\eta_{LU\tilde{P}}$ commutes in Set:

$$\begin{tikzcd}
LP \arrow[r, equal] \arrow[dd, Rightarrow, "\delta"'] & LU\tilde{P} \arrow[rr, Rightarrow, "LU\tilde{\sigma}=L\sigma"] \arrow[d, Rightarrow, "\eta_{LU\tilde{P}}"'] & & LU\tilde{P}\mathsf{T} \arrow[d, Rightarrow, "\eta_{LU\tilde{P}\mathsf{T}}"'] \arrow[dr, Rightarrow, "\delta_{\mathsf{T}}"] & \\
& U\mathsf{Free}LU\tilde{P} \arrow[rr, Rightarrow, "U\mathsf{Free}LU\tilde{\sigma}"] \arrow[d, Rightarrow, "U\tilde{\delta}"'] & & U\mathsf{Free}LU\tilde{P}\mathsf{T} \arrow[r, Rightarrow, "U\tilde{\delta}_{\mathsf{T}}"] & U\tilde{P}F\mathsf{T} \\
PF \arrow[r, equal] & U\tilde{P}F \arrow[rrr, Rightarrow, "U\tilde{\sigma}_F=\sigma_F"'] & & & U\tilde{P}\mathsf{T}F \arrow[u, Rightarrow, "U\tilde{P}\lambda=P\lambda"']
\end{tikzcd}$$

This, in turn, is a direct consequence of Lemma 5.11. ◀

▶ **Theorem 5.24.** *Let $\delta : LP \Rightarrow PF$, $\lambda : F\mathsf{T} \Rightarrow \mathsf{T}F$ and $\sigma : P \Rightarrow P\mathsf{T}$ be as in Lemma 5.23. Also, let (X,γ) be a $\mathsf{T}F$-coalgebra and let $V : \mathcal{V} \to PX$ be a valuation. For $\varphi \in u\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$, $[\![\varphi]\!]_\gamma^{V^\sharp} = (\!|\varphi|\!)_\gamma^{V^\sharp}$.*

Proof. Exactly the same as the proof of Theorem 5.13, except that Lemma 5.23 is used instead of Lemma 5.11. ◀

6 Concluding Remarks

This paper showed how to incorporate propositional operators arising canonically from the branching monad T into the linear time logics proposed in [2]. This involved moving to the Eilenberg-Moore category of T. The addition of arbitrary propositional operators to the logics appears to be incompatible with their step-wise semantics, and our results provide operators that *can be* safely added to the logics. The paper also provided an alternative, equivalent path-based semantics for the uniform modal fragments of the logics in loc. cit., as well as of their enhancements with canonical propositional operators (assuming canonical choices for both the branching and the linear time modalities), and explored conditions under which non-canonical choices for the modalities do not disrupt the equivalence result.

Future work will investigate extending the path-based semantics proposed here to the full $\mathcal{L}^{\mathcal{V}}$ and $\tilde{\mathcal{L}}^{\mathsf{Free}(\mathcal{V})}$ in the first instance, and subsequently also to $\mathcal{L}_\mu$ and $\tilde{\mathcal{L}}_\mu$. We also plan to investigate the relationship between our logics and recent work on graded monads and associated trace logics [17].

Acknowledgements. Thanks are due to Ichiro Hasuo and Toby Wilkinson for stimulating discussions, and to the anonymous referees for useful comments. The challenge of incorporating arbitrary propositional operators into the step-wise semantics was the subject of extensive discussions with Ichiro Hasuo. The encoding of the $\bigcirc$ modality in Example 4.4 was suggested by Toby Wilkinson. The observation in Remark 2.4 was made by one of the anonymous referees.

References

1 Corina Cîrstea. From branching to linear time, coalgebraically. In *Proceedings, FICS 2013*, pages 11–27, 2013.
2 Corina Cîrstea. A coalgebraic approach to linear-time logics. In *Proceedings, FOSSACS 2014*, pages 426–440, 2014.
3 Corina Cîrstea and Dirk Pattinson. Modular construction of complete coalgebraic logics. *Theor. Comput. Sci.*, 388(1-3):83–108, 2007.
4 Dion Coumans and Bart Jacobs. Scalars, monads, and categories. In *Quantum Physics and Linguistics. A Compositional, Diagrammatic Discourse*, pages 184–216. Oxford Univ. Press, 2013.
5 B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order (2. ed.)*. Cambridge University Press, 2002.
6 Ichiro Hasuo. Generic weakest precondition semantics from monads enriched with order. In *Proceedings, CMCS 2014*, pages 10–32, 2014.
7 Ichiro Hasuo, Bart Jacobs, and Ana Sokolova. Generic trace semantics via coinduction. *Logical Methods in Computer Science*, 3(4), 2007.
8 Michael Huth and Marta Z. Kwiatkowska. Quantitative analysis and model checking. In *Proceedings, LICS 1997*, pages 111–122, 1997.
9 Bart Jacobs. Introduction to coalgebra. Towards mathematics of states and observations. Draft.
10 Bart Jacobs, Alexandra Silva, and Ana Sokolova. Trace semantics via determinization. In *Proceedings, CMCS 2012*, pages 109–129, 2012.
11 Christian Kissig and Alexander Kurz. Generic trace logics. *CoRR*, abs/1103.3239, 2011.
12 Bartek Klin and Jurriaan Rot. Coalgebraic trace semantics via forgetful logics. In *Proceedings, FOSSACS 2015*, pages 151–166, 2015.
13 Anders Kock. Monads and extensive quantities, 2011. arXiv:1103.6009.

14 Anders Kock. Commutative monads as a theory of distributions. *Theory and Applications of Categories*, 26(4):97–131, 2012.

15 Clemens Kupke, Alexander Kurz, and Dirk Pattinson. Algebraic semantics for coalgebraic logics. *Electr. Notes Theor. Comput. Sci.*, 106:219–241, 2004.

16 Clemens Kupke, Alexander Kurz, and Yde Venema. Stone coalgebras. *Theor. Comput. Sci.*, 327(1-2):109–134, 2004.

17 Stefan Milius, Dirk Pattinson, and Lutz Schröder. Generic trace semantics and graded monads. This volume.