Anchored Rectangle and Square Packings*

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Abstract

For points $p_1, \ldots, p_n$ in the unit square $[0,1]^2$, an *anchored rectangle packing* consists of interior-disjoint axis-aligned empty rectangles $r_1, \ldots, r_n \subseteq [0,1]^2$ such that point p_i is a corner of the rectangle r_i for $i = 1, \ldots, n$ (r_i is *anchored* at p_i). We show that for every set of n points in $[0,1]^2$, there is an anchored rectangle packing of area at least $7/12 - O(1/n)$, and for every $n \in \mathbb{N}$, there are point sets for which the area of every anchored rectangle packing is at most $2/3$. The maximum area of an anchored *square* packing is always at least $5/32$ and sometimes at most $7/27$.

The above constructive lower bounds immediately yield constant-factor approximations, of $7/12 - \varepsilon$ for rectangles and $5/32$ for squares, for computing anchored packings of maximum area in $O(n \log n)$ time. We prove that a simple greedy strategy achieves a $9/47$-approximation for anchored square packings, and $1/3$ for lower-left anchored square packings. Reductions to maximum weight independent set (MWIS) yield a QPTAS and a PTAS for anchored rectangle and square packings in $n^{O(1/\varepsilon)}$ and $\exp(\text{poly}(\log(n/\varepsilon)))$ time, respectively.

1998 ACM Subject Classification I.3.5 Computational Geometry and Object Modeling

Keywords and phrases Rectangle packing, anchored rectangle, greedy algorithm, charging scheme, approximation algorithm

Digital Object Identifier 10.4230/LIPIcs.SoCG.2016.13

1 Introduction

Let $P = \{p_1, \ldots, p_n\}$ be a finite set of points in an axis-aligned bounding rectangle U. An *anchored rectangle packing* for P is a set of axis-aligned empty rectangles $r_1, \ldots, r_n$ that lie in U, are interior-disjoint, and p_i is one of the four corners of r_i for $i = 1, \ldots, n$; rectangle r_i is said to be *anchored* at p_i. For a given point set $P \subset U$, we wish to find the maximum total area $A(P)$ of an anchored rectangle packing of P. Since the ratio between areas is an affine invariant, we may assume that $U = [0,1]^2$. However, if we are interested in the maximum area of an *anchored square packing*, we *must* assume that $U = [0,1]^2$ (or that the aspect ratio of U is bounded from below by a constant; otherwise, with an *arbitrary* rectangle U, the guaranteed area is zero).

* This work was partially supported by the NSF awards CCF-1422311 and CCF-1423615.

32nd International Symposium on Computational Geometry (SoCG 2016).
Editors: Sándor Fekete and Anna Lubiw; Article No. 13; pp. 13:1–13:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

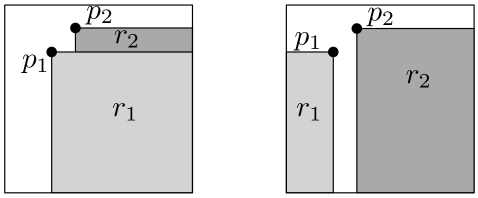

Figure 1 For $P = \{p_1, p_2\}$, with $p_1 = (\frac{1}{4}, \frac{3}{4})$ and $p_2 = (\frac{3}{8}, \frac{7}{8})$, a greedy algorithm selects rectangles of area $\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{8} \cdot \frac{5}{8} = \frac{41}{64}$ (left), less than the area $\frac{1}{4} \cdot \frac{3}{4} + \frac{5}{8} \cdot \frac{7}{8} = \frac{47}{64}$ of the packing on the right.

Table 1 Table of results for the four variants studied in this paper. The last two columns refer to lower-left anchored rectangles and lower-left anchored squares, respectively.

Anchored packing with	rectangles	squares	LL-rect.	LL-sq.
Guaranteed max. area	$\frac{7}{12} - O(\frac{1}{n}) \leq A(n) \leq \frac{2}{3}$	$\frac{5}{32} \leq A_{\text{sq}}(n) \leq \frac{7}{27}$	0	0
Greedy approx. ratio	$7/12 - \varepsilon$	$9/47$	0.091 [20]	$1/3$
Approximation scheme	QPTAS	PTAS	QPTAS	PTAS

Finding the maximum area of an anchored rectangle packing of n given points is suspected but not known to be NP-hard. Balas and Tóth [8] observed that the number of distinct rectangle packings that attain the maximum area, $A(P)$, can be exponential in n. From the opposite direction, the same authors [8] proved an exponential upper bound on the number of maximum area configurations, namely $2^n C_n = \Theta(8^n/n^{3/2})$, where $C_n = \frac{1}{n+1}\binom{2n}{n} = \Theta(4^n/n^{3/2})$ is the nth Catalan number. Note that a greedy strategy may fail to find $A(P)$; see Fig. 1.

Variants and generalizations. We consider three additional variants of the problem. An *anchored square packing* is an anchored rectangle packing in which all rectangles are squares; a *lower-left anchored rectangle packing* is a rectangle packing where each point $p_i \in r_i$ is the lower-left corner of r_i; and a *lower-left anchored square packing* has both properties.

We suspect that all variants, with rectangles or with squares, are NP-hard. Here, we put forward several approximation algorithms, while it is understood that the news regarding NP-hardness can occur at any time or perhaps take some time to establish.

Contributions. Our results are summarized in Table 1. Due to space limitations, some proofs are omitted; the reader is referred to [7] for details.

(i) We first deduce upper and lower bounds on the maximum area of an anchored rectangle packing of n points in $[0,1]^2$. For $n \in \mathbb{N}$, let $A(n) = \inf_{|P|=n} A(P)$. We prove that $\frac{7}{12} - O(1/n) \leq A(n) \leq \frac{2}{3}$ for all $n \in \mathbb{N}$ (Sections 2 and 3).

(ii) Let $A_{\text{sq}}(P)$ be the maximum area of an anchored square packing for a point set P, and $A_{\text{sq}}(n) = \inf_{|P|=n} A_{\text{sq}}(P)$. We prove that $\frac{5}{32} \leq A_{\text{sq}}(n) \leq \frac{7}{27}$ for all n (Sections 2 and 4).

(iii) The above constructive lower bounds immediately yield constant-factor approximations for computing anchored packings of maximum area ($7/12 - \varepsilon$ for rectangles and $5/32$ for squares) in $O(n \log n)$ time (Sections 3 and 4). In Section 5 we show that a (natural) greedy algorithm for anchored square packings achieves a better approximation ratio, namely $9/47 = 1/5.22\ldots$, in $O(n^2)$ time. By refining some of the tools developed for this bound, in Section 6 we prove a tight bound of $1/3$ for the approximation ratio of a greedy algorithm for lower-left anchored square packings.

(iv) We obtain a polynomial-time approximation scheme (PTAS) for the maximum area anchored square packing problem, and a quasi-polynomial-time approximation scheme (QPTAS) for the maximum area anchored rectangle packing problem, via a reduction to the maximum weight independent set (MWIS) problem for axis-aligned squares [16] and rectangles [2], respectively. Given n points, an $(1-\varepsilon)$-approximation for the anchored square packing of maximum area can be computed in time $n^{O(1/\varepsilon)}$; and for the anchored rectangle packing of maximum area, in time $\exp(\text{poly}(\log(n/\varepsilon)))$. Both results extend to the lower-left anchored variants; see [7, Section 7].

Motivation and related work. Packing axis-aligned rectangles in a rectangular container, albeit without anchors, is the unifying theme of several classic optimization problems. The 2D knapsack problem, strip packing, and 2D bin packing involve arranging a set of given rectangles in the most economic fashion [2, 9]. The maximum area independent set (MAIS) problem for rectangles (squares, or disks, etc.), is that of selecting a maximum area packing from a given set [3]; see classic papers such as [5, 28, 29, 30, 31] and also more recent ones [10, 11, 20] for quantitative bounds and constant approximations. These optimization problems are NP-hard, and there is a rich literature on approximation algorithms. Given an axis-parallel rectangle U in the plane containing n points, the problem of computing a maximum-area empty axis-parallel sub-rectangle contained in U is one of the oldest problems studied in computational geometry [4, 17]; the higher dimensional variant has been also studied [19]. In contrast, our problem setup is fundamentally different: the rectangles (one for each anchor) have variable sizes, but their location is constrained by the anchors.

Map labeling problems in geographic information systems (GIS) [24, 25, 27] call for choosing interior-disjoint rectangles that are incident to a given set of points in the plane. GIS applications often impose constraints on the label boxes, such as aspect ratio, minimum and maximum size, or priority weights. Most optimization problems of such variants are known to be NP-hard [21, 22, 23, 26]. In this paper, we focus on maximizing the total area of an anchored rectangle packing.

In a restricted setting where each point p_i is the lower-left corner of the rectangle r_i and $(0,0) \in P$, Allen Freedman [32, 33] conjectured almost 50 years ago that there is a lower-left anchored rectangle packing of area at least 1/2. The current best lower bound on the area under these conditions is (about) 0.091, as established in [20]. The analogous problem of estimating the total area for lower-left anchored square packings is much easier. If P consists of the n points $(i/n, i/n)$, $i = 0, 1, \ldots, n-1$, then the total area of the n anchored squares is at most $1/n$, and so it tends to zero as n tends to infinity. A looser anchor restriction, often appearing in map labeling problems with square labels, requires the anchors to be contained in the boundaries of the squares, however the squares need to be congruent; see e.g., [34].

In the context of *covering* (as opposed to *packing*), the problem of covering a given polygon by disks with given centers such that the sum of areas of the disks is minimized has been considered in [1, 14]. In particular, covering $[0,1]^2$ with ℓ_∞-disks of given centers and minimal area as in [12, 13] is dual to the anchored square packings studied here.

Notation. Given an n-element point set P contained in $U = [0,1]^2$, denote by $\text{OPT} = \text{OPT}(P)$ a packing (of rectangles or squares, as the case may be) of maximum total area. An algorithm for a packing problem has approximation ratio α if the packing it computes, Π, satisfies $\text{area}(\Pi) \geq \alpha\,\text{area}(\text{OPT})$, for some $\alpha \leq 1$. A set of points is *in general position* if no two points have the same x- or y-coordinate. The boundary of a planar body B is denoted by ∂B, and its interior by $\text{int}(B)$.

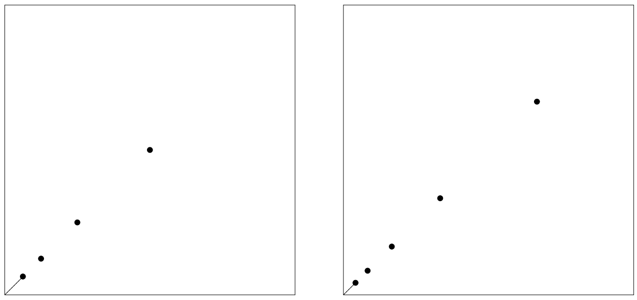

Figure 2 Left: 2/3 upper bound construction for anchored rectangles. Right: 7/27 upper bound construction for anchored squares.

2 Upper Bounds

▶ **Proposition 1.** *For every $n \in \mathbb{N}$, there exists a point set P_n such that every anchored rectangle packing for P_n has area at most $\frac{2}{3}$. Consequently, $A(n) \le \frac{2}{3}$.*

Proof. Consider the point set $P = \{p_1, \dots, p_n\}$, where $p_i = (x_i, y_i) = (2^{-i}, 2^{-i})$, for $i = 1, \dots, n$; see Fig. 2 (left). Let $R = \{r_1, \dots, r_n\}$ be an anchored rectangle packing for P. Since $p_1 = (\frac{1}{2}, \frac{1}{2})$, any rectangle anchored at p_1 has height at most $\frac{1}{2}$, width at most $\frac{1}{2}$, and hence area at most $\frac{1}{4}$.

For $i = 2, \dots, n$, the x-coordinate of p_i, x_i, is halfway between 0 and x_{i-1}, and y_i is halfway between 0 and y_{i-1}. Consequently, if p_i is the lower-right, lower-left or upper-left corner of r_i, then $\text{area}(r_i) \le (\frac{1}{2^i})(1 - \frac{1}{2^i}) = \frac{1}{2^i} - \frac{1}{4^i}$. If, p_i is the upper-right corner of r_i, then $\text{area}(r_i) \le \frac{1}{4^i}$. Therefore, in all cases, we have $\text{area}(r_i) \le \frac{1}{2^i} - \frac{1}{4^i}$. The total area of an anchored rectangle packing is bounded from above as follows:

$$A(P) \le \sum_{i=1}^{n} \left(\frac{1}{2^i} - \frac{1}{4^i} \right) = \left(1 - \frac{1}{2^n} \right) - \frac{1}{3} \left(1 - \frac{1}{4^n} \right) = \frac{2}{3} - \frac{1}{2^n} + \frac{1}{3 \cdot 4^n} \le \frac{2}{3}. \qquad ◀$$

▶ **Proposition 2.** *For every $n \in \mathbb{N}$, there exists a point set P_n such that every anchored square packing for P_n has area at most $\frac{7}{27}$. Consequently, $A_{\text{sq}}(n) \le \frac{7}{27}$.*

Proof. Consider the point set $P = \{p_1, \dots, p_n\}$, where $p_i = (\frac{4}{3} \cdot 2^{-i}, \frac{4}{3} \cdot 2^{-i})$, for $i = 1, \dots, n$; see Fig. 2 (right). Let $S = \{s_1, \dots, s_n\}$ be an anchored square packing for P. Since $p_1 = (\frac{2}{3}, \frac{2}{3})$ and $p_2 = (\frac{1}{3}, \frac{1}{3})$, any square anchored at p_1 or at p_2 has side-length at most $\frac{1}{3}$, hence area at most $\frac{1}{9}$. For $i = 3, \dots, n$, the x-coordinate of p_i, x_i, is halfway between 0 and x_{i-1}, and y_i is halfway between 0 and y_{i-1}. Hence any square anchored at p_i has side-length at most $x_i = y_i = \frac{4}{3 \cdot 2^i}$, hence area at most $\frac{16}{9 \cdot 4^i}$. The total area of an anchored square packing is bounded from above as follows:

$$A_{\text{sq}}(P) \le \frac{2}{9} + \frac{1}{9} \sum_{j=1}^{n-1} \frac{1}{4^j} < \frac{2}{9} + \frac{1}{9} \sum_{j=1}^{\infty} \frac{1}{4^j} = \frac{2}{9} + \frac{1}{9} \cdot \frac{1}{3} = \frac{7}{27}. \qquad ◀$$

▶ Remark. Stronger upper bounds hold for small n, e.g., $n \in \{1, 2\}$. Specifically, $A(1) = A_{\text{sq}}(1) = 1/4$ attained for the center $(\frac{1}{2}, \frac{1}{2}) \in [0, 1]^2$, and $A(2) = 4/9$ and $A_{\text{sq}}(2) = 2/9$ attained for $P = \{(\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})\}$.

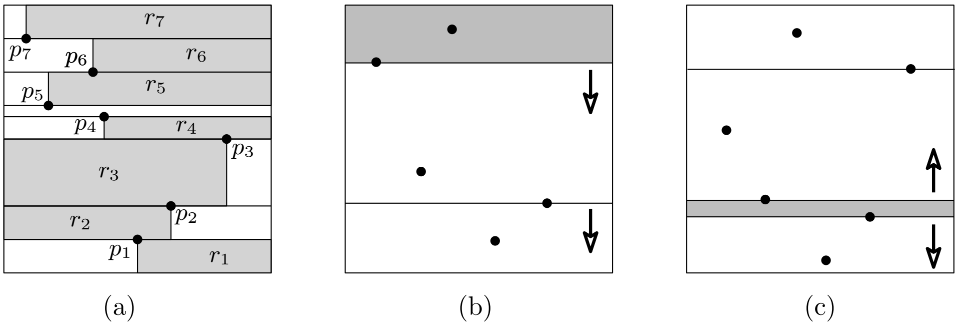

Figure 3 Left: Horizontal strips with anchored rectangles for 7 points. Middle: an example of the partition for odd n (here $n = 5$). Right: an example of the partition for even n (here $n = 6$). The strip that is discarded is shaded in the drawing.

3 Lower Bound for Anchored Rectangle Packings

In this section, we prove that for every set P of n points in $[0,1]^2$, we have $A(P) \geq \frac{7n-2}{12(n+1)}$. Our proof is constructive; we give a divide & conquer algorithm that partitions U into horizontal strips and finds n anchored rectangles of total area bounded from below as required. We start with a weaker lower bound, of about $1/2$, and then sharpen the argument to establish the main result of this section, a lower bound of about $7/12$.

▶ **Proposition 3.** *For every set of n points in the unit square $[0,1]^2$, an anchored rectangle packing of area at least $\frac{n}{2(n+1)}$ can be computed in $O(n \log n)$ time.*

Proof. Let $P = \{p_1, \ldots, p_n\}$ be a set of points in the unit square $[0,1]^2$ sorted by their y-coordinates. Draw a horizontal line through each point in P; see Fig. 3 (left). These lines divide $[0,1]^2$ into $n+1$ horizontal strips. A strip can have zero width if two points have the same x-coordinate. We leave a narrowest strip empty and assign the remaining strips to the n points such that each rectangle above (resp., below) the chosen narrowest strip is assigned to a point of P on its bottom (resp., top) edge. For each point divide the corresponding strip into two rectangles with a vertical line through the point. Assign the larger of the two rectangles to the point.

The area of narrowest strip is at most $\frac{1}{n+1}$. The rectangle in each of the remaining n strips covers at least $\frac{1}{2}$ of the strip. This yields a total area of at least $\frac{n}{2(n+1)}$. ◀

A key observation allowing a stronger lower bound is that for *two* points in a horizontal strip, one can always pack two anchored rectangles in the strip that cover strictly more than half the area of the strip. Specifically, we have the following easy-looking statement with 2 points in a rectangle (however, we do not have an easy proof!); details are in [7].

▶ **Lemma 4.** *Let $P = \{p_1, p_2\}$ be two points in an axis-parallel rectangle R such that p_1 lies on the bottom side of R. Then there exist two empty rectangles in R anchored at the two points of total area at least $\frac{7}{12}\,\mathrm{area}(R)$, and this bound is the best possible.*

In order to partition the unit square into strips that contain two points, one on the boundary, we need to use parity arguments. Let P be a set of n points in $[0,1]^2$ with y-coordinates $0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq 1$. Set $y_0 = 0$ and $y_{n+1} = 1$. For $i = 1, \ldots, n+1$, put $h_i = y_i - y_{i-1}$, namely h_i is the ith vertical gap. Obviously, we have

$$h_i \geq 0 \text{ for all } i = 1, \ldots, n+1, \text{ and } \sum_{i=1}^{n+1} h_i = 1. \tag{1}$$

Parity considerations are handled by the following lemma.

▶ **Lemma 5.**

(i) *If n is odd, at least one of the following $(n+1)/2$ inequalities is satisfied:*

$$h_i + h_{i+1} \leq \frac{2}{n+1}, \text{ for (odd) } i = 1, 3, \ldots, n-2, n. \tag{2}$$

(ii) *If n is even, at least one of the following $n+2$ inequalities is satisfied:*

$$h_1 \leq \frac{2}{n+2}, \qquad h_{n+1} \leq \frac{2}{n+2}, \qquad h_i + h_{i+1} \leq \frac{2}{n+2}, \text{ for } i = 1, 2, \ldots, n. \tag{3}$$

Proof. Assume first that n is odd. Put $a = \frac{2}{n+1}$ and assume that none of the inequalities in (2) is satisfied. Summation would yield $\sum_{i=1}^{n+1} h_i > \frac{n+1}{2}\, a = 1$, an obvious contradiction.

Assume now that n is even. Put $a = \frac{2}{n+2}$ and assume that none of the inequalities in (3) is satisfied. Summation would yield $2\sum_{i=1}^{n+1} h_i > (n+2)a = 2$, again a contradiction. ◀

We can now prove the main result of this section.

▶ **Theorem 6.** *For every set of n points in the unit square $[0,1]^2$, an anchored rectangle packing of area at least $\frac{7(n-1)}{12(n+1)}$ when n is odd and $\frac{7n}{12(n+2)}$ when n is even can be computed in $O(n \log n)$ time.*

Proof. Let $P = \{p_1, \ldots, p_n\}$ be a set of points in the unit square $[0,1]^2$ sorted by their y-coordinates with the notation introduced above. By Lemma 5, we find a horizontal strip corresponding to one of the inequalities that is satisfied.

Assume first that n is odd. Draw a horizontal line through each point in $p_j \in P$, for j even, as shown in Fig. 3. These lines divide $[0,1]^2$ into $\frac{n+1}{2}$ rectangles (horizontal strips). Suppose now that the satisfied inequality is $h_i + h_{i+1} \leq \frac{2}{n+2}$ for some odd i. Then we leave a rectangle between $y = y_{i-1}$ and $y = y_{i+1}$ empty, i.e., r_i is a rectangle of area 0. For the remaining rectangles, we assign two consecutive points of P such that each strip above $y = y_{i+1}$ (resp., below $y = y_{i-1}$) is assigned a point of P on its bottom (resp., top) edge. Within each rectangle R, we can choose two anchored rectangles of total area at least $\frac{7}{12}\,\text{area}(R)$ by Lemma 4. By Lemma 5(i), the area of the narrowest strip is at most $\frac{2}{n+1}$. Consequently, the area of the anchored rectangles is at least $\frac{7}{12}(1 - \frac{2}{n+1}) = \frac{7(n-1)}{12(n+1)}$.

Assume now that n is even. If the selected horizontal strip corresponds to the inequality $h_1 \leq \frac{2}{n+2}$, then divide the unit square along the lines $y = y_i$, where i is odd. We leave the strip of height h_1 empty, and assign pairs of points to all remaining strips such that one of the two points lies on the top edge of the strip. We proceed analogously if the inequality $h_{n+1} \leq \frac{2}{n+2}$ is satisfied. Suppose now that the satisfied inequality is $h_i + h_{i+1} \leq \frac{2}{n+2}$. If i is odd, we leave the strip of height $h_i \leq \frac{2}{n+2}$ (between $y = y_{i-1}$ and $y = y_i$) empty; if i is even, we leave the strip of height $h_{i+1} \leq \frac{2}{n+2}$ (between $y = y_i$ and $y = y_{i+1}$) empty. Above and below the empty strip, we can form a total of $n/2$ strips, each containing two points of P, with one of the two points lying on the bottom or the top edge of the strip. By Lemma 5(i), the area of the empty strip is at most $\frac{2}{n+2}$. Consequently, the area of the anchored rectangles is at least $\frac{7}{12}(1 - \frac{2}{n+2}) = \frac{7n}{12(n+2)}$, as claimed. ◀

4 Lower Bound for Anchored Square Packings

Given a set $P \subset U = [0,1]^2$ of n points, we show there is an anchored square packing of large total area. The proof we present is constructive; we give a recursive partitioning algorithm (as an inductive argument) based on a quadtree subdivision of U that finds n anchored squares of total area at least $5/32$. We need the following easy fact:

▶ **Observation 7.** *Let $u, v \subseteq U$ be two congruent axis-aligned interior-disjoint squares sharing a common edge such that $u \cap P \neq \emptyset$ and $\mathrm{int}(v) \cap P = \emptyset$. Then $u \cup v$ contains an anchored empty square whose area is at least $\mathrm{area}(u)/4$.*

Proof. Let a denote the side-length of u (or v). Assume that v lies right of u. Let $p \in P$ be the rightmost point in u. If p lies in the lower half-rectangle of u then the square of side-length $a/2$ whose lower-left anchor is p is empty and has area $a^2/4$. Similarly, if p lies in the higher half-rectangle of u then the square of side-length $a/2$ whose upper-left anchor is p is empty and has area $a^2/4$. ◀

▶ **Theorem 8.** *For every set of n points in $U = [0,1]^2$, where $n \geq 1$, an anchored square packing of total area at least $5/32$ can be computed in $O(n \log n)$ time.*

Proof. We first derive a lower bound of $1/8$ and then sharpen it to $5/32$. We proceed by induction on the number of points n contained in U and assigned to U; during the subdivision process, the rôle of U is taken by any subdivision square. If all points in P lie on U's boundary, ∂U, pick one arbitrarily, say, $(x, 0)$ with $x \leq 1/2$. (All assumptions in the proof are made without loss of generality.) Then the square $[x, x + 1/2] \times [0, 1/2]$ is empty and its area is $1/4 > 5/32$, as required. Otherwise, discard the points in $P \cap \partial U$ and continue on the remaining points.

If $n = 1$, we can assume that $x(p), y(p) \leq 1/2$. Then the square of side-length $1/2$ whose lower-left anchor is p is empty and contained in U, as desired; hence $A_{\mathrm{sq}}(P) \geq 1/4$. If $n = 2$ let x_1, x_2, x_3 be the widths of the 3 vertical strips determined by the two points, where $x_1 + x_2 + x_3 = 1$. We can assume that $0 \leq x_1 \leq x_2 \leq x_3$; then there are two anchored empty squares with total area at least $x_2^2 + x_3^2 \geq 2/9 > 5/32$, as required.

Assume now that $n \geq 3$. Subdivide U into four congruent squares, $U_1, \ldots, U_4$, labeled counterclockwise around the center of U according to the quadrant containing the square. Partition P into four subsets $P_1, \ldots, P_4$ such that $P_i \subset U_i$ for $i = 1, \ldots, 4$, with ties broken arbitrarily. We next derive the lower bound $A_{\mathrm{sq}}(P) \geq 1/8$. We distinguish 4 cases, depending on the number of empty sets P_i.

Case 1: precisely one of $P_1, \ldots, P_4$ is empty. We can assume that $P_1 = \emptyset$. By Observation 7, $U_1 \cup U_2$ contains an empty square anchored at a point in $P_1 \cup P_2$ of area at least $\mathrm{area}(U_1)/4 = 1/16$. By induction, U_3 and U_4 each contain an anchored square packing of area at least $c \cdot \mathrm{area}(U_3) = c \cdot \mathrm{area}(U_4)$. Overall, we have $A_{\mathrm{sq}}(P) \geq 2c/4 + 1/16 \geq c$, which holds for $c \geq 1/8$.

Case 2: precisely two of $P_1, \ldots, P_4$ are empty. We can assume that the pairs $\{P_1, P_2\}$ and $\{P_3, P_4\}$ each consist of one empty and one nonempty set. By Observation 7, $U_1 \cup U_2$ and $U_3 \cup U_4$, respectively, contain a square anchored at a point in $P_1 \cup P_2$ and $P_3 \cup P_4$ of area at least $\mathrm{area}(U_1)/4 = 1/16$. Hence $A_{\mathrm{sq}}(P) \geq 2 \cdot \frac{1}{16} = 1/8$.

Case 3: precisely three of $P_1, \ldots, P_4$ are empty. We can assume that $P_3 \neq \emptyset$. Let $(a, b) \in P$ be a maximal point in the product order (e.g., the sum of coordinates is maximum). Then $s = [a, a + \frac{1}{2}] \times [b, b + \frac{1}{2}]$ is a square anchored at (a, b), $s \subseteq [0,1]^2$ since $(a, b) \in U_3$, and $\mathrm{int}(s) \cap P = \emptyset$. Hence $A_{\mathrm{sq}}(P) \geq \mathrm{area}(s) = 1/4$.

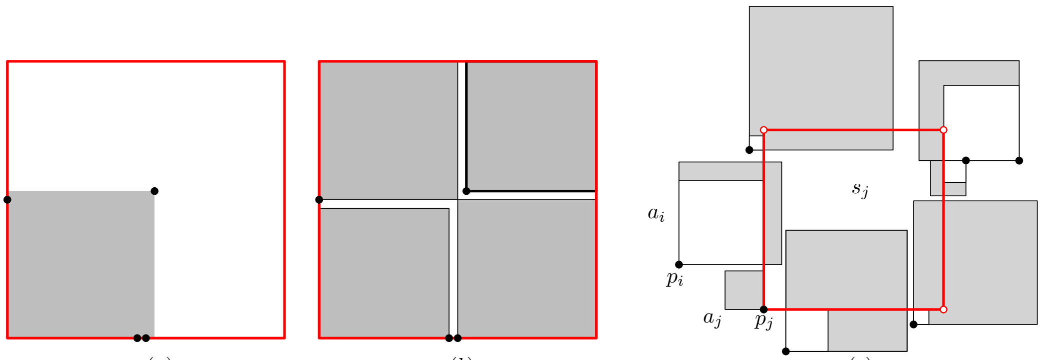

Figure 4 (a–b) A $1/4$ upper bound for the approximation ratio of Algorithm 9. (c) Charging scheme for Algorithm 9. Without loss of generality, the figure illustrates the case when s_j is a lower-left anchored square.

Case 4: $P_i \neq \emptyset$ for every $i = 1, \ldots, 4$. Note that $A_{\text{sq}}(P) \geq \sum_{i=1}^{4} A_{\text{sq}}(P_i)$, where the squares anchored at P_i are restricted to U_i. Induction completes the proof in this case.

In all four cases, we have verified that $A_{\text{sq}}(P) \geq 1/8$, as claimed. The inductive proof can be turned into a recursive algorithm based on a quadtree subdivision of the n points, which can be computed in $O(n \log n)$ time [6, 18]. In addition, computing an extreme point (with regard to a specified axis-direction) in any subsquare over all needed such calls can be executed within the same time bound. Note that the bound in Case 3 is at least $5/32$ and Case 4 is inductive. Sharpening the analysis of Cases 1 and 2 yields an improved bound $5/32$; since $5/32 < 1/4$, the value $5/32$ is not a bottleneck for Cases 3 and 4. Details are given in [7]; the running time remains $O(n \log n)$. ◀

5 Constant-Factor Approximations for Anchored Square Packings

In this section we investigate better approximations for square packings. Given a finite point set $P \subset [0,1]^2$, perhaps the most natural greedy strategy for computing an anchored square packing of large area is the following.

▶ **Algorithm 9.** Set $Q = P$ and $S = \emptyset$. While $Q \neq \emptyset$, repeat the following. For each point $q \in Q$, compute a *candidate* square $s(q)$ such that (i) $s(q) \subseteq [0,1]^2$ is *anchored* at q, (ii) $s(q)$ is empty of points from P in its interior, (iii) $s(q)$ is interior-disjoint from all squares in S, and (iv) $s(q)$ has maximum area. Then choose a largest candidate square $s(q)$, and a corresponding point q, and set $Q \leftarrow Q \setminus \{q\}$ and $S \leftarrow S \cup \{s(q)\}$. When $Q = \emptyset$, return the set of squares S.

▶ Remark. Let ρ_9 denote the approximation ratio of Algorithm 9, if it exists. The construction in Fig. 4(a–b) shows that $\rho_9 \leq 1/4$. For a small $\varepsilon > 0$, consider the point set $P = \{p_1, \ldots, p_n\}$, where $p_1 = (1/2 + \varepsilon, 1/2 + \varepsilon)$, $p_2 = (1/2, 0)$, $p_3 = (0, 1/2)$, and the rest of the points lie on the lower side of U in the vicinity of p_2, i.e., $x_i \in (1/2 - \varepsilon/2, 1/2 + \varepsilon/2)$ and $y_i = 0$ for $i = 4, \ldots, n$. The packing generated by Algorithm 9 consists of a single square of area $(1/2 + \varepsilon)^2$, as shown in Fig. 4(a), while the packing in Fig. 4(b) has an area larger than $1 - \varepsilon$. By letting ε be arbitrarily small, we deduce that $\rho_9 \leq 1/4$.

We first show that Algorithm 9 achieves a ratio of $1/6$ (Theorem 12) using a charging scheme that compares the greedy packing with an optimal packing. We then refine our analysis and sharpen the approximation ratio to $\frac{9}{47} = 1/5.22\ldots$ (Theorem 17).

Charging scheme for the analysis of Algorithm 9. Label the points in $P = \{p_1, \ldots, p_n\}$ and the squares in $S = \{s_1, \ldots, s_n\}$ in the order in which they are processed by Algorithm 9 with $q = p_i$ and $s_i = s(q)$. Let $G = \sum_{i=1}^n \text{area}(s_i)$ be the area of the greedy packing, and let OPT denote an optimal packing with $A = \text{area(OPT)} = \sum_{i=1}^n \text{area}(a_i)$, where a_i is the square anchored at p_i.

We employ a charging scheme, where we distribute the area of every optimal square a_i with $\text{area}(a_i) > 0$ among some greedy squares; and then show that the total area of the optimal squares charged to each greedy square s_j is at most $6\,\text{area}(s_j)$ for all $j = 1, \ldots, n$. (Degenerate optimal squares, i.e., those with $\text{area}(a_i) = 0$ do not need to be charged). For each step $j = 1, \ldots, n$ of Algorithm 9, we shrink some of the squares $a_1, \ldots, a_n$, and charge the area-decrease to the greedy square s_j. By the end (after the nth step), each of the squares $a_1, \ldots, a_n$ will be reduced to a single point.

Specifically in step j, Algorithm 9 chooses a square s_j, and: (1) we shrink square a_j to a single point; and (2) we shrink every square a_i, $i > j$ that intersects s_j in its interior until it no longer does so. This procedure ensures that no square a_i, with $i < j$, intersects s_j in its interior in step j. Refer to Fig. 4(c). Observe three important properties of the above iterative process:

(i) After step j, the squares $s_1, \ldots, s_j, a_1, \ldots, a_n$ have pairwise disjoint interiors.

(ii) After step j, we have $\text{area}(a_j) = 0$ (since a_j was shrunk to a single point).

(iii) At the beginning of step j, if a_i intersects s_j in its interior (and so $i \geq j$), then $\text{area}(a_i) \leq \text{area}(s_j)$ since s_j is feasible for p_i when a_j is selected by Algorithm 9 due to the greedy choice.

▶ **Lemma 10.** *Suppose there exists a constant $\varrho \geq 1$ such that for every $j = 1, \ldots, n$, square s_j receives a charge of at most $\varrho\,\text{area}(s_j)$. Then Algorithm 9 computes an anchored square packing whose area G is at least $1/\varrho$ times the optimal.*

Proof. Overall, each square s_j receives a charge of at most $\varrho\,\text{area}(s_j)$ from the squares in an optimal solution. Consequently, $A = \text{area(OPT)} = \sum_{i=1}^n \text{area}(a_i) \leq \varrho \sum_{j=1}^n \text{area}(s_j) = \varrho\, G$, and thus $G \geq A/\varrho$, as claimed. ◀

In the remainder of this section, we bound the charge received by one square s_j, for $j = 1, \ldots, n$. We distinguish two types of squares a_i, $i > j$, whose area is reduced by s_j:

- $\mathcal{Q}_1 = \{a_i : i > j$, the area of a_i is reduced by s_j, and $\text{int}(a_i)$ contains no corner of $s_j\}$,
- $\mathcal{Q}_2 = \{a_i : i > j$, the area of a_i is reduced by s_j, and $\text{int}(a_i)$ contains a corner of $s_j\}$.

It is clear that if the insertion of s_j reduces the area of a_i, $i > j$, then a_i is in either $\mathcal{Q}_1$ or $\mathcal{Q}_2$. Note that the area of a_j is also reduced to 0, but it is in neither $\mathcal{Q}_1$ nor $\mathcal{Q}_2$.

▶ **Lemma 11.** *Each square s_j receives a charge of at most $6\,\text{area}(s_j)$.*

Proof. Consider the squares in $\mathcal{Q}_1$. Assume that a_i intersects the interior of s_j, and it is shrunk to a_i'. The area-decrease $a_i \setminus a_i'$ is an L-shaped region, at least half of which lies inside s_j; see Fig. 4. By property (i), the L-shaped regions are pairwise interior-disjoint; and hence the sum of their areas is at most $2\,\text{area}(s_j)$. Consequently, the area-decrease caused by s_j in squares in $\mathcal{Q}_1$ is at most $2\,\text{area}(s_j)$.

Consider the squares in $\mathcal{Q}_2$. There are at most three squares a_i, $i > j$, that can contain a corner of s_j since the anchor of s_j is not contained in the interior of any square a_i. Since the area of each square in $\mathcal{Q}_2$ is at most $\text{area}(s_j)$ by property (iii), the area decrease is at most $3\,\text{area}(s_j)$, and so is the charge received by s_j from squares.

Finally, $\text{area}(a_j) \leq \text{area}(s_j)$ by property (iii), and this is also charged to s_j. Overall s_j receives a charge of at most $6\,\text{area}(s_j)$. ◀

The combination of Lemmas 10 and 11 readily implies the following.

▶ **Theorem 12.** *Algorithm 9 computes an anchored square packing whose area is at least* $1/6$ *times the optimal.*

Refined analysis of the charging scheme. We next improve the upper bound for the charge received by s_j; we assume for convenience that $s_j = U = [0,1]^2$. For the analysis, we use only a few properties of the optimal solution. Specifically, assume that $a_1, \ldots, a_m$ are interior-disjoint squares such that each a_i: (a) intersects the interior of s_j; (b) has at least a corner in the exterior of s_j; (c) does not contain $(0,0)$ in its interior; and (d) $\text{area}(a_i) \leq \text{area}(s_j)$.

The intersection of any square a_i with ∂U is a polygonal line on the boundary ∂U, consisting of one or two segments. Since the squares a_i form a packing, these intersections are interior-disjoint.

Let $\Delta_1(x)$ denote the maximum area-decrease of a set of squares a_i in $\mathcal{Q}_1$, whose intersections with ∂U have total length x. Similarly, let $\Delta_2(x)$ denote the maximum area-decrease of a set of squares a_i in $\mathcal{Q}_2$, whose intersections with ∂U have total length x. By adding suitable squares to $\mathcal{Q}_1$, we can assume that $4-x$ is the total length of the intersections $a_i \cap \partial U$ over squares in $\mathcal{Q}_2$ (i.e., the squares in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ cover the entire boundary of U). Consequently, the maximum total area-decrease is given by

$$\Delta(x) = \Delta_1(x) + \Delta_2(4-x), \text{ and } \Delta = \sup_{0 \leq x \leq 4} \Delta(x). \tag{4}$$

We now derive upper bounds for $\Delta_1(x)$ and $\Delta_2(x)$ independently, and then combine these bounds to optimize $\Delta(x)$. Since the total perimeter of U is 4, the domain of $\Delta(x)$ is $0 \leq x \leq 4$.

▶ **Lemma 13.** *The following inequalities hold:*

$$\Delta_1(x) \leq 2, \tag{5}$$

$$\Delta_1(x) \leq x, \tag{6}$$

$$\Delta_1(x) \leq 1 + (x-1)^2, \text{ for } 1 \leq x \leq 2, \tag{7}$$

$$\Delta_1(x) \leq 1 + \frac{\lfloor x \rfloor}{4} + \frac{(x - \lfloor x \rfloor)^2}{4}, \text{ for } 0 \leq x \leq 4. \tag{8}$$

Proof. Inequality (5) was explained in the proof of Theorem 12. Inequalities (6) and (7) follow from the fact that the side-length of each square a_i is at most 1 and from the fact that the area-decrease is at most the area (of respective squares); in addition, we use the inequality $\sum x_i^2 \leq (\sum x_i)^2$, for $x_i \geq 0$ and $\sum x_i \leq 1$, and the inequality $x^2 + y^2 \leq 1 + (x+y-1)^2$, for $0 \leq x, y \leq 1$, and $x + y > 1$. Write

$$\Delta_1(x) = \Delta_1^{\text{in}}(x) + \Delta_1^{\text{out}}(x), \tag{9}$$

where $\Delta_1^{\text{in}}(x)$ and $\Delta_1^{\text{out}}(x)$ denote the maximum area-decrease contained in U and the complement of U, respectively, of a set of squares in $\mathcal{Q}_1$ whose intersections with ∂U have total length x, where $0 \leq x \leq 4$. Obviously, $\Delta_1^{\text{in}}(x) \leq \text{area}(U) = 1$. We next show that

$$\Delta_1^{\text{out}}(x) \leq \frac{\lfloor x \rfloor}{4} + \frac{(x - \lfloor x \rfloor)^2}{4},$$

and thereby establish inequality (8).

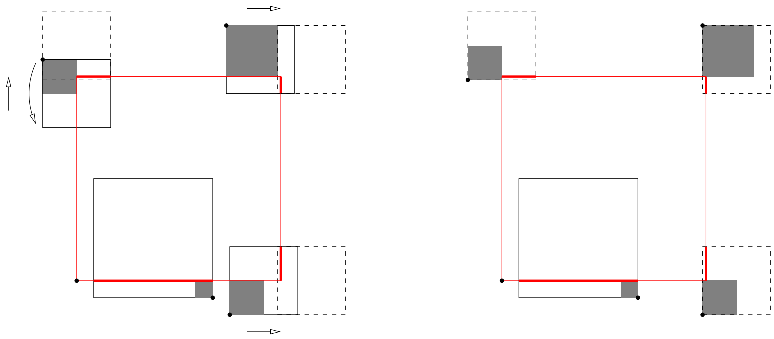

Figure 5 Bounding the area-decrease; moving the squares in $\mathcal{Q}_2$ into canonical position. The parts of ∂U covered by each square (after transformations) are drawn in thick red lines.

Consider a square a_i of side-length $x_i \le 1$ in $\mathcal{Q}_1$. Let z_i denote the length of the shorter side of the rectangle $a_i \setminus U$. The area-decrease outside U equals $x_i z_i - z_i^2$ and so it is bounded from above by $x_i^2/4$ (equality is attained when $z_i = x_i/2$).

Consequently,

$$\Delta_1^{\text{out}}(x) \le \sup \sum_{\substack{0 \le x_i \le 1 \\ \sum x_i = x}} \frac{x_i^2}{4} = \frac{\lfloor x \rfloor}{4} + \frac{(x - \lfloor x \rfloor)^2}{4},$$

where the last equality follows from a standard weight-shifting argument, and equality is attained when x is subdivided into $\lfloor x \rfloor$ unit length intervals and a remaining shorter interval of length $x - \lfloor x \rfloor$. ◀

Let $k \le 3$ be the number of squares a_i in $\mathcal{Q}_2$, where $i > j$. We can assume that exactly 3 squares a_i, with $i > j$, are in $\mathcal{Q}_2$, one for each corner except the lower-left anchor corner of U, that is, $k = 3$; otherwise the proof of Lemma 11 already yields an approximation ratio of 1/5. Clearly, we have $\Delta_2(x) \le k \le 3$, for any x.

We first bring the squares in $\mathcal{Q}_2$ into *canonical position*: x monotonically decreases, $\Delta(x)$ does not decrease, and properties (a–d) listed earlier are maintained. Specifically, we transform each square $a_i \in \mathcal{Q}_2$ as follows (refer to Fig. 5):

- Move the anchor of a_i to another corner if necessary so that one of its coordinates is contained in the interval $(0, 1)$;
- translate a_i horizontally or vertically so that $a_i \cap U$ decreases to a skinny rectangle of width ε, for some small $\varepsilon > 0$.

▶ **Lemma 14.** *The following inequality holds:*

$$\Delta_2(x) \le 2x - \frac{x^2}{3}, \text{ for } 0 \le x \le 4. \tag{10}$$

Proof. Assume that the squares in $\mathcal{Q}_2$ are in canonical position. Let y_i denote the side-length of a_i, let x_i denote the length of the longer side of the rectangle $a_i \cap U$ and z_i denote the length of the shorter side of the rectangle $a_i \setminus U$, $i = 1, 2, 3$. Since the squares in $\mathcal{Q}_2$ are in canonical position, we have $x_i + z_i = y_i \le 1$, for $i = 1, 2, 3$. We also have $\sum_{i=1}^{3} x_i = x - O(\varepsilon)$. Letting $\varepsilon \to 0$, we have $\sum_{i=1}^{3} x_i = x$.

$$\Delta_2(x) = \sup_{x_i+z_i=y_i\le 1} \sum_{i=1}^{3}(y_i^2 - z_i^2) = \sup_{0\le x_1,x_2,x_3\le 1} \sum_{i=1}^{3}(1-(1-x_i^2))$$
$$= \sup_{0\le x_1,x_2,x_3\le 1} \sum_{i=1}^{3}(2x_i - x_i^2) = 2x - \inf_{0\le x_1,x_2,x_3\le 1} \sum_{i=1}^{3} x_i^2 = 2x - \frac{x^2}{3}. \qquad \blacktriangleleft$$

Observe that the inequality $\Delta_2(x) \le 3$, for every $0 \le x \le 4$, is implied by (10). Putting together the upper bounds in Lemmas 13 and 14 yields Lemma 15 (refer to [7] for the proof):

▶ **Lemma 15.** *The following inequality holds:*

$$\Delta \le \frac{38}{9}. \tag{11}$$

From the opposite direction, $\Delta \ge 4$ holds even in a geometric setting, i.e., as implied by several constructions with squares.

▶ **Lemma 16.** *Each square s_j receives a charge of at most $\frac{47}{9}\,\mathrm{area}(s_j)$.*

Proof. By Lemma 15, the area-decrease is at most $38/9\,\mathrm{area}(s_j)$, and so is the charge received by s_j from squares in $\mathcal{Q}_1$ and from squares in $\mathcal{Q}_2$ with the exception of the case $i = j$. Adding this last charge yields a total charge of at most $\left(1 + \frac{38}{9}\right)\mathrm{area}(s_j) = \frac{47}{9}\,\mathrm{area}(s_j)$. ◀

The combination of Lemmas 10 and 16 now yields the following.

▶ **Theorem 17.** *Algorithm 9 computes an anchored square packing whose area is at least $9/47$ times the optimal.*

6 Constant-Factor Approximations for Lower-Left Anchored Square Packings

The following greedy algorithm, analogous to Algorithm 9, constructs a lower-left anchored square packing, given a finite point set $P \subset [0,1]^2$.

▶ **Algorithm 18.** Set $Q = P$ and $S = \emptyset$. While $Q \neq \emptyset$, repeat the following. For each point $q \in Q$, compute a *candidate* square $s(q)$ such that (i) $s(q) \subseteq [0,1]^2$ has q as its *lower-left* anchor, (ii) $s(q)$ is empty of points from P in its interior, (iii) $s(q)$ is interior-disjoint from all squares in S, and (iv) $s(q)$ has maximum area. Then choose a largest candidate square $s(q)$, and a corresponding point q, and set $Q \leftarrow Q \setminus \{q\}$ and $S \leftarrow S \cup \{s(q)\}$. When $Q = \emptyset$, return the set of squares S.

▶ Remark. Let ρ_{18} denote the approximation ratio of Algorithm 18. The construction in Fig. 6 shows that $\rho_{18} \le 1/3$. Specifically, for $\varepsilon > 0$, with $\varepsilon^{-1} \in \mathbb{N}$, consider the point set $P = \{(\varepsilon,\varepsilon),(0,\frac{1}{2}),(\frac{1}{2},0)\} \cup \{(\frac{1}{2}+k\varepsilon,\frac{1}{2}+k\varepsilon) : k = 1,\ldots,1/(2\varepsilon)-1\}$. Then the area of the packing in Fig. 6 (right) is $\frac{3}{4} - O(\varepsilon)$, but Algorithm 18 returns the packing shown in Fig. 6 (left) of area $\frac{1}{4} + O(\varepsilon)$.

We next demonstrate that Algorithm 18 achieves approximation ratio 1/3. According to the above example, this is the best possible for this algorithm.

▶ **Theorem 19.** *Algorithm 18 computes a lower-left anchored square packing whose area is at least $1/3$ times the optimal.*

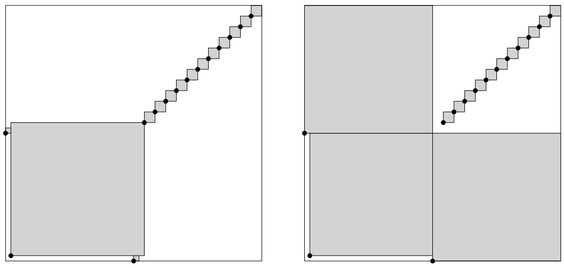

Figure 6 A 1/3 upper bound for the approximation ratio of Algorithm 18.

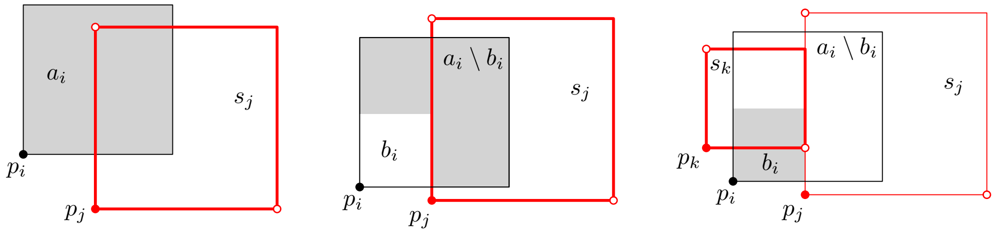

Figure 7 Left: a_i contains the upper-left corner of s_j; and area(a_i) is charged to s_j. Middle and Right: a_i contains no corner of s_j, but it contains the lower-right corner of s_k. Then area$(a_i \setminus b_i)$ is charged to s_j and area(b_i) is charged to s_k.

Proof. Label the points in $P = \{p_1, \ldots, p_n\}$ and the squares in $S = \{s_1, \ldots, s_n\}$ in the order in which they are processed by Algorithm 18 with $q = p_i$ and $s_i = s(q)$. Let $G = \sum_{i=1}^{n} \text{area}(s_i)$ be the area of the greedy packing, and let OPT denote an optimal packing with $A = \text{area}(\text{OPT}) = \sum_{i=1}^{n} \text{area}(a_i)$, where a_i is the square anchored at p_i.

We charge the area of every optimal square a_i to one or two greedy squares s_ℓ; and then show that the total area charged to s_ℓ is at most $3\,\text{area}(s_\ell)$ for all $\ell = 1, \ldots, n$. Consider a square a_i, $1 \leq i \leq n$, with area$(a_i) > 0$. Let $j = j(i)$ be the minimum index such that s_j intersects the interior of a_i. Let b_i denote the candidate square associated to p_i in step $j+1$ of Algorithm 18. Note that $b_i \subset a_i$, thus area$(b_i) <$ area(a_i). If area$(b_i) > 0$, then let $k = k(i)$ be the minimum index such that s_k intersects the interior of b_i.

We can now describe our *charging scheme*: If a_i contains the upper-left or lower-right corner of s_j, then charge area(a_i) to s_j (Fig. 7, left). Otherwise, charge area$(a_i \setminus b_i)$ to s_j, and charge area(b_i) to s_k (Fig. 7, middle-right).

We first argue that the charging scheme is well-defined, and the total area of a_i is charged to one or two squares (s_j and possibly s_k). Indeed, if no square s_ℓ, $\ell < i$, intersects the interior of a_i, then $a_i \subseteq s_i$, and $j(i) = i$; and if $a_i \not\subseteq s_j$ and no square s_ℓ, $j < \ell < i$, intersects the interior of b_i, then $b_i \subseteq s_i$ and $k(i) = i$.

Note that if area(a_i) is charged to s_j, then area$(a_i) \leq$ area(s_j). Indeed, if area$(a_i) >$ area(s_j), then a_i is entirely free at step j, so Algorithm 18 would choose a square at least as large as a_i instead of s_j, which is a contradiction. Analogously, if area(b_i) is charged to s_k, then area$(b_i) \leq$ area(s_k). Moreover, if area(b_i) is charged to s_k, then the upper-left or lower-right corner of s_k is on the boundary of b_i, and so this corner is contained in a_i; refer to Fig. 7 (right).

Fix $\ell \in \{1, \ldots, n\}$. We show that the total area charged to s_ℓ is at most $3\,\text{area}(s_\ell)$. If a square a_i, $i = 1, \ldots, n$, sends a positive charge to s_ℓ, then $\ell = j(i)$ or $\ell = k(i)$. We distinguish two types of squares a_i that send a positive charge to s_ℓ; refer to Fig. 8:

T1 a_i contains the upper-left or lower-right corner of s_ℓ in its interior.

T2 a_i contains neither the upper-left nor the lower-right corner of s_ℓ.

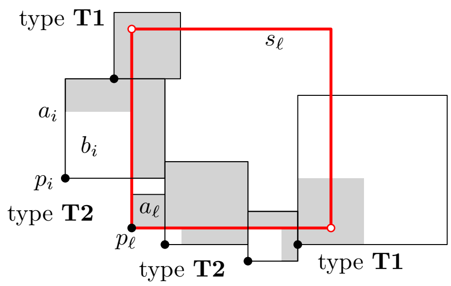

Figure 8 The shaded areas are charged to square s_ℓ.

Since OPT is a packing, at most one optimal square contains each corner of s_ℓ. Consequently, there is at most two squares a_i of type **T1**. Since $\text{area}(a_i) \leq \text{area}(s_\ell)$, the charge received from the squares of type **T1** is at most $2\,\text{area}(s_\ell)$.

By [7, Lemma 9], s_ℓ receives a charge of at most $\text{area}(s_\ell)$ from squares of type **T2**. It follows that each s_ℓ received a charge of at most $3\,\text{area}(s_\ell)$. Consequently,

$$A = \text{area}(\text{OPT}) = \sum_{i=1}^{n} \text{area}(a_i) \leq 3 \sum_{\ell=1}^{n} \text{area}(s_\ell) = 3\,G, \text{ and thus } G \geq A/3. \qquad \blacktriangleleft$$

7 Conclusion

We conclude with a few open problems:

1. Is the problem of computing the maximum-area anchored rectangle (respectively, square) packing NP-hard?
2. Is there a polynomial-time approximation scheme for the problem of computing an anchored rectangle packing of maximum area?
3. What lower bound on $A(n)$ can be obtained by extending Lemma 4 concerning rectangles from 2 to 3 points? Is there a short proof of Lemma 4?
4. Does Algorithm 9 for computing an anchored square packing of maximum area achieve a ratio of $1/4$? By Theorem 17 and the construction in Fig. 4, the approximation ratio is between $9/47 = 1/5.22\ldots$ and $1/4$. Improvements beyond the $1/5$ ratio are particularly exciting.
5. Is $A(n) = \frac{2}{3}$? Is $A_{\text{sq}}(n) = \frac{7}{27}$?
6. What upper and lower bounds on $A(n)$ and $A_{\text{sq}}(n)$ can be established in higher dimensions?
7. A natural variant of anchored squares is one where the anchors must be the centers of the squares. What approximation can be obtained in this case?

Acknowledgment. The authors are thankful to René Sitters for constructive feedback on the problems discussed in this paper. In particular, the preliminary approximation ratio 1/6 in Theorem 12 incorporates one of his ideas.

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