The Non-Uniform k-Center Problem

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Abstract

In this paper, we introduce and study the Non-Uniform k-Center (NUkC) problem. Given a finite metric space (X, d) and a collection of balls of radii $\{r_1 \geq \cdots \geq r_k\}$, the NUkC problem is to find a placement of their centers on the metric space and find the minimum dilation α, such that the union of balls of radius $\alpha \cdot r_i$ around the ith center covers all the points in X. This problem naturally arises as a min-max vehicle routing problem with fleets of different speeds, or as a wireless router placement problem with routers of different powers/ranges.

The NUkC problem generalizes the classic k-center problem when all the k radii are the same (which can be assumed to be 1 after scaling). It also generalizes the k-center with outliers (kCwO for short) problem when there are k balls of radius 1 and ℓ balls of radius 0. There are 2-approximation and 3-approximation algorithms known for these problems respectively; the former is best possible unless P=NP and the latter remains unimproved for 15 years.

We first observe that no $O(1)$-approximation is to the optimal dilation is possible unless P=NP, implying that the NUkC problem is more non-trivial than the above two problems. Our main algorithmic result is an $(O(1), O(1))$-*bi-criteria* approximation result: we give an $O(1)$-approximation to the optimal dilation, however, we may open $\Theta(1)$ centers of each radii. Our techniques also allow us to prove a simple (uni-criteria), optimal 2-approximation to the kCwO problem improving upon the long-standing 3-factor. Our main technical contribution is a connection between the NUkC problem and the so-called firefighter problems on trees which have been studied recently in the TCS community. We show NUkC is as hard as the firefighter problem. While we don't know if the converse is true, we are able to adapt ideas from recent works [4, 1] in non-trivial ways to obtain our constant factor bi-criteria approximation.
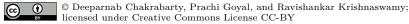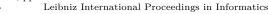
1   Introduction

Source location and vehicle routing problems are extremely well studied [20, 24, 10] in operations research. Consider the following location+routing problem: we are given a set of k ambulances with speeds $s_1, s_2, \ldots, s_k$ respectively, and we have to find the depot locations for these vehicles in a metric space (X, d) such that any point in the space can be served by some ambulance as fast as possible. If all speeds were the same, then we would place the ambulances in locations S such that $\max_{v \in X} d(v, S)$ is minimized – this is the famous
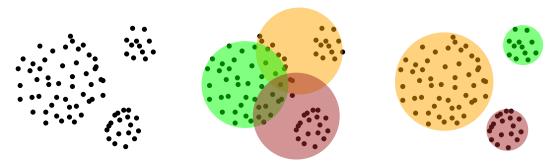
Figure 1 The left figure shows the dataset, the middle figure shows a traditional k-center clustering, and the right figure depicts a non-uniform clustering

k-center problem. Differing speeds, however, leads to non-uniformity, thus motivating the titular problem we consider.

▶ **Definition 1** (The Non-Uniform k-Center Problem (NUkC))**.** The input to the problem is a metric space (X, d) and a collection of k balls of radii $\{r_1 \geq r_2 \geq \cdots \geq r_k\}$. The objective is to find a placement $C \subseteq X$ of the centers of these balls, so as to minimize the dilation parameter α such that the union of balls of radius $\alpha \cdot r_i$ around the ith center covers all of X. Equivalently, we need to find centers $\{c_1, \ldots, c_k\}$ to minimize $\max_{v \in X} \min_{i=1}^{k} \frac{d(v, c_i)}{r_i}$.

As mentioned above, when all r_i's are the same (and equal to 1 by scaling), we get the k-center problem. The k-center problem was originally studied by Gonzalez [11] and Hochbaum and Shmoys [14] as a clustering problem of partitioning a metric space into different clusters to minimize maximum intra-cluster distances. One issue (see Figure 1 for an illustration and refer to [12] for a more detailed explanation) with k-center (and also k-median/means) as an objective function for clustering is that it favors clusters of similar sizes with respect to cluster radii. However, in presence of qualitative information on the differing cluster sizes as is often the case in certain applications, the non-uniform versions of the problem can arguably provide more nuanced solutions. One extreme special case was considered as the "clustering with outliers" problem [8] where a fixed number/fraction of points in the metric space need not be covered by the clusters. In particular, Charikar et al [8] consider (among many problems) the k-center with outlier problem (kCwO, for short) and show a 3-approximation for this problem. It is easy to see that kCwO is a special case of NUkC when there are k balls of radius 1 and ℓ (the number of outliers) balls of radius 0.

Motivated by the aforementioned reasons (both from facility location as well as from clustering settings) In this paper, we investigate the worst-case complexity of the NUkC problem. Gonzalez [11] and Hochbaum and Shmoys [14] give 2-approximations for the k-center problem, and also show that no better factor is possible unless P = NP. Charikar et al [8] give a 3-approximation for the kCwO problem, and this has been the best factor known for 15 years. Given these algorithms, it is natural to wonder if a simple $O(1)$-approximation exists for the NUkC problem. In fact, our first result shows a qualitative distinction between NUkC and these problems: constant-approximations are impossible for NUkC unless P=NP.

▶ **Theorem 2.** *For any constant $c \geq 1$, the NUkC problem does not admit a c-factor approximation unless $P = NP$, even when the underlying metric is a tree metric.*

The hardness result is by a reduction from the so-called *resource minimization for fire containment* problem on trees (RMFC-T, in short), a variant of the firefighter problem. To

circumvent the above hardness, we give the following bi-criteria approximation algorithm which is the main result of the paper, and which further highlights the connections with RMFC-T since our algorithms heavily rely on the recent algorithms for RMFC-T [4, 1]. An (a, b)-factor bi-criteria algorithm for NUkC returns a solution which places at most a balls of each type (thus in total it may use as many as $a \cdot k$ balls), and the dilation is at most b times the optimum dilation for the instance which places exactly one ball of each type.

▶ **Theorem 3.** *There is an $(O(1), O(1))$-factor bi-criteria algorithm for the NUkC problem.*

Furthermore, as we elucidate below, our techniques also give uni-criteria results when the number of distinct radii is 2. In particular, we get a 2-approximation for the kCwO problem and a $(1 + \sqrt{5})$-approximation when there are only two distinct types of radii.

▶ **Theorem 4.** *There is a 2-approximation for the kCwO problem.*

▶ **Theorem 5.** *There is a $(1 + \sqrt{5})$-approximation for the NUkC problem when the number of distinct radii is at most 2.*

1.1 Discussion on Techniques

Our proofs of Theorems 2 and 3 shows a strong connection between NUkC and the so-called *resource minimization for fire containment* problem on trees (RMFC-T, in short). This connection is one of the main findings of the paper, so we first formally define this problem.

▶ **Definition 6** (Resource Minimization for Fire Containment on Trees (RMFC-T))**.** Given a rooted tree T as input, the goal is to select a collection of non-root nodes N from T such that (a) every root-leaf path has at least one vertex from N, and (b) $\max_t |N \cap L_t|$ is minimized, where L_t is the tth-*layer* of T, that is, the vertices of T at exactly distance t from the root.

To understand the reason behind the name, consider a fire starting at the root spreading to neighboring vertices each day; the RMFC-T problem minimizes the number of firefighters needed per day so as to prevent the fire spreading to the leaves of T.

It is NP-hard to decide if the optimum of RMFC-T is 1 or not [9, 18]. Given any RMFC-T instance and any $c > 1$, we construct an NUkC instance on a tree metric such that in the "yes" case there is always a placement with dilation $= 1$ which covers the metric, while in the "no" case even a dilation of c doesn't help. Upon understanding our hardness construction, the inquisitive reader may wonder if the reduction also works in the other direction, i.e., whether we can solve NUkC using a reduction to RMFC-T problem. Unfortunately, we do not know if this is true even for two types of radii. However, as we explain below we still can use positive results for the RMFC-T problem to design good algorithms for the NUkC problem.

Indeed, we start off by considering the natural LP relaxation for the NUkC problem and describe an *LP-aware reduction* of NUkC to RMFC-T. More precisely, given a feasible solution to the LP-relaxation for the given NUkC instance $\mathcal{I}$, we describe a procedure to obtain an instance $\mathcal{I}'$ of RMFC-T, *and* also a feasible fractional solution for the natural LP relaxation of the RMFC-T problem on $\mathcal{I}'$. Moreover, given any feasible *integral* solution to the $\mathcal{I}'$, we can obtain a feasible solution to $\mathcal{I}$ which dilates the radii by a constant factor. An *LP-based* ρ-approximation to RMFC-T would then imply $(\rho, O(1))$-bi-criteria approximation algorithms for NUkC. Plugging in the result of Chalermsook and Chuzhoy [4], we directly obtain an $(O(\log^* n), O(1))$-bi-criteria approximation for NUkC. We can also obtain Theorem 4 and Theorem 5 since the corresponding RMFC-T instances have no integrality gap.

Here we reach a technical bottleneck: Chalermsook and Chuzhoy [4] also show that the integrality gap of the natural LP relaxation for RMFC-T is $\Omega(\log^* n)$. When combined with

our hardness reduction in Theorem 2 , this also implies a $(\Omega(\log^* n), c)$ integrality gap for any constant $c > 1$ for the natural LP relaxation for NUkC. That is, even if we allow a violation of c in the radius dilation, there is a $\Omega(\log^* n)$-integrality gap in terms of the violation in number of balls opened of each type. For RMFC-T though, Adjiashvili, Baggio and Zenklusen [1] recently showed an improved $O(1)$-approximation bypassing the LP integrality gap. At a very high level, the main technique in [1] is to carefully and efficiently "guess" a subset of the optimum solution, such that the natural LP-relaxation for covering only the uncovered leaves has $O(1)$-integrality gap. However, this guessing procedure crucially uses the tree structure of the given RMFC-T instance. Unfortunately for us though, we get the RMFC-T tree only after solving the LP for NUkC, which already has an $\Omega(\log^* n)$-gap! Nevertheless, inspired by the ideas in [1], we show that we can also efficiently "guess" the positions of a certain number of balls in an optimum solution, such that the standard LP-relaxation for covering the uncovered points has $O(1)$-gap. We can then invoke our reduction to RMFC-T to solve our problem. This is quite delicate, and is the most technically involved part of the paper.

1.2   Related Work and Open Questions

The k-center problem [11, 14] and the k-center with outliers [8] probems are classic problems in approximation algorithms and clustering. These problems have also been investigated under various settings such as the incremental model [6, 23], streaming model [5, 23], and more recently in the map-reduce model [15, 22]. Similarly, the k-median [7, 16, 21, 2] and k-means [16, 17, 13, 19] problems are also classic problems studied extensively in approximation algorithms and clustering. The generalization of k-median to a routing+location problem was also studied recently [10]. It would be interesting to explore the complexity of the non-uniform versions of these problems. Another direction would be to explore if the new non-uniform model can be useful in solving clustering problems arising in practice.

2   Hardness Reduction

In this section, we prove Theorem 2 based on the following NP-hardness [18] for RMFC-T.

▶ **Theorem 7** ([18]). *Given a tree T whose leaves are at the same distance from the root, it is NP-hard to distinguish between the following two cases.*
YES: *There is a solution to the* RMFC-T *instance of value* 1.
NO: *All solutions to the* RMFC-T *instance has value* 2.

Given an RMFC-T instance defined by tree T, we now describe the construction of our NUkC instance. Let h be the height of the tree, and let L_t denote the vertices of the tree at distance exactly t from the root. So, the leaves constitute L_h since all leaves are at the same distance from the root. The NUkC instance, $\mathcal{I}(T)$, is defined by the metric space (X, d), and a collection of balls. The points in our metric space will correspond to the leaves of the tree, i.e., $X = L_h$. To define the metric, we assign a weight $d(e) = (2c + 1)^{h-i+1}$ for each edge whose one endpoint is in L_i and the other in L_{i-1}; we then define d be the shortest-path metric on X induced by this weighted tree. Finally, we set $k = h$, and define the k radii $r_1 \geq r_2 \geq \ldots \geq r_k$ iteratively as follows: define $r_k := 0$, and for $k \geq i > 1$, set $r_{i-1} := (2c + 1) \cdot r_i + 2(2c + 1)$. This completes the NUkC instance. Before proceeding we make the simple observation: for any two leaves u and u' with lca $v \in L_t$, we have $d(u, u') = 2(2c + 1 + (2c + 1)^2 + \cdots + (2c + 1)^{h-t}) = r_t$. The following lemma proves Theorem 2.

▶ **Lemma 8.** *If T is the YES case of Theorem 7, then $\mathcal{I}(T)$ has optimum dilation $= 2$. If T is the NO case of 7, then $\mathcal{I}(T)$ has optimum dilation $\geq 2c$.*

Proof. Suppose T is in the YES case, and there is a solution to RMFC-T which selects at most 1 node from each level L_t. If $v \in L_t$ is selected, then select a center c_v *arbitrarily* from any leaf in the sub-tree rooted at v and open the ball of radius r_t. We now need to show all points in $X = L_h$ are covered by these balls. Let u be any leaf; there must be a vertex v in some level L_t in u's path to the root such that a ball of radius r_t is opened at c_v. However, $d(u, c_v) \leq d(u, v) + d(v, c_v) = 2r_t$ and so the ball of radius $2r_t$ around c_v covers u.

Now suppose T is in the NO case, and the NUkC instance has a solution with optimum dilation $< 2c$. We build a good solution for the RMFC-T instance N as follows: suppose the NUkC solution opens the radius $< 2c \cdot r_t$ ball around center u. Let v be the vertex on the u-root path appearing in level L_t. We then pick this node in N. Observe two things: first, this ball covers all the leaves in the sub-tree rooted at v since $r_t \geq d(u, u')$ for any such u'. Furthermore, since the NUkC solution has only one ball of each radius, we get that $|N \cap L_t| \leq 1$. Finally, since $d(u, w) \geq 2c \cdot r_t$ for all leaves w not in the sub-tree rooted at v, the ball of radius r_t around u doesn't contain any leaves other than those rooted at v. Contra-positively, since all leaves w are covered in some ball, every leaf must lie in the sub-tree of some vertex picked in N. That is, N is a solution to RMFC-T with value $= 1$ contradicting the NO case. ◀

3 LP-aware reduction from NUkC to RMFC-T

For reasons which will be apparent soon, we consider instances $\mathcal{I}$ of NUkC counting multiplicites. That is, we consider an instance to be a collection of tuples $(k_1, r_1), \ldots, (k_h, r_h)$ to indicate there are k_i balls of radius r_i. So we have $r_1 \geq r_2 \ldots \geq r_h$ and $\sum_{t=1}^{h} k_t = k$. Intuitively, the reason we do this is that if two radii r_t and r_{t+1} are "close-by" then it makes sense to round up r_{t+1} to r_t and increase k_t, losing only a constant-factor loss in the dilation.

LP-relaxation for NUkC. We now state the natural LP relaxation for a given NUkC instance $\mathcal{I}$. For each $p \in X$ and radius type r_i, we have a variable $x_{p,i} \geq 0$ denoting the extent to which we place a ball of radius r_i centered at p. By doing a binary search on the optimal dilation and scaling, we may assume that the optimum dilation is 1. Then, the following linear program must be feasible. In what follows, define $B(q, r_i) = \{p : d(p, q) \leq r_i\}$.

$$\forall p \in X, \qquad \sum_{t=1}^{h} \sum_{q \in B(p, r_t)} x_{q,t} \geq 1 \qquad \text{(NUkC LP)}$$

$$\forall t \in 1, \cdots, h \qquad \sum_{p \in X} x_{p,t} \leq k_t$$

LP-relaxation for RMFC-T. Since we reduce fractional NUkC to fractional RMFC-T, we now state the natural LP relaxation for RMFC-T on a tree T of depth $h + 1$. In fact, we will work with the following budgeted-version of RMFC-T (which is equivalent to the original RMFC-T problem — for a proof, see [1]): Instead of minimizing the maximum number of "firefighters" at any level t (that is $|N \cap L_t|$ where N is the chosen solution), suppose we specify a budget limit of k_t on $|N \cap L_t|$. The goal is the minimize the maximum dilation of these budgets. Then the following is a natural LP relaxation for the budgeted RMFC-T problem on trees. Here $L = L_h$ is the set of leaves, and L_t are the layer t-nodes. For a leaf

node v, we let P_v denote the vertex set of the unique leaf-root path excluding the root.

$$\min \alpha$$

$$\forall v \in L, \qquad \sum_{u \in P_v} y_u \quad \geq 1 \qquad\qquad\qquad\qquad \text{(RMFC-T LP)}$$

$$\forall t \in 1, \cdots, h \qquad \sum_{u \in L_t} y_u \quad \leq \alpha \cdot k_t$$

The LP-aware Reduction to Tree metrics.   We now describe our main reduction algorithm, which takes as input an NUkC instance $\mathcal{I} = \{(X, d); (k_1, r_1), \ldots, (k_h, r_h)\}$ and a feasible solution x to NUkC LP, and returns a budgeted RMFC-T instance $\mathcal{I}_T$ defined by a tree T along with budgets for each level, and a feasible solution y to RMFC-T LP with dilation 1. The tree we construct will have height $h + 1$ and the budgeted RMFC-T instance will have budgets precisely k_t at level $1 \leq t \leq h$, and the budget for the leaf level is 0. For clarity, throughout this section we use the word *points* to denote elements of the metric space in $\mathcal{I}$, and the word *vertices/nodes* to denote the tree nodes in the RMFC-T instance that we construct. We build the tree T in a bottom-up manner, where in each round, we pick a set of far-away representative points (the distance scale increases as we move up the tree) and cluster all points to their nearest representative. This is similar to a so-called clustering step in many known algorithms for facility location (see e.g., [7]), but whereas an arbitrary set of far-away representatives would suffice in the facility location algorithms, we need to be careful in how we choose this set to make the overall algorithm work.

Formally, each vertex of the tree T is mapped to some point in X, and we denote the mapping of the vertices at level t by $\psi_t : L_t \to X$. We will maintain that each ψ_t will be injective, so $\psi_t(u) \neq \psi_t(v)$ for $u \neq v$ in L_t. So, ψ_t^{-1} is well defined for the range of ψ_t.

The complete algorithm runs in rounds $h + 1$ to 2 building the tree one level per round. To begin with, the ψ_{h+1} mapping is an arbitrary bijective mapping between $L := L_{h+1}$, the set of leaves of the tree, and the points of X (so, in particular, $|L| = |X|$). We may assume it to be the identity bijection. In each round t, the range of the mappings become progressively smaller, that is[1], $\psi_t(L_t) \supseteq \psi_{t-1}(L_{t-1})$. We call $\psi_t(L_t)$ as the **winners at level** t. We now describe round t. Let $\mathsf{Cov}_t(p) := \sum_{q \in B(p, r_t)} x_{q,t}$ denote the fractional amount the point p is covered by radius r_t balls in the solution x. Also define $\mathsf{Cov}_{\geq t}(p) := \sum_{s \geq t} \mathsf{Cov}_s(p)$ denoting the fractional amount p is covered by radius r_t or smaller balls. Let $\mathsf{Cov}_{h+1}(p) = 0$ for all p.

Finally, we add a root vertex and connect it to all vertices in L_1. This gives us the final tree T and a solution y which assigns a value to all non-leaf, non-root vertices of the tree T. The following claim asserts well-definedness of the algorithm.

▶ **Lemma 9.** *The solution y is a feasible solution to* RMFC-T *LP on $\mathcal{I}_T$ with dilation* 1.

Proof. The proof is via two claims for the two different set of inequalities.

▶ **Claim 1.** *For all $1 \leq t \leq h$, we have $\sum_{w \in L_t} y_w \leq k_t$.*

Proof. Fix t. Let $W_t \subseteq X$ denote the winners at level t, that is, $W_t = \psi_t(L_t)$. By definition of the algorithm, $\sum_{w \in L_t} y_w = \sum_{p \in W_t} \mathsf{Cov}_t(p)$. Now note that for any two points $p, q \in W_t$, we have $B(p, r_t) \cap B(q, r_t) = \emptyset$. To see this, consider the first point which enters A in the $(t + 1)$th round when L_t was being formed. If this is p, then all points in the radius $2r_t$ ball

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[1] We are using the notation $\psi(X) := \bigcup_{x \in X} \psi(x)$.

Algorithm 1 Round t of the LP-aware Reduction.

    Input: Level L_t, subtrees below L_t, the mappings $\psi_s : L_s \to X$ for all $t \leq s \leq h$.
    Output: Level L_{t-1}, the connections between L_{t-1} and L_t, and the mapping ψ_{t-1}.
    Define $A = \psi_t(L_t)$ the set of points who are winners at level t.
    while $A \neq \emptyset$ **do**
        (a) Choose the point $p \in A$ with *minimum coverage* $\mathsf{Cov}_{\geq t}(p)$.
        (b) Let $N(p) := \{q \in A : d(p,q) \leq 2r_{t-1}\}$ be the set of all nearby points in A to p.
        (c) Create a new tree vertex $w \in L_{t-1}$ corresponding to p and set $\psi_{t-1}(w) := p$. *Call p
        a winner at level $t-1$, and each $q \in N(p) \subseteq A$ a loser to p at this level.*
        (d) Create edge (w,v) for tree vertices $v \in \psi_t^{-1}(N(p))$ associated with $N(p)$ at level t.
        (e) Set $A \leftarrow A \setminus (N(p))$.
        (f) Set $y_w = \mathsf{Cov}_{t-1}(p)$.
    end while

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are deleted from A. Since the balls are disjoint, the second inequality of NUkC LP implies $\sum_{p \in W_t} \sum_{q \in B(p,r_t)} x_{q,t} \leq k_t$. The second summand in the LHS is precisely $\mathsf{Cov}_t(p)$. ◄

▶ **Claim 2.** *For any leaf node $w \in L$, we have $\sum_{v \in P_w} y_v \geq 1$.*

Proof. We start with an observation. Fix a level t and a winner point $p \in W_t$. Let $u \in L_t$ such that $\psi_t(u) = p$. Since $W_t \subseteq W_{t+1} \subseteq \cdots \subseteq W_h$, there is a leaf v in the subtree rooted at u corresponding to p. Moreover, by the way we formed our tree edges in step (d), we have that $\psi_s(w') = p$ for all w' in the (u,v)-path and hence $\sum_{w' \in [u,v]\text{-path}} y_{w'} = \mathsf{Cov}_{\geq t}(p)$.

Now, for contradiction, suppose there is some leaf corresponding to, say point p, such that the root-leaf path has total y-assignment less than 1. Then, pick the point, among all such unsatisfied points p, who appears in a winning set W_t with t *as small as possible.*

By the preceding observation, the total y-assignment p receives on its path from level h to level t is exactly $\mathsf{Cov}_{\geq t}(p)$. Moreover, suppose p loses to q at level $t-1$, i.e., $\psi_t^{-1}(p)$ is a child of $\psi_{t-1}^{-1}(q)$. In particular, it means that q has also been a winner up to level t and so the total y-assignment on q's path upto level t is also precisely $\mathsf{Cov}_{\geq t}(q)$. Additionally, since $\psi_{t-1}^{-1}(q)$ became the parent node for $\psi_t^{-1}(p)$, we know that $\mathsf{Cov}_{\geq t}(q) \leq \mathsf{Cov}_{\geq t}(p)$ due to the way we choose winners in step (a) of the while loop. Finally, by our maximality assumption on p, we know that q is fractionally satisfied by the y-solution. Therefore, there is fractional assignment of at least $(1 - \mathsf{Cov}_{\geq t}(q))$ on q's path from nodes in level $t-1$ to level 1. Putting these observations together, we get that the total fractional assignment on p's root-leaf path is at least $\mathsf{Cov}_{\geq t}(p) + (1 - \mathsf{Cov}_{\geq t}(q)) \geq 1$, which results in the desired contradiction. ◄

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The following lemma shows that any good *integral* solution to the RMFC-T instance $\mathcal{I}_T$ can be converted to a good integral solution for the NUkC instance $\mathcal{I}$.

▶ **Lemma 10.** *Suppose there exists a feasible solution N to $\mathcal{I}_T$ such that for all $1 \leq t \leq h$, $|N \cap L_t| \leq \alpha k_t$. Then there is a solution to the NUkC instance $\mathcal{I}$ that opens, for each $1 \leq t \leq h$, at most αk_t balls of radius $\leq 2r_{\geq t}$, where $r_{\geq t} := r_t + r_{t+1} + \cdots + r_h$.*

Proof. Construct the NUkC solution as follows: for level $1 \leq t \leq h$ and every vertex $w \in N \cap L_t$, place the center at $\psi_t(w)$ of radius $2r_{\geq t}$. We claim that every point in X is covered by some ball. Indeed, for any $p \in X$, look at the leaf $v = \psi_{h+1}(p)$, and let $w \in N$ be a node in the root-leaf path. Let $w \in L_t$ for some t. Now observe that $d(p, \psi_t(w)) \leq 2r_{\geq t}$;

this is because for any edge (u', v') in the tree where u' is in L_t and is the parent of v', we have that $d(\psi_{t+1}(v'), \psi_{t+1}(u')) < 2r_t$. ◀

This completes the reduction, and we now prove a few results using this.

▶ **Theorem 11.** *There is a polynomial time $(\mathcal{O}(\log^* n), 8)$-bi-criteria algorithm for* NUkC.

Proof. Given any instance $\mathcal{I}$ of NUkC, we first club the radii to the nearest power of 2 to get an instance $\mathcal{I}'$ with radii $(k_1, r_1), \cdots, (k_h, r_h)$ such that an (a, b)-factor solution for $\mathcal{I}'$ is an $(a, 2b)$-solution for $\mathcal{I}$. Now, by scaling, we assume that the optimal dilation for $\mathcal{I}'$ is 1; we let x be the feasible solution to the NUkC LP. Then, using Algorithm 1, we can construct the tree $\mathcal{I}'_T$ and a feasible solution y to the RMFC-T LP. We can now use the following theorem of Chalermsook and Chuzhoy [4]: given any feasible solution to the RMFC-T LP, we can obtain a feasible set N covering all the leaves such that for all t, $|N \cap L_t| \leq \mathcal{O}(\log^* n)k_t$. Finally, we can apply Lemma 10 to obtain a $(\mathcal{O}(\log^* n), 4)$ solution to $\mathcal{I}'$ (since $r_{\geq t} \leq 2r_t$). ◀

Proof of Theorem 4 and Theorem 5. We use the following claim regarding the integrality gap of RMFC-T LP for depth 2 trees.

▶ **Claim 3.** *When $h = 2$ and k_t's are integers, given any fractional solution to* RMFC-T *LP, we can find a feasible integral solution as well.*

Proof. Given a feasible solution y to RMFC-T LP, we need to find a set N such that $|N \cap L_t| \leq k_t$ for $t = 1, 2$. There must exist at least one vertex $w \in L_1$ such that $y_w \in (0, 1)$ for otherwise the solution y is trivially integral. If only one vertex $w \in L_1$ is fractional, then since k_1 is an integer, we can raise this y_w to be an integer as well. So at least two vertices w and w' in L_1 are fractional. Now, without loss of generality, let us assume that $|C(w)| \geq |C(w')|$, where $C(w)$ is the set of children of w. Now for some small constant $0 < \epsilon < 1$, we do the following: $y'_w := y_w + \epsilon$, $y'_{w'} := y_{w'} - \epsilon$, $\forall c \in C(w)$, $y'_c := y_c - \epsilon$, and $\forall c \in C(w')$, $y'_c := y_c + \epsilon$. Note that $y(L_1)$ remains unchanged, $y(L_2)$ can only decrease, and root-leaf paths still add to at least 1. We repeat this till we rule out all fractional values. ◀

To see the proof of Theorem 4, note that an instance of the k-center with outliers problem is an NUkC instance with $(k, 1), (\ell, 0)$, that is, $r_1 = 1$ and $r_2 = 0$. We solve the LP relaxation and obtain the tree and an RMFC-T solution. The above claim implies a feasible integral solution to RMFC-T since $h = 2$, and finally note that $r_{\geq 1} = r_1$ for kCwO, implying we get a 2-factor approximation.

The proof of Theorem 5 is similar. If $r_1 < \theta r_2$ where $\theta = (\sqrt{5} + 1)/2$, then we simply run k-center with $k = k_1 + k_2$. This gives a $2\theta = \sqrt{5} + 1$-approximation. Otherwise, we apply Lemma 10 to get a $2(1 + \frac{1}{\theta}) = \sqrt{5} + 1$-approximation. ◀

We end this section with a general theorem, which is an improvement over Lemma 10 in the case when many of the radius types are close to each other, in which case $r_{\geq t}$ could be much larger than r_t. Indeed, the natural way to overcome this would be to group the radius types into geometrically increasing values as we did in the proof of Theorem 11. However, for some technical reasons we will not be able to bucket the radius types in the following section, since we would instead be bucketing the *number of balls* of each radius type in a geometric manner. Instead, we can easily modify Algorithm 1 to build the tree by focusing only on radius types where the radii grow geometrically.

▶ **Theorem 12.** *Given an* NUkC *instance* $\mathcal{I} = \{M = (X, d), (k_1, r_1), (k_2, r_2), \ldots, (k_h, r_h)\}$ *and an LP solution x for* NUkC *LP, there is an efficient reduction which generates an* RMFC-T *instance* $\mathcal{I}_T$ *and an LP solution y to* RMFC-T *LP, such that the following holds:*

(i) *For any two tree vertices $w \in L_t$ and $v \in L_{t'}$ where w is an ancestor of v (which means $t \leq t'$), suppose p and q are the corresponding points in the metric space, i.e., $p = \psi_t(w)$ and $q = \psi_{t'}(v)$, then it holds that $d(p, q) \leq 8 \cdot r_t$.*

(ii) *Suppose there exists a feasible solution N to $\mathcal{I}_T$ such that for all $1 \leq t \leq h$, $|N \cap L_t| \leq \alpha k_t$. Then there is a solution to the NUkC instance $\mathcal{I}$ that opens, for each $1 \leq t \leq h$, at most αk_t balls of radius at most $8 \cdot r_t$.*

4 Getting an $(O(1), O(1))$-approximation algorithm

In this section, we improve our approximation factor on the number of clusters from $O(\log^* n)$ to $O(1)$, while maintaining a constant-approximation in the radius dilation. As mentioned in the introduction, this requires more ideas since using NUkC LP one cannot get any factor better than $(O(\log^* n), O(1))$-bi-criteria approximation since any integrality gap for RMFC-T LP translates to a $(\Omega(\log^* n), \Omega(1))$ integrality gap for NUkC LP.

Our algorithm is heavily inspired by the recent paper of Adjiashvili et al [1] who give an $O(1)$-approximation for the RMFC-T problem. In fact, the structure of our algorithms follows the same three "steps" of their algorithm. Given an RMFC-T instance, in [1] the authors first "compress" the input tree to get a new tree whose depth is only logarithmic; next, they give a partial rounding result which saves "bottom heavy" leaves, that is, leaves which are fractionally covered to at least a constant fraction by low level tree nodes; and finally, they give a clever *partial enumeration algorithm* for guessing the nodes from the top levels chosen by the optimum solution. We also proceed in these three steps with the first two being very similar. However, the enumeration step requires new ideas for our problem. In particular, the enumeration procedure in [1] crucially uses the tree structure of the firefighter instance, and the way our reduction generates the tree for the RMFC-T instance is by using the optimal LP solution for the NUkC instance, which in itself suffers from the $\Omega(\log^* n)$ integrality gap. Therefore, we need to devise a more sophisticated enumeration scheme inspired by the one in [1]. Throughout this section, we do not optimize for the constants.

4.1 Part I: Radii Reduction

In this part, we describe a preprocessing step which decreases the number of types of radii. This is similar to Theorem 5 in [1].

▶ **Theorem 13.** *Let $\mathcal{I}$ be an NUkC instance with radii $\{r_1, r_2, \cdots, r_k\}$. We can efficiently compute instance $\widehat{\mathcal{I}}$ with radii multiplicities $(k_0, \widehat{r}_0), ..., (k_L, \widehat{r}_L)$ and $L = \Theta(\log k)$ such that:*

(i) *$k_i := 2^i$ for all $0 \leq i < L$ and $k_L \leq 2^L$.*

(ii) *If the NUkC instance $\mathcal{I}$ has a feasible solution, then there exists a feasible solution for $\widehat{\mathcal{I}}$.*

(iii) *Given an (α, β)-bi-criteria solution to $\widehat{\mathcal{I}}$, we can efficiently obtain a $(3\alpha, \beta)$-bi-criteria solution to $\mathcal{I}$.*

Proof. For an instance $\mathcal{I}$, we construct the compressed instance $\widehat{\mathcal{I}}$ as follows. Partition the radii into $\Theta(\log k)$ classes by defining barriers at $\widehat{r}_i = r_{2^i}$ for $0 \leq i \leq \lfloor \log k \rfloor$. Now to create instance $\widehat{\mathcal{I}}$, we simply round up all the radii r_j for $2^i \leq j < 2^{i+1}$ to the value $\widehat{r}_i = r_{2^i}$. Notice that the multiplicity of $\widehat{r}_i$ is precisely 2^i (except maybe for the last bucket, where there might be fewer radii rounded up than the budget allowed).

Property (i) follows by construction. Property (ii) follows from the way we rounded up the radii. Indeed, if the optimal solution for $\mathcal{I}$ opens a ball of radius r_j around a point p,

then we can open a ball of radius $\widehat{r}_i$ around p, where i is such that $2^i \leq j < 2^{i+1}$. Clearly the number of balls of radius $\widehat{r}_i$ is at most 2^i since OPT uses at most one ball of each radius r_j.

For property (iii), suppose we have a solution $\widehat{S}$ for $\widehat{\mathcal{I}}$ which opens $\alpha 2^i$ clusters of radius $\beta \widehat{r}_i$ for all $0 \leq i \leq L$. Construct a solution S for $\mathcal{I}$ as follows. For each $1 \leq i \leq L$, let C_i denote the set of centers where $\widehat{S}$ opens balls of radius $\beta \widehat{r}_i$. In the solution S, we also open balls at precisely these centers with 2α balls of radius r_j for every $2^{i-1} \leq j < 2^i$. Since $|C_i| \leq \alpha \cdot 2^i$, we can open a ball at every point in C_i; furthermore, since $j < 2^i$, we have $r_j \geq \widehat{r}_i$ and so we cover whatever the balls from $\widehat{S}$ covered.

Finally, we also open the α clusters (corresponding to $i = 0$) of radius $\beta r_1 = \beta \widehat{r}_0$ at the respective centers C_0 where $\widehat{S}$ opens centers of radius $\widehat{r}_0$. Therefore, the total number of clusters of radius type is at most 2α with the exception of r_1, which may have 3α clusters. ◄

4.2 Part II: Satisfying Bottom Heavy Points

One main reason why the above height reduction step is useful, is the following theorem from [1] for RMFC-T instances on trees; we provide a proof sketch for completeness.

▶ **Theorem 14** ([1]). *Given a tree T of height h and a feasible solution y to (*RMFC-T *LP), we can find a feasible integral solution N to* RMFC-T *such that for all $1 \leq t \leq h$, $|N \cap L_t| \leq k_t + h$.*

Proof. Let y be a basic feasible solution of (RMFC-T LP). Call a vertex v of the tree *loose* if $y_v > 0$ and the sum of y-mass on the vertices from v to the root (inclusive of v) is < 1. Let V_L be the set of loose vertices of the tree, and let V_I be the set of vertices with $y_v = 1$. Clearly $N = V_L \cup V_I$ is a feasible solution: every leaf-to-root path either contains an integral vertex or at least two fractional vertices with the vertex closer to root being loose. Next we claim that $|V_L| \leq h$; this proves the theorem since $|N \cap L_t| \leq |V_I \cap L_t| + |V_L| \leq k_t + |V_L|$.

The full proof can be found in Lemma 6, [1] – here is a high level sketch. There are $|L| + h$ inequalities in (RMFC-T LP), and so the number of fractional variables is at most $|L| + h$. We may assume there are no $y_v = 1$ vertices. Now, in every leaf-to-root path there must be at least 2 fractional vertices, and the one closest to the leaf must be non-loose. If the closest fractional vertex to each leaf was unique, then that would account for $|L|$ fractional non-loose vertices implying the number of loose vertices must be $\leq h$. This may not be true; however, if we look at *linearly independent* set of inequalities that are tight, we can argue uniqueness as a clash can be used to exhibit linear dependence between the tight constraints. ◄

▶ **Theorem 15.** *Suppose we are given an* NUkC *instance $\widehat{\mathcal{I}}$ with radii multiplicities $(k_0, \widehat{r}_0), (k_1, \widehat{r}_1), \ldots, (k_L, \widehat{r}_L)$ with budgets $k_i = 2^i$ for radius type $\widehat{r}_i$, and an LP solution x to (*NUkC LP*) for $\widehat{\mathcal{I}}$. Let $\tau = \log \log L$, and suppose $X' \subseteq X$ be the points covered mostly by small radii, that is, let $\mathsf{Cov}_{\geq \tau}(p) \geq \frac{1}{2}$ for every $p \in X'$. Then, there is an efficient procedure* round *which opens at most $O(k_t)$ balls of radius $O(\widehat{r}_t)$ for $\tau \leq t \leq L$, and covers all of X'.*

Proof. The procedure round works as follows: we partition the points of X' into two sets, one set X_U in which the points receive at least $\frac{1}{4}$ of the coverage by clusters of radius $\widehat{r}_i$ where $i \in \{\log \log L, \log \log L + 1, \ldots, \log L\}$, and another set X_B in which the points receive $\frac{1}{4}$ coverage from clusters of levels $t \in \{\log L + 1, \log L + 2, \ldots, L\}$. More precisely, $X_U := \{p \in X' : \sum_{t=\tau}^{\log L} \mathsf{Cov}_t(p) \geq 1/4\}$, and $X_B = X' \setminus X_B$.

Now consider the following LP-solution to (NUkC LP) for $\widehat{\mathcal{I}}$ restricted to X_U: we scale x by a factor 4 and zero-out x on radii type $\widehat{r}_i$ for $i \notin \{\log \log L, \ldots, \log L\}$. By definition of X_U this is a feasible fractional solution; furthermore, the LP-reduction algorithm described

in Section 3 will lead to a tree T of height $\leq \log L$ and fractional solution y for (RMFC-T LP) on T were each $k_i \geq 2^{\log \log L} = \log L$. Applying Theorem 14, we can find an integral solution N with at most $O(k_i)$ vertices at levels $i \in \{\log \log L, \ldots, \log L\}$. We can then translate this solution back using Theorem 12 to NUkC and find $O(k_t)$ clusters of radius $O(\widehat{r}_t)$ to cover all the points X_U. A similar argument, when applied to the smaller radius types $\widehat{r}_t$ for $t \in \{\log L, \ldots, L\}$ can cover the points in X_B. ◄

We now show how we can immediately also get a (very weakly) quasi-polynomial time $O(1)$-approximation for NUkC. Indeed, if we could enumerate the set of clusters of radii $\widehat{r}_t$ for $0 \leq t < \log \log L$, we can then explicitly solve an LP where all the uncovered points need to be fractionally covered by only clusters of radius type $\widehat{r}_t$ for $t \geq \log \log L$. We can then round this solution using Corollary 15 to obtain the desired $O(1)$-approximation for the NUkC instance. Moreover, the time complexity of enumerating the optimal clusters of radii $\widehat{r}_t$ for $0 \leq t < \log \log L$ is $n^{O(\log L)} = n^{O(\log \log k)}$, since the number of clusters of radius at least $\widehat{r}_{\log \log L}$ is at most $O(2^{\log \log L}) = O(\log L)$. Finally, there was nothing special in the proof of Corollary 15 about the choice of $\tau = \log \log L$ — we could set $t = \log^{(q)} L$ to be the qth iterated logarithm of L, and obtain an $O(q)$-approximation. As a result, we get the following corollary. Note that this gives an alternate way to prove Theorem 11.

▶ **Corollary 16.** *For any $q \geq 1$, there exists an $(O(q), O(1))$-factor bi-criteria algorithm for* NUkC *which runs in $n^{O(\log^{(q)} k)}$ time.*

4.3 Part III: Clever Enumeration of Large Radii Clusters

In this section, we show how to obtain the $(O(1), O(1))$-factor bi-criteria algorithm. At a high level, our algorithm tries to "guess" the centers[2] A of large radius, that is $\widehat{r}_i$ for $i \leq \tau := \log \log L = \log \log \log k$, which the optimum solution uses. However, this guessing is done in a cleverer way than in Corollary 16. In particular, given a guess which is consistent with the optimum solution (the initial "null set" guess is trivially consistent), our enumeration procedure generates a list of candidate additions to A of size *at most* $2^\tau \approx \operatorname{poly} \log \log k$ (instead of n), one of which is a consistent enhancement of the guessed set A. This reduction in number of candidates also requires us to maintain a guess D of points where the optimum solution *doesn't* open centers. Furthermore, we need to argue that the "depth of recursion" is also bounded by $\operatorname{poly} \log \log k$; this crucially uses the technology developed in Section 3. Altogether, we get the total time is at most $(\operatorname{poly} \log \log k)^{\operatorname{poly} \log \log k} = o(k)$ for large k. In this extended abstract with page limits, we omit all proofs in this subsection, and point the reader to the full version of our paper [3].

We start with some definitions. Throughout, A and D represent sets of tuples of the form (p, t) where $p \in X$ and $t \in \{0, 1, \ldots, \tau\}$. Given such a set A, we associate a partial solution S_A which opens a ball of radius $22\widehat{r}_t$ at the point p for all p s.t. $(p, t) \in A$. For the sake of analysis, fix an optimum solution OPT. We say the set A is **consistent** with OPT if for all $(p, t) \in A$, there exists a *unique* $q \in X$ such that OPT opens a ball of radius $\widehat{r}_t$ at q and $d(p, q) \leq 11\widehat{r}_t$. In particular, this implies that S_A covers all points which this OPT-ball covers. We say the set D is **consistent** with OPT if for all $(q, t) \in D$, OPT *doesn't* open a radius $\widehat{r}_t$ ball at q (it may open a different radius ball at q though). Given a pair of sets

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2 Actually, we end up guessing centers "close" to the optimum centers, but for this introductory paragraph this intuition is adequate.

(A,D), we define the minLevel of each point p to be the type of largest radius consistent with our guesses which can cover it, i.e.,

$$\mathsf{minLevel}_{A,D}(p) := 1 + \arg\max_{t}\{(q,t) \in D \ \text{ for all } \ q \in B(p,\widehat{r}_t)\}$$

If (A, D) is a consistent pair and $\mathsf{minLevel}_{A,D}(p) = t$, then this implies in the OPT solution, p is covered by a ball of radius $\widehat{r}_t$ or smaller.

Next, we describe a nuanced LP-relaxation for NUkC. Fix a pair of sets (A, D) as described above. Let X_G be the subset of points in X covered by the partial solution S_A. Fix a subset $Y \subseteq X \setminus X_G$ of points. Define the following LP.

$$\forall p \in Y, \qquad \sum_{t=\mathsf{minLevel}(p)}^{L} \sum_{q \in B(p,\widehat{r}_t)} x_{q,t} \ \geq 1 \qquad\qquad (\text{LP}_{\mathsf{NUkC}}(Y, A, D))$$

$$\forall t \in 1, \cdots, h \qquad \sum_{q \in Y} x_{q,t} \ \leq \ k_t$$

$$\forall (p,t) \in A, \qquad x_{p,t} = 1$$

The following claim encapsulates the utility of the above relaxation.

▶ **Claim 4.** *If (A, D) is consistent with* OPT*, then* $LP_{\mathsf{NUkC}}(X \setminus X_G, A, D)$ *is feasible.*

Finally, for convenience, we define a **forbidden set** $F := \{(p, i) : p \in X, 1 \leq i \leq \tau\}$ which if added to D disallows any large radii balls to be placed anywhere.

Now we are ready to describe the enumeration Algorithm 2. We start with A and D being null, and thus vacuously consistent with OPT. The enumeration procedure ensures that: given a consistent (A, D) tuple, either it finds a good solution using LP rounding (Step 10), or generates candidate additions (Steps 18–20) to A or D ensuring that one of them leads to a larger consistent tuple.

Define $\gamma_0 := 4 \log \log k \cdot \log \log \log k$. The algorithm is run with $\mathsf{Enum}(\emptyset, \emptyset, \gamma_0)$. The proof that we get a polynomial time $(O(1), O(1))$-bi-criteria approximation algorithm follows from three lemmas. Lemma 17 shows that if Step 10 is true with a consistent pair (A, D), then the output in Step 13 is a $(O(1), O(1))$-approximation. Lemma 18 shows that indeed Step 10 is true for γ_0 as set. Finally, Lemma 19 shows that the algorithm runs in polynomial time.

▶ **Lemma 17.** *If (A, D) is a consistent pair such that Step 10 is true, then the solution returned is an $(O(1), O(1))$-approximation algorithm.*

▶ **Lemma 18.** $\mathsf{Enum}(\emptyset, \emptyset, \gamma_0)$ *finds consistent (A, D) such that Step 10 is true.*

▶ **Lemma 19.** $\mathsf{Enum}(\emptyset, \emptyset, \gamma_0)$ *runs in polynomial time for large enough k.*

Algorithm 2 $\mathsf{Enum}(A, D, \gamma)$

1: Let $X_G = \{p : \exists\, (q, i) \in A \text{ s.t } d(p, q) \leq 22\widehat{r_i}\}$ denote points covered by S_A.
2: **if** there is **no** feasible solution to $LP_{\mathsf{NUkC}}(X \setminus X_G, A, D)$ **then**
3:    **Abort.** // *Claim 4 implies (A, D) is not consistent.*
4: **else**
5:    Let x^* be a feasible solution to $LP_{\mathsf{NUkC}}(X \setminus X_G, A, D)$.
6: **end if**
7: Let $X_B = \{u \in X \setminus X_G : \mathsf{Cov}_{\geq \tau}(u) \geq \frac{1}{2}\}$ denote bottom-heavy points in x^*
8: Let S_B be the solution implied by Corollary 15.
   // *This solution opens $O(k_t)$ balls of radius $O(\widehat{r_t})$ for $\tau \leq t \leq L$ and covers all of X_B.*
9: Let $X_T = X \setminus (X_G \cup X_B)$ denote the top heavy points in x^*
10: **if** $LP_{\mathsf{NUkC}}(X_T, A, F \cup D)$ has a feasible solution x_T **then**
11:    By definition of F, in x_T we have $\mathsf{Cov}_{\geq \tau}(u) = 1$ for all $u \in X_T$.
12:    Let S_T be the solution implied by Corollary 15.
   // *This solution opens $O(k_t)$ balls of radius $O(\widehat{r_t})$ for $\tau \leq t \leq L$ and covers all of X_T.*
13:    Output $(S_A \cup S_B \cup S_T)$. // *This is a $(O(1), O(1))$-approximation for the NUkC instance.*
14: **else**
15:    **for** every level $0 \leq t \leq \tau$ **do**
16:       Let $C_t = \{p \in X_T \text{ s.t } \mathsf{minLevel}_{A,D}(p) = t\}$, the set of points in X_T with $\mathsf{minLevel}\ t$.
17:       Use the LP-aware reduction from Section 3 using x^* and the set of points C_t to create tree T_t.
18:       **for** every winner p at level t in T_t **do**
19:          $\mathsf{Enum}(A \cup \{(p, t)\}, D, \gamma - 1)$
20:          $\mathsf{Enum}(A, D \cup \bigcup_{p' \in B(p, 11\widehat{r_t})} \{(p', t)\}), \gamma - 1)$
21:       **end for**
22:    **end for**
23: **end if**

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