Robust Bell Inequalities from Communication Complexity*

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Abstract

The question of how large Bell inequality violations can be, for quantum distributions, has been the object of much work in the past several years. We say a Bell inequality is normalized if its absolute value does not exceed 1 for any classical (i.e. local) distribution. Upper and (almost) tight lower bounds have been given in terms of number of outputs of the distribution, number of inputs, and the dimension of the shared quantum states. In this work, we revisit normalized Bell inequalities together with another family: inefficiency-resistant Bell inequalities. To be inefficiency-resistant, the Bell value must not exceed 1 for any local distribution, including those that can abort. Both these families of Bell inequalities are closely related to communication complexity lower bounds. We show how to derive large violations from any gap between classical and quantum communication complexity, provided the lower bound on classical communication is proven using these lower bounds. This leads to inefficiency-resistant violations that can be exponential in the size of the inputs. Finally, we study resistance to noise and inefficiency for these Bell inequalities.

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1 Introduction

The question of achieving large Bell violations has been studied since Bell's seminal paper in 1964 [6]. In one line of investigation, proposals have been made to exhibit families of

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Table 1 Bounds on quantum violations of bipartite normalized Bell inequalities, in terms of the dimension d of the local Hilbert space, the number of settings (or inputs) N and the number of outcomes K (or outputs) per party. In the fourth column, we compare ad hoc results to the recent constructions of [10] (Theorem 7) which gives a lower bound of $\frac{\sqrt{c}}{q}$, where c (resp. q) stands for the classical (resp. quantum) communication complexity of simulating a distribution. We give upper bounds on their construction in terms of the parameters d, N, K.

Parameter	Upper bound	Ad hoc lower bounds	Best possible lower bound from [10]
Number of inputs N	$2^c \leq N$ [29, 12, 21]	$\frac{\sqrt{N}}{\log(N)}$ [19]	$\frac{\sqrt{c}}{q} \leq \log(N)$
Number of outputs K	$O(K)$ [19]	$\Omega\left(\frac{K}{(\log(K))^2}\right)$ [11]	$\leq \log(K)$
Dimension d	$O(d)$ [21]	$\Omega\left(\frac{d}{(\log(d))^2}\right)$ [11]	$\leq \log\log(d)$

distributions which admit unbounded violations [33, 28, 34, 36]. In another, various measures of nonlocality have been studied, such as the amount of communication necessary and sufficient to simulate quantum distributions classically [32, 7, 42, 43, 37, 12], or the resistance to detection inefficiencies and noise. More recently, focus has turned to giving upper and lower bounds on violations achievable, in terms of various parameters: number of players, number of inputs, number of outputs, dimension of the quantum state, and amount of entanglement [12, 21, 19].

Up until quite recently, violations were studied in the case of specific distributions (measuring Bell states), or families of distributions. Buhrman *et al.* [11] gave a construction that could be applied to several problems which had efficient quantum protocols (in terms of communication) and for which one could show a trade-off between communication and error in the classical setting. This still required an *ad hoc* analysis of communication problems. Recently Buhrman *et al.* [10] proposed the first general construction of quantum states along with Bell inequalities from any communication problem. The quantum states violate the Bell inequalities when there is a sufficiently large gap between quantum and classical communication complexity (a super-quadratic gap is necessary, unless a quantum protocol without local memory exists).

Table 1 summarizes the best known upper and lower bounds on quantum violations achievable with normalized Bell inequalities.

1.1 Our results

We revisit the question of achieving large Bell violations by exploiting known connections with communication complexity. Strong lower bounds in communication complexity, equivalent to the partition bound, amount to finding *inefficiency-resistant Bell inequalities* [27]. These are Bell functionals that are bounded above by 1 on all local distributions *that can abort.*

First, we study the resistance of normalized Bell inequalities to inefficiency. We show that, up to a constant factor in the value of the violation, any normalized Bell inequality can be made resistant to inefficiency while maintaining the normalization property (**Theorem 6**).

Second, we show how to derive large Bell violations from any communication problem for which the partition bound is bounded below and the quantum communication complexity is bounded above. The problems studied in communication complexity are far beyond the quantum set, but we show how to easily derive a quantum distribution from a quantum

Table 2 Comparison of the Bell violations obtained by the general construction of Buhrman *et al.* [10] for normalized Bell violations (second column) and this work, for inefficiency-resistant Bell violations (see Propositions 13, 14, 15, and 16). The parameter n is the size of the input (typically, $N = 2^n$.) Explicit Bell inequalities are given in the Appendix. The construction of Buhrman *et al.* only yields a violation when the gap between classical and quantum complexities is more than quadratic. In the case where the gap is too small to prove a violation, we indicate this with "N/A".

Problem	Normalized Bell violations [10]	Inefficiency-resistant Bell violations (this work)
VSP [38, 24]	$\Omega\left(\sqrt[6]{n}/\sqrt{\log n}\right)$ $d = 2^{\Theta(n\log n)}, K = 2^{\Theta(n)}$	$2^{\Omega(\sqrt[3]{n})-O(\log n)}$ $d = 2^{O(\log n)}, K = 3$
DISJ [39, 40, 1]	N/A	$2^{\Omega(n)-O(\sqrt{n})}$ $d = 2^{O(\sqrt{n})}, K = 3$
TRIBES [18, 9]	N/A	$2^{\Omega(n)-O(\sqrt{n}\log^2 n)}$ $d = 2^{O(\sqrt{n}\log^2 n)}, K = 3$
ORT [41, 9]	N/A	$2^{\Omega(n)-O(\sqrt{n}\log n)}$ $d = 2^{O(\sqrt{n}\log n)}, K = 3$

protocol. The Bell value we obtain is 2^{c-2q}, where c is the partition lower bound on the classical communication complexity of the problem considered, and q is an upper bound on its quantum communication complexity (**Theorem 8 and Corollary 9**). The quantum distribution has one extra output per player compared to the original distribution and uses the same amount of entanglement as the quantum protocol plus as many EPR pairs as needed to teleport the quantum communication in the protocol. We show that these Bell violations can be made noise-resistant, at the cost of a 2^{2q} factor in the number of outcomes per player (**Theorem 10**).

Finally, we provide tools to build Bell inequalities from communication lower bounds in the literature. Lower bounds used in practice to separate classical from quantum communication complexity are usually achieved using corruption bounds and its variants. In **Theorem 12**, we give an explicit construction which translates these bounds into a suitable Bell functional. Table 2 summarizes the new results or the improvements that we obtain in this work.

1.2 Related work

The study of the maximum violation of Bell inequalities began with Tsirelson [44], who showed that for two-outcome correlation Bell inequalities, the maximum violation is bounded above by Grothendieck's constant. Tsirelson also raised the question of whether one can have unbounded violations of Bell inequalities. More precisely, he asked whether there exist families of Bell inequalities for which the amount of the violation grows arbitrarily large.

The first answer to this question came from Mermin [33], who gave a family Bell inequalities for which a violation exponential in the number of parties is achieved. In the years that followed, several new constructions appeared for number of parties and number of inputs [3, 30, 28, 34, 36].

The study of upper bounds on violations of normalized Bell inequalities resumed in [12], where an upper bound of $O(K^2)$ (with K the number of outputs per player) and of $2^c \leq N$ (with c the communication complexity and N the number of inputs per player) were proven. In [21] the authors proved a bound of $O(d)$ in terms of the dimension d of the local Hilbert space, and in [19], the bound in terms of the number of outputs was improved to $O(K)$.

In [19], Bell inequalities are constructed for which a near optimal, but probabilistic, violation of order $\Omega(\sqrt{m}/\log m)$, with $N = K = d = m$, is proven. In [11], the same violation, although requiring $N = 2^m$ inputs, is achieved for a family of Bell inequalities and quantum distributions built using the quantum advantage in one-way communication complexity for the Hidden Matching problem (with $K = d = m$). In the same paper, a violation of order $\Omega(m/(\log m)^2)$, with $K = d = m$ and $N = 2^m/m$ is achieved with the Khot-Vishnoi game. Recently, an asymmetric version of that game was introduced to allow one of the parties to only make dichotomic measurements, with a smaller (although almost optimal for this scenario) violation $\Omega(\sqrt{m}/(\log m)^2)$ [35].

For *inefficiency-resistant* Bell inequalities, the bounds in [19] do not apply. In fact, Laplante *et al.* proved in [27] a violation exponential in the dimension and the number of outputs for this type of Bell functionals, achieved by a quantum distribution built, as in [11], from the Hidden Matching communication complexity problem.

The connection exhibited in [11] between Bell violations and communication complexity is generalized by Buhrman *et al.* in [10] where a fully general construction is given to go from a quantum communication protocol for a function f to a Bell inequality and a quantum distribution which achieves a violation of order $\Omega\left(\frac{\sqrt{R_{1/3}(f)}}{Q_{1/3}(f)}\right)$. The downside to this construction is that the quantum distribution has a double exponential (in the communication) number of outputs and the protocol to implement it uses an additional double exponential amount of entanglement. Also, this result does not apply for quantum advantages in a zero-error setting.

2 Preliminaries

2.1 Quantum nonlocality

Local, quantum, and nonsignaling distributions have been widely studied in quantum information theory since the seminal paper of Bell [6]. In an experimental setting, two players share an entangled state and each player is given a measurement to perform. The outcomes of the measurements are predicted by quantum mechanics and follow some probability distribution $p(a, b|x, y)$, where a is the outcome of Alice's measurement x, and b is the outcome of Bob's measurement y.

We consider bipartite distribution families of the form $\mathbf{p} = (p(\cdot, \cdot|x, y))_{(x,y)\in\mathcal{X}\times\mathcal{Y}}$ with inputs $(x, y) \in \mathcal{X} \times \mathcal{Y}$ determining a probability distribution $p(\cdot, \cdot|x, y)$ over the outcomes $(a, b) \in \mathcal{A}\times\mathcal{B}$, with the usual positivity and normalization constraints. The set of probability distribution families is denoted by $\mathcal{P}$. For simplicity, we call simply "distributions" such probability distribution families. The expression "Alice's marginal" refers to her marginal output distribution, that is $\sum_b p(\cdot, b|x, y)$ (and similarly for Bob).

The *local deterministic distributions*, denoted $\mathcal{L}_{\text{det}}$, are the ones where Alice outputs according to a deterministic strategy, i.e., a (deterministic) function of x, and Bob independently outputs as a function of y, without communicating. The *local distributions* $\mathcal{L}$ are obtained by taking distributions over the local deterministic strategies. Operationally, this corresponds to protocols with shared randomness and no communication. Geometrically, $\mathcal{L}$ is the convex hull of $\mathcal{L}_{\text{det}}$.

A *Bell test* [6] consists of estimating all the probabilities $p(a, b|x, y)$ and computing a *Bell functional*, or linear function, on these values. The Bell functional B is chosen together with a threshold τ so that any local classical distribution ℓ verifies the *Bell inequality* $B(\ell) \leq \tau$, but the chosen distribution $\mathbf{p}$ exhibits a *Bell violation*: $B(\mathbf{p}) > \tau$. By normalizing, we can assume without loss of generality that ℓ verifies $B(\ell) \leq 1$ for any $\ell \in \mathcal{L}$, and $B(\mathbf{p}) > 1$.

In this paper, we will also consider strategies that are allowed to abort the protocol with some probability. When they abort, they output the symbol $\perp$ ($\perp$ denotes a new symbol which is not in $\mathcal{A} \cup \mathcal{B}$). We will use the notation $\mathcal{L}^{\perp}_{\text{det}}$ and $\mathcal{L}^{\perp}$ to denote local strategies that can abort, where $\perp$ is added to the possible outputs for both players. When $\ell \in \mathcal{L}^{\perp}_{\text{det}}$ or $\mathcal{L}^{\perp}$, $\ell(a, b|x, y)$ is *not* conditioned on $a, b \neq \perp$ since $\perp$ is a valid output for such distributions.

The *quantum distributions*, denoted $\mathcal{Q}$, are the ones that result from applying measurements x, y to their part of a shared entangled bipartite state. Each player outputs his or her measurement outcome (a for Alice and b for Bob). In communication complexity terms, these are zero-communication protocols with shared entanglement. If the players are allowed to abort, then the corresponding set of distributions is denoted $\mathcal{Q}^{\perp}$.

Boolean (and other) functions can be cast as sampling problems. Consider a boolean function $f : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$ (non-boolean functions and relations can be handled similarly). First, we split the output so that if $f(x, y) = 0$, Alice and Bob are required to output the same bit, and if $f(x, y) = 1$, they output different bits. Let us further require Alice's marginal distribution to be uniform, likewise for Bob, so that the distribution is well defined. Call the resulting distribution $\mathbf{p}_f$, that is, for any $a, b \in \{0, 1\}$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we have $p_f(a, b|x, y) = 1/2$ if $a \oplus b = f(x, y)$, and $p_f(a, b|x, y) = 0$ otherwise, $\oplus$ being the 1-bit XOR.

If $\mathbf{p}_f$ were local, f could be computed with one bit of communication using shared randomness: Alice sends her output to Bob, and Bob XORs it with his output. If $\mathbf{p}_f$ were quantum, there would be a 1-bit protocol with shared entanglement for f. In communication complexity, we are usually interested in distributions having nontrivial communication complexity, and lie well beyond these sets.

Finally, a distribution is *nonsignaling* if for each player, its marginal output distributions, given by $p_A(a|x, y) = \sum_b p(a, b|x, y)$, for Alice, and $p_B(b|x, y) = \sum_a p(a, b|x, y)$, for Bob, do not depend on the other player's input. When this is the case, we write the marginals as $p_A(a|x)$ and $p_B(b|y)$. Operationally, this means that each player cannot influence the statistics of what the other player observes with his own choice of input. We note with $\mathcal{C}$ the set of nonsignaling distributions, also referred to as the *causal* set, and we note $\mathcal{C}^{\perp}$ when we allow aborting. The well-known inclusion relations between these sets are $\mathcal{L} \subset \mathcal{Q} \subset \mathcal{C} \subset \mathcal{P}$.

For any Boolean function f, the distribution $\mathbf{p}_f$ is nonsignaling since the marginals are uniform. A fundamental question of quantum mechanics has been to establish experimentally whether nature is truly *nonlocal*, as predicted by quantum mechanics, or whether there is a purely classical (i.e., *local*) explanation to the phenomena that have been predicted by quantum theory and observed in the lab.

2.2 Measures of nonlocality

We have described nonlocality as a yes/no property, but some distributions are somehow more nonlocal than others. To have a robust measure of nonlocality, it should verify some common sense properties: for a fixed distribution, the measure should be bounded; it should also be convex, since sampling from the convex combination of two distributions can be done by first picking randomly one of the two distributions using shared randomness, and then sampling from that distribution. We also expect such a measure of nonlocality to have various equivalent formulations. Several measures have been proposed and studied: resistance to noise [22, 2, 36, 20], resistance to inefficiency [30, 31, 27], amount of communication necessary to reproduce them [32, 7, 42, 43, 37, 12], information-theoretic measures [8, 14, 13], etc.

In the form studied in this paper, normalized Bell inequalities were first studied in [12], where they appeared as the dual of the linear program for a well-studied lower bound on communication complexity, known as the nuclear norm ν [29] (the definition is given in Section 2.3). There are many equivalent formulations of this bound. For distributions arising from boolean functions, it has the mathematical properties of a norm, and it is related to winning probabilities of XOR games. It can also be viewed as a gauge, that is, a quantity measuring by how much the local set must be expanded in order to contain the distribution considered. For more general nonsignaling distributions, besides having a geometrical interpretation in terms of affine combinations of local distributions, it has also been shown to be equivalent to the amount of local noise that can be tolerated before the distribution becomes local [21].

A subsequent paper [27] studied equivalent formulations of the partition bound, one of the strongest lower bounds in communication complexity [17]. This bound also also has several formulations: the primal formulation can be viewed as resistance to detector inefficiency, and the dual formulation is given in terms of inefficiency-resistant Bell inequality violations.

In this paper, we show how to deduce large violations on quantum distributions from large violations on nonsignaling distributions, provided there are efficient quantum communication protocols for the latter.

2.3 Communication complexity and lower bounds

In classical communication complexity (introduced by [45]), two players each have a share of the input, and wish to compute a function on the full input. Communication complexity measures the number of bits they need to exchange to solve this problem in the worst case, over all inputs of a given size n. In this paper we consider a generalization of this model, where instead of computing a function, they each produce an output, say a and b, which should follow, for each (x, y), some prescribed distribution $p(a, b|x, y)$ (which depends on their inputs x, y). We assume that the order in which the players speak does not depend on the inputs. This is without loss of generality at a cost of a factor of 2 in the communication.

We use the following notation for communication complexity of distributions. $R_\epsilon(\mathbf{p})$ is the minimum number of bits exchanged in the worst case to output with the distribution $\mathbf{p}$, up to ϵ in total variation distance for all x, y. We call total variation distance between distributions the distance denoted by $|.|_1$, and defined as $|\mathbf{p}-\mathbf{p}'|_1 = \max_{x,y} \sum_{a,b} |p(a,b|x,y) - p'(a,b|x,y)|$.

We use Q to denote quantum communication complexity (see [47]), and we use the superscript $*$ to denote the presence of shared entanglement. For randomized communication, we assume shared randomness.

To give upper bounds on communication complexity it suffices to give a protocol and analyze its complexity. Proving lower bounds is often a more difficult task, and many techniques have been developed to achieve this. The methods we describe here are complexity measures which can be applied to any function. To prove a lower bound on communication, it suffices to give a lower bound on one of these complexity measures, which are bounded above by communication complexity for any function. We describe here most of the complexity measures relevant to this work.

The nuclear norm ν, given here in its dual formulation and extended to nonsignaling distributions, is expressed by the following linear program [29, 12]. (There is a quantum analogue, γ_2, which is not needed in this work. We refer the interested reader to the definition for distributions in [12]).

▶ **Definition 1** ([29, 12])**.** The nuclear norm ν of a nonsignaling distribution $\mathbf{p} \in \mathcal{C}$ is given by

$$\begin{aligned} \nu(\mathbf{p}) = \max_{B} \quad & B(\mathbf{p}) \\ \text{subject to} \quad & |B(\ell)| \leq 1 \quad \forall \ell \in \mathcal{L}_{det}. \end{aligned}$$

With error ϵ, $\nu_\epsilon(\mathbf{p}) = \min_{\mathbf{p}' \in \mathcal{C} : |\mathbf{p}'-\mathbf{p}|_1 \leq \epsilon} \nu(\mathbf{p}')$. We call any Bell functional that satisfies the constraint in the above linear program *normalized Bell functional.*

In this definition and in the rest of the paper, unless otherwise specified (in particular in Lemma 19), B ranges over vectors of real coefficients $B_{a,b,x,y}$ and $B(\mathbf{p})$ denotes $\sum_{a,b,x,y} B_{a,b,x,y} p(a,b|x,y)$, where a,b ranges over the non-abort outputs and x,y ranges over the inputs. So even when B and $\mathbf{p}$ have coefficients on the abort events, we do not count them. Table 1 summarizes the known upper and lower bounds on ν for various parameters. The (log of the) nuclear norm is a lower bound on classical communication complexity.

▶ **Proposition 2** ([29, 12])**.** *For any nonsignaling distribution* $\mathbf{p} \in \mathcal{C}$*,* $\mathrm{R}_\epsilon(\mathbf{p}) + 1 \geq \log(\nu_\epsilon(\mathbf{p}))$*, and for any boolean function* f*,* $\mathrm{R}_\epsilon(f) \geq \log(\nu_\epsilon(\mathbf{p}_f))$*.*

As lower bounds on communication complexity of Boolean functions go, ν is one of the weaker bounds, equivalent to the smooth discrepancy [17], and no larger than the approximate nonnegative rank and the smooth rectangle bounds [25]. More significantly for this work, up to small multiplicative constants, for boolean functions, (the log of) ν is a lower bound on quantum communication, so it is useless to establish gaps between classical and quantum communication complexity. (This limitation, with the upper bound in terms of the number of outputs on normalized Bell violations, is a consequence of Grothendieck's theorem [15].)

The classical and quantum efficiency measures, given here in their dual formulations, are expressed by the following two convex optimization programs. The classical bound is a generalization to distributions of the partition bound of communication complexity [17, 27]. This bound is one of the strongest lower bounds known, and can be exponentially larger than ν (an example is the Vector in Subspace problem). It is always as least as large as the relaxed partition bound which is in turn always at least as large as the smooth rectangle bound [17, 23]. Its weaker variants have been used to show exponential gaps between classical and quantum communication complexity. The definition we give here is a stronger formulation than the one given in [27]. We show they are equivalent in Appendix D.

▶ **Definition 3** ([27])**.** The ϵ-error efficiency bound of a distribution $\mathbf{p} \in \mathcal{P}$ is given by

$$\begin{aligned} eff_\epsilon(\mathbf{p}) = \max_{B,\beta} \quad & \beta \\ \text{subject to} \quad & B(\mathbf{p}') \geq \beta && \forall \mathbf{p}' \in \mathcal{P} \text{ s.t. } |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon, \\ & B(\ell) \leq 1 && \forall \ell \in \mathcal{L}_{det}^{\perp}. \end{aligned}$$

We call any Bell functional that satisfies the second constraint in the above linear program *inefficiency-resistant Bell functional.* The ϵ-error quantum efficiency bound of a $\mathbf{p} \in \mathcal{P}$ is

$$\begin{aligned} eff^*_\epsilon(\mathbf{p}) = \max_{B,\beta} \quad & \beta \\ \text{subject to} \quad & B(\mathbf{p}') \geq \beta && \forall \mathbf{p}' \in \mathcal{P} \text{ s.t. } |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon, \\ & B(\mathbf{q}) \leq 1 && \forall \mathbf{q} \in \mathcal{Q}^{\perp}. \end{aligned}$$

We denote $eff = eff_0$ and $eff^* = eff^*_0$ the 0-error bounds.

For any given distribution $\mathbf{p}$, its classical communication complexity is bounded below by the (log of the) efficiency. For randomized communication complexity with error ϵ, the bound is $\log(eff_\epsilon)$ and for quantum communication complexity, the bound is $\log(eff^*_\epsilon)$. Note that for any $\mathbf{p} \in \mathcal{Q}$, the quantum communication complexity is 0 and the eff^* bound is 1. For any function f, the efficiency bound $eff_\epsilon(\mathbf{p}_f)$ is equivalent to the partition bound [17, 27].

▶ **Proposition 4** ([27])**.** *For any* $\mathbf{p} \in \mathcal{P}$ *and any* $0 \leq \epsilon < 1/2$*,* $\mathrm{R}_\epsilon(\mathbf{p}) \geq \log(eff_\epsilon(\mathbf{p}))$ *and* $Q_\epsilon(\mathbf{p}) \geq \frac{1}{2}\log(eff^*_\epsilon(\mathbf{p}))$*. For any* $\mathbf{p} \in \mathcal{C}$ *and any* $0 \leq \epsilon \leq 1$*,* $\nu_\epsilon(\mathbf{p}) \leq 2 eff_\epsilon(\mathbf{p})$*.*

Theorem 8 below involves upper bounds on the quantum efficiency bound. To give an upper bound on the quantum efficiency of a distribution $\mathbf{p}$, it is more convenient to use the primal formulation, and upper bounds can be given by exhibiting a local (or quantum) distribution with abort which satisfies the following two properties: the probability of aborting should be the same on all inputs x, y, and conditioned on not aborting, the outputs of the protocol should reproduce the distribution $\mathbf{p}$. The efficiency is inverse proportional to the probability of not aborting, so the goal is to abort as little as possible.

▶ **Proposition 5** ([27]). *For any distribution $\mathbf{p} \in \mathcal{P}$, $eff^*(\mathbf{p}) = 1/\eta^*$, with η^* the optimal value of the following optimization problem (non-linear, because $\mathcal{Q}^\perp$ is not a polytope).*

$$\begin{aligned} &\max_{\zeta, \mathbf{q} \in \mathcal{Q}^\perp} && \zeta \\ &\textit{subject to} && q(a,b|x,y) = \zeta p(a,b|x,y) \quad \forall x,y,a,b \in \mathcal{X}\times\mathcal{Y}\times\mathcal{A}\times\mathcal{B} \end{aligned}$$

Moreover, for any $0 \leq \epsilon \leq 1$, $eff^*_\epsilon(\mathbf{p}) = \min_{\mathbf{p}' \in \mathcal{P}: |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon} eff^*(\mathbf{p}')$.

3 Properties of Bell inequalities

Syntactically, there are two differences between the normalized Bell functionals (Definition 1) and the inefficiency-resistant ones (Definition 3). The first difference is that the normalization constraint is relaxed: for inefficiency-resistant functionals, the lower bound on the Bell value for local distributions is removed. Since this is a maximization problem, this relaxation allows for larger violations. This difference alone would not lead to a satisfactory measure of nonlocality, since one could obtain unbounded violations by shifting and dilating the Bell functional. The second difference prevents this. The upper bound is required to hold not only for local distributions, but also those that can abort. This is a much stronger condition. Notice that a local distribution can selectively abort on configurations that would otherwise tend to keep the Bell value small, making it harder to satisfy the constraint.

In this section, we show that normalized Bell violations can be modified to be resistant to local distributions that abort, while preserving the violation on any nonsignaling distribution, up to a factor of 3. This means that we can add the stronger constraint of resistance to local distributions that abort to Definition 1, incurring a loss of just a factor of 3, and the only remaining difference between the resulting linear programs is the relaxation of the lower bound (dropping the absolute value) for local distributions that abort.

▶ **Theorem 6.** *Let B be a normalized Bell functional on $\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}$ and $\mathbf{p} \in \mathcal{C}$ a nonsignaling distribution such that $B(\mathbf{p}) \geq 1$. Then there exists a normalized Bell functional B^* on $(\mathcal{A}\cup\{\perp\})\times(\mathcal{B}\cup\{\perp\})\times\mathcal{X}\times\mathcal{Y}$ with 0 coefficients on the $\perp$ outputs such that : $\forall \mathbf{p} \in \mathcal{C}$, $B^*(\mathbf{p}) \geq \frac{1}{3}B(\mathbf{p}) - \frac{2}{3}$, and $\forall \ell \in \mathcal{L}^\perp_{\mathrm{det}}$, $|B^*(\ell)| \leq 1$.*

The formal proof of Theorem 6 is deferred to Appendix A, and we will only give its high-level structure in this part of the paper. First, we show (see Observation 17) how to rescale a normalized Bell functional so that it saturates its normalization constraint. Then, Definition 18 adds weights to abort events to make the Bell functional resistant to inefficiency. Finally, Lemma 19 removes the weights on the abort events of a Bell functional while keeping it bounded on the local set with abort, without dramatically changing the values it takes on the nonsignaling set. Our techniques are similar to the ones used in [31].

4 Exponential violations from communication bounds

Recently, Buhrman *et al.* gave a general construction to derive normalized Bell inequalities from any sufficiently large gap between classical and quantum communication complexity.

▶ **Theorem 7** ([10])**.** *For any function f for which there is a quantum protocol using q qubits of communication but no prior shared entanglement, there exists a quantum distribution $\mathbf{q} \in \mathcal{Q}$ and a normalized Bell functional B such that $B(\mathbf{q}) \geq \frac{\sqrt{R_{1/3}(f)}}{6\sqrt{3}q}(1-2^{-q})^{2q}$.*

Their construction is quite involved, requiring protocols to be memoryless, which they show how to achieve in general, and uses multiport teleportation to construct a quantum distribution. The Bell inequality they construct expresses a correctness constraint.

In this section, we show how to obtain large inefficiency-resistant Bell violations for quantum distributions from gaps between quantum communication and classical communication lower bounds. We first prove the stronger of two statements, which gives violations of $\frac{eff_\epsilon(\mathbf{p})}{eff^\star_{\epsilon'}(\mathbf{p})}$. For any problem for which a classical lower bound c is given using the efficiency or partition bound or any weaker method (including the rectangle bound and its variants), and any upper bound q on quantum communication complexity, it implies a violation of 2^{c-2q}.

▶ **Theorem 8.** *For any distribution $\mathbf{p} \in \mathcal{P}$ and any $0 \leq \epsilon' \leq \epsilon \leq 1$, if (B, β) is a feasible solution to the dual of $eff_\epsilon(\mathbf{p})$ and $(\zeta, \mathbf{q})$ is a feasible solution to the primal for $eff^\star_{\epsilon'}(\mathbf{p})$, then there is a quantum distribution $\overline{\mathbf{q}} \in \mathcal{Q}$ such that $B(\overline{\mathbf{q}}) \geq \zeta\beta$ and $B(\ell) \leq 1, \forall \ell \in \mathcal{L}^\perp_{det}$, and in particular, if both are optimal solutions, then $B(\overline{\mathbf{q}}) \geq \frac{eff_\epsilon(\mathbf{p})}{eff^\star_{\epsilon'}(\mathbf{p})}$. The distribution $\overline{\mathbf{q}}$ has one additional output per player compared to the distribution $\mathbf{p}$.*

Proof. Let (B, β) be a feasible solution to the dual of $eff_\epsilon(\mathbf{p})$, $\mathbf{p}'$ be such that $eff^\star_{\epsilon'}(\mathbf{p}) = eff^\star(\mathbf{p}')$ with $|\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon'$, and $(\zeta, \mathbf{q})$ be a feasible solution to the primal for $eff^\star(\mathbf{p}')$. From the constraints, we have $\mathbf{q} \in \mathcal{Q}^\perp$, $q(a, b|x, y) = \zeta p'(a, b|x, y)$ for all $(a, b, x, y) \in \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y}$, $B(\ell) \leq 1$ for all $\ell \in \mathcal{L}^\perp_{det}$, and $B(\mathbf{p}'') \geq \beta$ for all $\mathbf{p}''$ s.t. $|\mathbf{p}'' - \mathbf{p}|_1 \leq \epsilon$. Then $B(\mathbf{q}) = \zeta B(\mathbf{p}') \geq \zeta\beta$. However, $\mathbf{q} \in \mathcal{Q}^\perp$ but technically we want a distribution in $\mathcal{Q}$ (not one that aborts). So we add a new (valid) output 'A' to the set of outputs of each player, and they should output 'A' instead of aborting whenever $\mathbf{q}$ aborts. The resulting distribution, say $\overline{\mathbf{q}} \in \mathcal{Q}$ (with additional outcomes 'A' on both sides), is such that $B(\overline{\mathbf{q}}) = B(\mathbf{q})$ (since the Bell functional B does not have any weight on $\perp$ or on 'A'). ◀

Theorem 7 and Theorem 8 are both general constructions, but there are a few significant differences. Firstly, Theorem 8 requires a lower bound on the partition bound in the numerator, whereas Theorem 7 only requires a lower bound on communication complexity (which could be exponentially larger). Secondly, Theorem 7 requires a quantum communication protocol in the denominator, whereas our theorem only requires an upper bound on the quantum efficiency (which could be exponentially smaller). Thirdly, our bound is exponentially larger than Buhrman *et al.*'s for most problems considered here, and applies to subquadratic gaps, but their bounds are of the more restricted class of normalized Bell inequalities.

Theorem 8 gives an explicit Bell functional provided an explicit solution to the efficiency (partition) bound is given and the quantum distribution is obtained from a solution to the primal of $eff^\star$ (Proposition 5). Recall that a solution to the primal of $eff^\star$ is provided by a quantum zero-communication protocol that can abort, which conditioned on not aborting, outputs following $\mathbf{p}$. We can also start from a quantum protocol, as we show below. From the quantum protocol, we derive a quantum distribution using standard techniques.

▶ **Corollary 9.** *For any distribution $\mathbf{p} \in \mathcal{P}$ and any $0 \leq \epsilon' \leq \epsilon \leq 1$ such that $R_\epsilon(\mathbf{p}) \geq \log(eff_\epsilon(\mathbf{p})) \geq c$ and $Q_{\epsilon'}(\mathbf{p}) \leq q$, there exists an explicit inefficiency-resistant B derived from the efficiency lower bound, and an explicit quantum distribution $\overline{\mathbf{q}} \in \mathcal{Q}$ derived from the quantum protocol such that $B(\overline{\mathbf{q}}) \geq 2^{c-2q}$.*

Proof. Let (B, β) be an optimal solution to $eff_\epsilon(\mathbf{p})$ and let c be such that $eff_\epsilon(\mathbf{p}) = \beta \geq 2^c$. By optimality of B, we have $B(\mathbf{p}') \geq 2^c$ for any $\mathbf{p}'$ such that $|\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon$. Since $Q_{\epsilon'}(\mathbf{p}) \leq q$, there exists a q-qbit quantum protocol for some distribution $\mathbf{p}'$ with $|\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon$. Then, we can use teleportation to obtain a $2q$ classical bit, entanglement-assisted protocol for $\mathbf{p}'$. We can simulate it without communication by picking a shared $2q$-bit random string and running the protocol but without sending any messages. If the measurements do not match the string, output a new symbol 'A' (not in the output set of the quantum protocol and different from $\perp$). We obtain a quantum distribution $\overline{\mathbf{q}}$ such that $B(\overline{\mathbf{q}}) = B(\mathbf{p}')/2^{2q} \geq 2^{c-2q}$. ◀

Most often, communication lower bounds are not given as efficiency or partition bounds, but rather using variants of the corruption bound. We show in Section 6.1 how to map a corruption bound to explicit Bell coefficients.

5 Noise-resistant violations from communication bounds

Normalized Bell inequalities are naturally resistant to any local noise: if the observed distribution is $\tilde{\mathbf{p}} = (1-\varepsilon)\mathbf{p} + \varepsilon\ell$ for some $\ell \in \mathcal{L}$, then $B(\tilde{\mathbf{p}}) \geq (1-\varepsilon)B(\mathbf{p}) - \varepsilon$ since $|B(\ell)| \leq 1$. In inefficiency-resistant Bell inequalities, relaxing the absolute value leads to the possibility that $B(\ell)$ has a large negative value for some local ℓ. (Indeed, such large negative values are inherent to large gaps between ν and eff.) If this distribution were used as adversarial noise, the observed distribution, $(1-\varepsilon)\mathbf{p} + \varepsilon\ell$, could have a Bell value much smaller than 1. This makes inefficiency-resistant Bell inequalities susceptible to adversarial local noise.

Our construction from Theorem 8 is susceptible to uniform noise since most of the time, the output is 'A'. Uniform noise will disproportionately hit the non-'A' outputs, destroying the structure of the distribution. In Theorem 10, we show that our construction can be made resistant to uniform noise, by including a (possible) transcript from the protocol in the outputs. (Notice that this leads to a much larger output set.) Since the transcripts in our construction are teleportation measurements, they follow a uniform distribution, making the modified distribution resistant to uniform noise. The tolerance to noise comes from the error parameter in the classical communication lower bound.

Let $N_\varepsilon(\mathbf{p}) = \{(1-\delta)\mathbf{p} + \delta\mathbf{u},\ \delta \in [0, \varepsilon]\} \subseteq \mathcal{P}$ be the ε-noisy neighbourhood of $\mathbf{p}$, where $\mathbf{u}$ the uniform noise distribution, that is: $u(a,b|x,y) = \frac{1}{|\mathcal{A}|\cdot|\mathcal{B}|}$ for all $(a,b) \in \mathcal{A} \times \mathcal{B}$.

▶ **Theorem 10.** *For any distribution $\mathbf{p} \in \mathcal{P}$ and any $0 \leq \epsilon' \leq \epsilon \leq 1$ such that $R_\epsilon(\mathbf{p}) \geq \log(eff_\epsilon(\mathbf{p})) \geq c$ and $Q_{\epsilon'}(\mathbf{p}) \leq q$, there exists an explicit inefficiency-resistant $\tilde{B}$ derived from the efficiency lower bound, and an explicit quantum distribution $\overline{\mathbf{q}} \in \mathcal{Q}$ derived from the quantum protocol such that $\tilde{B}(\mathbf{q}') \geq 2^{c-2q}$ for any $\mathbf{q}' \in N_{\varepsilon-\varepsilon'}(\overline{\mathbf{q}})$.*

Proof. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be Alice's (resp. Bob's) possible outputs for $\mathbf{p}$. From a quantum communication protocol for $\mathbf{p}'$ with $|\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon'$ using q qubits of communication, we construct an entanglement-assisted protocol using $2q$ bits of communication and teleportation. Let $\mathcal{M}_A$ (resp. $\mathcal{M}_B$) be the set of possible transcripts for Alice (resp. Bob), with $|\mathcal{M}_A| = M_A$ (resp. $|\mathcal{M}_B| = M_B$), and note that $\log M_A + \log M_B = 2q$.

We define the quantum distribution $\overline{\mathbf{q}}$ where Alice's possible outputs are $\mathcal{A} \times \mathcal{M}_A$ and Bob's possible outputs are $\mathcal{B} \times \mathcal{M}_B$. Alice proceeds as follows (Bob proceeds similarly):

1. She runs the quantum protocol for $\mathbf{p}'$ as if all bits received from Bob were 0.
2. She outputs (a, μ_A), where μ_A is the transcript of the messages she would have sent to Bob and a is the output she would have produced in the original protocol.

By definition, this distribution is such that, for all a, b, x, y, $\overline{q}(a, 0, b, 0|x, y) = \frac{1}{2^{2q}} p'(a, b|x, y)$.

Let $eff_\varepsilon(\mathbf{p}) \geq 2^c$ be achieved by the Bell functional B. By definition, we have $B(\ell) \leq 1$ for all $\ell \in \mathcal{L}^{\perp}_{det}$, and $B(\mathbf{p}'') \geq 2^c$ for all $\mathbf{p}''$ such that $|\mathbf{p}'' - \mathbf{p}|_1 \leq \varepsilon$. In particular for any $\mathbf{p}'' \in N_{\varepsilon-\varepsilon'}(\mathbf{p})$, that is, $\mathbf{p}'' = (1-\delta)\mathbf{p} + \delta \mathbf{u}$ for some $\delta \in [0, \varepsilon - \varepsilon']$, we have $|\mathbf{p}'' - \mathbf{p}|_1 \leq \varepsilon$ and therefore $B(\mathbf{p}'') = (1-\delta)B(\mathbf{p}') + \delta B(\mathbf{u}) \geq 2^c$, where $B(\mathbf{u}) = \frac{1}{AB}\sum_{a,b,x,y} B_{a,b,x,y}$.

Let the Bell functional $\tilde{B}$ for distributions over $(\mathcal{A} \times \mathcal{M}_A) \times (\mathcal{B} \times \mathcal{M}_B)$ be defined as follows: $\tilde{B}_{(a,\mu_A),(b,\mu_B),x,y} = B_{a,b,x,y}$ if $\mu_A = \mu_B = 0$, and $\tilde{B}_{(a,\mu_A),(b,\mu_B),x,y} = 0$ otherwise.

Let $\tilde{\mathcal{L}}^{\perp}_{det}$ be the local set for distributions over $(\mathcal{A} \times \mathcal{M}_A) \times (\mathcal{B} \times \mathcal{M}_B)$. Then $\tilde{B}$ satisfies $\tilde{B}(\ell) \leq 1$ for all $\ell \in \tilde{\mathcal{L}}^{\perp}_{det}$ (by assimilating any event with $\mu_A \neq 0$ or $\mu_B \neq 0$ to a $\perp$ event), as well as $\tilde{B}(\overline{\mathbf{q}}) = \frac{1}{2^{2q}} B(\mathbf{p}')$. Hence, $\forall \delta \in [0, \varepsilon - \varepsilon']$, we also have $(1-\delta)\tilde{B}(\overline{\mathbf{q}}) + \delta \tilde{B}(\mathbf{u}) =$
$= (1-\delta)\frac{1}{2^{2q}} B(\mathbf{p}') + \delta \frac{1}{ABM_AM_B} \sum_{a,\mu_A,b,\mu_B,x,y} \tilde{B}_{(a,\mu_A),(b,\mu_B),x,y}$
$= \frac{1}{2^{2q}} \left[(1-\delta)B(\mathbf{p}') + \delta \frac{1}{AB}\sum_{a,b,x,y} B_{a,b,x,y}\right] = \frac{1}{2^{2q}} \left[(1-\delta)B(\mathbf{p}') + \delta B(\mathbf{u})\right].$

Therefore, for all $\mathbf{q}' \in N_{\varepsilon-\varepsilon'}(\overline{\mathbf{q}})$, $\tilde{B}(\mathbf{q}') \geq 2^{c-2q}$, as claimed. ◀

6 Explicit constructions

6.1 From corruption bound to Bell inequality violation

We now explain how to construct an explicit Bell inequality violation from the corruption bound. The corruption bound, introduced by Yao in [46], is a very useful lower bound technique. It has been used for instance in [39] to get a tight $\Omega(n)$ lower bound on the randomized communication complexity of Disjointness (whereas the approximate rank, for example, can only show a lower bound of $\Theta(\sqrt{n})$). Let us recall that a rectangle R of $\mathcal{X} \times \mathcal{Y}$ is a subset of that set of the form $R_A \times R_B$, where $R_A \subseteq \mathcal{X}$ and $R_B \subseteq \mathcal{Y}$.

▶ **Theorem 11** (Corruption bound [46, 4, 26])**.** *Let f be a (possibly partial) Boolean function on $\mathcal{X} \times \mathcal{Y}$. Given $\gamma, \delta \in (0,1)$, suppose that there is a distribution μ on $\mathcal{X} \times \mathcal{Y}$ such that for every rectangle $R \subseteq \mathcal{X} \times \mathcal{Y}$*

$$\mu(R \cap f^{-1}(1)) > \gamma\mu(R \cap f^{-1}(0)) - \delta$$

Then, for every $\epsilon \in (0,1)$, $2^{R_\epsilon(f)} \geq \frac{1}{\delta}\left(\mu(f^{-1}(0)) - \frac{\epsilon}{\gamma}\right)$.

See, e.g., Lemma 3.5 in [5] for a rigorous treatment. For several problems, such a μ is already known. In Theorem 12 below, whose proof we defer to Appendix B, we show how to construct a Bell inequality violation from this type of bound.

▶ **Theorem 12.** *Let f be a (possibly partial) Boolean function on $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y} \subseteq \{0,1\}^n$. Fix $z \in \{0,1\}$. Let μ be an input distribution, and $(U_i)_{i \in I}$ (resp. $(V_j)_{j \in J}$) be a family of pairwise nonoverlapping subsets of $f^{-1}(\bar{z})$ (resp. of $f^{-1}(z)$). Assume that there exists $g : \mathbb{N} \to (0, +\infty)$ such that, for any rectangle $R \subseteq \mathcal{X} \times \mathcal{Y}$*

$$\sum_{i \in I} u_i \mu(R \cap U_i) \geq \sum_{j \in J} v_j \mu(R \cap V_j) - g(n). \tag{1}$$

Then, the Bell functional B given by the following coefficients: for all $a, b, x, y \in \{0,1\} \times$

$\{0,1\} \times \mathcal{X} \times \mathcal{Y}$,

$$B_{a,b,x,y} = \begin{cases} 1/2(-u_i g(n)^{-1}\mu(x,y)) & \text{if } (x,y) \in U_i \text{ and } a \oplus b = z, \\ 1/2(v_j g(n)^{-1}\mu(x,y)) & \text{if } (x,y) \in V_j \text{ and } a \oplus b = z, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

satisfies

$$B(\ell) \leq 1, \quad \forall \ell \in \mathcal{L}_{det}^{\perp}, \tag{3}$$

$$B(\mathbf{p}_f) = \frac{1}{2 \cdot g(n)} \sum_j v_j \mu(V_j) \tag{4}$$

and for any $\mathbf{p}' \in \mathcal{P}$ *such that* $|\mathbf{p}' - \mathbf{p}_f|_1 \leq \epsilon$ *:*

$$B(\mathbf{p}') \geq \frac{1}{2 \cdot g(n)} \left[\sum_j v_j \mu(V_j) - \epsilon \left(\sum_j |v_j| \mu(V_j) + \sum_i |u_i| \mu(U_i) \right) \right]. \tag{5}$$

For many other problems in the literature, such as Vector in Subspace and Tribes, stronger variants of the corruption bound are needed to obtain good lower bounds. These stronger variants have been shown to be no stronger than the partition bound (more specifically, the relaxed partition bound) [23]. The generalization in Theorem 12 of the hypothesis of Theorem 11, which the reader might have notice, allow us to construct explicit Bell functionals also for these problems.

6.2 Some specific examples

Using Corollary 9 and the construction to go from a corruption bound (or its variants) to a Bell inequality (Theorem 12), we give explicit Bell inequalities and violations for several problems studied in the literature. Since our techniques also apply to small gaps, we include problems for which the gap between classical and quantum communication complexity is polynomial.

Vector in Subspace

In the Vector in Subspace Problem $\mathrm{VSP}_{0,n}$, Alice is given an $n/2$ dimensional subspace of an n dimensional space over $\mathbb{R}$, and Bob is given a vector. This is a partial function, and the promise is that either Bob's vector lies in the subspace, in which case the function evaluates to 1, or it lies in the orthogonal subspace, in which case the function evaluates to 0. Note that the input set of $\mathrm{VSP}_{0,n}$ is continuous, but it can be discretized by rounding, which leads to the problem $\widetilde{\mathrm{VSP}}_{\theta,n}$ (see [24] for details). Klartag and Regev [24] show that the VSP can be solved with an $O(\log n)$ quantum protocol, but the randomized communication complexity of this problem is $\Omega(n^{1/3})$. As shown in [23], this is also a lower bound on the relaxed partition bound. Hence Corollary 9 yields the following.

▶ **Proposition 13.** *There exists a Bell inequality* B *and a quantum distribution* $\overline{\mathbf{q}}_{VSP} \in \mathcal{Q}$ *such that* $B\left(\overline{\mathbf{q}}_{VSP}\right) \in 2^{\Omega(n^{1/3})-O(\log n)}$ *and for all* $\ell \in \mathcal{L}_{det}^{\perp}$, $B(\ell) \leq 1$.

Note that the result of [24] (Lemma 4.3) is not of the form needed to apply Theorem 12. It is yet possible to obtain an explicit Bell functional following the proof of Lemma 5.1 in [23].

Disjointness

In the Disjointness problem, the players receive two sets and have to determine whether they are disjoint or not. More formally, the Disjointness predicate is defined over $\mathcal{X} = \mathcal{Y} = \mathcal{P}([n])$ by $\mathrm{DISJ}_n(x, y) = 1$ iff x and y are disjoint. It is also convenient to see this predicate as defined over length n inputs, where $\mathrm{DISJ}_n(x, y) = 1$ for $x, y \in \{0, 1\}^n$ if and only if $|\{i : x_i = 1 = y_i\}| = 0$. The communication complexity for DISJ_n is $\Omega(n)$ using a corruption bound [39] and there is a quantum protocol using $O(\sqrt{n})$ communication [1]. Combining these results with ours, we obtain the following.

▶ **Proposition 14.** *There is a quantum distribution $\overline{\mathbf{q}}_{\mathrm{DISJ}} \in \mathcal{Q}$ and an explicit Bell inequality B satisfying: $B(\overline{\mathbf{q}}_{\mathrm{DISJ}}) = 2^{\Omega(n)-O(\sqrt{n})}$, and for all $\ell \in \mathcal{L}^{\perp}_{det}$, $B(\ell) \leq 1$.*

The proof is deferred to the Appendix (see Section C.1).

Tribes

Let $r \geq 2$, $n = (2r + 1)^2$. Let $\mathrm{TRIBES}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ be defined as: $\mathrm{TRIBES}_n(x, y) := \bigwedge_{i=1}^{\sqrt{n}} \left(\bigvee_{j=1}^{\sqrt{n}} (x_{(i-1)\sqrt{n}+j} \wedge y_{(i-1)\sqrt{n}+j}) \right)$. The Tribes function has an $\Omega(n)$ classical lower bound [16] using the smooth rectangle bound and a $O(\sqrt{n}(\log n)^2)$ quantum protocol [9]. Combining these results with ours, we obtain the following.

▶ **Proposition 15.** *There is a quantum distribution $\overline{\mathbf{q}}_{\mathrm{TRIBES}} \in \mathcal{Q}$ and an explicit Bell inequality B satisfying: $B(\overline{\mathbf{q}}_{\mathrm{TRIBES}}) = 2^{\Omega(n)-O(\sqrt{n}(\log n)^2)}$, and for all $\ell \in \mathcal{L}^{\perp}_{det}$, $B(\ell) \leq 1$.*

The proof is deferred to the Appendix (see Section C.2).

Gap Orthogonality

The Gap Orthogonality (ORT) problem was introduced by Sherstov as an intermediate step to prove a lower bound for the Gap Hamming Distance (GHD) problem [41]. We derive an explicit Bell inequality for ORT from Sherstov's lower bound of $\Omega(n)$, shown in [23] to be a relaxed partition bound. (Applying Corollary 9 also gives a (non-explicit) violation for GHD.) The quantum upper bound is $O(\sqrt{n} \log n)$ by the general result of [9]. In the ORT problem, the players receive vectors and need to tell whether they are nearly orthogonal or far from orthogonal. More formally, we consider the input space $\{-1, +1\}^n$ (to stick to the usual notations for this problem), and we denote $\langle \cdot, \cdot \rangle$ the scalar product on $\{-1, +1\}^n$. Let $\mathrm{ORT}_n : \{-1, +1\}^n \times \{-1, +1\}^n \to \{-1, +1\}$ be the partial function defined as in [41] by: $\mathrm{ORT}_n(x, y) = -1$ if $|\langle x, y \rangle| \leq \sqrt{n}$, and $\mathrm{ORT}_n(x, y) = +1$ if $|\langle x, y \rangle| \geq 2\sqrt{n}$. Combining the results mentioned above with ours, we obtain the following.

▶ **Proposition 16.** *There is a quantum distribution $\overline{\mathbf{q}}_{\mathrm{ORT}} \in \mathcal{Q}$ and an explicit Bell inequality B satisfying: $B(\overline{\mathbf{q}}_{\mathrm{ORT}}) = 2^{\Omega(n)-O(\sqrt{n}\log n)}$, and for all $\ell \in \mathcal{L}^{\perp}_{det}$, $B(\ell) \leq 1$.*

The proof is deferred to the Appendix (see Section C.3).

7 Discussion

We have given three main results. First, we showed that normalized Bell inequalities can be modified to be bounded in absolute value on the larger set of local distributions that can abort without significantly changing the value of the violations achievable with nonsignaling distributions. Then, we showed how to derive large inefficiency-resistant Bell violations from any gap between the partition bound and the quantum communication complexity of some given distribution $\mathbf{p}$. The distributions $\mathbf{q}$ achieving the large violations are relatively simple (only 3 outputs for boolean distributions $\mathbf{p}$) and can be made resistant to uniform noise at the expense of an increase in the number of outputs exponential in $Q(\mathbf{p})$. Finally, we showed how to construct explicit Bell inequalities when the separation between classical and quantum communication complexity is proven via the corruption bound.

From a practical standpoint, the specific Bell violations we have studied are probably not feasible to implement, because the parameters needed are still impractical or the quantum states are infeasible to implement. However, our results suggest that we could consider functions with small gaps in communication complexity, in order to find practical Bell inequalities that are robust against uniform noise and detector inefficiency. Let us consider an experimental setup with non-abort probability η per side, and ε uniform noise. Suppose we have a Boolean function with a lower bound of $c > 3\log(1/\eta^2)$ on classical communication with ε' error, and an $(\varepsilon' - \varepsilon)$-correct quantum protocol, with $\varepsilon' > \varepsilon$, using $q = \log(1/\eta^2)$ qubits. Our construction gives an inefficiency-resistant Bell violation of $2^{c-2q} > 1/\eta^2$, which is robust against ε uniform noise. (The number of outcomes per side increases to $\frac{2}{\eta^2}$.) Factoring in the inefficiency, the observed violation would still be $\eta^2 2^{c-2q} > 1$.

Regarding upper bounds, since (the log of) efficiency is a lower bound on communication complexity, inefficiency-resistant Bell violations are bounded above by the number of inputs per side. For dimension d and number of outcomes K, we obtain the upper bound $eff_\epsilon(\mathbf{q}) \leq 2^{O((\frac{Kd}{\epsilon})^2 \log^2(K))}$ for quantum distributions, by combining known bounds. Indeed, we know that $R_\epsilon(\mathbf{p}) \leq O((\frac{K\nu(\mathbf{p})}{\epsilon})^2 \log^2(K))$ for any $\mathbf{p} \in \mathcal{C}$ (see [12]). Combining this with the bounds $eff_\epsilon(\mathbf{p}) \leq 2^{R_\epsilon(\mathbf{p})}$ (Proposition 4), and $\nu(\mathbf{q}) \leq O(d)$ for any $\mathbf{q} \in \mathcal{Q}$ (see [21]), gives the desired upper bound. Hence unbounded violations are possible for $K = 3$ outputs per side.

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A Proof of Theorem 6

▶ Observation 17. Let B be a non-constant normalized Bell functional and $\mathbf{p} \in \mathcal{C}$ such that $B(\mathbf{p}) \geq 1$. Consider $\ell^- \in \mathcal{L}_{\text{det}}$ such that $B(\ell^-) = m = \min\{B(\ell) | \ell \in \mathcal{L}_{\text{det}}\}$ and $\ell^+ \in \mathcal{L}_{\text{det}}$ such that $B(\ell^+) = M = \max\{B(\ell) | \ell \in \mathcal{L}_{\text{det}}\}$. We have $m < M$ because B is non-constant. The Bell functional $\tilde{B}$ defined by $\tilde{B}(\cdot) = \frac{1}{M-m}(2B(\cdot) - M - m)$, is such that $\tilde{B}(\ell^+) = 1$, $\tilde{B}(\ell^-) = -1$, $|\tilde{B}(\ell)| \leq 1$ for all $\ell \in \mathcal{L}_{\text{det}}^{\perp}$, and $\tilde{B}(\mathbf{p}) \geq B(\mathbf{p})$.

▶ **Definition 18.** For any two families of distributions, $\mathbf{m}_A = (m_A(\cdot|x))_{x \in \mathcal{X}}$ over outcomes in $\mathcal{A}$ for Alice and $\mathbf{m}_B = (m_B(\cdot|y))_{y \in \mathcal{Y}}$ over outcomes in $\mathcal{B}$ for Bob, $f_{\mathbf{m}_A,\mathbf{m}_B} : \mathcal{C}^{\perp} \to \mathcal{C}$ replaces abort events on Alice's (resp. Bob's) side by a sample from $\mathbf{m}_A$ (resp. $\mathbf{m}_B$).

For B a normalized Bell functional with coefficients only on non-abort events, the Bell functional $B^{\perp}_{\mathbf{m}_A,\mathbf{m}_B}$ on $(\mathcal{A}\cup\{\perp\})\times(\mathcal{B}\cup\{\perp\})\times\mathcal{X}\times\mathcal{Y}$ is given by

$$(B^{\perp}_{\mathbf{m}_A,\mathbf{m}_B})_{a,b,x,y} = B_{a,b,x,y} + \chi_{\{\perp\}}(a)\sum_{a'\neq\perp} m_A(a'|x)B_{a',b,x,y}$$
$$+\chi_{\{\perp\}}(b)\sum_{b'\neq\perp} m_B(b'|y)B_{a,b',x,y}+\chi_{\{\perp\}}(a)\chi_{\{\perp\}}(b)\sum_{a',b'\neq\perp} m_A(a'|x)m_B(b'|y)B_{a',b',x,y}$$

where $\chi_{\mathcal{S}}$ is the indicator function for set $\mathcal{S}$ taking value 1 on $\mathcal{S}$ and 0 everywhere else.

Note that $f_{\mathbf{m}_A,\mathbf{m}_B}$ preserves locality, and $B^{\perp}_{\mathbf{m}_A,\mathbf{m}_B}(\mathbf{p}) = B(f_{\mathbf{m}_A,\mathbf{m}_B}(\mathbf{p}))$, $\forall \mathbf{p}\in\mathcal{C}^{\perp}$, so $B^{\perp}_{\mathbf{m}_A,\mathbf{m}_B}(\mathbf{p}) = B(\mathbf{p})$, for all $\mathbf{p}\in\mathcal{C}$, and $|B^{\perp}_{\mathbf{m}_A,\mathbf{m}_B}(\ell)|\leq 1$, for all $\ell\in\mathcal{L}^{\perp}$.

▶ **Lemma 19.** *Let B' be a normalized Bell functional on $(\mathcal{A}\cup\{\perp\})\times(\mathcal{B}\cup\{\perp\})\times\mathcal{X}\times\mathcal{Y}$. (possibly with weights on $\perp$.) Then the Bell functional B'' on the same set defined by*

$$B''_{a,b,x,y} = B'_{a,b,x,y} - B'_{a,\perp,x,y} - B'_{\perp,b,x,y} + B'_{\perp,\perp,x,y}, \tag{6}$$

for all $(a,b,x,y)\in(\mathcal{A}\cup\{\perp\})\times(\mathcal{B}\cup\{\perp\})\times\mathcal{X}\times\mathcal{Y}$ satisfies :
1. *If $B''_{a,b,x,y} = 0$, then $a=\perp$ or $b=\perp$,*
2. *$\forall\mathbf{p}\in\mathcal{C}, B''(\mathbf{p}) = B'(\mathbf{p}) - B'(\mathbf{p}_{A,\perp}) - B'(\mathbf{p}_{\perp,B}) + B'(\mathbf{p}_{\perp,\perp})$,*
where $\mathbf{p}_{A,\perp}\in\mathcal{L}^{\perp}$ (resp. $\mathbf{p}_{\perp,B}\in\mathcal{L}^{\perp}$) is the local distribution obtained from $\mathbf{p}$ if Bob (resp. Alice) replaces any of his (resp. her) outputs by $\perp$, and $\mathbf{p}_{\perp,\perp}\in\mathcal{L}^{\perp}$ is the local distribution where Alice and Bob always output $\perp$. In Item 2 above, for any $\mathbf{p}'$, $B'(\mathbf{p}') = \sum_{a,b,x,y} B'_{a,b,x,y}\mathbf{p}'(a,b|x,y)$ where the sum is also over the abort events.

Proof. Item 1 follows from (6). We prove Item 2. For $\mathbf{p}\in\mathcal{C}^{\perp}$ with marginals $\mathbf{p}_A$ and $\mathbf{p}_B$, we have: for all $y\in Y$, $p_A(a|x) = \sum_{b\in\mathcal{B}\cup\{\perp\}} p(a,b|x,y)$, and for all $x\in X$, $p_B(b|y) = \sum_{a\in\mathcal{A}\cup\{\perp\}} p(a,b|x,y)$. For the remainder of this proof, summations involving a (resp. b) are over $a\in\mathcal{A}\cup\{\perp\}$ (resp. $b\in\mathcal{B}\cup\{\perp\}$). By definition, $p_{A,\perp}(a,b|x,y) = p_A(a|x)\chi_{\{\perp\}}(b)$, $p_{\perp,B}(a,b|x,y) = \chi_{\{\perp\}}(a)p_B(b|y)$, and $p_{\perp,\perp}(a,b|x,y) = \chi_{\{\perp\}}(a)\chi_{\{\perp\}}(b)$. We have:

$$\begin{aligned}
B''(\mathbf{p}) &= \sum_{a,b,x,y}\left[B'_{a,b,x,y} - B'_{a,\perp,x,y} - B'_{\perp,b,x,y} + B'_{\perp,\perp,x,y}\right]p(a,b|x,y)\\
&= \sum_{a,b,x,y} B'_{a,b,x,y}p(a,b|x,y) - \sum_{a,x,y} B'_{a,\perp,x,y}\sum_b p(a,b|x,y)\\
&\quad - \sum_{b,x,y} B'_{\perp,b,x,y}\sum_a p(a,b|x,y) + \sum_{x,y} B'_{\perp,\perp,x,y}\sum_{a,b} p(a,b|x,y)\\
&= B'(\mathbf{p}) - \sum_{a,x,y} B'_{a,\perp,x,y}p_A(a|x) - \sum_{b,x,y} B'_{\perp,b,x,y}p_B(b|y) + \sum_{x,y} B'_{\perp,\perp,x,y}\\
&= B'(\mathbf{p}) - B'(\mathbf{p}_{A,\perp}) - B'(\mathbf{p}_{\perp,B}) + B'(\mathbf{p}_{\perp,\perp}).
\end{aligned}$$ ◀

We are now ready to prove Theorem 6.

Proof of Theorem 6. From Observation 17, we can assume that there exists $\ell^-,\ell^+\in\mathcal{L}_{\text{det}}$ such that $B(\ell^-) = -1$ and $B(\ell^+) = 1$ (otherwise, we replace B by its saturated version $\tilde{B}$). Since ℓ^- and ℓ^+ are deterministic distributions, we have: $\ell^- = \ell_A^-\otimes\ell_B^-$ and $\ell^+ = \ell_A^+\otimes\ell_B^+$, for some marginals $\ell_A^-,\ell_B^-,\ell_A^+$, and ℓ_B^+. We consider the two replacing Bell functionals from Definition 18 constructed from (B,ℓ_A^-,ℓ_B^-) on one hand, and (B,ℓ_A^+,ℓ_B^+) on the other hand. Taking $B' = \frac{1}{2}(B^{\perp}_{\ell_A^-,\ell_B^-} + B^{\perp}_{\ell_A^+,\ell_B^+})$, we have $|B'(\ell)|\leq 1$, $\forall\ell\in\mathcal{L}^{\perp}$, and therefore we can apply

Lemma 19 to B' to get B''. Since $B'(\mathbf{p}_{\perp,\perp}) = \frac{1}{2}(B^{\perp}_{\ell_A^-,\ell_B^-}(\mathbf{p}_{\perp,\perp}) + B^{\perp}_{\ell_A^+,\ell_B^+}(\mathbf{p}_{\perp,\perp})) = \frac{1}{2}(B(\ell^-) + B(\ell^+)) = 0$, we have for all $\mathbf{p} \in \mathcal{C}^{\perp}$, $B''(\mathbf{p}) = B'(\mathbf{p}) - B'(\mathbf{p}_{A,\perp}) - B'(\mathbf{p}_{\perp,B})$. Therefore $B^* = \frac{1}{3}B''$ satisfies all the required properties. In particular, since $|B'(\ell)| \leq 1$ $\forall \ell \in \mathcal{L}^{\perp}$, we have for any $\mathbf{p} \in \mathcal{C}$, $B^*(\mathbf{p}) \geq \frac{1}{3}B'(\mathbf{p}) - \frac{1}{3}|B'(\mathbf{p}_{A,\perp})| - \frac{1}{3}|B'(\mathbf{p}_{\perp,B})| \geq \frac{1}{3}B'(\mathbf{p}) - \frac{2}{3}$, and for any $\ell \in \mathcal{L}^{\perp}$, $|B^*(\ell)| \leq \frac{1}{3}|B'(\ell)| + \frac{1}{3}|B'(\ell_{A,\perp})| + \frac{1}{3}|B'(\ell_{\perp,B})| \leq 1$. ◀

B Proof of Theorem 12

Proof. Let us first set $B_{z,x,y} = B_{a,b,x,y}$ for all $a \oplus b = z$. Let $\ell \in \mathcal{L}^{\perp}_{det}$. Then, we have:

$$B(\ell) = \sum_{(x,y)\in R} B_{z,x,y} + \sum_{(x,y)\in S} B_{z,x,y}$$

where R and S are the two rectangles where ℓ outputs z. Let us take a rectangle R. Then :

$$\sum_{(x,y)\in R} B_{z,x,y} = \frac{1}{2\cdot g(n)} \left(\sum_j v_j \mu(V_j \cap R) - \sum_i u_i \mu(U_i \cap R) \right) \leq 1/2$$

with the inequality following from (1). This proves (3).

Let us now compute $B(\mathbf{p}_f)$. By linearity of B and the definition of its coefficients, we have:

$$\begin{aligned} B(\mathbf{p}_f) &= \sum_{a,b,x,y} B_{a,b,x,y} \mathbf{p}_f(a,b|x,y) \\ &= \frac{1}{2} \sum_{(x,y)\in f^{-1}(z),a,b} B_{a,b,x,y} \chi_{\{z\}}(a \oplus b) + \frac{1}{2} \sum_{(x,y)\in f^{-1}(\bar{z}),a,b} B_{a,b,x,y} \chi_{\{\bar{z}\}}(a \oplus b) \\ &= 1/2 \sum_j \sum_{(x,y)\in V_j} v_j g(n)^{-1} \mu(x,y) \\ &= \frac{1}{2\cdot g(n)} \sum_j v_j \mu(V_j) \end{aligned}$$

(for the third equality we used the fact that $B_{a,b,x,y} = 0$ when $a \oplus b = \bar{z}$). This proves (4).

Moreover, for any family of additive error terms $\Delta(a,b|x,y) \in [-1,1]$ such that

$$\sum_{a,b} |\Delta(a,b|x,y)| \leq \epsilon \qquad \forall x,y \in \mathcal{X} \times \mathcal{Y},$$

denoted collectively as $\mathbf{\Delta}$, we have

$$\begin{aligned} |B(\mathbf{\Delta})| &= \left| \sum_{a,b,x,y} B_{a,b,x,y} \Delta(a,b|x,y) \right| \\ &= \frac{1}{2\cdot g(n)} \left| \sum_{a,b\,:\,a\oplus b = z} \left[\sum_i \sum_{(x,y)\in U_i} (-u_i)\mu(x,y)\Delta(a,b|x,y) \right. \right. + \\ &\qquad \left. \left. \sum_j \sum_{(x,y)\in V_j} v_j \mu(x,y) \Delta(a,b|x,y) \right] \right| \end{aligned}$$

$$\leq \frac{1}{2\cdot g(n)}\left[\sum_i \sum_{(x,y)\in U_i} |u_i|\mu(x,y)\left(\sum_{a,b}|\Delta(a,b|x,y)|\right) + \sum_j \sum_{(x,y)\in V_j} |v_j|\mu(x,y)\left(\sum_{a,b}|\Delta(a,b|x,y)|\right)\right]$$

$$\leq \frac{\epsilon}{2\cdot g(n)}\left[\sum_i |u_i|\mu(U_i) + \sum_j |v_j|\mu(V_j)\right]$$

From this calculation and (4), we obtain, for $\mathbf{p}' = \mathbf{p}_f + \mathbf{\Delta}$:

$$B(\mathbf{p}') = B(\mathbf{p}_f) + B(\mathbf{\Delta}) \geq \frac{1}{2\cdot g(n)}\left[\sum_j v_j\mu(V_j) - \epsilon\left(\sum_j |v_j|\mu(V_j) + \sum_i |u_i|\mu(U_i)\right)\right],$$

which proves (5). ◀

C Explicit examples

Let us formulate a special case of Theorem 12 that will be useful in the examples. Here there is just one subset in $f^{-1}(0)$ and one in $f^{-1}(1)$.

▶ **Corollary 20.** *Let f be a (possibly partial) Boolean function on $\mathcal{X}\times\mathcal{Y}$, where $\mathcal{X},\mathcal{Y}\subseteq\{0,1\}^n$. Given $\gamma\in(0,1)$ and $g:\mathbb{N}\to(0,1)$, suppose that there is a distribution μ on $\mathcal{X}\times\mathcal{Y}$ such that: for any rectangle $R\subseteq\mathcal{X}\times\mathcal{Y}$,*

$$\mu(R\cap f^{-1}(1)) > \gamma\mu(R\cap f^{-1}(0)) - g(n). \tag{7}$$

Then μ satisfies (1) with $z=0$, $i=j=1$, $U_1=f^{-1}(1)$, $V_1=f^{-1}(0)$, $u_1=1$, $v_1=\gamma$. Let B be defined by (2), that is: for all $a,b,x,y\in\{0,1\}\times\{0,1\}\times\mathcal{X}\times\mathcal{Y}$,

$$B_{a,b,x,y} = \begin{cases} -\frac{1}{2\cdot g(n)}\mu(x,y) & \text{if } f(x,y)=1 \text{ and } a\oplus b=0 \\ \frac{\gamma}{2\cdot g(n)}\mu(x,y) & \text{if } f(x,y)=0 \text{ and } a\oplus b=0 \\ 0 & \text{otherwise.}\end{cases}$$

Then, B satisfies

$$B(\ell)\leq 1, \quad \forall \ell\in\mathcal{L}_{det}^{\perp},$$
$$B(\mathbf{p}_f) = \frac{\gamma}{2\cdot g(n)}\mu(f^{-1}(0))$$

and for any $\mathbf{p}'\in\mathcal{P}$ such that $|\mathbf{p}'-\mathbf{p}_f|_1\leq\epsilon$:

$$B(\mathbf{p}') \geq \frac{1}{2\cdot g(n)}\Big[\gamma\mu(f^{-1}(0)) - \epsilon\big(\gamma\mu(f^{-1}(0)) + \mu(f^{-1}(1))\big)\Big].$$

C.1 Disjointness

In [39], Razborov proved the following.

▶ **Lemma 21** ([39]). *There exist two distributions μ_0 and μ_1 with $\mathrm{supp}(\mu_0)\subseteq\mathrm{DISJ}_n^{-1}(1)$ and $\mathrm{supp}(\mu_1)\subseteq\mathrm{DISJ}_n^{-1}(0)$, such that: for any rectangle R in the input space,*

$$\mu_1(R)\geq\Omega(\mu_0(R)) - 2^{\Omega(n)}.$$

Following his proof, one can check that we actually have:

$$\mu_1(R) \geq \frac{1}{45}\mu_0(R) - 2^{-\epsilon n+\log_2(2/9)}.$$

So, letting $\mu := (\mu_0 + \mu_1)/2$,

$$\mu(R \cap f^{-1}(0)) \geq \frac{1}{45}\mu(R \cap f^{-1}(1)) - 2^{-\epsilon n+\log_2(4/9)}. \tag{8}$$

▶ Remark. Actually, $\mathrm{supp}(\mu_1) = A_1 := \{(x,y) : |x| = |y| = m, |x \cap y| = 1\} \subseteq \mathrm{DISJ}_n^{-1}(0)$.

Note that by this construction, $\mu(f^{-1}(0)) = \mu(f^{-1}(1)) = 1/2$. Combining (8) with Corollary 20 (with $g(n) = 2^{-\epsilon n+\log_2(4/9)}$), we obtain:

▶ **Corollary 22.** *There exists a Bell inequality B satisfying: $\forall \ell \in \mathcal{L}_{det}^{\perp}$, $B(\ell) \leq 1$,*

$$B(\mathbf{p}_{\mathrm{DISJ}_n}) = \frac{1}{90}2^{\epsilon n-\log_2(4/9)},$$

and for any distribution $\mathbf{p}' \in \mathcal{P}$ such that $|\mathbf{p}' - \mathbf{p}_{\mathrm{DISJ}_n}|_1 \leq \varepsilon$,

$$B(\mathbf{p}') \geq 2^{\epsilon n-\log_2(4/9)}\frac{1-46\epsilon}{90}.$$

More precisely, Theorem 12 gives an explicit construction of such a Bell inequality: we can define B as:

$$B_{a,b,x,y} = \begin{cases} -2^{\epsilon n-\log_2(4/9)}\mu(x,y) & \text{if } \mathrm{DISJ}_n(x,y) = 0 \text{ and } a \oplus b = 1 \\ \frac{1}{45}2^{\epsilon n-\log_2(4/9)}\mu(x,y) & \text{if } \mathrm{DISJ}_n(x,y) = 1 \text{ and } a \oplus b = 1 \\ 0 & \text{otherwise.} \end{cases}$$

To obtain Proposition 14, we use Corollary 9 together with the fact that $Q_{\varepsilon'}(\mathrm{DISJ}_n) = O(\sqrt{n})$.

C.2 Tribes

Let $n = (2r+1)^2$ with $r \geq 2$ and let $\mathrm{TRIBES}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be defined as:

$$\mathrm{TRIBES}_n(x,y) := \bigwedge_{i=1}^{\sqrt{n}} \left(\bigvee_{j=1}^{\sqrt{n}} (x_{(i-1)\sqrt{n}+j} \text{ and } y_{(i-1)\sqrt{n}+j}) \right).$$

In [16][Sec. 3] the following is proven:

▶ **Lemma 23.** *There exists a probability distribution μ on $\{0,1\}^n \times \{0,1\}^n$ for which there exist numbers $\alpha, \lambda, \gamma, \delta > 0$ such that for sufficiently large n and for any rectangle R in the input space:*

$$\gamma\mu(U_1 \cap R) \geq \alpha\mu(V_1 \cap R) - \lambda\mu(V_2 \cap R) - 2^{-\delta n/2+1}$$

where $U_1 = \mathrm{TRIBES}_n^{-1}(0)$, $\{V_1, V_2\}$ forms a partition of $\mathrm{TRIBES}_n^{-1}(1)$ and $\mu(U_1) = 1 - 7\beta^2/16$, $\mu(V_1) = 6\beta^2/16$, $\mu(V_2) = \beta^2/16$ with $\beta = \frac{r+2}{r+1}$.

In [16], the coefficients are $\alpha = 0.99, \lambda = \frac{16}{3(0.99)^2}$ and $\gamma = \frac{16}{(0.99)^2}$ (the authors say these values have not been optimized).

Combining this result with our Theorem 12 (taking $z = 1, i = 1, j = 2$, U_1, V_1, V_2 as in Lemma 23, $u_1 = \gamma, v_1 = \alpha, v_2 = -\lambda$, and $g(n) = 2^{-\delta n/2+1}$), we obtain:

▶ **Corollary 24.** *There exists a Bell inequality satisfying:* $\forall \ell \in \mathcal{L}_{det}^{\perp}$, $B(\ell) \leq 1$,

$$B(\mathbf{p}_{\mathrm{TRIBES}_n}) = 2^{\delta n/2-1}\frac{\beta^2}{16}(6\alpha - \lambda),$$

and for any distribution $\mathbf{p}' \in \mathcal{P}$ *such that* $|\mathbf{p}' - \mathbf{p}_{\mathrm{TRIBES}_n}|_1 \leq \varepsilon$,

$$B(\mathbf{p}') \geq 2^{\delta n/2-1}\left[\frac{\beta^2}{16}(6\alpha - \lambda) - \epsilon(\gamma(1 - 7\beta^2/16) + \lambda\beta^2/16 + \alpha 6\beta^2/16)\right].$$

More precisely, Theorem 12 provides a Bell inequality B yielding this bound, defined as:

$$B_{a,b,x,y} = \begin{cases} -\gamma 2^{\delta n/2-1}\mu(x,y) & \text{if } (x,y) \in U_1 \text{ and } a \oplus b = 1 \\ \alpha 2^{\delta n/2-1}\mu(x,y) & \text{if } (x,y) \in V_1 \text{ and } a \oplus b = 1 \\ -\lambda 2^{\delta n/2-1}\mu(x,y) & \text{if } (x,y) \in V_2 \text{ and } a \oplus b = 1 \\ 0 & \text{otherwise.} \end{cases}$$

To obtain Proposition 15, we use Corollary 9 together with the fact that $Q_{\varepsilon'}(\mathrm{TRIBES}_n) = O(\sqrt{n}(\log n)^2)$.

C.3 Gap Orthogonality

Let f_n be the partial functions over $\{-1,+1\}^n \times \{-1,+1\}^n$ by $f_n(x,y) = \mathrm{ORT}_{64n}(x^{64}, y^{64})$, that is:

$$f_n(x,y) = \begin{cases} -1 & \text{if } |\langle x,y\rangle| \leq \sqrt{n}/8 \\ +1 & \text{if } |\langle x,y\rangle| \geq \sqrt{n}/4. \end{cases}$$

In [41], Sherstov proves the following result.

▶ **Lemma 25** ([41])**.** *Let* $\delta > 0$ *be a sufficiently small constant and* μ *the uniform measure over* $\{0,1\}^n \times \{0,1\}^n$*. Then,* $\mu(f_n^{-1}(+1)) = \Theta(1)$ *and for all rectangle* R *in* $\{0,1\}^n \times \{0,1\}^n$ *such that* $\mu(R) > 2^{-\delta n}$,

$$\mu(R \cap f_n^{-1}(+1)) \geq \delta\mu(R \cap f_n^{-1}(-1)).$$

This implies that if we put uniform weight on inputs of ORT_{64n} of the form (x^{64}, y^{64}) and put 0 weight on the others, we get a distribution μ' satisfying the constraints of Corollary 20 for ORT_{64n} together with $\gamma = \delta$ from Lemma 4 and $g(64n) = 2^{\delta n}$.

To get a distribution satisfying the constraints of Corollary 20 on inputs of ORT_{64n+l} for all $0 \leq l \leq 63$ we extend μ' as follows:

$$\tilde{\mu}(xu, yv) = \begin{cases} \mu'(x,y) & \text{if } u = +1^l,\ v = -1^l \text{ and } \left(\langle x,y\rangle < -\sqrt{64n} \text{ or } 0 \leq \langle x,y\rangle \leq \sqrt{64n}\right) \\ \mu'(x,y) & \text{if } u = +1^l,\ v = +1^l \text{ and } \left(-\sqrt{64n} \leq \langle x,y\rangle < 0 \text{ or } \langle x,y\rangle > \sqrt{64n}\right) \\ 0 & \text{otherwise} \end{cases}$$

Using this distribution $\tilde{\mu}$ together with $\gamma = \delta$ from Lemma 25 and with $g(n) = 2^{-\delta n}$ we obtain, from Corollary 20, a Bell inequality violation for ORT_{64n+l} for all $0 \leq l \leq 63$:

▶ **Corollary 26.** *There exists a Bell inequality* B *satisfying:* $\forall \ell \in \mathcal{L}_{det}^{\perp}$, $B(\ell) \leq 1$,

$$B(\mathbf{p}_{\mathrm{ORT}_{64n+l}}) = 2^{\delta n}\delta\tilde{\mu}(\mathrm{ORT}_{64n+l}^{-1}(-1)),$$

and for any distribution $\mathbf{p}' \in \mathcal{P}$ *such that* $|\mathbf{p}' - \mathbf{p}_{\mathrm{ORT}_{64n+l}}|_1 \leq \varepsilon$,

$$B(\mathbf{p}') \geq 2^{\delta n}\left(\delta\tilde{\mu}(\mathrm{ORT}_{64n+l}^{-1}(-1)) - \epsilon\left[\delta\tilde{\mu}(\mathrm{ORT}_{64n+l}^{-1}(-1)) + \tilde{\mu}(\mathrm{ORT}_{64n+l}^{-1}(+1))\right]\right).$$

More precisely, Theorem 12 gives an explicit construction of such a Bell inequality: we can define B as:

$$B_{a,b,x,y} = \begin{cases} -2^{\delta n}\tilde{\mu}(x,y) & \text{if } (x,y) \in \mathrm{ORT}^{-1}_{64n+l}(+1) \text{ and } a \oplus b = -1 \\ \delta 2^{\delta n}\tilde{\mu}(x,y) & \text{if } (x,y) \in \mathrm{ORT}^{-1}_{64n+l}(-1) \text{ and } a \oplus b = -1 \\ 0 & \text{otherwise.} \end{cases}$$

To obtain Proposition 16, we use Corollary 9 together with the fact that $Q_{\varepsilon'}(\mathrm{ORT}_n) = O(\sqrt{n}\log n)$.

D Equivalent formulations of the efficiency bounds

In [27], the zero-error efficiency bound was defined in its primal and dual forms as follows

▶ **Definition 27** ([27]). The efficiency bound of a distribution $\mathbf{p} \in \mathcal{P}$ is given by

$$\begin{aligned} eff(\mathbf{p}) = \min_{\zeta,\mu_\ell \geq 0} \quad & \frac{1}{\zeta} \\ \text{subject to} \quad & \sum_{\ell \in \mathcal{L}^{\perp}_{det}} \mu_\ell \ell(a,b|x,y) = \zeta p(a,b|x,y) \quad \forall (a,b,x,y) \in \mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y} \\ & \sum_{\ell \in \mathcal{L}^{\perp}_{det}} \mu_\ell = 1 \\ = \max_{B} \quad & B(\mathbf{p}) \\ \text{subject to} \quad & B(\ell) \leq 1 \quad \forall \ell \in \mathcal{L}^{\perp}_{det} \end{aligned}$$

The ϵ-error efficiency bound was in turn defined as $\min_{\mathbf{p}' \in \mathcal{P} | |\mathbf{p}'-\mathbf{p}|_1 \leq \epsilon} eff(\mathbf{p}')$. In this appendix, we show that this is equivalent to the definition used in the present article (Definition 3). In the original definition, the Bell functional could depend on the particular $\mathbf{p}'$. We show that it is always possible to satisfy the constraint with the same Bell functional for all $\mathbf{p}'$ close to $\mathbf{p}$.

In order to prove this, we will need the following notions.

▶ **Definition 28.** A *distribution error* $\mathbf{\Delta}$ is a family of additive error terms $\Delta(a,b|x,y) \in [-1,1]$ for all $(a,b,x,y) \in \mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}$ such that

$$\sum_{a,b} \Delta(a,b|x,y) = 0 \qquad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

For any $0 \leq \epsilon \leq 1$, the set Δ_ϵ is the set of distribution errors $\mathbf{\Delta}$ such that

$$\sum_{a,b} |\Delta(a,b|x,y)| \leq \epsilon \qquad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}.$$

This set is a polytope, so it admits a finite set of extremal points. We denote this set by Δ^{ext}_ϵ.

We will use the following properties of Δ_ε.

▶ Fact 29. For any distribution $\mathbf{p} \in \mathcal{P}$, we have

$$\{\mathbf{p}' \in \mathcal{P} |\ |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon\} \subseteq \{\mathbf{p} + \mathbf{\Delta} |\ \mathbf{\Delta} \in \Delta_\varepsilon\}$$

The reason why the set on the right hand side might be larger is that $\mathbf{p}+\mathbf{\Delta}$ might not be a valid distribution. In order to ensure that this is the case, it is sufficient to impose that all obtained purposed probabilities are nonnegative, leading to the following property.

▶ Fact 30. For any distribution $\mathbf{p} \in \mathcal{P}$, we have

$$\{\mathbf{p}' \in \mathcal{P} | \ |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon\} = \{\mathbf{p}+\mathbf{\Delta} | \ \mathbf{\Delta} \in \Delta_\varepsilon \ \& \ p(a,b|x,y) + \Delta(a,b|x,y) \geq 0 \ \forall a,b,x,y\}$$

We are now ready to prove the following theorem.

▶ **Theorem 31.** *Let $\mathbf{p} \in \mathcal{P}$ be a distribution, $eff_\varepsilon(\mathbf{p})$ be defined as in Definition 3 and $eff(\mathbf{p})$ be defined as in Definition 27. Then, we have*

$$eff_\varepsilon(\mathbf{p}) = \min_{\mathbf{p}' \in \mathcal{P} : |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon} eff(\mathbf{p}').$$

Proof. Let $\overline{eff}_\varepsilon(\mathbf{p}) = \min_{\mathbf{p}' \in \mathcal{P} : |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon} eff(\mathbf{p}')$. We first show that $eff_\varepsilon(\mathbf{p}) \leq \overline{eff}_\varepsilon(\mathbf{p})$. Let (B,β) be an optimal feasible point for $eff_\varepsilon(\mathbf{p})$, so that

$$\begin{aligned} eff_\varepsilon(\mathbf{p}) &= \beta, & \\ B(\mathbf{p}') &\geq \beta & \forall \mathbf{p}' \text{ s.t. } |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon, \\ B(\ell) &\leq 1 & \forall \ell \in \mathcal{L}_{det}^{\perp}. \end{aligned}$$

Therefore (B,β) is also a feasible point for $eff(\mathbf{p}')$ for all $\mathbf{p}' \in \mathcal{P}$ such that $|\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon$, so that $eff(\mathbf{p}') \geq \beta$ for all such $\mathbf{p}'$, and $\overline{eff}_\varepsilon(\mathbf{p}) \geq \beta = eff_\varepsilon(\mathbf{p})$.

It remains to show that $eff_\varepsilon(\mathbf{p}) \geq \overline{eff}_\varepsilon(\mathbf{p})$. In order to do so, we first use the primal form of $eff(\mathbf{p}')$ in Definition 27 to express $\overline{eff}_\varepsilon(\mathbf{p})$ as follows

$$\begin{aligned} \overline{eff}_\varepsilon(\mathbf{p}) = & \min_{\substack{\mathbf{p}' \in \mathcal{P} \\ \text{s.t } |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon}} && eff(\mathbf{p}') \\ = & \min_{\zeta, \mu_\ell \geq 0, \mathbf{p}' \in \mathcal{P}} && \frac{1}{\zeta} \\ & \text{subject to} && \sum_{\ell \in \mathcal{L}_{det}^{\perp}} \mu_\ell \ell(a,b|x,y) = \zeta p'(a,b|x,y) \quad \forall (a,b,x,y) \in \mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y} \\ & && \sum_{\ell \in \mathcal{L}_{det}^{\perp}} \mu_\ell = 1, \quad |\mathbf{p}' - \mathbf{p}|_1 \leq \epsilon \\ = & \min_{\zeta, \mu_\ell \geq 0, \mathbf{\Delta} \in \Delta_\varepsilon} && \frac{1}{\zeta} \\ & \text{subject to} && \sum_{\ell \in \mathcal{L}_{det}^{\perp}} \mu_\ell \ell(a,b|x,y) = \\ & && \quad \zeta[p(a,b|x,y) + \Delta(a,b|x,y)] \qquad \forall (a,b,x,y) \in \mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y} \\ & && \sum_{\ell \in \mathcal{L}_{det}^{\perp}} \mu_\ell = 1, \end{aligned}$$

where the last equality follows from Fact 30 and the fact that the first condition of the program imposes that $p(a,b|x,y) + \Delta(a,b|x,y)$ is nonnegative (since $\sum_\ell \mu_\ell \ell(a,b|x,y)$ is nonnegative).

Since Δ_ε is a polytope, $\overline{eff}_\varepsilon(\mathbf{p})$ can be expressed as the following linear program

$$\begin{aligned}
\overline{eff}_\varepsilon(\mathbf{p}) = \min_{\zeta,\mu_\ell\geq 0,\nu_\Delta\geq 0} \quad & \frac{1}{\zeta} \\
\text{subject to} \quad & \sum_{\ell\in\mathcal{L}_{det}^{\perp}} \mu_\ell \ell(a,b|x,y) = \zeta[p(a,b|x,y)+ \\
& \quad \sum_{\boldsymbol{\Delta}\in\Delta_\varepsilon^{ext}} \nu_\Delta \Delta(a,b|x,y)] \qquad \forall (a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y} \\
& \sum_{\ell\in\mathcal{L}_{det}^{\perp}} \mu_\ell = 1, \quad \sum_{\boldsymbol{\Delta}\in\Delta_\varepsilon^{ext}} \nu_\Delta = 1.
\end{aligned}$$

Note that this can be written in standard LP form via the change of variables $\mu_\ell = \zeta w_\ell$. By LP duality, we then obtain

$$\begin{aligned}
\overline{eff}_\varepsilon(\mathbf{p}) = \max_{B,\beta} \quad & \beta \\
\text{subject to} \quad & B(\mathbf{p}+\boldsymbol{\Delta}) \geq \beta \qquad \forall \boldsymbol{\Delta}\in\Delta_\varepsilon, \\
& B(\ell) \leq 1 \qquad \forall \ell\in\mathcal{L}_{det}^{\perp}.
\end{aligned}$$

Comparing this to the definition of $eff_\varepsilon(\mathbf{p})$ (Definition 3) and together with Fact 29, we therefore have $\overline{eff}_\varepsilon(\mathbf{p}) \leq eff_\varepsilon(\mathbf{p})$. ◀