Generalized Distance Domination Problems and Their Complexity on Graphs of Bounded mim-width

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Abstract

We generalize the family of (σ, ρ)-problems and locally checkable vertex partition problems to their distance versions, which naturally captures well-known problems such as distance-r dominating set and distance-r independent set. We show that these distance problems are XP parameterized by the structural parameter *mim-width*, and hence polynomial on graph classes where mim-width is bounded and quickly computable, such as k-trapezoid graphs, Dilworth k-graphs, (circular) permutation graphs, interval graphs and their complements, convex graphs and their complements, k-polygon graphs, circular arc graphs, complements of d-degenerate graphs, and H-graphs if given an H-representation. To supplement these findings, we show that many classes of (distance) (σ, ρ)-problems are W[1]-hard parameterized by mim-width + solution size.

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1 Introduction

Telle and Proskurowski [19] defined the (σ, ρ)-domination problems, and the more general *locally checkable vertex partitioning* problems (LCVP). In (σ, ρ)-domination problems, feasible

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solutions are vertex sets with constraints on how many neighbours each vertex of the graph has in the set. The framework generalizes important and well-studied problems such as Maximum Independent Set and Minimum Dominating Set, as well as Perfect Code, Minimum subgraph with minimum degree d and a multitude of other problems. See Table 1. Bui-Xuan, Telle and Vatshelle [6] showed that (σ, ρ)-domination and locally checkable vertex partitioning problems can be solved in time XP parameterized by *mim-width*, if we are given a corresponding decomposition tree. Roughly speaking, the structural parameter mim-width measures how easy it is to decompose a graph along vertex cuts inducing a bipartite graph with small maximum induced matching size [20].

In this paper, we consider distance versions of problems related to independence and domination, like Distance-r Independent Set and Distance-r Dominating Set. The Distance-r Independent Set problem, also studied under the names r-Scattered Set and r-Dispersion (see e.g. [2] and the references therein), asks to find a set of at least k vertices whose vertices have pairwise distance strictly longer than r. Agnarsson et al. [1] pointed out that it is identical to the original Independent Set problem on the r-th power graph G^r of the input graph G, and also showed that for fixed r, it can be solved in linear time for interval graphs, and circular arc graphs. The Distance-r Dominating Set problem was introduced by Slater [18] and Henning et al. [11]. They also discussed that it is identical to solve the original Dominating Set problem on the r-th power graph. Slater presented a linear-time algorithm to solve Distance-r Dominating Set problem on forests.

We generalize all of the (σ, ρ)-domination and LCVP problems to their distance versions, which naturally captures Distance-r Independent Set and Distance-r Dominating Set. Where the original problems put constraints on the size of the immediate neighborhood of a vertex, we consider the constraints to be applied to the ball of radius r around it. Consider for instance the Minimum Subgraph with Minimum Degree d problem; where the original problem is asking for the smallest (number of vertices) subgraph of minimum degree d, we are instead looking for the smallest subgraph such that for each vertex there are at least d vertices at distance at least 1 and at most r. In the Perfect Code problem, the target is to choose a subset of vertices such that each vertex has exactly one chosen vertex in its closed neighbourhood. In the distance-r version of the problem, we replace the closed neighbourhood by the closed r-neighbourhood. This problem is known as Perfect r-Code, and was introduced by Biggs [4] in 1973. Similarily, for every problem in Table 1 its distance-r generalization either introduces a new problem or is already well-known.

We show that all these distance problems are XP parameterized by mim-width if a decomposition tree is given. The main result of the paper is of structural nature, namely that for any positive integer r the mim-width of a graph power G^r is at most twice the mim-width of G. It follows that we can reduce the distance-r version of a (σ, ρ)-domination problem to its non-distance variant by taking the graph power G^r, whilst preserving small mim-width.

The downside to showing results using the parameter mim-width, is that we do not know an XP algorithm computing mim-width. Computing a decomposition tree with optimal mim-width is NP-complete in general and W[1]-hard parameterized by itself. Determining the optimal mim-width is not in APX unless NP = ZPP, making it unlikely to have a polynomial-time constant-factor approximation algorithm [17], but saying nothing about an XP algorithm. However, for several graph classes we are able to find a decomposition tree of constant mim-width in polynomial time, using the results of Belmonte and Vatshelle [3]. These include; permutation graphs, convex graphs and their complements, interval graphs and their complements (all of which have *linear* mim-width 1); (circular k-) trapezoid graphs, circular permutation graphs, Dilworth-k graphs, k-polygon graphs, circular arc graphs and complements of d-degenerate graphs. Fomin, Golovach and Raymond [10] show that we can find linear decomposition trees of constant mim-width for the very general class of H-graphs, see Definition 10, in polynomial time, *given*[2] an H-representation of the input graph. For all of the above graph classes, our results imply that the distance-r (σ, ρ)-domination and LCVP problems become polynomial time solvable.

Graphs represented by intersections of objects in some model are often closed under taking powers. For instance, interval graphs, and generally d-trapezoid graphs [9, 1], circular arc graphs [16, 1], and leaf power graphs (by definition) are such graphs. We refer to [5, Chapter 10.6] for a survey of such results. For these classes, we already know that the distance-r version of a (σ, ρ)-domination problem can be solved in polynomial time. However, this closure property does not always hold; for instance, permutation graphs are not closed under taking powers. Our result provides that to obtain such algorithmic results, we do not need to know that these classes are closed under taking powers; it is sufficient to know that classes have bounded mim-width. To the best of our knowledge, for the most well-studied distance-r (σ, ρ)-domination problem, Distance-r Dominating Set, we obtain the first polynomial time algorithms on Dilworth k-graphs, convex graphs and their complements, complements of interval graphs, k-polygon graphs, H-graphs (given an H-representation of the input graph), and complements of d-degenerate graphs.

The natural question to ask after obtaining an XP algorithm, is whether we can do better, e. g. can we show that for all fixed r, the distance-r (σ, ρ)-domination problems are in FPT? Fomin et al. [10] answered this in the negative by showing that (the standard, i.e. distance-1 variants of) Maximum Independent Set, Minimum Dominating Set and Minimum Independent Dominating Set problems are W[1]-hard parameterized by (linear) mim-width + solution size. We modify their reductions to extend these results to several families of (σ, ρ)-domination problems, including the maximization variants of Induced Matching, Induced d-Regular Subgraph and Induced Subgraph of Max Degree $\leq d$, the minimization variants of Total Dominating Set and d-Dominating Set and both the maximization and the minimization variant of Dominating Induced Matching.

The remainder of the paper is organized as follows. In Section 2 we introduce the (σ, ρ) problems and define their distance-r generalization. In Section 3 we introduce mim-width, and state previously known results. In Section 4 we show that the mim-width of a graph grows by at most a factor 2 when taking (arbitrary large) powers and give algorithmic consequences. We discuss LCVP problems, their distance-r versions and algorithmic consequences regarding them in Section 5 and in Section 6 we present the above mentioned lower bounds. Finally, we give some concluding remarks in Section 7. Proofs of statements marked with '★' are deferred to the full version [12].

2 Distance-r (σ, ρ)-Domination Problems

Let σ and ρ be finite or co-finite subsets of the natural numbers $\sigma, \rho \subseteq \mathbb{N}$. Furthermore, for a graph G, one of its vertices $v \in V(G)$, and a positive integer r, let $N^r(v)$ denote the *ball of radius r around v*, i. e. $N^r(v) := \{w \in V(G) \setminus \{v\} \mid \textsc{dist}_G(v, w) \leq r\}$. A vertex set $S \subseteq V(G)$ is called a *distance-r (σ, ρ)-dominating set*, if

- for each vertex $v \in S$ it holds that $|N^r(v) \cap S| \in \sigma$, and

[2] We would like to remark that it is NP-complete to decide whether a graph is an H-graph whenever H is not a cactus [7].

Table 1 Some vertex subset properties expressible as (σ, ρ) sets, with $\mathbb{N} = \{0, 1, ...\}$ and $\mathbb{N}^+ = \{1, 2, ...\}$. Column d shows $d = \max(d(\sigma), d(\rho))$. For each problem, at least one of the minimization, the maximization and the existence problem is NP-complete. For problems marked with $\star$ (resp., $\star\star$), W[1]-hardness of the maximization (resp., minimization) problem parameterized by mim-width + solution size is shown in the present paper. For problems marked with $*$ (resp., $**$) the W[1]-hardness of maximization (resp., minimization) in the same parameterization was shown by Fomin et al. [10].

σ	ρ	d	Standard name
$\{0\}$	$\mathbb{N}$	1	Independent set $*$
$\mathbb{N}$	$\mathbb{N}^+$	1	Dominating set $**$
$\{0\}$	$\mathbb{N}^+$	1	Maximal Independent set $**$
$\mathbb{N}^+$	$\mathbb{N}^+$	1	Total Dominating set $\star\star$
$\{0\}$	$\{0, 1\}$	2	Strong Stable set or 2-Packing
$\{0\}$	$\{1\}$	2	Perfect Code or Efficient Dom. set
$\{0, 1\}$	$\{0, 1\}$	2	Total Nearly Perfect set
$\{0, 1\}$	$\{1\}$	2	Weakly Perfect Dominating set
$\{1\}$	$\{1\}$	2	Total Perfect Dominating set
$\{1\}$	$\mathbb{N}$	2	Induced Matching $\star$
$\{1\}$	$\mathbb{N}^+$	2	Dominating Induced Matching $\star$, $\star\star$
$\mathbb{N}$	$\{1\}$	2	Perfect Dominating set
$\mathbb{N}$	$\{d, d+1, ...\}$	d	d-Dominating set $\star\star$
$\{d\}$	$\mathbb{N}$	$d+1$	Induced d-Regular Subgraph $\star$
$\{d, d+1, ...\}$	$\mathbb{N}$	d	Subgraph of Min Degree $\geq d$
$\{0, 1, ..., d\}$	$\mathbb{N}$	$d+1$	Induced Subg. of Max Degree $\leq d$ $\star$

- for each vertex $v \in V(G) \setminus S$ it holds that $|N^r(v) \cap S| \in \rho$.

For the special case of $r = 1$, we call a distance-1 (σ, ρ)-dominating set simply a (σ, ρ)-dominating set or (σ, ρ) set. This recovers many well-studied naturally defined vertex sets of graphs. For instance, a $(\{0\}, \mathbb{N})$ set is an independent set as there are no edges inside of the set, and we do not care about adjacencies between S and $V(G) \setminus S$; and a $(\mathbb{N}, \mathbb{N}^+)$ set is a dominating set since each vertex in $V(G) \setminus S$ has to have at least one neighbor in S.

There are three types of distance-r (σ, ρ)-domination problems: minimization, maximization, and existence. For $r = 1$, we denote the problem of finding a minimum (maximum) (σ, ρ) set as the Min-(σ, ρ) (Max-(σ, ρ)) problem, see Table 1 for examples.

The *d-value* of a distance-r (σ, ρ) problem is a constant which will ultimately affect the runtime of the algorithm. For a set $\mu \subseteq \mathbb{N}$, the value $d(\mu)$ should be understood as the highest value in $\mathbb{N}$ we need to enumerate in order to describe μ. Hence, if μ is finite, it is simply the maximum value in μ, and if μ is co-finite, it is the maximum natural number *not* in μ (1 is added for technical reasons).

▶ **Definition 1** (d-value)**.** Let $d(\mathbb{N}) = 0$. For every non-empty finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max\{x \mid x \in \mu\}, \max\{x \mid x \in \mathbb{N} \setminus \mu\})$.

For a given distance-r (σ, ρ) problem $\Pi_{\sigma,\rho}$, its d-value is defined as $d(\Pi_{\sigma,\rho}) := \max\{d(\sigma), d(\rho)\}$, see column d in Table 1.

3 Mim-width and Applications

Maximum induced matching width, or *mim-width* for short, was introduced in the Ph. D. thesis of Vatshelle [20], used implicitly by Belmonte and Vatshelle [3], and is a structural graph parameter described over *decomposition trees* (sometimes called *branch decompositions*), similar to graph parameters such as *rank-width* and *module-width*. Decomposition trees naturally appear in divide and conquer style algorithms where one recursively partitions the pieces of a problem into two parts. When the algorithm is at the point where it combines solutions of its subproblems to form a full solution, the structure of the cuts are (unsurprisingly) important to the runtime; this is especially true of dynamic programming when one needs to store multiple sub-solutions at each intermediate node. We will briefly introduce the necessary machinery here, but for a more comprehensive introduction we refer the reader to [20].

A graph of maximum degree at most 3 is called *subcubic*. A *decomposition tree* for a graph G is a pair (T, δ) where T is a subcubic tree and $\delta : V(G) \to L(T)$ is a bijection between the vertices of G and the leaves of T. Each edge $e \in E(T)$ naturally splits the leaves of the tree in two groups depending on their connected component when e is removed. In this way, each edge $e \in E(T)$ also represent a partition of $V(G)$ into two partition classes A_e and $\overline{A}_e$. One way to measure the cut structure is by the *maximum induced matching* across a cut of (T, δ). A set of edges M is called an *induced matching* if no pair of edges in M shares an endpoint and if the subgraph induced by the endpoints of M does not contain any additional edges.

▶ **Definition 2** (mim-width). Let G be a graph, and let (T, δ) be a decomposition tree for G. For each edge $e \in E(T)$ and corresponding partition of the vertices $A_e, \overline{A}_e$, we let $cutmim_G(A_e, \overline{A}_e)$ denote the size of a maximum induced matching of the bipartite graph on the edges crossing the cut. Let the *mim-width of the decomposition tree* be

$$mimw_G(T, \delta) = \max_{e \in E(T)} \{cutmim(A_e, \overline{A}_e)\}$$

The *mim-width of the graph G*, denoted $mimw(G)$, is the minimum value of $mimw_G(T, \delta)$ over all possible decompositions trees (T, δ). The *linear mim-width of the graph G* is the minimum value of $mimw_G(T, \delta)$ over all possible decompositions trees (T, δ) where T is a caterpillar.

In previous work, Bui-Xuan et al. [6] and Belmonte and Vatshelle [3] showed that all (σ, ρ) problems can be solved in time $n^{\mathcal{O}(w)}$ where w denotes the mim-width of a decomposition tree that is provided as part of the input. More precisely, they show the following.[3]

▶ **Proposition 3** ([3, 6]). *There is an algorithm that given a graph G and a decomposition tree (T, δ) of G with $w := mimw_G(T, \delta)$ solves each (σ, ρ) problem Π with $d := d(\Pi)$*
(i) *in time $\mathcal{O}(n^{4+2d \cdot w})$, if T is a caterpillar, and*
(ii) *in time $\mathcal{O}(n^{4+3d \cdot w})$, otherwise.*

4 Mim-width on Graph Powers

▶ **Definition 4** (Graph power). Let $G = (V, E)$ be a graph. Then the k-th power of G, denoted G^k, is a graph on the same vertex set where there is an edge between two vertices if and only if the distance between them is at most k in G. Formally, $V(G^k) = V(G)$ and $E(G^k) = \{uv \mid \textsc{dist}_G(u, v) \leq k\}$.

[3] We would like to remark that the original results in [6] are stated in terms of the number of d-neighborhood equivalence classes across the cuts in the decomposition tree ($nec_d(T, \delta)$) giving a runtime of $n^4 \cdot nec_d(T, \delta)^c$ (where $c = 2$ if the given decomposition is a caterpillar and $c = 3$ otherwise). In [3, Lemma 2], Belmonte and Vatshelle show that $nec_d(T, \delta) \leq n^{d \cdot mimw_G(T, \delta)}$.

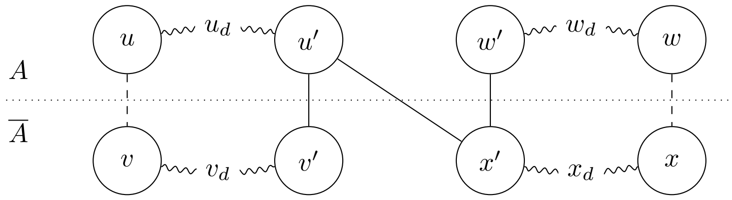

Figure 1 Structure of two paths P_{uv} and P_{wx} when the edge $u'x'$ exists in G. Dashed edges appear in G^k, solid edges appear in G, squiggly lines are (shortest) paths existing in G (possibly of length 0, and possibly crossing back and forth across the cut).

▶ **Theorem 5.** *For any graph G and positive integer k, mimw$(G^k) \leq 2 \cdot$ mimw(G).*

Proof. Assume that there is a decomposition tree of mim-width w for the graph G. We show that the same decomposition tree has mim-width at most $2w$ for G^k.

We consider a cut $A, \overline{A}$ of the decomposition tree. Let M be a maximum induced matching across the cut for G^k. To prove our claim, it suffices to construct an induced matching across the cut M' in G such that $|M'| \geq \frac{|M|}{2}$.

We begin by noticing that for an edge $uv \in M$, the distance between u and v is at most k in G. For each such edge $uv \in M$, we let P_{uv} denote some shortest path between u and v in G (including the endpoints u and v).

▶ **Claim 5.1.** *Let $uv, wx \in M$ be two distinct edges of the matching. Then P_{uv} and P_{wx} are vertex disjoint.*

Proof. We may assume that $u, w \in A$ and $v, x \in \overline{A}$. Now assume for the sake of contradiction there exists a vertex $y \in P_{uv} \cap P_{wx}$. Because both paths have length at most k, we have that $\textsc{dist}_G(u, y) + \textsc{dist}_G(y, v) \leq k$, and $\textsc{dist}_G(w, y) + \textsc{dist}_G(y, x) \leq k$. Adding these together, we get

$$\textsc{dist}_G(u, y) + \textsc{dist}_G(y, v) + \textsc{dist}_G(w, y) + \textsc{dist}_G(y, x) \leq 2k.$$

Since uv and wx are both in M, there can not exist edges ux and wv in G^k. Hence, their distance in G is strictly greater than k, i.e. $\textsc{dist}_G(u, y) + \textsc{dist}_G(y, x) \geq \textsc{dist}_G(u, x) > k$, and $\textsc{dist}_G(w, y) + \textsc{dist}_G(y, v) > k$. Putting these together, we obtian our contradiction:

$$\textsc{dist}_G(u, y) + \textsc{dist}_G(y, x) + \textsc{dist}_G(w, y) + \textsc{dist}_G(y, v) > 2k$$

This concludes the proof of the claim. ⌟

Our next observation is that for each $uv \in M$, the path P_{uv} starts (without loss of generality) in A, and ends in $\overline{A}$. There must hence exist at least one point at which the path cross from A to $\overline{A}$. For each $uv \in M$, we can thus safely let $u'v' \in E(P_{uv})$ denote an edge in G such that $u' \in A$ and $v' \in \overline{A}$.

We plan to construct our matching M' by picking a subset of such edges. However, we can not simply take all of them, since some pairs may be incompatible in the sense that they will not form an induced matching across the cut $A, \overline{A}$. We examine the structures that arise when two such edges $u'v'$ and $w'x'$ are incompatible, and can not both be included in the same induced matching across the cut. For easier readability, we let α_d be a shorthand notation for $\textsc{dist}_G(\alpha, \alpha')$ for $\alpha \in \{u, v, w, x\}$.

▶ **Claim 5.2.** *Let $uv, wx \in M$ be two distinct edges of M and let $u'v'$ and $w'x'$ be edges on the shortest paths as defined above. If there is an edge $u'x' \in E(G)$, then all of the following hold. See Figure 1.*

(a) $u_d + x_d = k$

(b) $u_d + v_d = w_d + x_d = k - 1$

(c) $w_d = u_d - 1$

Proof. (a) Since ux is not an edge in G^k, the distance between u and x must be at least $k+1$ in G, and so $u_d + x_d$ must be at least k. It remains to show that $u_d + x_d \leq k$ for equality to hold. Similarly to the proof of Claim 5.1, we know that P_{uv} and P_{wx} both are of length at most k. We get

$$u_d + v_d + w_d + x_d \leq 2k - 2 \tag{1}$$

The -2 at the end is because we do not include the length contributed by edges $u'v'$ and $w'x'$ in our sum. Now assume for the sake of contradiction that $u_d + x_d \geq k + 1$. Then we get that

$$v_d + w_d \leq 2k - 2 - k - 1 = k - 3$$

Because $\text{DIST}_G(v', w') \leq 3$ (follow the edges $u'v' \to u'x' \to w'x'$), this implies that $\text{DIST}_G(v, w) \leq k$, and the edge vw would hence exist in G^k. This contradicts that uv and wx were both in the same induced matching M.

(b) Assume for the sake of contradiction that $u_d + v_d \leq k - 2$. Then, rather than Equation 1, we get the following bound

$$u_d + v_d + w_d + x_d \leq 2k - 3$$

By (a) we know that $u_d + x_d = k$, so by a similar argument as above we get that $v_d + x_d \leq k - 3$, obtaining a contradiction. An anolgous argument holds for $w_d + x_d$.

(c) This follows immidiately by substituting (a) into (b). ⌟

We will now construct our induced matching M'. We construct two candidates for M', and we will pick the biggest one. First, we construct M'_0 by including $u'v'$ for each edge $uv \in M$ where $\text{DIST}_G(u, u')$ is even. Symetrically, M'_1 is constructed by including $u'v'$ if $\text{DIST}_G(u, u')$ is odd. Clearly, at least one of M'_0, M'_1 contains $\geq \frac{|M|}{2}$ egdes. It remains to show that M' indeed forms an induced matching across the cut $A, \overline{A}$ in G.

Consider two distinct edges $u'v'$ and $w'x'$ from M'. By Claim 5.1, the two edges are vertex disjoint. If there is an edge violating that $u'v'$ and $w'x'$ are both in the same induced matching, it must be either $u'x'$ or $v'w'$. Without loss of generality we may assume it is an edge of the type $u'x'$. By Claim 5.2 (c), we then have that the parities of $\text{DIST}_G(u, u')$ and $\text{DIST}_G(w, w')$ are different. But by how M' was constructed, this is not possible. This concludes the proof. ◀

▶ **Observation 6.** *For a positive integer r, a graph G and a vertex $u \in V(G)$, the r-neighbourhood of u is equal to the neighbourhood of u in G^r, i.e. $N^r_G(u) = N_{G^r}(u)$.*

The observation above shows that solving a distance-r (σ, ρ) problem on G is the same as solving the same standard distance-1 variation of the problem on G^r. Hence, we may reduce our problem to the standard version by simply computing the graph power. Combining Theorem 5 with the algorithms provided in Proposition 3, we have the following consequence.

▶ **Corollary 7** (★). *There is an algorithm that for all $r \in \mathbb{N}$, given a graph G and a decomposition tree (T, δ) of G with $w := mimw_G(T, \delta)$ solves each distance-r (σ, ρ) problem Π with $d := d(\Pi)$*

(i) *in time $\mathcal{O}(n^{4+4d\cdot w})$, if T is a caterpillar, and*

(ii) *in time $\mathcal{O}(n^{4+6d\cdot w})$, otherwise.*

5 LCVP Problems

A generalization of (σ, ρ) problems are the *locally checkable vertex partitioning* (LCVP) problems. A *degree constraint matrix* D is a $q \times q$ matrix where each entry is a finite or co-finite subset of $\mathbb{N}$. For a graph G and a partition of its vertices $\mathcal{V} = \{V_1, V_2, \dots V_q\}$, we say that it is a D-partition if and only if, for each $i, j \in [q]$ and each vertex $v \in V_i$, it holds that $|N(v) \cap V_j| \in D[i, j]$. Empty partition classes are allowed.

For instance, if a graph can be partitioned according to the 3×3 matrix whose diagonal entries are $\{0\}$ and the non-diagonal ones are $\mathbb{N}$, then the graph is 3-colorable. Typically, the natural algorithmic questions associated with LCVP properties are existential.[4] Interesting problems which can be phrased in such terms include the H-Covering and Graph H-Homomorphism problems where H is fixed, as well as q-coloring, Perfect Matching Cut and more. We refer to [19] for an overview.

We generalize LCVP properties to their distance-r version, by considering the ball of radius r around each vertex rather than just the immediate neighbourhood.

▶ **Definition 8** (Distance-r neighbourhood constraint matrix). A distance-r neighbourhood constraint matrix D is a $q \times q$ matrix where each entry is a finite or co-finite subset of $\mathbb{N}$. For a graph G and a partition of its vertices $\mathcal{V} = \{V_1, V_2, \dots V_q\}$, we say that it is a D-distance-r-partition if and only if, for each $i, j \in [q]$ and each vertex $v \in V_i$, it holds that $|N^r(v) \cap V_j| \in D[i, j]$. Empty partition classes are allowed.

We say that an algorithmic problem is a *distance-r* LCVP *problem* if the property in question can be described by a distance-r neighbourhood constraint matrix. For example, the distance-r version of a problem such as q-coloring can be interpreted as an assignment of at most q colours to vertices of a graph such that no two vertices are assigned the same colour if they are at distance r or closer.

For a given distance-r LCVP problem Π, its d-value $d(\Pi)$ is the maximum d-value over all the sets in the corresponding neighbourhood constraint matrix.

As in the case of (σ, ρ) problems, combining Theorem 5 with Observation 6 and the works [3, 6] we have the following result.

▶ **Corollary 9.** *There is an algorithm that for all $r \in \mathbb{N}$, given a graph G and a decomposition tree (T, δ) of G with $w := mimw_G(T, \delta)$ solves each distance-r LCVP problem Π with $d := d(\Pi)$*

(i) *in time $\mathcal{O}(n^{4+4qd\cdot w})$, if T is a caterpillar, and*

(ii) *in time $\mathcal{O}(n^{4+6qd\cdot w})$, otherwise.*

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[4] Note however that each (σ, ρ) problem can be stated as an LCVP problem via the matrix $D_{(\sigma,\rho)} = \begin{bmatrix} \sigma & \mathbb{N} \\ \rho & \mathbb{N} \end{bmatrix}$, so maximization or minimization of some block of the partition can be natural as well.

6 Lower Bounds

We show that several (σ, ρ)-problems are W[1]-hard parameterized by linear mim-width plus solution size. Our reductions are based on two recent reductions due to Fomin, Golovach and Raymond [10] who showed that Independent Set and Dominating Set are W[1]-hard parameterized by linear mim-width plus solution size. In fact they show hardness for the above mentioned problems on *H-graphs* (the parameter being the number of edges in H plus solution size) which we now define formally.

▶ **Definition 10** (H-Graph)**.** Let X be a set and $\mathcal{S}$ a family of subsets of X. The *intersection graph* of $\mathcal{S}$ is a graph with vertex set $\mathcal{S}$ such that $S, T \in \mathcal{S}$ are adjacent if and only if $S \cap T \neq \emptyset$. Let H be a (multi-) graph. We say that G is an *H-graph* if there is a subdivision H' of H and a family of subsets $\mathcal{M} := \{M_v\}_{v \in V(G)}$ (called an *H-representation*) of $V(H')$ where $H'[M_v]$ is connected for all $v \in V(G)$, such that G is isomorphic to the intersection graph of $\mathcal{M}$.

All of the hardness results presented in this section are obtained via reductions to the respective problems on H-graphs, and the hardness for linear mim-width follows from the following proposition.

▶ **Proposition 11** (Theorem 2 in [10])**.** *Let G be an H-graph. Then, G has linear mim-width at most $2 \cdot ||H||$ and a corresponding decomposition tree can be computed in polynomial time given an H-representation of G.*

The first lower bound concerns several maximization problems that can be expressed in the (σ, ρ) framework. Recall that the Independent Set problem can be formulated as Max-$(\{0\}, \mathbb{N})$. The following result states that a class of problems that generalize the Independent Set problem where each vertex in the solution is allowed to have at most some fixed number of d neighbors of the solution, and several variants thereof, is W[1]-hard on H-graphs parameterized by $||H||$ plus solution size.

▶ **Theorem 12.** *For any fixed $d \in \mathbb{N}$ and $x \le d + 1$, the following holds. Let $\sigma^* \subseteq \mathbb{N}_{\le d}$ with $d \in \sigma^*$. Then,* Max-$(\sigma^*, \mathbb{N}_{\ge x})$ Domination *is* W[1]*-hard on H-graphs parameterized by the number of edges in H plus solution size, and the hardness holds even if an H-representation of the input graph is given.*

Proof. To prove the theorem, we provide a reduction from Multicolored Clique where given a graph G and a partition $V_1, \ldots, V_k$ of $V(G)$, the question is whether G contains a clique of size k using precisely one vertex from each V_i ($i \in [k]$). This problem is known to be W[1]-complete [8, 14].

Let $(G, V_1, \ldots, V_k)$ be an instance of Multicolored Clique. We can assume that $k \ge 2$ and that $|V_i| = p$ for $i \in [k]$. If the second assumption does not hold, let $p := \max_{i \in [k]} |V_i|$ and add $p - |V_i|$ isolated vertices to V_i, for each $i \in [k]$. (Note that adding isolated vertices does not change the answer to the problem.) For $i \in [k]$, we denote by $v_1^i, \ldots, v_p^i$ the vertices of V_i. We first describe the reduction of Fomin et al. [10] and then explain how to modify it to prove the theorem.

The Construction of Fomin, Golovach and Raymond [10]. The graph H is obtained as follows.

1. Construct k nodes $u_1, \ldots, u_k$.

2. For every $1 \le i < j \le k$, construct a node $w_{i,j}$ and two pairs of parallel edges $u_i w_{i,j}$ and $u_j w_{i,j}$.

We then construct the subdivision H' of H by first subdividing each edge p times. We denote the subdivision nodes for 4 edges of H constructed for each pair $1 \le i < j \le k$ in Step 2 by $x_1^{(i,j)}, \ldots, x_p^{(i,j)}$, $y_1^{(i,j)}, \ldots, y_p^{(i,j)}$, $x_1^{(j,i)}, \ldots, x_p^{(j,i)}$, and $y_1^{(j,i)}, \ldots, y_p^{(j,i)}$. To simplify notation, we assume that $u_i = x_0^{(i,j)} = y_0^{(i,j)}$, $u_j = x_0^{(j,i)} = y_0^{(j,i)}$ and $w_{i,j} = x_{p+1}^{(i,j)} = y_{p+1}^{(i,j)} = x_{p+1}^{(j,i)} = y_{p+1}^{(j,i)}$.

We now construct the H-graph G'' by defining its H-representation $\mathcal{M} = \{M_v\}_{v \in V(G'')}$ where each M_v is a connected subset of $V(H')$. (Recall that G denotes the graph of the Multicolored Clique instance.)

1. For each $i \in [k]$ and $s \in [p]$, construct a vertex z_s^i with model

$$M_{z_s^i} := \bigcup_{j \in [k], j \neq i} \left(\left\{ x_0^{(i,j)}, \ldots, x_{s-1}^{(i,j)} \right\} \cup \left\{ y_0^{(i,j)}, \ldots, y_{p-s}^{(i,j)} \right\} \right).$$

2. For each edge $v_s^i v_t^j \in E(G)$ for $s, t \in [p]$ and $1 \le i < j \le k$, construct a vertex $r_{s,t}^{(i,j)}$ with:

$$\begin{aligned} M_{r_{s,t}^{(i,j)}} := &\left\{ x_s^{(i,j)}, \ldots, x_{p+1}^{(i,j)} \right\} \cup \left\{ y_{p-s+1}^{(i,j)}, \ldots, y_{p+1}^{(i,j)} \right\} \\ &\cup \left\{ x_t^{(j,i)}, \ldots, x_{p+1}^{(j,i)} \right\} \cup \left\{ y_{p-t+1}^{(j,i)}, \ldots, y_{p+1}^{(j,i)} \right\}. \end{aligned}$$

Throughout the following, for $i \in [k]$ and $1 \le i < j \le k$, respectively, we use the notation

$$Z(i) := \bigcup_{s \in [p]} \left\{ z_s^i \right\} \text{ and } R(i,j) := \bigcup_{\substack{v_s^i v_t^j \in E(G),\\ s,t \in [p]}} \left\{ r_{s,t}^{(i,j)} \right\}.$$

We now observe the crucial property of G''.

▶ **Observation 12.1** (Claim 18 in [10]). *For every $1 \le i < j \le k$, a vertex $z_h^i \in V(G')$ (a vertex $z_h^j \in V(G')$) is* not *adjacent to a vertex $r_{s,t}^{(i,j)} \in V(G')$ corresponding to the edge $v_s^i v_t^j \in E(G)$ if and only if $h = s$ ($h = t$, respectively).*

The New Gadget. We now describe how to obtain from G'' a graph G' that will be the graph of the instance of Max-$(\sigma^*, \mathbb{N}_{\ge x})$ Domination. We do so by adding a gadget to each set $Z(i)$ and $R(i,j)$ (for all $1 \le i < j \le k$). We first describe the gadget and then explain how to modify H' to a new graph K' such that G' is a K-graph (where K denotes the graph obtained from K' by undoing the above described subdivisions that were made in H to obtain H'). Let X be any set of vertices of G''. The gadget $\mathfrak{B}(X)$ is a complete bipartite graph on $2d - 1$ vertices and bipartition $(\{\beta_{1,1}, \ldots, \beta_{1,d}\}, \{\beta_{2,1}, \ldots, \beta_{2,d-1}\})$. such that for $h \in [d]$, each vertex $\beta_{1,h}$ is additionally adjacent to each vertex in X. For $1 \le i < j \le k$, we use the notation $\mathfrak{B}(i) := \mathfrak{B}(Z(i))$ and $\mathfrak{B}(i,j) := \mathfrak{B}(R(i,j))$ and we denote their vertices by $\beta^i_{\cdot,\cdot}$ and $\beta^{(i,j)}_{\cdot,\cdot}$, respectively.

We obtain K' by 'hardcoding' each gadget $\mathfrak{B}(\cdot)$ into H'. That is, for $i \in [k]$, we add the graph $\mathfrak{B}(i)$ and connect it to the remaining vertices via the edges $u_i \beta^i_{1,h}$ for $h \in [d]$. For $1 \le i < j \le k$, we proceed analogously in encoding $\mathfrak{B}(i,j)$ into H'. For an illustration of the graph K, see Figure 2. We observe that $|K| = 2d\left(k + \binom{k}{2}\right) = kd(k+1)$ and

$$||K|| = 4\binom{k}{2} + \left(k + \binom{k}{2}\right) \cdot (d + d(d-1)) = \frac{1}{2}k\left(d^2(k+1) + 4(k-1)\right). \qquad (2)$$

We subdivide all newly introduced edges, i.e. all edges in $E(K') \setminus E(H')$ and for an edge $xy \in E(K') \setminus E(H')$, we denote the resulting vertex by $s(x,y)$. We are now ready to describe (the K-representation of) G'.

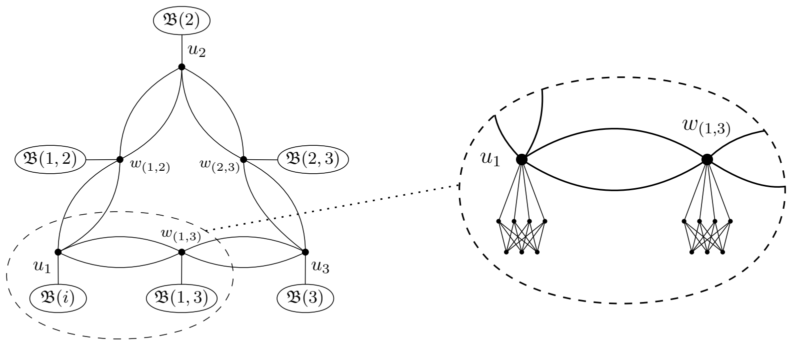

Figure 2 The graph K with respect to which the graph G' constructed in the proof of Theorem 12 is a K-graph. In this example, we have $k = 3$ and $d = 4$.

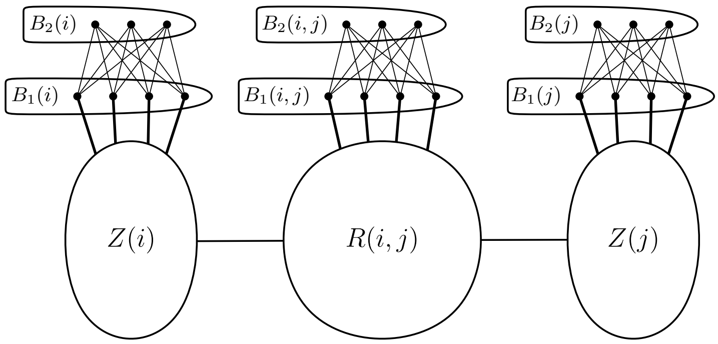

Figure 3 A part of the graph G', where $1 \le i < j \le k$ and $d = 4$.

1. For all $i \in [k]$ and $s \in [p]$, we add the vertices $s(u_i, \beta^i_{1,h})$ (where $h \in [d]$) to the model of z^i_s. For all $1 \le i < j \le k$ and $s, t \in [p]$ with $v^i_s v^j_t \in E(G)$, we add the vertices $s(w_{(i,j)}, \beta^{(i,j)}_{1,h})$ for $h \in [d]$ to the model of $r^{(i,j)}_{s,t}$.
2. For all $i \in [k]$ and $h \in [d]$, we add a vertex $b^i_{1,h}$ with model $\{\beta^i_{1,h}, s(u_i, \beta^i_{1,h})\} \cup \bigcup_{h' \in [d-1]} \{s(\beta^i_{1,h}, \beta^i_{2,h'})\}$.
3. For all $i \in [k]$ and $h \in [d-1]$, we add a vertex $b^i_{2,h}$ with model $\{\beta^i_{2,h}\} \cup \bigcup_{h' \in [d]} \{s(\beta^i_{2,h}, \beta^i_{1,h'})\}$.
4. For all $v^i_s v^j_t \in E(G)$ (where $1 \le i < j \le k$ and $s, t \in [p]$) and $h \in [d]$, we add a vertex $b^{(i,j)}_{1,h}$ with model $\{\beta^{(i,j)}_{1,h}, s(w_{(i,j)}, \beta^{(i,j)}_{1,h})\} \cup \bigcup_{h' \in [d-1]} \{s(\beta^{(i,j)}_{1,h}, \beta^{(i,j)}_{2,h'})\}$.
5. For all $v^i_s v^j_t \in E(G)$ (where $1 \le i < j \le k$ and $s, t \in [p]$) and $h \in [d-1]$, we add a vertex $b^{(i,j)}_{1,h}$ with model $\{\beta^{(i,j)}_{2,h}\} \cup \bigcup_{h' \in [d]} \{s(\beta^{(i,j)}_{2,h}, \beta^{(i,j)}_{1,h'})\}$.

One can verify that these five steps introduce the above described vertices to G'. For an illustration of G', see Figure 3. The correctness proof of the reduction is given in the appendix; it is essentially a proof of the following claim.

▶ **Claim 12.2 (★).** *G has a multicolored clique if and only if G' has a $(\sigma^*, \mathbb{N}_{\ge x})$ set of size $k' = 2d \cdot (k + \binom{k}{2})$.*

We observe that $|V(G')| = \mathcal{O}(|V(G)| + d^2 \cdot k^2)$ and clearly, G' can be constructed from G in time polynomial in $|V(G)|$, d and k as well. Furthermore, by (2), $||K|| = \mathcal{O}(d^2 \cdot k^2)$ and the theorem follows. ◀

By Proposition 11, the previous theorem implies

▶ **Corollary 13.** *For any fixed $d \in \mathbb{N}$ and $x \leq d+1$, the following holds. Let $\sigma^* \subseteq \mathbb{N}_{\leq d}$ with $d \in \sigma^*$. Then,* Max-$(\sigma^*, \mathbb{N}_{\geq x})$ Domination *is* W[1]*-hard parameterized by linear mim-width plus solution size, and the hardness holds even if a corresponding decomposition tree is given.*

We now turn to hardness of minimization problems that can be expressed in (σ, ρ) notation. First, with a slight modification of the reduction due to Fomin et al. [10], we obtain hardness for problems such as Total Dominating Set and Dominating Induced Matching.

▶ **Theorem 14 (★).** *For $\sigma^* \subseteq \mathbb{N}^+$ with $1 \in \sigma^*$ and $\rho^* \subseteq \mathbb{N}^+$ with $\{1,2\} \subseteq \rho^*$,* Min-(σ^*, ρ^*) Domination *is* W[1]*-hard on H-graphs parameterized by the number of edges in H plus solution size, and the hardness holds even when an H-representation of the input graph is given.*

As a somewhat orthogonal result to Theorem 12, we now show hardness of several problems related to the d-Dominating Set problem, where each vertex that is not in the solution set has to be dominated by at least some fixed number of d neighbors in the solution.

▶ **Theorem 15 (★).** *For any fixed $d \in \mathbb{N}_{\geq 2}$,[5] the following holds. Let $\sigma^* \subseteq \mathbb{N}$ with $\{0, 1, d-1\} \subseteq \sigma^*$ and $\rho^* \subseteq \mathbb{N}_{\geq d}$ with $\{d, d+1\} \subseteq \rho^*$. Then,* Min-$(\sigma^*, \rho^*)$ Domination *is* W[1]*-hard on H-graphs parameterized by the number of edges in H plus solution size, and the hardness even holds when an H-representation of the input graph is given.*

Similarly to above, a combination of the previous two theorems with Proposition 11 yields the following hardness results for (σ, ρ) mimization problems on graphs of bounded linear mim-width.

▶ **Corollary 16.** *Let $\sigma^* \subseteq \mathbb{N}$ and $\rho^* \subseteq \mathbb{N}$. Then,* Min-$(\sigma^*, \rho^*)$ Domination *is* W[1]*-hard parameterized by linear mim-width plus solution size, if one of the following holds.*
(i) *$\sigma^* \subseteq \mathbb{N}^+$ with $1 \in \sigma^*$ and $\rho^* \subseteq \mathbb{N}^+$ with $\{1, 2\} \subseteq \rho^*$.*
(ii) *For some fixed $d \in \mathbb{N}_{\geq 2}$, $\{0, 1, d-1\} \subseteq \sigma^*$ and $\rho^* \subseteq \mathbb{N}_{\geq d}$ with $\{d, d+1\} \subseteq \rho^*$.*
Furthermore, the hardness holds even if a corresponding decomposition tree is given.

7 Concluding Remarks

We have introduced the class of distance-r (σ, ρ) and LCVP problems. This generalizes well-known graph distance problems like distance-r domination, distance-r independence, distance-r coloring and perfect r-codes. It also introduces many new distance problems for which the standard distance-1 version naturally captures a well-known graph property.

Using the graph parameter mim-width, we showed that all these problems are solvable in polynomial time for many interesting graph classes. These meta-algorithms will have runtimes which can likely be improved significantly for a particular problem on a particular graph class. For instance, blindly applying our results to solve Distance-r Dominating Set on permutation graphs yields an algorithm that runs in time $\mathcal{O}(n^8)$: Permutation graphs have linear mim-width 1 (with a corresponding decomposition tree that can be computed in linear time) [3, Lemmas 2 and 5], so we can apply Corollary 7(i). However, there is an algorithm that solves Distance-r Dominating Set on permutation graphs in time $\mathcal{O}(n^2)$ [15]; a much faster runtime.

We would like to draw attention to the most important and previously stated [13, 17, 20] open question regarding the mim-width parameter: Is there an XP approximation algorithm

[5] Note that the analogous statement for $d = 1$ follows from the reduction given in [10].

for computing mim-width? An important first step could be to devise a polynomial-time algorithm deciding if a graph has mim-width 1, or even linear mim-width 1.

Regarding lower bounds, we expanded on the previous results by Fomin et al. [10] and showed that many (σ, ρ) problems are W[1]-hard parameterized by mim-width. However, it remains open whether there exists a problem which is NP-hard in general, yet FPT by mim-width. In particular, there are currently no hardness results when σ and ρ are both finite. Even so, we conjecture that every NP-hard (distance) (σ, ρ) problem is W[1]-hard parameterized by mim-width.

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