# Packing Directed Circuits Quarter-Integrally


## Tomáš Masařík

Department of Applied Mathematics, Charles University, Prague, Czech Republic & Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warsaw, Poland
masarik@kam.mff.cuni.cz

## Irene Muzi

Technische Universität Berlin, Germany
irene.muzi@tu-berlin.de

## Marcin Pilipczuk

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warsaw, Poland
malcin@mimuw.edu.pl

## Paweł Rzążewski

Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland
p.rzazewski@mini.pw.edu.pl

## Manuel Sorge

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warsaw, Poland
manuel.sorge@mimuw.edu.pl


## Abstract


The celebrated Erdős-Pósa theorem states that every undirected graph that does not admit a family of $k$ vertex-disjoint cycles contains a feedback vertex set (a set of vertices hitting all cycles in the graph) of size $\mathcal{O}(k \log k)$. After being known for long as Younger's conjecture, a similar statement for directed graphs has been proven in 1996 by Reed, Robertson, Seymour, and Thomas. However, in their proof, the dependency of the size of the feedback vertex set on the size of vertex-disjoint cycle packing is not elementary.

We show that if we compare the size of a minimum feedback vertex set in a directed graph with *quarter-integral* cycle packing number, we obtain a polynomial bound. More precisely, we show that if in a directed graph $G$ there is no family of $k$ cycles such that every vertex of $G$ is in at most *four* of the cycles, then there exists a feedback vertex set in $G$ of size $\mathcal{O}(k^4)$. On the way there we prove a more general result about quarter-integral packing of subgraphs of high directed treewidth: for every pair of positive integers $a$ and $b$, if a directed graph $G$ has directed treewidth $\Omega(a^6 b^8 \log^2(ab))$, then one can find in $G$ a family of $a$ subgraphs, each of directed treewidth at least $b$, such that every vertex of $G$ is in at most four subgraphs.



**2012 ACM Subject Classification** Mathematics of computing → Graph algorithms

**Keywords and phrases** Directed graphs, graph algorithms, linkage, Erdős–Pósa property

**Digital Object Identifier** 10.4230/LIPIcs.ESA.2019.72

**Related Version** A full version of the paper is available at `http://arxiv.org/abs/1907.02494`.

**Funding** This research is a part of projects that have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme Grant Agreements 648527 (IM) and 714704 (all authors).
*Tomáš Masařík*: Also supported by Charles University, student grant number SVV–2017–260452.

**Acknowledgements** We thank Stephan Kreutzer (TU Berlin) for interesting discussions on the topic and for pointing out Lemma 3.




27th Annual European Symposium on Algorithms (ESA 2019).
Editors: Michael A. Bender, Ola Svensson, and Grzegorz Herman; Article No. 72; pp. 72:1–72:13
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

# 1 Introduction

The theory of graph minors, developed over the span of over 20 years by Robertson and Seymour, had a tremendous impact on the area of graph algorithms. Arguably, one of the cornerstone contributions is the notion of *treewidth* [19] and the deep understanding of obstacles to small treewidth, primarily in the form of the *excluded grid theorem* [5, 20, 21].

Very tight relations of treewidth and the size of the largest grid as a minor in sparse graph classes, such as planar graphs or graphs excluding a fixed graph as a minor, led to the rich and fruitful theory of bidimensionality [10]. In general graphs, fine understanding of the existence of well-behaved highly-connected structures (not necessarily grids) in graphs of high treewidth has been crucial to the development of efficient approximation algorithms for the Disjoint Paths problem [9].

In undirected graphs, one of the first theorems that gave some well-behaved structure in a graph that is in some sense highly connected is the famous Erdős-Pósa theorem [11] linking the feedback vertex set number of a graph (the minimum number of vertices one needs to delete to obtain an acyclic graph) and the cycle packing number (the maximum possible size of a family of vertex-disjoint cycles in a graph). The Erdős-Pósa theorem states that a graph that does not contain a family of $k$ vertex-disjoint cycles has feedback vertex set number bounded by $\mathcal{O}(k \log k)$.

A similar statement for directed graphs, asserting that a directed graph without a family of $k$ vertex-disjoint cycles has feedback vertex set number at most $f(k)$, has been long known as the Younger's conjecture until finally proven by Reed, Robertson, Seymour, and Thomas in 1996 [17]. However, the function $f$ obtained in [17] is not elementary; in particular, the proof relies on the Ramsey theorem for $\Theta(k)$-regular hypergraphs. This is in contrast with the (tight) $\Theta(k \log k)$ bound in undirected graphs.

Our main result is that if one compares the feedback vertex set number of a directed graph to the *quarter-integral* cycle packing number (i.e., the maximum size of a family of cycles in $G$ such that every vertex lies on at most four cycles), one obtains a polynomial bound.

▶ **Theorem 1.** *If a directed graph $G$ does not contain a family of $k$ cycles such that every vertex in $G$ is contained in at most four cycles, then there exists a feedback vertex set in $G$ of size $\mathcal{O}(k^4)$.*

We remark that if one relaxes the condition even further to a *fractional cycle packing*,[1] Seymour [22] proved that a graph without a fractional cycle packing of size at least $k$ admits a feedback vertex set of size $\mathcal{O}(k \log k \log \log k)$.

*Directed treewidth* is a directed analog of the successful notion of treewidth, introduced in [13, 16]. An analog of the excluded grid theorem for directed graphs has been conjectured by Johnson, Robertson, Seymour, and Thomas [13] in 2001 and finally proven by Kawarabayashi and Kreutzer in 2015 [15]. Similarly as in the case of the directed Erdős-Pósa property, the relation between the directed treewidth of a graph and a largest directed grid as a minor in [15] is not elementary.

For a directed graph $G$, let $\mathrm{fvs}(G)$, $\mathrm{dtw}(G)$, and $\mathrm{cp}(G)$ denote the feedback vertex set number, directed treewidth, and the cycle packing number of $G$, respectively. The following lemma is a restatement of the result of Amiri, Kawarabayashi, Kreutzer, and Wollan [1, Lemma 4.2]:

[1] A *fractional cycle packing* assigns to every cycle $C$ in $G$ a non-negative real weight $w(C)$ such that for every $v \in V(G)$ the total weight of all cycles containing $v$ is at most 1. The *size* of a fractional cycle packing $w$ is the total weight of all cycles in the packing.

▶ **Lemma 2** ([1, Lemma 4.2]). *Let $G$ be a directed graph with $\operatorname{dtw}(G) \leq w$. For each strongly connected directed graph $H$, the graph $G$ has either $k$ disjoint copies of $H$ as a topological minor, or contains a set $T$ of at most $k \cdot (w+1)$ vertices such that $H$ is not a topological minor of $G - T$.*

Note that the authors of [1] prove Lemma 2 for topological and butterfly minors, but the previous restatement is sufficient for our purposes. By taking $H$ as the directed 2-cycle it is easy to derive the following bound:

▶ **Lemma 3.** *For a directed graph $G$ it holds that $\operatorname{fvs}(G) \leq (\operatorname{dtw}(G)+1)\operatorname{cp}(G)$.*

In the light of Lemma 3 and since a directed grid minor of size $k$ contains $k$ vertex-disjoint cycles, the directed grid theorem of Kawarabayashi and Kreutzer [15] is a generalization of the directed Erdős-Pósa property due to Reed, Robertson, Seymour, and Thomas [17].

Theorem 1 is a direct corollary of Lemma 3 and the following statement that we prove.

▶ **Theorem 4.** *If a directed graph $G$ does not contain a family of $k$ cycles such that every vertex in $G$ is contained in at most four cycles, then $\operatorname{dtw}(G) = \mathcal{O}(k^3)$.*

Furthermore, if one asks not for a cycle packing, but a packing of subgraphs of large directed treewidth, we prove the following packing result.

▶ **Theorem 5.** *There exists an absolute constant $c$ with the following property. For every pair of positive integers $a$ and $b$, and every directed graph $G$ of directed treewidth at least $c \cdot a^6 \cdot b^8 \cdot \log^2(ab)$, there are directed graphs $G_1, G_2, \ldots, G_a$ with the following properties:*

1. *each $G_i$ is a subgraph of $G$,*
2. *each vertex of $G$ belongs to at most four graphs $G_i$, and*
3. *each graph $G_i$ has directed treewidth at least $b$.*

Note that by setting $b = 2$ in Theorem 5, one obtains Theorem 4 with a slightly weaker bound of $\mathcal{O}(k^6 \log^2 k)$ and, consequently, Theorem 1 with a weaker bound of $\mathcal{O}(k^7 \log^2 k)$.

Theorem 5 should be compared to its undirected analog of Chekuri and Chuzhoy [4] that asserts that in an undirected graph $G$ of treewidth at least $c\min(ab^2, a^3b)$ one can find $a$ vertex-disjoint subgraphs of treewidth at least $b$. While we still obtain a polynomial bound, we can only prove the existence of a quarter-integral (as opposed to integral, i.e., vertex-disjoint) packing of subgraphs of high directed treewidth.

In the Disjoint Paths problem, given a graph $G$ and a set of terminal pairs $(s_i, t_i)_{i=1}^k$, we ask to find an as large as possible collection of vertex-disjoint paths such that every path in the collection connects some $s_i$ with $t_i$. Let OPT be the number of paths in the optimum solution; we say that a family $\mathcal{P}$ is a *congestion-c polylogarithmic approximation* if every path in $\mathcal{P}$ connects a distinct pair $(s_i, t_i)$, each vertex of $V(G)$ is contained in at most $c$ paths of $\mathcal{P}$, and $|\mathcal{P}| \geq \mathrm{OPT}/\mathrm{polylog}(\mathrm{OPT})$. The successful line of research of approximation algorithms for the Disjoint Paths problem in undirected graphs leading in particular to a congestion-2 polylogarithmic approximation algorithm of Chuzhoy and Li [9] for the edge-disjoint version, would not be possible without a fine understanding of well-behaved well-connected structures in a graph of high treewidth. Of central importance to such *routing* algorithms is the notion of a *crossbar*: a crossbar of order $k$ and congestion $c$ is a subgraph $C$ of $G$ with an *interface* $I \subseteq V(C)$ of size $k$ such that for every matching $M$ on $I$, one can connect the endpoints of the matching edges with paths in $C$ such that every vertex is in at most $c$ paths. Most of the known approximation algorithms for Disjoint Paths find a crossbar $(C, I)$ with a large set of disjoint paths between $I$ and the set of terminals $s_i$ and $t_i$. While one usually does not control how the paths connect the terminals $s_i$ and $t_i$ to interface vertices of $I$, the ability of the crossbar to connect *any* given matching on the interface leads to a solution.

To obtain a polylogarithmic approximation algorithm, one needs the order of the crossbar to be comparable to the number of terminal pairs, which – by well-known tools such as *well-linked decompositions* [8] – is of the order of treewidth of the graph. At the same time, we usually allow constant congestion (every vertex can appear in a constant number of paths of the solution, instead of just one). Thus, the milestone graph-theoretic result used in approximation algorithms for Disjoint Paths is the existence of a congestion-2 crossbar of order $k$ in a graph of treewidth $\Omega(k\text{polylog}(k))$.

While the existence of similar results for the general Disjoint Paths problem in directed graphs is implausible [2], Chekuri, and Ene proposed to study the case of *symmetric demands* where one asks for a path from $s_i$ to $t_i$ and a path from $t_i$ to $s_i$ for a terminal pair $(s_i, t_i)$. First, they provided an analog of the well-linked decomposition for this case [6], and then with Pilipczuk [7] showed an existence of an analog of a crossbar and a resulting approximation algorithm for Disjoint Paths with symmetric demands in planar directed graphs. Later, this result has been lifted to arbitrary proper minor-closed graph classes [3]. However, the general case remains widely open.

As discussed above, for applications in approximation algorithms for Disjoint Paths, it is absolutely essential to squeeze as much as possible from the bound linking directed treewidth of a graph with the order of the crossbar, while the final congestion is of secondary importance (but we would like it to be a small constant). We think of Theorem 5 as a step in this direction: sacrificing integral packings for quarter-integral ones, we obtain much stronger bounds than the non-elementary bounds of [17]. Furthermore, such a step seems necessary, as it is hard to imagine a crossbar of order $k$ that would not contain a constant-congestion (i.e., every vertex used in a constant number of cycles) packing of $\Omega(k)$ directed cycles.

On the technical side, the proof of Theorem 5 borrows a number of technical tools from the recent work of Hatzel, Kawarabayashi, and Kreutzer that proved polynomial bounds for the directed grid minor theorem in planar graphs [12]. We follow their general approach to obtain a directed treewidth sparsifier [12, Section 5] and modify it in a number of places for our goal. The main novelty comes in different handling of the case when two linkages intersect a lot. Here we introduce a new partitioning tool (see Section 3) which we use in the crucial moment where we separate subgraphs $G_i$ from each other.

**Organization.** After brief preliminaries in Section 2, we prove Theorem 5 in Sections 3–5: Section 3 introduces the new partitioning tool, Section 4 handles the most complicated "dense case" in the analysis, while Section 5 wraps up the argument. Discussions on the the adaptation of the arguments of Section 5 to obtain the improved bound of Theorem 4 and some argumentation from Section 4 that directly follows the arguments of [12] can be found in the full version of the paper.

## 2 Preliminaries

Let $G = (V(G), E(G))$ be a directed graph and let $A, B$ be subsets of $V(G)$ with $|A| = |B|$. A *linkage* from $A$ to $B$ in $G$ is a set $\mathcal{L}$ of $|A|$ pairwise vertex-disjoint paths in $G$, each with a starting vertex in $A$ and ending vertex in $B$. The *order* of $\mathcal{L}$ is $|\mathcal{L}| = |A|$. For $X, Y \subseteq V(G)$ and a linkage $\mathcal{L}$ from $X$ to $Y$, we denote $A(\mathcal{L}) := X$ and $B(\mathcal{L}) := Y$. For a path or a walk $P$, by $\mathsf{start}(P)$ and $\mathsf{end}(P)$ we denote the starting and ending vertex of $P$, respectively.

Let $\mathcal{L}$ and $\mathcal{K}$ be linkages. The *intersection graph* of $\mathcal{L}$ and $\mathcal{K}$, denoted by $I(\mathcal{L}, \mathcal{K})$, is the bipartite graph with the vertex set $\mathcal{L} \cup \mathcal{K}$ and an edge between a vertex in $\mathcal{L}$ and a vertex in $\mathcal{K}$ if the corresponding paths share at least one vertex.

A vertex set $W \subseteq V(G)$ is *well-linked* if for all subsets $A, B \subseteq W$ with $|A| = |B|$ there is a linkage $\mathcal{L}$ of order $|A|$ from $A$ to $B$ in $G \setminus (W \setminus (A \cup B))$.

Let $\mathcal{P}$ be a family of walks in $G$ and let $c$ be a positive integer. We say that $\mathcal{P}$ is *of congestion* $c$ if for every $v \in V(G)$, the total number of times the walks in $\mathcal{P}$ visit $v$ is at most $c$; here, if a walk $W \in \mathcal{P}$ visits $v$ multiple times, we count each visit separately. A family of paths $\mathcal{P}$ is a *half-integral* (*quarter-integral*) if it is of congestion 2 (resp. 4).

We call two linkages $\mathcal{L}$ and $\mathcal{L}^{\mathsf{back}}$ *dual* to each other if $A(\mathcal{L}) = B(\mathcal{L}^{\mathsf{back}})$ and $A(\mathcal{L}^{\mathsf{back}}) = B(\mathcal{L})$. For two dual linkages $\mathcal{L}$ and $\mathcal{L}^{\mathsf{back}}$ in a graph $G$, we define an auxiliary directed graph $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ as follows. We take $V(\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})) = \mathcal{L}$ and for every path $P \in \mathcal{L}^{\mathsf{back}}$ that starts in a vertex $\mathsf{start}(P) = \mathsf{end}(L)$ for some $L \in \mathcal{L}$ and ends in a vertex $\mathsf{end}(P) = \mathsf{start}(L')$ for some $L' \in \mathcal{L}$, we put an arc $(L, L')$ to $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$. Note that it may happen that $L = L'$. When the backlinkage $\mathcal{L}^{\mathsf{back}}$ is clear from the context, we abbreviate $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ to $\mathsf{Aux}(\mathcal{L})$. Observe that in $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ every node is of in- and out-degree exactly one and thus this graph is a disjoint union of directed cycles.

With every arc $(L, L')$ of $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ we can associate the walk from $\mathsf{start}(L)$ to $\mathsf{start}(L')$ that first goes along $L$ and then follows the path $P \in \mathcal{L}^{\mathsf{back}}$ that gives raise to the arc $(L, L')$. Consequently, with every collection of pairwise disjoint paths and cycles in $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ there is an associated collection of walks (closed walks for cycles) in $G$ that is of congestion 2 as it originated from two linkages. Note that the same construction works if $\mathcal{L}$ and $\mathcal{L}^{\mathsf{back}}$ are half-integral linkages, and then the walks in $G$ corresponding to a family of paths and cycles in $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ would be of congestion 4.

Furthermore, with a pair of dual linkages $\mathcal{L}$ and $\mathcal{L}^{\mathsf{back}}$ we can associate a *backlinkage-induced order* $\mathcal{L} = \{L_1, L_2, \ldots, L_{|\mathcal{L}|}\}$ as follows. If $C_1, C_2, \ldots, C_r$ are the cycles of the graph $\mathsf{Aux}(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$ in an arbitrary order, then $L_1, L_2, \ldots, L_{|C_1|}$ are the vertices of $C_1$ in the order of their appearance on $C_1$, and $L_{|C_1|+1}, \ldots, L_{|C_1|+|C_2|}$ are the vertices of $C_2$ in the order of their appearance on $C_2$, etc. That is, we order the elements of $\mathcal{L}$ first according to the cycle of $\mathsf{Aux}(\mathcal{L})$ they lie on, and then, within one cycle, according to the order around this cycle.

We will also need the following operation on a pair of dual linkages $\mathcal{L}$ and $\mathcal{L}^{\mathsf{back}}$. Let $\mathcal{P} \subseteq \mathcal{L}$ be a sublinkage. For every $P \in \mathcal{P}$, construct a walk $Q(P)$ as follows. Start from the path $Q_0 \in \mathcal{L}^{\mathsf{back}}$ with $\mathsf{start}(Q_0) = \mathsf{end}(P)$ and set $Q(P) = Q_0$. Given $Q_i \in \mathcal{L}^{\mathsf{back}}$ for $i \geq 0$, proceed as follows. Let $P_{i+1} \in \mathcal{L}$ be the path with $\mathsf{end}(Q_i) = \mathsf{start}(P_{i+1})$. If $P_{i+1} \in \mathcal{P}$, then stop. Otherwise, define $Q_{i+1} \in \mathcal{L}^{\mathsf{back}}$ to be the path with $\mathsf{end}(P_{i+1}) = \mathsf{start}(Q_{i+1})$. Append $P_{i+1}$ and $Q_{i+1}$ at the end of $Q(P)$ and repeat. Finally, we shortcut $Q(P)$ to a path $Q'(P)$ with the same endpoints. In this manner, $\mathcal{Q} := \{Q'(P) \mid P \in \mathcal{P}\}$ is a half-integral linkage with $A(\mathcal{P}) = B(\mathcal{Q})$ and $A(\mathcal{Q}) = B(\mathcal{P})$. We call $\mathcal{Q}$ the *backlinkage induced by* $\mathcal{P}$ *on* $(\mathcal{L}, \mathcal{L}^{\mathsf{back}})$. Furthermore, we can perform the same construction if $\mathcal{L}$ and $\mathcal{L}^{\mathsf{back}}$ are half-integral linkages, obtaining a quarter-integral linkage $\mathcal{Q}$.

We say that $G$ is $d$-degenerate if and only if every subgraph of $G$ contains a vertex of degree at most $d$.

In this paper we do not need the exact definition of directed treewidth. Instead, we rely on the following two results.

▶ **Lemma 6** ([16]). *Every directed graph $G$ of directed treewidth $k$ contains a well-linked set of size $\Omega(k)$.*

▶ **Lemma 7** ([14, 15]). *There is an absolute constant $c'$ with the following property. Let $\alpha, \beta \geq 1$ be integers and let $G$ be a digraph of $\mathrm{dtw}(G) \geq c' \cdot \alpha^2 \beta^2$. Then there exists a set of $\alpha$ vertex-disjoint paths $P_1, \ldots, P_\alpha$ and sets $A_i, B_i \subseteq V(P_i)$, where $A_i$ appears before $B_i$ on $P_i$, both $|A_i|, |B_i| = \beta$, and $\bigcup_{i=1}^{\alpha} A_i \cup B_i$ is well-linked.*

We also need the following two auxiliary results. Note that a coloring in Lemma 8 can be arbitrary and is not necessarily proper.

▶ **Lemma 8** ([18, Lemma 4.3]). *Let $r \geq 2$, $d$ be a real, and $H$ be an $r$-colored graph with color classes $V_1, \dots, V_r$, such that for every $i$ it holds that $|V_i| \geq 4e(r-1)d$ and for every $i \neq j$ the graph $H[V_i \cup V_j]$ is $d$-degenerate. Then there exists an independent set $\{x_1, \dots, x_r\}$ such that $x_i \in V_i$ for every $i \in [r]$.*

▶ **Lemma 9** ([12, Lemma 5.5]). *Let $G$ be a digraph and $P_1, \dots, P_k$ be disjoint paths such that each $P_i$ consists of two subpaths $A_i$ and $B_i$, where $A_i$ precedes $B_i$. Furthermore, let $\{L_{i,j} : i, j \in [k], i \neq j\}$ be a set of pairwise disjoint paths, such that $L_{i,j}$ starts in $B_i$ and ends in $A_j$. Then*

$$\mathrm{dtw}\Big(\bigcup_i P_i \cup \bigcup_{i \neq j} L_{i,j}\Big) \geq \frac{k}{8}.$$

## 3 Partitioning Lemma

In this section, we develop a main technical tool that we use in the proof of Theorem 5. Intuitively, in a subcase of the proof, we will have a bipartite graph of large minimum degree which we partition into subgraphs induced by pairs of vertex sets $(U_i, W_i)$. These subgraphs will define the $G_i$ from the statement of Theorem 5. To obtain a lower bound on the directed treewidth of $G_i$, we need that the parts $(U_i, W_i)$ each induce a subgraph of large average degree. This will be achieved using the following lemma.

▶ **Lemma 10.** *Let $h \geq 0$ and $n$ be integers, $d$ be a positive real such that $d \cdot 4^{h+1} - 1 > 2$, and let $G$ be a bipartite graph with bipartition classes $X = \{x_1, x_2, \dots, x_a\}$ and $Y = \{y_1, y_2, \dots, y_b\}$, such that $a + b \leq n$ and $|E(G)| \geq (d \cdot 4^{h+1} - 1) \cdot n$. Then in $[a]$ we can find $k := 2^h$ pairwise disjoint sets $I_1, I_2, \dots, I_k$, and in $[b]$ we can find $k$ pairwise disjoint sets $J_1, J_2, \dots, J_k$, such that:*

1. *for every $i \in [k]$ the set $I_i$ is a segment of $[a]$ and the set $J_i$ is a segment of $[b]$,*
2. *for every $i \in [k]$, the number of edges between $\{x_i : i \in I_i\}$ and $\{y_i : i \in J_i\}$ is at least $d \cdot n$.*

**Proof.** For $I \subseteq [a]$ and $J \subseteq [b]$, let $e(I, J)$ we denote the number of edges $x_i y_j$ of $G$, such that $i \in I$ and $j \in J$. Observe that $|E(G)| > 2n$.

We prove the lemma by induction on $h$. Note that for $h = 0$ the claim is trivially satisfied by taking $I_1 = X$ and $J_1 = Y$, as $d \cdot 4^{h+1} - 1 > 2$ and $h \geq 0$ implies $d \cdot 4^{h+1} - 1 \geq d$. So now assume that $h \geq 1$ and the claim holds for $h - 1$. Let $s \in [a]$ be the minimum integer, for which $\sum_{i=1}^{s} \deg x_i \geq |E(G)|/2$, and let $t \in [b]$ be the minimum integer, for which $\sum_{i=1}^{t} \deg y_i \geq |E(G)|/2$. We observe that $d \cdot 4^{h+1} - 1 > 2$ implies that $1 < s < a$ and $1 < t < b$. Define $X^1 := \{1, 2, \dots, s-1\}$ and $X^2 := \{s+1, \dots, a\}$, and $Y^1 := \{1, 2, \dots, t-1\}$ and $Y^2 := \{t+1, \dots, b\}$.

We aim to show that the number of edges joining $X^1$ and $Y^1$ is roughly the same as the number of edges joining $X^2$ and $Y^2$, and the number of edges joining $X^1$ and $Y^2$ is roughly the same as the number of edges joining $X^2$ and $Y^1$. Since $\deg x_s \leq b < n$ and $\deg y_t \leq a < n$, by the choice of $s$ and $t$ we obtain the following set of inequalities.

$$\begin{aligned}
e(X,Y)/2 - \deg x_s &\le e(X^1,Y) \le e(X,Y)/2\\
e(X,Y)/2 - \deg x_s &\le e(X^2,Y) \le e(X,Y)/2\\
e(X,Y)/2 - \deg y_t &\le e(X,Y^1) \le e(X,Y)/2\\
e(X,Y)/2 - \deg y_t &\le e(X,Y^2) \le e(X,Y)/2.
\end{aligned} \tag{1}$$

Observe that

$$\begin{aligned}
e(X^1,Y^1)+e(X^1,Y^2) \le e(X^1,Y) &= e(X^1,Y^1)+e(X^1,Y^2)+e(X^1,\{t\})\\
&\le e(X^1,Y^1)+e(X^1,Y^2)+\deg y_t
\end{aligned}$$

(and analogously for each of the remaining inequalities in (1)). Thus we obtain:

$$\begin{aligned}
e(X,Y)/2 - n &\le e(X^1,Y^1)+e(X^1,Y^2) \le e(X,Y)/2\\
e(X,Y)/2 - n &\le e(X^2,Y^1)+e(X^2,Y^2) \le e(X,Y)/2\\
e(X,Y)/2 - n &\le e(X^1,Y^2)+e(X^2,Y^1) \le e(X,Y)/2\\
e(X,Y)/2 - n &\le e(X^1,Y^2)+e(X^2,Y^2) \le e(X,Y)/2.
\end{aligned} \tag{2}$$

By subtracting appropriate pairs of inequalities in (2), we obtain the following bounds.

$$\begin{aligned}
-n &\le e(X^1,Y^1)-e(X^2,Y^2) \le n\\
-n &\le e(X^1,Y^2)-e(X^2,Y^1) \le n
\end{aligned} \tag{3}$$

Recall that

$$\begin{aligned}
e(X,Y) &= e(X^1,Y^1)+e(X^1,Y^2)+e(X^2,Y^1)+e(X^2,Y^2)+\deg x_s+\deg y_t\\
&\le e(X^1,Y^1)+e(X^1,Y^2)+e(X^2,Y^1)+e(X^2,Y^2)+n.
\end{aligned}$$

Thus, by the pigeonhole principle, at least one of the following holds:

$$\begin{aligned}
e(X^1,Y^1)+e(X^2,Y^2) &\ge e(X,Y)/2-n/2\\
e(X^1,Y^2)+e(X^2,Y^1) &\ge e(X,Y)/2-n/2.
\end{aligned} \tag{4}$$

Suppose that the first case holds. Define $G^1 = G[X^1 \cup Y^1]$ and $G^2 = G[X^2 \cup Y^2]$. Combining (3) and (4), we obtain that

$$\begin{aligned}
|E(G^1)| &= e(X^1,Y^1) \ge e(X,Y)/4 - 3n/4 \ge (d\cdot 4^{h+1}-1)n/4 - 3n/4 = (d\cdot 4^h - 1)n\\
|E(G^2)| &= e(X^2,Y^2) \ge e(X,Y)/4 - 3n/4 \ge (d\cdot 4^h - 1)n.
\end{aligned} \tag{5}$$

We observe that graphs $G^1, G^2$ satisfy the inductive assumption (for $h-1$), so in the vertex set of $G^1$ we can find two families of $k/2$ pairwise corresponding segments $I^1_1, I^1_2, \ldots, I^1_{k/2}$ and $J^1_1, J^1_2, \ldots, J^1_{k/2}$, and in the vertex set of $G^2$ we can find two families of $k/2$ pairwise corresponding segments $I^2_1, I^2_2, \ldots, I^2_{k/2}$ and $J^2_1, J^2_2, \ldots, I^2_{k/2}$. We obtain the desired subsegments of $X$ and $Y$ by setting:

$$I_i = \begin{cases} I^1_i & \text{if } i \le k/2,\\ I^2_{i-k/2} & \text{if } i > k/2,\end{cases} \qquad J_i = \begin{cases} J^1_i & \text{if } i \le k/2,\\ J^2_{i-k/2} & \text{if } i > k/2.\end{cases}$$

If the second case in (4) holds, we take $G^1 = G[X^1, Y^2]$ and $G^2 = G[X^2, Y^1]$, and the rest of the proof is analogous. ◀

The following statement brings the technical statement of Lemma 10 into a more easily applicable form.

▶ **Lemma 11.** *Let $k, r \geq 1$ be two integers and let $G$ be a bipartite graph with bipartition classes $X = \{x_1, x_2, \dots, x_a\}$ and $Y = \{y_1, y_2, \dots, y_b\}$ and minimum degree at least $1200 \cdot r \cdot k$. Then there are $k$ sets $U_1, U_2, \dots, U_k$, and $k$ sets $W_1, W_2, \dots, W_k$, such that:*

1. *for each $i \in [k]$ the set $U_i$ is a segment of $[a]$ and the set $W_i$ is a segment of $[b]$,*
2. *for each distinct $i, j \in [k]$ we have $U_i \cap U_j = \emptyset$ and $W_i \cap W_j = \emptyset$,*
3. *for every $i \in [k]$, the average degree of the graph $G[U_i \cup W_i]$ is at least $r$.*

**Proof.** Let $h$ be the minimum integer, such that $k' := 2^h \geq 3k$; note that $k' < 6k$. Also, define $d = 2r/k$ and $n = a + b$. We have

$$d \cdot 4^{h+1} - 1 = 4d(k')^2 - 1 \geq \frac{8r}{k} \cdot (3k)^2 - 1 = 72 \cdot r \cdot k - 1 > 2.$$

Observe that the number of edges in $G$ is at least

$$n \cdot r \cdot 600k > (16r/k \cdot (6k)^2)n > (4d(k')^2)n > (d \cdot 4^{h+1} - 1)n.$$

Thus $G$ satisfies the assumptions of Lemma 10 for $h$, $n$, and $d$. Let $I_1, I_2, \dots, I_{k'}$ be the disjoint segments in $X$, and $J_1, J_2, \dots, J_{k'}$ be the disjoint segments in $Y$, whose existence is guaranteed by Lemma 10.

A segment $I_i$ ($J_i$, resp.) is called *large* if $|I_i| \geq 3n/k'$ ($|J_i| \geq 3n/k'$, resp.). A pair $(I_i, J_i)$ is *large* if at least one of $I_i, J_i$ is large, otherwise the pair is *small*. Note that there are at most $n/(3n/k') = k'/3$ large segments $I_i$ and at most $k'/3$ large segments $J_i$, so the number of large pairs is at most $2k'/3$. Thus the number of small pairs is at least $k'/3 \geq k$. We obtain the segments $(U_i, W_i)$ by taking the first $k$ small pairs $(I_i, J_i)$. Clearly these segments satisfy conditions 1. and 2. of the lemma.

Now take any $i \in [k]$ and let us compute the average degree of the graph $G_i := G[U_i, W_i]$. By Lemma 10, $|E(G_i)| \geq d \cdot n$. On the other hand, since $(U_i, W_i)$ is a small pair, we have that $|V(G_i)| = |U_i \cup W_i| < 6n/k'$. Thus we obtain that the average degree of $G_i$ is

$$\frac{|E(G_i)|}{|V(G_i)|} > \frac{d \cdot n}{6n/k'} = \frac{dk'}{6} \geq d\,\frac{3k}{6} = \frac{2r}{k} \cdot \frac{k}{2} = r.$$

This completes the proof. ◀

## 4 The Dense Case

In this section, we prove Theorem 5 roughly in the case when there are two linkages $\mathcal{L}$ and $\mathcal{K}$ such that their set $A(\mathcal{L}) \cup A(\mathcal{K}) \cup B(\mathcal{L}) \cup B(\mathcal{K})$ of endpoints is well linked and such that the paths in $\mathcal{L}$ and $\mathcal{K}$ intersect a lot. The formal statement proved in this section is as follows.

▶ **Lemma 12.** *Let $a, b \in \mathbb{N}^+$. Let $D$ be a directed graph and $\mathcal{L}$ and $\mathcal{K}$ be two linkages in $D$ such that $A(\mathcal{L}) \cup B(\mathcal{L}) \cup A(\mathcal{K}) \cup B(\mathcal{K})$ is well-linked in $D$. Suppose that the intersection graph $I(\mathcal{L}, \mathcal{K})$ has degeneracy more than $384\,000 \cdot a \cdot b \cdot \log_2(|\mathcal{L}|/b)$. Then there are directed graphs $D_1, D_2, \dots, D_a$ with the following properties:*

- **(i)** *each $D_i$ is a subgraph of $D$,*
- **(ii)** *each vertex of $D$ belongs to at most four graphs $D_i$, and*
- **(iii)** *each graph $D_i$ has directed treewidth at least $b$.*

**Proof Outline.** The basic idea of the proof of Lemma 12 is as follows. We first fix a pair of linkages $\mathcal{L}^{\mathsf{back}}$ and $\mathcal{K}^{\mathsf{back}}$ which are dual to $\mathcal{L}$ and $\mathcal{K}$, respectively. (This is possible because of well-linkedness of the endpoints.) The subgraphs $D_i$ that we construct will subpartition the vertex set of each of the four linkages $\mathcal{L}, \mathcal{L}^{\mathsf{back}}, \mathcal{K}, \mathcal{K}^{\mathsf{back}}$ and hence each vertex of $G$ is in at most four subgraphs $D_i$. To construct the desired subgraphs $D_i$, we consider the backlinkage-induced order $\Pi_{\mathcal{L}}$ on $\mathcal{L}$ and $\Pi_{\mathcal{K}}$ on $\mathcal{K}$. Using these orderings of the paths of $\mathcal{L}$ and $\mathcal{K}$, we can apply the partitioning lemma (Lemma 11) to the intersection graph of $\mathcal{L}$ and $\mathcal{K}$, obtaining a subpartition $I_1, \ldots, I_k$ of $\mathcal{L}$ and a subpartition $J_1, \ldots, J_k$ of $\mathcal{K}$. These subpartitions have the nice property that each intersection graph $I(I_i, J_i)$ induced by a pair $I_i, J_i$ contains many edges (representing intersections between the corresponding paths) and that only a constant number of cycles of $\mathsf{Aux}(\mathcal{L})$ and $\mathsf{Aux}(\mathcal{K})$ cross $I_i$ or $J_i$. By closing each of these crossing cycles by introducing an artificial new path, we obtain a pair of dual linkages $I_i, I_i'$, and a pair of dual of linkages $J_i, J_i'$. Using then Lemma 13 below, we will obtain a lower bound on the directed treewidth of the graph induced by $I_i \cup J_i \cup I_i' \cup J_i'$, which constitute our desired subgraph $D_i$.

**Treewidth Lower Bound.** For technical reasons, we will have to work with half-integral linkages. The intersection graph for a pair of half-integral linkages is defined in the same way as for ordinary linkages.

▶ **Lemma 13.** *Let $k, d \in \mathbb{N}^+$ and $\mathcal{P}, \mathcal{P}^{\mathsf{back}}, \mathcal{Q}, \mathcal{Q}^{\mathsf{back}}$ be four half-integral linkages in a directed graph such that $\mathcal{P}$ and $\mathcal{P}^{\mathsf{back}}$ are dual to each other and $\mathcal{Q}$ and $\mathcal{Q}^{\mathsf{back}}$ are dual to each other. Let the intersection graph $I(\mathcal{P}, \mathcal{Q})$ have minimum degree at least $d$ where $d \geq 8k \log_{\frac{4}{3}}(\frac{|\mathcal{P}|}{24k}) + 24k + 4$. Then the graph $\bigcup(\mathcal{P} \cup \mathcal{P}^{\mathsf{back}} \cup \mathcal{Q} \cup \mathcal{Q}^{\mathsf{back}})$ has directed treewidth at least $k$.*

The proof of Lemma 13 is inspired by the proof of Lemma 5.4 in [12]. We could use Lemma 5.4 here as well, but its proof, unfortunately, contains errors. Nevertheless, we derive an incomparable bound which is much better for our use since the lower bound claimed in Lemma 5.4 [12] is $k^2$. Also, we adapt the constants in the lemma for half-integral linkages. We postpone the proof of Lemma 13 to the full version of the paper.

**Main Proof of the Dense Case.** We are now ready to prove the main lemma of this section.

**Proof of Lemma 12.** Let $d = 384\,000 \cdot a \cdot b \cdot \log_2(|\mathcal{L}|/b)$. Since $I(\mathcal{L}, \mathcal{K})$ is not $d$-degenerate, it contains an induced subgraph $I'$ of minimum degree larger than $d$. Redefine $\mathcal{L}$ and $\mathcal{K}$ to be the sublinkages of $\mathcal{L}$ and $\mathcal{K}$ contained in this subgraph $I'$, that is, $\mathcal{L} := \mathcal{L} \cap V(I')$ and $\mathcal{K} := \mathcal{K} \cap V(I')$. Note that $|\mathcal{L}| > d$, $|\mathcal{K}| > d$, the size of $\mathcal{L}$ only decreases, and that $A(\mathcal{L}) \cup B(\mathcal{L}) \cup A(\mathcal{K}) \cup B(\mathcal{K})$ remains well-linked.

Let $\mathcal{L}^{\mathsf{back}}$ be a linkage in $D$ from $B(\mathcal{L})$ to $A(\mathcal{L})$ and let $\mathcal{K}^{\mathsf{back}}$ be a linkage in $D$ from $B(\mathcal{K})$ to $A(\mathcal{K})$. Note that $\mathcal{L}^{\mathsf{back}}$ and $\mathcal{K}^{\mathsf{back}}$ exist because $A(\mathcal{L}) \cup B(\mathcal{L}) \cup A(\mathcal{K}) \cup B(\mathcal{K})$ is well linked.

We focus on $\mathsf{Aux}(\mathcal{L})$ and $\mathsf{Aux}(\mathcal{K})$. Take backlinkage-induced orderings $(L_1, \ldots, L_{|\mathcal{L}|})$ of $\mathcal{L}$ and $(K_1, \ldots, K_{|\mathcal{K}|})$ of $\mathcal{K}$. Apply Lemma 11 with $k = a$, $r = 320b \log_2(|\mathcal{L}|/b)$, $G = I(\mathcal{L}, \mathcal{K})$, $X = \{L_1, \ldots, L_{|\mathcal{L}|}\}$, and $Y = \{K_1, \ldots, K_{|\mathcal{K}|}\}$, obtaining $a$ sets $U_1, \ldots, U_a$ and $a$ sets $W_1, \ldots, W_a$ with the corresponding properties. To see that Lemma 11 is applicable, observe that $I(\mathcal{L}, \mathcal{K})$ has minimum degree at least $384\,000 \cdot a \cdot b \log_2(|\mathcal{L}|/b) = 1200 \cdot 320b \log_2(|\mathcal{L}|/b) \cdot a = 1200 \cdot r \cdot k$. Observe for later on that, for each $i \in [a]$, the intersection graph $I(U_i, W_i)$ of the two linkages $U_i$ and $W_i$ has average degree at least $320b \log_2(|\mathcal{L}|/b)$ by property 3 of Lemma 11.

Now define, for each $i \in [a]$, a graph $D_i$ as follows. Initially, take the union of all paths in $U_i$ and $W_i$. Then, for each edge $(L, L')$ of $\mathsf{Aux}(\mathcal{L})$ such that $L, L' \in U_i$, add to $D_i$ the unique path $P \in \mathcal{L}^{\mathsf{back}}$ that connects $L$ and $L'$, that is, $\mathsf{end}(L) = \mathsf{start}(P)$ and $\mathsf{end}(P) = \mathsf{start}(L')$. Similarly, for each edge $(K, K')$ of $\mathsf{Aux}(\mathcal{K})$ such that $K, K' \in W_i$, add to $D_i$ the unique path $Q \in \mathcal{K}^{\mathsf{back}}$ with $\mathsf{end}(K) = \mathsf{start}(Q)$ and $\mathsf{end}(Q) = \mathsf{start}(K')$. In formulas:

$$\begin{aligned} U_i' \;:=\; \{P \in \mathcal{L}^{\mathsf{back}} \mid \exists (L, L') \in E(\mathsf{Aux}(\mathcal{L}))\colon \\ L, L' \in U_i \wedge \mathsf{end}(L) = \mathsf{start}(P) \wedge \mathsf{end}(P) = \mathsf{start}(L')\} \end{aligned}$$

and

$$\begin{aligned} W_i' \;:=\; \{Q \in \mathcal{K}^{\mathsf{back}} \mid \exists (K, K') \in E(\mathsf{Aux}(\mathcal{K}))\colon \\ K, K' \in W_i \wedge \mathsf{end}(K) = \mathsf{start}(Q) \wedge \mathsf{end}(Q) = \mathsf{start}(K')\}. \end{aligned}$$

We set

$$D_i \;:=\; \bigcup (U_i \cup W_i \cup U_i' \cup W_i').$$

We claim that $D_i$ satisfies the required properties. Clearly, $D_i$ is a subgraph of $D$, giving property (i). To see property (ii), consider a linkage $\mathcal{P} \in \{\mathcal{L}, \mathcal{L}^{\mathsf{back}}, \mathcal{K}, \mathcal{K}^{\mathsf{back}}\}$. We claim that no two subgraphs $D_i$, $D_j$ contain the same path of $\mathcal{P}$. This claim follows indeed from property 2. of Lemma 11, stating that $U_i \cap U_j = \emptyset$ and $W_i \cap W_j = \emptyset$ and inspecting the definition of $D_i$ and $D_j$. Thus, $\{V(D_i) \mid i \in [a]\}$ is a partition of a subset of the vertex set $V(\mathcal{P})$ of the paths in $\mathcal{P}$. Thus, each vertex $v \in V(D)$ occurs in at most four subgraphs $D_i$, showing property (ii).

It remains to show property (iii), the lower bound on the directed treewidth of $D_i$. We aim to modify $D_i$, increasing the directed treewidth by at most a constant, to obtain a graph $D_i^{(2)}$ which is the union of two pairs of dual half-integral linkages such that two linkages contained in distinct pairs intersect a lot. Then we can apply Lemma 13, giving a lower bound on the directed treewidth of $D_i^{(2)}$ which then implies a lower bound on the directed treewidth of $D_i$.

We first modify $D_i$ to obtain a graph $D_i^{(1)}$ which is the union of two pairs of dual linkages. Recall the orderings $\vec{\mathcal{L}} := (L_1, \ldots, L_{|\mathcal{L}|})$ and $\vec{\mathcal{K}} := (K_1, \ldots, K_{|\mathcal{K}|})$ on $\mathcal{L}$ and $\mathcal{K}$, respectively, which we have defined above. By property 1. of Lemma 11, $U_i$ is a segment of $\vec{\mathcal{L}}$ and $W_i$ is a segment of $\vec{\mathcal{K}}$. Hence, by the way we have defined $\vec{\mathcal{L}}$, there are at most two cycles $C$ in $\mathsf{Aux}(\mathcal{L})$ which are not contained in $U_i$ or disjoint with $U_i$, that is $V(C) \setminus U_i \neq \emptyset$ and $V(C) \cap U_i \neq \emptyset$. Call such a cycle *broken.* Similarly, there are at most two cycles $C$ in $\mathsf{Aux}(\mathcal{K})$ such that $V(C) \setminus W_i \neq \emptyset$ and $V(C) \cap W_i \neq \emptyset$. Call such a cycle *broken* as well. For each broken cycle $C$, do the following operation on $D_i$ to obtain $D_i^{(1)}$. If $C$ is in $\mathsf{Aux}(\mathcal{L})$, let $L^C_{\mathsf{out}}$ be the vertex of outdegree zero in the subgraph $\mathsf{Aux}(\mathcal{L})[V(C) \cap U_i]$ and let $L^C_{\mathsf{in}}$ be the vertex of indegree zero. Add the directed edge $(\mathsf{end}(L^C_{\mathsf{out}}), \mathsf{start}(L^C_{\mathsf{in}}))$ to $D_i$. Proceed analogously if $C$ is in $\mathsf{Aux}(\mathcal{K})$: Let $K^C_{\mathsf{out}}$ be the vertex of outdegree zero in the subgraph $\mathsf{Aux}(\mathcal{K})[V(C) \cap W_i]$ and let $K^C_{\mathsf{in}}$ be the vertex of indegree zero, and add the directed edge $(\mathsf{end}(K^C_{\mathsf{out}}), \mathsf{start}(K^C_{\mathsf{in}}))$ to $D_i$. In this way, we add at most four edges to $D_i$, obtaining $D_i^{(1)}$. Note that adding an edge increases the directed treewidth by at most one[2], and hence $\mathrm{dtw}(D_i^{(1)}) \leq \mathrm{dtw}(D_i) + 4$.

[2] In the corresponding robber-cop game (see [13]), we can always guard the new edge with an additional cop.

We claim that $D_i^{(1)}$ is the union of two pairs of dual linkages. To see this, note first that $U_i$ and $W_i$ are linkages in $D_i^{(1)}$. Now consider

$$U_i^b \; := \; U_i' \cup \{(\mathsf{end}(L_{\mathsf{out}}^C), \mathsf{start}(L_{\mathsf{in}}^C)) \mid C \text{ a broken cycle in } \mathsf{Aux}(\mathcal{L})\}$$

and

$$W_i^b := W_i' \cup \{(\mathsf{end}(K_{\mathsf{out}}^C), \mathsf{start}(K_{\mathsf{in}}^C)) \mid C \text{ a broken cycle in } \mathsf{Aux}(\mathcal{K})\},$$

wherein $L_{\mathsf{in}}^C, L_{\mathsf{out}}^C, K_{\mathsf{in}}^C$, and $K_{\mathsf{out}}^C$ are defined as above. Clearly, $D_i^{(1)} = \bigcup(U_i \cup W_i \cup U_i^b \cup W_i^b)$. Moreover, both $U_i^b$ and $W_i^b$ are linkages because $U_i'$ and $W_i'$ are linkages and because $L_{\mathsf{in}}^C, L_{\mathsf{out}}^C, K_{\mathsf{in}}^C$, and $K_{\mathsf{out}}^C$ have indegree or outdegree zero in $\mathsf{Aux}(\mathcal{L})[V(C)]$ or $\mathsf{Aux}(\mathcal{K})[V(C)]$, respectively. Finally, by definition, $U_i$ and $U_i^b$ are dual to each other and $W_i$ and $W_i^b$ are dual to each other. Thus, $D_i^{(1)}$ is the union of two pairs of dual linkages, as claimed.

In order to apply Lemma 13, we need a pair of linkages whose intersection graph has a large minimum degree. So far, the linkages which define $D_i^{(1)}$ guarantee only large average degree (via property 3. of Lemma 11). We now derive a subgraph $D_i^{(2)}$ of $D_i^{(1)}$ such that $D_i^{(2)}$ is the union of two pairs of dual half-integral linkages $(\mathcal{P}, \mathcal{P}^{\mathsf{back}}), (\mathcal{Q}, \mathcal{Q}^{\mathsf{back}})$ and $I(\mathcal{P}, \mathcal{Q})$ has large minimum degree. To achieve this, recall that the intersection graph $I(U_i, W_i)$ of the two linkages $U_i$, $W_i$ in $D_i^{(1)}$ has average degree at least $320b\log_2(|\mathcal{L}|/b)$. Hence, there is a subgraph $I'$ of $I(U_i, W_i)$ with minimum degree at least $320b\log_2(|\mathcal{L}|/b)$. Let $\mathcal{P} \subseteq U_i$ be the sublinkage of $U_i$ contained in $I'$, that is $\mathcal{P} = U_i \cap V(I')$. Similarly, let $\mathcal{Q} = W_i \cap V(I')$.

We define $\mathcal{P}^{\mathsf{back}}$ to be the backlinkage induced by $\mathcal{P}$ on $(U_i, U_i^b)$ and $\mathcal{Q}^{\mathsf{back}}$ to be the backlinkage induced by $\mathcal{Q}$ on $(W_i, W_i^b)$. Note that $\mathcal{P}^{\mathsf{back}}$ and $\mathcal{Q}^{\mathsf{back}}$ are half-integral and dual to $\mathcal{P}$ and $\mathcal{Q}$, respectively.

Take now the subgraph $D_i^{(2)}$ to be the union $\bigcup(\mathcal{P} \cup \mathcal{P}^{\mathsf{back}} \cup \mathcal{Q} \cup \mathcal{Q}^{\mathsf{back}})$. Then apply Lemma 13 to $\mathcal{P}, \mathcal{P}^{\mathsf{back}}, \mathcal{Q}, \mathcal{Q}^{\mathsf{back}}$ with $k = b + 4$ and $d = 320b\log_2(|\mathcal{L}|/b)$. To see that the preconditions of Lemma 13 are satisfied, first recall that the intersection graph $I(\mathcal{P}, \mathcal{Q})$ has minimum degree at least $320b\log_2(|\mathcal{L}|/b)$. Furthermore,

$$\begin{aligned} d = 320b\log_2\frac{|\mathcal{L}|}{b} \geq 200b\log_2\frac{|\mathcal{L}|}{b} + 120b + 4 \geq \frac{5\cdot 40b}{2}\log_2\frac{|\mathcal{L}|}{b} + 120b + 4 \geq \\ \frac{8\cdot 5b}{\log_2(4/3)}\log_2\frac{|\mathcal{L}|}{b} + 24(5b) + 4 \geq 8\cdot(b+4)\log_{4/3}\frac{|\mathcal{L}|}{24(b+4)} + 24(b+4) + 4 = \\ 8k\log_{4/3}\frac{|\mathcal{L}|}{24k} + 24k + 4, \end{aligned}$$

and thus indeed the preconditions of Lemma 13 are satisfied. Thus, the directed treewidth of $D_i^{(2)}$ is at least $b + 4$. Since $D_i^{(2)}$ is a subgraph of $D_i^{(1)}$ and $\operatorname{dtw}(D_i) \geq \operatorname{dtw}(D_i^{(1)}) - 4$, we have $\operatorname{dtw}(D_i) \geq b$, as required. ◀

## 5 Wrapping up the Proof of Theorem 5

**Proof of Theorem 5.** Let $G$ be a directed graph of $\operatorname{dtw}(G) \geq c \cdot a^6 b^8 \log^2(ab)$, where $c$ is a large constant, whose value will follow from the reasoning below. First, we invoke Lemma 7 with $\beta = 2^{37}a^2b^3\log(ab)$ and $\alpha = 8ab$ (here we assume that $c$ is sufficiently large so that the assumption is satisfied). We obtain a set of vertex-disjoint paths $P_1, \ldots, P_{8ab}$ and sets $A_i, B_i \subseteq V(P_i)$, where $A_i$ appears before $B_i$ on $P_i$, and $|A_i| = |B_i| = 2^{37}a^2b^3\log(ab)$, and the set $\bigcup_{i=1}^{8ab} A_i \cup B_i$ is well-linked. Denote by $\mathcal{L}_{i,j}$ a linkage from $B_i$ to $A_j$.

We split the $8ab$ paths $P_i$ into $a$ segments, each consisting of $8b$ paths. Formally, for every $\iota \in [a]$ we define $I_\iota = \{j \mid 8(\iota-1)b < j \leq 8\iota b\}$.

Now we set $r = 64ab^2$ and create an auxiliary $r$-colored graph $H$, whose vertices will be paths of appropriately chosen linkages $\mathcal{L}_{i,j}$. More specifically, for every $\iota \in [a]$, and every $i, j \in I_\iota$, we introduce a vertex for every path in $\mathcal{L}_{i,j}$ and color it $(i, j)$. Two vertices of $H$ are adjacent if and only if their corresponding paths share a vertex in $G$. Note that for two linkages $\mathcal{L}_{i,j}$ and $\mathcal{L}_{i',j'}$, the graph $H[\mathcal{L}_{i,j} \cup \mathcal{L}_{i',j'}]$ is precisely the intersection graph $I(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'})$.

We set $d := 2^{27}ab\log(ab)$ and consider two cases:

**(i)** for all $i, j, i', j'$ the graph $I(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'})$ is $d$-degenerate.

**(ii)** there exist $i, j, i', j'$, for which the graph $I(\mathcal{L}_{i,j}, \mathcal{L}_{i',j'})$ is not $d$-degenerate.

An intuition behind case (i) is that for each subgraph of $H$ there is always a path (in $G$) such that it shares a vertex with at most $d$ paths from all used linkages back.

**Case (i)** We use Lemma 8 on $H$. Graph $H$ has $64ab^2$ color classes such that for each $(i, j) \neq (i', j')$ the graph $H[\mathcal{L}_{i,j} \cup \mathcal{L}_{i',j'}]$ is $d$-degenerate. Note that $|\mathcal{L}_{i,j}| = 2^{37}a^2b^3\log(ab) \geq 4e(r-1)d$ is sufficiently large to satisfy the last assumption of the lemma. We are given an independent set $x_1, \ldots, x_{64ab^2}$ that represents pairwise disjoint paths $L_{i,j}$ from $B_i$ to $A_j$ for all $i, j \in I_\iota$. We also recall that $A_i$ and $B_i$ lie on $P_i$ and all $P_i$'s are pairwise disjoint.

Let $G_\iota$ consist of all paths $P_i$ for $i \in I_\iota$ and $L_{i,j}$ for $i, j \in I_\iota$. By Lemma 9 for $k = 8b$ we obtain $\mathrm{dtw}(G_\iota) \geq b$ while each vertex is in at most 2 such subgraphs. Indeed, each vertex can appear only once on some $P_i$ and once on some $L_{i,j}$.

**Case (ii)** The claim follows from Lemma 12. Since $|\mathcal{L}| = 2^{37}a^2b^3\log(ab)$ then $d = 2^{27}ab\log(ab) > 2^{19}ab\log(2^{37}a^2b^2\log(ab))$. ◀

## 6 Conclusions

We have shown that if one relaxes the disjointness constraint to quarter-integral packing (i.e., every vertex used at most four times), then the Erdős-Pósa property in directed graphs admits a polynomial bound between the cycle packing number and the feedback vertex set number. A natural question would be to decrease the dependency further, even at the cost of higher congestion (but still a constant). More precisely, we pose the following question: Does there exist a constant $c$ and a polynomial $p$ such that for every integer $k$ if a directed graph $G$ does not contain a family of $k$ cycles such that every vertex of $G$ is in at most $c$ of the cycles, then the directed treewidth of $G$ is at most $kp(\log k)$?

One of the sources of polynomial blow-up in the proof of Theorem 5 is the quadratic blow-up in Lemma 7. Lemma 7 is a direct corollary of another result of [14] that asserts that a directed graph $G$ of directed treewidth $\Omega(k^2)$ contains a path $P$ and a set $A \subseteq V(P)$ that is well-linked and of size $k$. Is this quadratic blow-up necessary? Can we improve it, even at the cost of some constant congestion in the path $P$ (i.e., allow $P$ to visit every vertex a constant number of times)? We remark that the essence of the improvement from $\mathcal{O}(k^6\log^2 k)$ (obtained by setting $b = 2$ in Theorem 5) to $\mathcal{O}(k^3)$ asserted by Theorem 4 is to avoid the usage of Lemma 7 and to replace it with a simple well-linkedness trick. However, this trick fails in the general setting of Theorem 5.

## References


1 Saeed Akhoondian Amiri, Ken-ichi Kawarabayashi, Stephan Kreutzer, and Paul Wollan. The Erdős-Pósa Property for Directed Graphs. *CoRR*, abs/1603.02504, 2016. arXiv:1603.02504.

2 Matthew Andrews, Julia Chuzhoy, Venkatesan Guruswami, Sanjeev Khanna, Kunal Talwar, and Lisa Zhang. Inapproximability of edge-disjoint paths and low congestion routing on undirected graphs. *Combinatorica*, 30(5):485–520, 2010.

3 Timothy Carpenter, Ario Salmasi, and Anastasios Sidiropoulos. Routing Symmetric Demands in Directed Minor-Free Graphs with Constant Congestion. *CoRR*, abs/1711.01692, 2017. arXiv:1711.01692.

4 Chandra Chekuri and Julia Chuzhoy. Large-treewidth graph decompositions and applications. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC 2013)*, pages 291–300. ACM, 2013.

5 Chandra Chekuri and Julia Chuzhoy. Polynomial Bounds for the Grid-Minor Theorem. *Journal of the ACM*, 63(5):40:1–40:65, 2016. doi:10.1145/2820609.

6 Chandra Chekuri and Alina Ene. The all-or-nothing flow problem in directed graphs with symmetric demand pairs. *Mathematical Programming*, pages 1–24, 2014.

7 Chandra Chekuri, Alina Ene, and Marcin Pilipczuk. Constant Congestion Routing of Symmetric Demands in Planar Directed Graphs. *SIAM Journal on Discrete Mathematics*, 32(3):2134–2160, 2018. doi:10.1137/17M1150694.

8 Chandra Chekuri, Sanjeev Khanna, and F. Bruce Shepherd. Multicommodity flow, well-linked terminals, and routing problems. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC 2005)*, pages 183–192. ACM, 2005.

9 Julia Chuzhoy and Shi Li. A Polylogarithmic Approximation Algorithm for Edge-Disjoint Paths with Congestion 2. *Journal of the ACM*, 63(5):45:1–45:51, 2016. doi:10.1145/2893472.

10 Erik D. Demaine and Mohammadtaghi Hajiaghayi. Linearity of grid minors in treewidth with applications through bidimensionality. *Combinatorica*, 28(1):19–36, 2008.

11 Pál Erdős and Lajos Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965.

12 Meike Hatzel, Ken-ichi Kawarabayashi, and Stephan Kreutzer. Polynomial Planar Directed Grid Theorem. In *Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2019)*, pages 1465–1484, 2019.

13 Thor Johnson, Neil Robertson, Paul D. Seymour, and Robin Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138–154, 2001.

14 Ken-ichi Kawarabayashi and Stephan Kreutzer. The Directed Grid Theorem. *CoRR*, abs/1411.5681, 2014. arXiv:1411.5681.

15 Ken-ichi Kawarabayashi and Stephan Kreutzer. The Directed Grid Theorem. In *Proceedings of the 47th Annual ACM on Symposium on Theory of Computing (STOC 2015)*, pages 655–664, 2015. doi:10.1145/2746539.2746586.

16 Bruce A. Reed. Introducing directed tree width. *Electronic Notes in Discrete Mathematics*, 3:222–229, 1999.

17 Bruce A. Reed, Neil Robertson, Paul D. Seymour, and Robin Thomas. Packing directed circuits. *Combinatorica*, 16(4):535–554, 1996.

18 Bruce A. Reed and David R. Wood. Polynomial treewidth forces a large grid-like-minor. *European Journal of Combinatorics*, 33(3):374–379, 2012. doi:10.1016/j.ejc.2011.09.004.

19 Neil Robertson and Paul D. Seymour. Graph minors. III. Planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984. doi:10.1016/0095-8956(84)90013-3.

20 Neil Robertson and Paul D. Seymour. Graph minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986. doi:10.1016/0095-8956(86)90030-4.

21 Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly Excluding a Planar Graph. *Journal of Combinatorial Theory, Series B*, 62(2):323–348, 1994. doi:10.1006/jctb.1994.1073.

22 Paul D. Seymour. Packing Directed Circuits Fractionally. *Combinatorica*, 15(2):281–288, 1995. doi:10.1007/BF01200760.