Minimization and Parameterized Variants of Vertex Partition Problems on Graphs

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Abstract

Let $\Pi_1, \Pi_2, \ldots, \Pi_c$ be graph properties for a fixed integer c. Then, $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is the problem of asking whether the vertex set of a given graph can be partitioned into c subsets $V_1, V_2, \ldots, V_c$ such that the subgraph induced by V_i satisfies the graph property Π_i for every $i \in \{1, 2, \ldots, c\}$. Minimization and parameterized variants of $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition have been studied for several specific graph properties, where the size of the vertex subset V_1 satisfying Π_1 is minimized or taken as a parameter. In this paper, we first show that the minimization variant is hard to approximate for any nontrivial additive hereditary graph properties, unless $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs. We then give FPT algorithms for the parameterized variant when restricted to the case where $c = 2$, Π_1 is a hereditary graph property, and Π_2 is the class of acyclic graphs.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms

Keywords and phrases Graph Algorithms, Approximability, Fixed-Parameter Tractability, Vertex Partition Problem, Feedback Vertex Set Problem

Digital Object Identifier 10.4230/LIPIcs.ISAAC.2020.40

Funding *Yuma Tamura*: Partially supported by JSPS KAKENHI Grant Number JP20J11259, Japan.
Takehiro Ito: Partially supported by JSPS KAKENHI Grant Numbers JP18H04091 and JP19K11814, Japan.
Xiao Zhou: Partially supported by JSPS KAKENHI Grant Number JP19K11813, Japan.

1 Introduction

Various combinatorial problems on graphs can be seen as problems of partitioning the vertex set of a given graph into a fixed number of vertex subsets satisfying prescribed properties. For example, c-Coloring is the problem of deciding whether the vertex set of a given graph can be partitioned into c independent sets (i.e., edgeless graphs). Another example is Near-Bipartiteness, which is the problem of deciding whether the vertex set of a given graph can be partitioned into two subsets such that one forms an independent set and the other forms an acyclic graph. These problems can be unified as the problem $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition for a fixed integer c, where $\Pi_1, \Pi_2, \ldots, \Pi_c$ denote graph properties: $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition, also known as Generalized Graph Coloring [1], is the problem of asking whether the vertex set of a given graph can be partitioned into c subsets $V_1, V_2, \ldots, V_c$ such that the subgraph induced by V_i satisfies the graph property Π_i for every $i \in \{1, 2, \ldots, c\}$. We call such a vertex partition a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-*coloring* of the graph. Minimization and parameterized variants of $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition have been also studied in the literature for several graph properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, where the size of the vertex subset V_1 satisfying Π_1 is minimized or taken as a parameter.

We here define some terms for graph properties. A *graph property*, or simply a *property*, is a property of graphs closed under isomorphism. We sometimes regard a graph property as a class of graphs (i.e., a set of all graphs) satisfying the property. A graph property Π is *hereditary* if, for any graph G satisfying Π, every induced subgraph of G also satisfies Π. A graph property Π is *additive* if, for any two graphs G and H satisfying Π, the disjoint union of G and H also satisfies Π, where the *disjoint union* of $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph whose vertex set is $V_G \cup V_H$ and edge set is $E_G \cup E_H$. A graph property Π is *nontrivial* if there exists at least one graph satisfying Π and there exists at least one graph which does not satisfy Π.

1.1 Related Results and Known Results

Farrugia [3] showed that $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is NP-hard for any fixed nontrivial additive hereditary graph properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, unless $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs. Notice that if $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs, then the problem is equivalent to 2-Coloring and hence it can be solved in linear time for general graphs.

Kanj et al. [6] widely studied the parameterized complexity of (Π_1, Π_2)-Partition. They mentioned that a simple branching technique yields a single-exponential FPT algorithm for Parameterized (Π_1, Π_2)-Partition if Π_1 and Π_2 are hereditary graph properties such that the membership of Π_1 can be decided in polynomial time and Π_2 can be characterized by a finite set of forbidden induced subgraphs.

Many FPT algorithms have been developed for various problems, which can be seen as Parameterized (Π_1, Π_2)-Partition with specific graph properties Π_1 and Π_2, such as Feedback Vertex Set [5], Independent Feedback Vertex Set [7, 11], and $\mathcal{G}$-Bipartization [10]. On the other hand, Parameterized (Π_1, Π_2)-Partition is fixed-parameter intractable even if Π_1 is the class of all graphs: the problem is $W[P]$-complete if Π_2 is the class of d-degenerate graphs for any $d \geq 2$ (this corresponds to d-Degenerate Vertex Deletion) [9], and the problem is $W[2]$-hard if Π_2 is the class of wheel-free graphs (this corresponds to Wheel-Free Deletion) [8].

From the viewpoint of approximation, there is a polynomial-time 2-approximation algorithm for Feedback Vertex Set [2], which is equivalent to Min (Π_1, Π_2)-Partition if Π_1 is the class of all graphs and Π_2 is the class of acyclic graphs. However, if we change Π_1 to the class of edgeless graphs, then the problem is equivalent to Independent Feedback Vertex Set and it is hard to approximate even for planar bipartite graphs [14].

1.2 Our Contribution

In this paper, we study the approximability of Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition and the fixed-parameter tractability of Parameterized (Π_1, Π_2)-Partition.

We first study the approximability. It is already NP-hard to decide if a given graph has at least one $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring for nontrivial additive hereditary graph properties $\Pi_1, \Pi_2, \ldots, \Pi_c$ [3]. In this paper, we give inapproximability results of Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition even for the case where we know that a given graph has at least one $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring. We show that Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition, any fixed $c \geq 2$, is hard to approximate for any fixed nontrivial additive hereditary graph properties, unless $c = 2$ and both Π_1 and Π_2 are classes of edgeless graphs. In addition, we show that Min (Π_1, Π_2)-Partition for planar bipartite graphs remains hard to approximate if each of Π_1 and Π_2 has a minimal forbidden induced subgraph that is planar and bipartite. Interestingly, as we will discuss in Section 3.2, Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition can be solved in polynomial time for bipartite graphs if $c \geq 3$ and $\Pi_1, \Pi_2, \ldots, \Pi_c$ are nontrivial additive hereditary graph properties. We note that various well-known graph properties are additive and hereditary: for example, the classes of acyclic graphs, interval graphs, planar graphs, and more generally, $\mathcal{H}$-free graphs for a graph family $\mathcal{H}$.

We then investigate the fixed-parameter tractability of Parameterized $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition when restricted to $c = 2$ and Π_2 is the class of acyclic graphs. We first develop an FPT algorithm for the problem if Π_1 is a hereditary graph property; we also show that the running time can be improved for bounded degeneracy graphs. Note that this result cannot be covered by [6], because the class of acyclic graphs is characterized by the infinite forbidden cycles. We then give an FPT algorithm for the case where Π_1 is the class of graphs with maximum degree Δ, for a fixed Δ. We also develop a faster FPT algorithm when restricted to $\Delta = 1$.

Proofs for the claims marked with $(*)$ are omitted from this extended abstract.

2 Preliminaries

In this paper, we assume that graphs are simple, finite, undirected, and unweighted. Let $G = (V, E)$ be a graph. We sometimes denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For a vertex subset V' of G, let $G[V']$ be the subgraph of G induced by V'. We denote simply by $G - V'$ the induced subgraph $G[V \setminus V']$. We say that an induced subgraph H of G is *proper* if $V(G) \setminus V(H) \neq \emptyset$. For a vertex v in G and a vertex subset $V' \subseteq V$, we denote by $N(v, V')$ the set of all neighbors of v in $G[V' \cup \{v\}]$, that is, $N(v, V') = \{w \in V' : vw \in E\}$.

We have already defined the terms *graph property*, *hereditary*, *additive*, and *nontrivial* in Introduction. Recall that we sometimes regard a graph property as a class of graphs (i.e., a set of all graphs) satisfying the property. For a property Π, a graph is said to be a *forbidden induced subgraph* for Π if it does not satisfy Π. A forbidden induced subgraph H is said to be *minimal* if any proper induced subgraph of H satisfies Π. A *minimal forbidden set* $\mathcal{F}(\Pi)$ of Π is a set of all minimal forbidden induced subgraphs for Π. Any additive hereditary property can be characterized by a (possibly infinite) minimal forbidden set $\mathcal{F}(\Pi)$ such that every graph in $\mathcal{F}(\Pi)$ is connected. Moreover, if the property is nontrivial, every graph in $\mathcal{F}(\Pi)$ has at least two vertices. For example, $\mathcal{F}(\Pi) = \{K_2\}$ if Π is the class of edgeless graphs, and $\mathcal{F}(\Pi') = \{C_3, C_4, C_5, \ldots\}$ if Π' is the class of acyclic graphs, where K_n is a complete graph of n vertices and C_n is a cycle of n vertices.

In the remainder of this paper, we regard a partition of the vertex set of a graph G as a (vertex) coloring of G. Let $C = \{1, 2, \ldots, c\}$ be a color set, where c is a positive integer. Then, a *coloring* of G is simply a mapping $f : V(G) \to C$. A vertex $v \in V(G)$ is said to be *assigned to the color* i if $v \in f^{-1}(i)$. For properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, a coloring f of G is called a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-*coloring* of G if $G[f^{-1}(i)]$ satisfies Π_i for every $i \in C$. We say that a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring f of G is *optimal* if $|f^{-1}(1)|$ is minimum among all $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-colorings of G. We define $\mathsf{OPT}(G)$ as follows:

$$\mathsf{OPT}(G) = \min\{|f^{-1}(1)| : f \text{ is a } (\Pi_1, \Pi_2, \ldots, \Pi_c)\text{-coloring of } G\}$$

if G has a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring; otherwise we let $\mathsf{OPT}(G) = +\infty$. For fixed properties $\Pi_1, \Pi_2, \ldots, \Pi_c$, we define Min $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition as the problem of computing $\mathsf{OPT}(G)$ for a given graph G. We also study the problem parameterized by the solution size k: Parameterized $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-Partition is the problem of determining whether $\mathsf{OPT}(G) \leq k$ or not.

3 Inapproximability

In this section, we study the inapproximability of MIN $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-PARTITION. We say that an algorithm for MIN $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-PARTITION is $\rho(n)$-*approximation* if it returns a value z for a given graph G with n vertices such that $z \leq \rho(n) \cdot \mathsf{OPT}(G)$ and G has a $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-coloring f satisfying $|f^{-1}(1)| = z$. Then, $\mathsf{OPT}(G) \leq z \leq \rho(n) \cdot \mathsf{OPT}(G)$ always holds, and hence the algorithm must compute $\mathsf{OPT}(G)$ if either $\mathsf{OPT}(G) = 0$ or $\mathsf{OPT}(G) = +\infty$ holds. In this section, we give inapproximability results that hold even if we know that a given graph G satisfies both $\mathsf{OPT}(G) \neq 0$ and $\mathsf{OPT}(G) \neq +\infty$. We say that a graph G is *promised* if both $\mathsf{OPT}(G) \neq 0$ and $\mathsf{OPT}(G) \neq +\infty$ hold.

3.1 General graphs

The main result of this subsection is the following theorem.

▶ **Theorem 1.** *Let Π_1 and Π_2 be any two fixed nontrivial additive hereditary graph properties. Let G be a promised graph of n vertices, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Under the assumption that $\mathrm{P} \neq \mathrm{NP}$, MIN (Π_1, Π_2)-PARTITION admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$ unless both Π_1 and Π_2 are classes of edgeless graphs.*

Note that if both Π_1 and Π_2 are classes of edgeless graphs, MIN (Π_1, Π_2)-PARTITION is solvable in polynomial time, because the problem is equivalent to 2-COLORING.

We can construct an approximation-preserving reduction from MIN (Π_1, Π_2)-PARTITION to MIN $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-PARTITION for any fixed $c \geq 3$, and obtain the following corollary.

▶ **Corollary 2** ($*$). *Let $c \geq 3$ be a fixed constant, and let $\Pi_1, \Pi_2, \ldots, \Pi_c$ be any fixed nontrivial additive hereditary graph properties. Let G be a promised graph of n vertices, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Under the assumption that $\mathrm{P} \neq \mathrm{NP}$, MIN $(\Pi_1, \Pi_2, \ldots, \Pi_c)$-PARTITION admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$.*

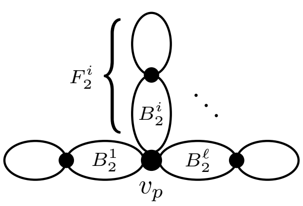
In the remainder of this subsection, we prove Theorem 1 by giving a gap-producing reduction from POSITIVE 1-IN-3-SAT. For this purpose, we define gadgets in Section 3.1.1 and explain how to construct a promised graph for our reduction in Section 3.1.2.

For a given 3-CNF formula ϕ, 1-IN-3-SAT is the problem of asking whether there exists a satisfying truth assignment of ϕ such that each clause in ϕ has exactly one `true` literal. The problem is called POSITIVE 1-IN-3-SAT if ϕ contains only positive literals. POSITIVE 1-IN-3-SAT is known to be NP-complete [13].

3.1.1 Gadgets

An *end-block* of a graph G is a maximal 2-connected component of G that contains at most one cut-vertex of G. If G has no cut-vertex, then G itself is an end-block. Let $\mathcal{F}(\Pi_1)$ be a minimal forbidden set for Π_1, and let B_1 be an end-block having the smallest number of vertices among all end-blocks of all graphs in $\mathcal{F}(\Pi_1)$. We denote by F_1 a minimal forbidden induced subgraph in $\mathcal{F}(\Pi_1)$ that contains B_1. Similarly, we define B_2 and F_2 for $\mathcal{F}(\Pi_2)$.

We first define a *forcing gadget* X^ℓ, which forces some particular vertex v_p to be assigned to the color 1. (See also Figure 1(a).) Let $\ell \geq 2$ be an integer, and $F_2^1, F_2^2, \ldots, F_2^\ell$ be ℓ copies of F_2. For $i \in \{1, 2, \ldots, \ell\}$, let v_i be a vertex that is not a cut-vertex of F_2^i and is chosen from the vertices in the end-block B_2^i of F_2^i. Note that such a vertex v_i exists, because B_2^i has at least two vertices and at most one cut-vertex of F_2^i. We identify $v_1, v_2, \ldots, v_\ell$ as a



(a) **(b)**

Figure 1 (a) The forcing gadget X^ℓ and (b) the forbidding gadget Y^ℓ, where some parts of gadgets are omitted.

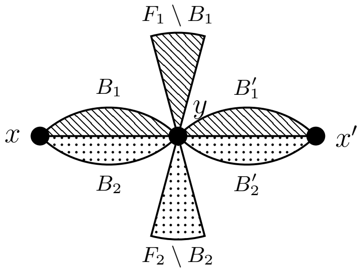


Figure 2 The relay gadget R^ℓ. The striped and dotted parts are constructed from F_1 and F_2, respectively. The vertex y is adjacent to the vertices x and x'.

single vertex v_p, that is, the resulting graph X^ℓ consists of ℓ copies of F_2 sharing v_p. We call v_p the *root* of X^ℓ. Then, we can force the root v_p to be assigned to the color 1, as in the following sense.

▶ **Proposition 3** (∗). *For any* (Π_1, Π_2)*-coloring* f *of the forcing gadget* X^ℓ *such that* $|f^{-1}(1)| < \ell$*,* v_p *is assigned to the color* 1*.*

We then define a *forbidding gadget* Y^ℓ, which forbids some particular vertex v_q to be assigned to the color 1. (See also Figure 1(b).) For a graph $G = (V, E)$ and a vertex subset $V' \subseteq V$, *planting* V' *with the forcing gadget* X^ℓ is the operation as follows: we first make $|V'|$ copies of X^ℓ, and then identify each vertex in V' with the root v_p of the copies. We choose an arbitrary vertex of F_1, say v_q, and we plant $V(F_1) \setminus \{v_q\}$ with X^ℓ. We define the resulting graph as the forbidding gadget Y^ℓ, and call v_q the *root* of Y^ℓ. Then, we can forbid the root v_q to be assigned to the color 1, as in the following sense.

▶ **Proposition 4** (∗). *For any* (Π_1, Π_2)*-coloring* f *of the forbidding gadget* Y^ℓ *such that* $|f^{-1}(1)| < \ell$*,* v_q *is assigned to the color* 2*.*

We now define a *relay gadget* R^ℓ, which was used in [3]. (See also Figure 2.) This gadget will be used to propagate the color assignment: the vertex x in Figure 2 is assigned to the color 1 if and only if the vertex x' in Figure 2 is assigned to the color 1. Let y_1 be the cut-vertex in B_1 of F_1; if B_1 has no cut-vertex, that is, $B_1 = F_1$, then y_1 is chosen arbitrarily from F_1. Let x_1 be an arbitrary vertex in B_1 which is adjacent to y_1. Let F_1' be the graph obtained by adding a copy of B_1, denoted by B_1', such that the copy of y_1 in B_1' is identified with y_1. (See the striped part in Figure 2, where $y = y_1$ and $x = x_1$.) In addition, let x_1' be the copy of x_1 in B_1'. In the same way, we define y_2, x_2, F_2', B_2' and x_2' for F_2. (See the dotted part in Figure 2, where $y = y_2$, $x = x_2$ and $x' = x_2'$.) We then merge F_1' and F_2' by identifying y_1 with y_2, x_1 with x_2, and x_1' with x_2', respectively. We label the identified

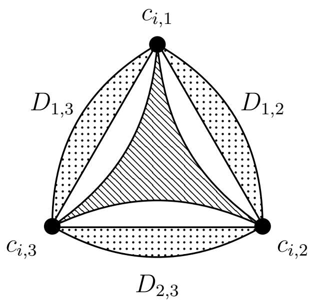
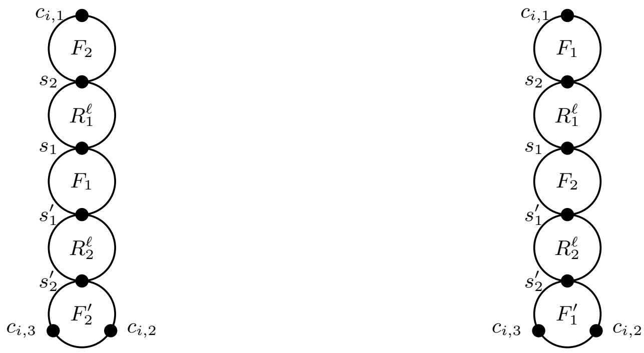


Figure 3 Image of the clause gadget C_i^ℓ, where the striped and dotted parts represent the inner and outer parts, respectively.



(a) The case of $|V(F_2)| > 2$.

(b) The case of $|V(F_2)| = 2$.

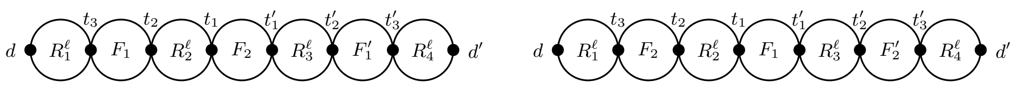
Figure 4 The inner part in the clause gadget C_i^ℓ.

vertices as y, x, and x'. For the resulting graph, we plant $V(F_1') \setminus \{y, x, x'\}$ with the forcing gadget X^ℓ, and plant $V(F_2') \setminus \{y, x, x'\}$ with the forbidding gadget Y^ℓ. This completes the construction of the relay gadget R^ℓ. The relay gadget propagates the color assignment, as in the following sense.

▶ **Proposition 5** (∗). *For any (Π_1, Π_2)-coloring f of the relay gadget R^ℓ such that $|f^{-1}(1)| < \ell$, x is assigned to the color 1 if and only if x' is assigned to the color 1.*

Finally, we define a *clause gadget* C_i^ℓ, which corresponds to a clause c_i of a given 3-CNF formula. The clause gadget contains three vertices $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$ which correspond to the three literals in c_i. (See also Figure 3.) The construction of C_i^ℓ differs between two cases $|V(F_2)| > 2$ and $|V(F_2)| = 2$. However, we will explain only the case of $|V(F_2)| > 2$ and give illustrations for the other case, because the case of $|V(F_2)| = 2$ can be obtained by simply swapping F_1 and F_2 for the case of $|V(F_2)| > 2$. (See also Figure 4 and 5.)

The clause gadget C_i^ℓ consists of an *inner part* and an *outer part*, as illustrated in Figure 3. The inner part is constructed as follows. (See also Figure 4(a).) Let s_1 and s_1' be any two



(a) The case of $|V(F_2)| > 2$.

(b) The case of $|V(F_2)| = 2$.

Figure 5 The gadget D which will be used in the outer part of the clause gadget C_i^ℓ.

distinct vertices of F_1, and let s_2 and $c_{i,1}$ be any two distinct vertices of F_2. In addition, we make a copy F_2' of F_2, and let s_2', $c_{i,2}$ and $c_{i,3}$ be any three distinct vertices of F_2'. We plant $V(F_1) \setminus \{s_1, s_1'\}$ with X^ℓ, $V(F_2) \setminus \{s_2, c_{i,1}\}$ with Y^ℓ, and $V(F_2') \setminus \{s_2', c_{i,2}, c_{i,3}\}$ with Y^ℓ, respectively. Then, we connect F_2, F_1 and F_2' via relay gadgets R_1^ℓ and R_2^ℓ as shown in Figure 4(a), where we identify s_2 with x' in R_1^ℓ, s_1 with x in R_1^ℓ, s_1' with x' in R_2^ℓ, and s_2' with x in R_2^ℓ.

The outer part consists of three copies $D_{1,2}, D_{1,3}, D_{2,3}$ of a gadget D, defined as follows. (See also Figure 5(a) for the construction of D.) Let t_1 and t_1' be any two distinct vertices of F_2, and let t_2 and t_3 be any two distinct vertices of F_1. We make a copy F_1' of F_1, and let t_2' and t_3' be the vertices of F_1' corresponding to t_2 and t_3, respectively. We plant $V(F_2) \setminus \{t_1, t_1'\}$ with Y^ℓ, $V(F_1) \setminus \{t_2, t_3\}$ with X^ℓ, and $V(F_1') \setminus \{t_2', t_3'\}$ with X^ℓ. Then, we connect F_1, F_2 and F_1' via four relay gadgets $R_1^\ell, \ldots, R_4^\ell$ as shown in Figure 5(a), in the same manner as the inner part, where x in R_1^ℓ is renamed d, and x' in R_4^ℓ is renamed d'. Let D be the resulting graph.

We are now ready to construct the clause gadget C_i^ℓ. (See also Figure 3.) We first prepare three copies $D_{1,2}, D_{1,3}, D_{2,3}$ of D, and then identify $c_{i,j}$ with d of $D_{j,k}$, and $c_{i,k}$ with d' of $D_{j,k}$, where $j, k \in \{1, 2, 3\}$ and $j < k$, respectively. Then, we have the following proposition.

▶ **Proposition 6** (∗). *Suppose that $|F_2| > 2$. For any (Π_1, Π_2)-coloring f of the clause gadget C_i^ℓ such that $|f^{-1}(1)| < \ell$, exactly one of $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$ is assigned to the color 1.*

Therefore, the vertex in $\{c_{i,1},\ c_{i,2},\ c_{i,3}\}$ assigned to the color 1 will correspond to the `true` literal of the clause for the case of $|F_2| > 2$. On the other hand, for the case of $|F_2| = 2$, this correspondence holds for the color 2: the vertex in $\{c_{i,1},\ c_{i,2},\ c_{i,3}\}$ assigned to the color 2 will correspond to the `true` literal of the clause.

▶ **Proposition 7** (∗). *Suppose that $|F_2| = 2$. For any (Π_1, Π_2)-coloring f of the clause gadget C_i^ℓ such that $|f^{-1}(1)| < \ell$, exactly one of $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$ is assigned to the color 2.*

3.1.2 Reduction

We construct the corresponding graph for Min (Π_1, Π_2)-Partition from a given instance ϕ of Positive 1-in-3-SAT. Let α and β be the numbers of variables and clauses in ϕ, respectively. We first prepare α vertices $v_1, v_2, \ldots, v_\alpha$, and β copies $C_1^\ell, C_2^\ell, \ldots, C_\beta^\ell$ of the clause gadget; the value ℓ will be defined later, but now we assume that ℓ is a polynomial in the input size of ϕ. Each vertex v_j corresponds to a variable x_j of ϕ, and each clause gadget C_i^ℓ corresponds to a clause c_i of ϕ. We next prepare 3β copies of the relay gadget R^ℓ. If a variable x_j appears as a k-th literal of a clause c_i, where $k \in \{1, 2, 3\}$, then we identify the vertex v_j with x in R^ℓ, and identify the vertex $c_{i,k}$ of C_i^ℓ with x' in R^ℓ. This completes the construction of the corresponding graph G_ϕ^ℓ for Min (Π_1, Π_2)-Partition. G_ϕ^ℓ can be constructed in polynomial time if ℓ is polynomial in the input size of ϕ.

Let $p = |V(F_1)|$ and $q = |V(F_2)|$. Note that both p and q are fixed constants, which do not depend on the given instance of Positive 1-in-3-SAT. Let γ be an arbitrary integer such that $\gamma \geq 80pq\beta$. We denote by $n_{\phi,\ell}$ the number of vertices in G_ϕ^ℓ.

▶ **Lemma 8** (∗). *G_ϕ^ℓ is promised, and it holds that $n_{\phi,\ell} \leq \gamma q \ell$.*

We now give the key lemma for our reduction.[1]

[1] As we will see later, $\gamma \leq \ell$ holds, and hence Lemma 9 implies that there exists no graph G_ϕ^ℓ such that $\gamma \leq \mathsf{OPT}(G_\phi^\ell) < \ell$.

▶ **Lemma 9** (∗). *The following* (I) *and* (II) *hold*:

(I) *if* $OPT(G^\ell_\phi) < \ell$, *then* ϕ *has a satisfying truth assignment*; *and*
(II) *if* $OPT(G^\ell_\phi) \geq \gamma$, *then* ϕ *has no satisfying truth assignment.*

We are ready to prove Theorem 1. We set

$$\ell = \gamma^{\lceil (2-\varepsilon)/\varepsilon \rceil} \cdot q^{\lceil (1-\varepsilon)/\varepsilon \rceil},$$

then ℓ is a polynomial in the input size of ϕ. Assume for a contradiction that Min (Π_1, Π_2)-Partition admits a polynomial-time approximation algorithm within a factor $n^{1-\varepsilon}$ for some fixed $0 < \varepsilon \leq 1$, where n is the number of vertices in a given graph. Let $\mathsf{APX}(G^\ell_\phi)$ be the value computed by the approximation algorithm. Then, we have

$$\mathsf{OPT}(G^\ell_\phi) \leq \mathsf{APX}(G^\ell_\phi) \leq n^{1-\varepsilon}_{\phi,\ell} \cdot \mathsf{OPT}(G^\ell_\phi). \tag{1}$$

We then give the following lemma, which will be used to yield a contradiction.

▶ **Lemma 10.** $APX(G^\ell_\phi) < \ell$ *if and only if* ϕ *has a satisfying truth assignment.*

Proof. We first prove the only-if direction. Suppose that $\mathsf{APX}(G^\ell_\phi) < \ell$ holds. By the left inequality of (1) we have $\mathsf{OPT}(G^\ell_\phi) < \ell$. Then, Lemma 9(I) says that ϕ has a satisfying truth assignment.

We then prove the if direction, by taking a contraposition. Suppose that $\mathsf{APX}(G^\ell_\phi) \geq \ell$ holds. By the right inequality of (1) we have $n^{1-\varepsilon}_{\phi,\ell} \cdot \mathsf{OPT}(G^\ell_\phi) \geq \ell$. Then, by Lemma 8, we have

$$\mathsf{OPT}(G^\ell_\phi) \geq \frac{\ell}{n^{1-\varepsilon}_{\phi,\ell}} \geq \frac{\ell}{(\gamma q \ell)^{1-\varepsilon}} = \frac{\ell^\varepsilon}{(\gamma q)^{1-\varepsilon}} \geq \frac{(\gamma^{(2-\varepsilon)/\varepsilon} \cdot q^{(1-\varepsilon)/\varepsilon})^\varepsilon}{(\gamma q)^{1-\varepsilon}} = \gamma.$$

Then, Lemma 9(II) says that ϕ has no satisfying truth assignment. ◀

We have assumed that $\mathsf{APX}(G^\ell_\phi)$ can be computed in polynomial time. Then, Lemma 10 yields a contradiction unless P = NP, because it implies that we can solve Positive 1-in-3-SAT in polynomial time. This completes the proof of Theorem 1. ◀

3.2 Planar Bipartite Graphs

In this subsection, we study Min (Π_1, Π_2)-Partition for planar bipartite graphs. Notice that any bipartite graph G has a (Π_1, Π_2)-coloring (i.e., $\mathsf{OPT}(G) \neq +\infty$) if both properties Π_1 and Π_2 are nontrivial, additive and hereditary.

The main result of this subsection is the following theorem, which can be obtained by modifying the arguments in Sections 3.1.1 and 3.1.2.

▶ **Theorem 11** (∗). *Let* Π_1 *and* Π_2 *be any two fixed nontrivial additive hereditary graph properties, each of which contains a minimal forbidden induced subgraph that is planar and bipartite. Let* G *be a planar bipartite graph of* n *vertices which is promised, and let* ε *be any fixed constant such that* $0 < \varepsilon \leq 1$. *Under the assumption that* P $\neq$ NP, *Min* (Π_1, Π_2)-*Partition admits no polynomial-time approximation algorithm for* G *within a factor* $n^{1-\varepsilon}$ *unless both* Π_1 *and* Π_2 *are classes of edgeless graphs.*

In contrast to Theorem 1, Theorem 11 cannot be generalized for $c \geq 3$. In fact, it always holds that $\mathsf{OPT}(G) = 0$ for any $c \geq 3$ and any bipartite graph G if $\Pi_1, \Pi_2, \ldots, \Pi_c$ are nontrivial additive hereditary properties, because G has a $(\Pi_2, \Pi_3, \ldots, \Pi_c)$-coloring.

Theorem 11 immediately yields the following corollary.

▶ **Corollary 12.** *Let Π_1 and Π_2 be any two classes of graphs listed below:*

- *edgeless graphs,*
- *cluster graphs (P_3-free graphs),*
- *cographs (P_4-free graphs),*
- *star graphs,*
- *path graphs,*
- *acyclic graphs,*
- *outerplanar graphs,*
- *series-parallel graphs,*
- *interval graphs,*
- *chordal graphs, or*
- *graphs of bounded maximum degree.*

Let G be a planar bipartite graph of n vertices which is promised, and let ε be any fixed constant such that $0 < \varepsilon \leq 1$. Then, under the assumption that $\mathrm{P} \neq \mathrm{NP}$, Min (Π_1, Π_2)-Partition admits no polynomial-time approximation algorithm for G within a factor $n^{1-\varepsilon}$ unless both Π_1 and Π_2 are classes of edgeless graphs.

4 FPT Algorithms

In this section, we focus on the fixed-parameter tractability of Parameterized (Π_1, Π_2)-Partition when the graph property Π_2 is the class of acyclic graphs.

4.1 Hereditary Properties

We first consider the case where the graph property Π_1 is hereditary.

▶ **Theorem 13.** *Let Π_1 be any hereditary graph property, and let Π_2 be the class of acyclic graphs. Given a graph G and a nonnegative integer k, suppose that one can decide in $t(k)$ time whether a subgraph H with at most k vertices of G satisfies Π_1. Then, Parameterized (Π_1, Π_2)-Partition for G can be solved in $2^{O(k^2)}(t(k) + n + m)$ time, where n and m are the numbers of vertices and edges in G, respectively.*

In this subsection, we also prove that the running time above can be improved for bounded degeneracy graphs. A graph G is *d-degenerate* if any subgraph of G has a vertex of degree at most d. It is known that many graph classes have bounded degeneracy: for example, planar graphs, graphs of bounded maximum degree, and bounded treewidth graphs.

▶ **Theorem 14.** *Let Π_1 be any hereditary graph property, and let Π_2 be the class of acyclic graphs. Given a d-degenerate graph G and a nonnegative integer k, suppose that one can decide in $t(k)$ time whether a subgraph H with at most k vertices of G satisfies Π_1. Then, Parameterized (Π_1, Π_2)-Partition for G can be solved in $2^{O(h(k,d))}(t(k) + n + m)$ time, where $h(k, d) = \max\{d^3 + 3d^2 + 3d, (d+1)\log k + \log(d+1)\} \cdot k$, and n and m are the numbers of vertices and edges in G, respectively.*

For many natural properties, one can decide in $k^{O(1)}$ or $2^{O(k)}$ time whether a subgraph H with at most k vertices satisfies Π_1: for example, the classes of edgeless graphs, planar graphs, and proper c-colorable graphs for a fixed c. Thus, Parameterized (Π_1, Π_2)-Partition is solvable in $2^{O(k^2)}(n + m)$ time for general graphs and in $2^{O(k \log k)}(n + m)$ time for bounded degeneracy graphs, when Π_1 is such a natural hereditary property and Π_2 is the class of acyclic graphs.

To prove Theorems 13 and 14, we use the idea of a *compact representation* of minimal feedback vertex sets [4, 12]. Recall that a feedback vertex set S of a graph G is a vertex subset of G such that $G - S$ is acyclic. A compact representation for a set of minimal feedback vertex sets of a graph G is a set $\mathcal{C}$ of pairwise disjoint subsets of $V(G)$ such that choosing exactly one vertex from every set in $\mathcal{C}$ results in a minimal feedback vertex set of G. We say that a minimal feedback vertex set S of G *is contained in* a compact representation $\mathcal{C}$ if S can be obtained from $\mathcal{C}$ by this operation. A compact representation $\mathcal{C}$ is called a *k-compact representation* if the number of sets in $\mathcal{C}$ is at most k. We can efficiently enumerate k-compact representations of minimal feedback vertex sets in G, as follows:

▶ **Theorem 15** ([12]). *Given a graph G with m edges and an integer k, there exists an algorithm which enumerates k-compact representations of G in $O(23.1^k m)$ time such that any minimal feedback vertex set of size at most k is contained in some k-compact representation. Moreover, the number of k-compact representations output by the algorithm is at most $O(23.1^k)$.*

An instance (G, k) of Parameterized (Π_1, Π_2)-Partition is a yes-instance if and only if there is a (Π_1, Π_2)-coloring f of G such that $f^{-1}(1)$ forms a minimal feedback vertex set of size at most k of G, because Π_1 is hereditary. Therefore, Parameterized (Π_1, Π_2)-Partition can be rephrased as the problem of asking whether there exists a minimal feedback vertex set S of G such that $|S| \leq k$ and $G[S]$ satisfies Π_1. A compact representation $\mathcal{C}$ is called *good* if $\mathcal{C}$ contains such a minimal feedback vertex set S. Given a graph and a k-compact representation $\mathcal{C}$, one can determine whether $\mathcal{C}$ is good or not, by the following lemma.

▶ **Lemma 16** (∗). *Let $G = (V, E)$ be a graph with m edges. Given a k-compact representation $\mathcal{C}$ of minimal feedback vertex sets in G, assume that each set in $\mathcal{C}$ has at most α vertices. Then, one can determine whether $\mathcal{C}$ is good in $O(\alpha^k(t(k) + m))$ time.*

Therefore, our strategy is to enumerate k-compact representations of minimal feedback vertex sets in G by Theorem 15, and then check whether each enumerated k-compact representation $\mathcal{C}$ is good. Note that, however, the number α of vertices of each set in $\mathcal{C}$ is not always bounded by a function of k. Therefore, we kernelize each enumerated k-compact representation $\mathcal{C}$ to prove Theorems 13 and 14.

We now explain how to kernelize a k-compact representation $\mathcal{C}$ of minimal feedback vertex sets in G. A set in $\mathcal{C}$ is said to be *singleton* if the set consists of exactly one vertex, otherwise *multiple*. Then, the following proposition holds.

▶ **Proposition 17** ([4]). *Let C_1 and C_2 be any two distinct multiple sets in a compact representation $\mathcal{C}$ of minimal feedback vertex sets in a graph G. Then, any two vertices $v_1 \in C_1$ and $v_2 \in C_2$ are not adjacent in G.*

Let X be the set of the vertices of all singleton sets in $\mathcal{C}$. For a multiple set C in $\mathcal{C}$ and a subset $X' \subseteq X$, let $C_{X'}$ be the subset of C such that $N(u, X) = X'$ holds (on G) for every vertex u in $C_{X'}$. We iterate the following reduction rule for $\mathcal{C}$ until the rule is not applicable.

Reduction Rule. If there is a multiple set C in $\mathcal{C}$ such that $|C_{X'}| \geq 2$ for some $X' \subseteq X$, then choose an arbitrary vertex u from $C_{X'}$ and remove all vertices of $C_{X'} \setminus \{u\}$ from C.

▶ **Lemma 18.** *Let $\mathcal{C}$ be a k-compact representation of minimal feedback vertex sets in a graph G. By applying Reduction Rule to $\mathcal{C}$, one can obtain a k-compact representation $\mathcal{C}^*$ of minimal feedback vertex sets in G such that*
(a) *each set in $\mathcal{C}^*$ has at most 2^k vertices of G; and*
(b) *$\mathcal{C}$ is good if and only if $\mathcal{C}^*$ is good.*

Proof. We first prove the claim (a). Suppose that $\mathcal{C}$ has a multiple set C with at least $2^k + 1$ vertices. Since $|X| \leq k$, two vertices $u, u' \in C$ exist such that $N(u, X) = N(u', X)$ on G.

Then, we apply Reduction Rule to $\mathcal{C}$ and obtain another k-compact representation. Thus, we can obtain a k-compact representation $\mathcal{C}^*$ such that each set in $\mathcal{C}^*$ has at most 2^k vertices by iterating Reduction Rule.

We next prove the claim (b). Let $\mathcal{C}'$ be a k-compact representation of G obtained by applying Reduction Rule to $\mathcal{C}$ once. It suffices to show that $\mathcal{C}$ is good if and only if $\mathcal{C}'$ is good. The if direction is straightforward, namely, if $\mathcal{C}'$ is good, then $\mathcal{C}$ is also good. We thus prove the only-if direction. Suppose that $\mathcal{C}$ is good, and let S be a minimal feedback vertex set of G such that S is contained in $\mathcal{C}$ and $G[S]$ satisfies Π_1. If $u \in S$, then $\mathcal{C}'$ also contains S and hence $\mathcal{C}'$ is good. Therefore, we suppose that $u \notin S$ and S has a vertex u' in $C_{X'} \setminus \{u\}$. Let $S' = (S \cup \{u\}) \setminus \{u'\}$. Then, S' is contained in $\mathcal{C}$, because u and u' are in the same set C in $\mathcal{C}$. Thus, S' is also contained in $\mathcal{C}'$. Moreover, from Proposition 17 and the assumption that $N(u, X) = N(u', X)$ holds, $G[S']$ is isomorphic to $G[S]$. Therefore, $G[S']$ satisfies Π_1, and hence $\mathcal{C}'$ is good. ◀

Proof of Theorem 13. Let (G, k) be an instance of Parameterized (Π_1, Π_2)-Partition, and let $n = |V(G)|$ and $m = |E(G)|$. Using Theorem 15, we first enumerate k-compact representations of all minimal feedback vertex sets in G in $O(23.1^k m)$ time. We then apply Reduction Rule to all enumerated k-compact representations. For each k-compact representation $\mathcal{C}$, by Lemma 18 we obtain a kernelized k-compact representation $\mathcal{C}^*$ such that each set in $\mathcal{C}^*$ has at most 2^k vertices of G; this can be done in $O(2^k kn + m)$ time. For each kernelized k-compact representation $\mathcal{C}^*$, by Lemma 16 we decide whether $\mathcal{C}^*$ is good in $O(2^{k^2} \cdot (t(k) + n + m))$ time. Theorem 15 says that there are at most $O(23.1^k)$ k-compact representations of G, and hence we produce kernelized k-compact representations in $O(23.1^k \cdot (2^k kn + m))$ time in total and determine whether there is a good k-compact representation of G in $O(23.1^k \cdot 2^{k^2} \cdot (t(k) + n + m))$ time in total. Therefore, the total running time of the algorithm is $2^{O(k^2)}(t(k) + n + m)$. This completes the proof of Theorem 13. ◀

We then prove Theorem 14. Suppose that a given graph G is d-degenerate for some integer $d \geq 1$. We apply the same algorithm (and hence the same Reduction Rule) to G. Using the fact that G is d-degenerate, we can estimate the size of each set in a kernelized compact representation more sharply, as follows.

▶ **Lemma 19** ($*$). *Suppose that a graph G is d-degenerate for some integer $d \geq 1$. Let $\mathcal{C}$ be a k-compact representation of minimal feedback vertex sets in G. By applying Reduction Rule to $\mathcal{C}$, one can obtain a k-compact representation $\mathcal{C}^*$ of minimal feedback vertex sets in G such that*

(a) *each set in $\mathcal{C}^*$ has at most $2^{d^3+3d^2+3d}$ vertices of G if $k \leq d^3 + 3d^2 + 3d$, otherwise it has less than $\sum_{i=0}^{d+1} \binom{k}{i}$ vertices of G; and*

(b) *$\mathcal{C}$ is good if and only if $\mathcal{C}^*$ is good.*

Using Lemma 19 (instead of Lemma 18), we can prove Theorem 14 by the similar arguments as in the proof of Theorem 13.

4.2 Graph Properties with Bounded Maximum Degree

The parameterized variant of Independent Feedback Vertex Set is equivalent to Parameterized (Π_1, Π_2)-Partition when Π_1 is the class of edgeless graphs and Π_2 is the class of acyclic graphs. Since the class of edgeless graphs is the class of graphs with maximum degree zero, it is natural to consider the case where Π_1 is the class of graphs with bounded maximum degree. In this subsection, we give the following theorem for such a case.

▶ **Theorem 20** (∗). *Let Π_1 be the class of graphs with maximum degree Δ for a fixed integer Δ, and let Π_2 be the class of acyclic graphs. Given a graph G with n vertices and m edges,* PARAMETERIZED (Π_1, Π_2)-PARTITION *can be solved in $O(23.1^k m) + 2^{O(\Delta k \log k)}(n+m)$ time.*

Our algorithm for Theorem 20 takes a similar strategy as in Section 4.1, but employs the following modified reduction rule to kernelize a k-compact representation $\mathcal{C}$ of minimal feedback vertex sets in a graph G. Recall that X denotes the set of the vertices of all singleton sets in $\mathcal{C}$.

Modified Reduction Rule.

Rule A: if there is a multiple set C in $\mathcal{C}$ containing a vertex u such that $|N(u, X)| \geq \Delta+1$, then remove u from C; and

Rule B: if there is a multiple set C in $\mathcal{C}$ such that $|C_{X'}| \geq 2$ for some $X' \subseteq X$, then choose an arbitrary vertex u from $C_{X'}$ and remove all vertices of $C_{X'} \setminus \{u\}$ from C.

We note that Rule B above is the same as Reduction Rule in Section 4.1. We omit the details and analysis of the algorithm from this extended abstract.

Finally, we note that the running time of the algorithm can be improved when $\Delta = 1$, as follows.

▶ **Theorem 21** (∗). *Let Π_1 be the class of graphs with maximum degree one, and let Π_2 be the class of acyclic graphs. Then,* PARAMETERIZED (Π_1, Π_2)-PARTITION *can be solved in $O(23.1^k(k^{2.5} + n + m))$ time.*

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