First-Order Transductions of Graphs

Patrice Ossona de Mendez
Centre d'Analyse et de Mathématique Sociales CNRS UMR 8557, Paris, France
Charles University, Prague, Czech Republic

Abstract

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1 Introduction

Logical methods in Computer Science have a long history, as witnessed e.g. by the relative longevity of SQL in relational database management. More recently, Courcelle's theorem, which combines second-order logic and tree decompositions of graphs, showed that a many NP-complete algorithmic problems in graph theory can be solved in polynomial time on graphs with bounded tree-width (and even on graphs with bounded clique-width). At the heart of the latter result is the notion of monadic second-order transductions, which are a way to encode a graph within a structure using coloring and monadic second-order logic formulas. In this presentation we consider first-order transductions, for which the formulas have to be first-order formulas. As a counterpart for this strong restriction, many algorithmic problems become fixed parameter tractable when restricted to nowhere dense classes, which include classes excluding a topological minor thus, in particular, classes of planar graphs and classes of graphs with bounded degrees.

In this setting, the main challenge is to extend results obtained in the sparse setting (for bounded expansion classes and nowhere dense classes) to the dense setting, in a similar way the results about monadic second-order model checking have been extended from classes with bounded tree-width to classes with bounded clique-width.

2 Sparse classes

The study of classes of sparse graphs has long been divided into two dual points of view: on the one hand, classes of graphs with bounded degrees – and particularly classes of regular graphs, enjoy strong connections with group theory and important combinatorial properties deriving from spectral properties. On the other hand, classes excluding a minor are strongly related to topological graph theory, as witnessed by Robertson and Seymour's Graph Structure Theorem [36], which is probably the most important result in structural graph theory. It was believed for a long time that at the source of this duality lied a fundamental gap between the notions of minor and topological minor. However, the Graph Structure Theorem has recently been extended to graphs excluding a topological minor by Grohe and Marx [17] (see also [9]), witnessing that both notions are closer than expected, even though this extension builds on two types of blocks, namely graphs almost embedded on a surface and graphs with almost bounded degrees.

Another unifying approach [21] was proposed fifteen years ago by Nešetřil and the author, based on the concepts of shallow minors [33], of low tree-depth decompositions [22], of generalized coloring numbers [19, 39], and of quasi-wideness [7]. This approach led to a taxonomy of classes of sparse graphs, with two main dividing lines, which respectively delineate classes with bounded expansion [23] and nowhere dense classes [24, 25]. These types of graph classes received numerous characterizations, and we shall use two of them as definitions.

For a graph G and a non-negative integer k, we denote by $\mathrm{TM}_k(G)$ (where TM stands for topological minor) the set of all graphs H such that some $\leq k$-subdivision of H is a subgraph of G and, by extension, for a class $\mathscr{C}$ we define $\mathrm{TM}_k(\mathscr{C}) = \bigcup_{G\in\mathscr{C}} \mathrm{TM}_k(G)$. A class of graphs $\mathscr{C}$ is *nowhere dense* if $\mathrm{TM}_k(\mathscr{C})$ has bounded clique number (or, equivalently, if $\mathrm{TM}_k(\mathscr{C})$ is not the class $\mathscr{G}$ of all graphs) for every integer k for every non-negative integer k. A class $\mathscr{C}$ has *bounded expansion* if, for each integer k, there is a uniform bound on the average degrees of the graphs in $\mathrm{TM}_k(\mathscr{C})$ (or, equivalently, if each of the class $\mathrm{TM}_k(\mathscr{C})$ is degenerate).

It is remarkable that bounded expansion and nowhere-denseness can be defined indifferently using shallow topological minors (as above), shallow minors, or shallow immersions (see [8]). Also, bounded expansion can be defined indifferently using the average degree, the degeneracy, the chromatic number [8], and even the fractional chromatic number [10]. For an in-depth study of bounded expansion and nowhere dense classes, we refer the reader to [26].

The main aspect of sparse classes is probably that vertices are easily separated. This property, which may be formalized in terms of neighborhood covers and uniform quasi-wideness, is the core of the model checking algorithm of Grohe, Kreutzer, and Siebertz, who proved the following result [16].

▶ **Theorem 2.1.** *For every nowhere dense class $\mathscr{C}$ and every $\varepsilon > 0$ there is an algorithm that checks in time $f(\theta)\, n^{1+\varepsilon}$ if a graph $G \in \mathscr{C}$ with n vertices satisfies a first-order sentence θ.*

For monotone classes of graphs, under standard assumptions from complexity theory, nowhere-denseness is actually a necessary condition for first-order model checking to be fixed-parameter tractable.

Also, one of the manifestations of these separability properties lies in the low neighborhood complexity of bounded expansion and nowhere dense classes. Precisely, for a class $\mathscr{C}$ and an integer d define the maximum number $\pi_d^{\mathscr{C}}$ of traces of the balls of radius d on subsets of vertices of size n in graphs in $\mathscr{C}$:

$$\pi_d^{\mathscr{C}}(n) = \sup_{G\in\mathscr{C}} \max_{A\subseteq V(G),|A|=n} \big|\{A \cap N_d(v) : v \in V(G)\}\big|.$$

Then we have the following characterizations.

▶ **Theorem 2.2.** *A monotone class $\mathscr{C}$ has bounded expansion if and only if $\pi_d^{\mathscr{C}}(n) = O(n)$ for every integer d and it is nowhere dense if and only if $\pi_d^{\mathscr{C}}(n) = O(n^{1+\varepsilon})$ for every integer d and every $\varepsilon > 0$.*

The above characterization of bounded expansion classes was proved by Reidl, Villaamil, and Stavropoulos [35]; the difficult direction of the characterization of nowhere dense classes was proved by Gajarskỳ et al. [13] for the case $d = 1$ and by Eickmeyer et al. [11] for the general case. These characterizations have been dramatically strengthened to bounds on the shattering functions of first-order definable families of subsets of vertices by Pilipczuk, Siebertz, and Toruńczyk [32], thus unveiling a deep connection between the notions of sparse classes and the model theoretical notion of classes with low VC-density.

Also, it follows from another separation property, namely uniform quasi-wideness, that for a monotone class of graphs, nowhere-denseness, stability and dependence are equivalent properties, where the last two refer to the model theoretical fundamental dividing lines identified by Shelah in its classification theory [37]. Precisely, based on a result of Podewski and Ziegler [34] Adler and Adler [1] proved the following collapse.

▶ **Theorem 2.3.** *For a monotone class of graphs $\mathscr{C}$ the following are equivalent:*

1. *$\mathscr{C}$ is nowhere dense;*
2. *$\mathscr{C}$ is stable;*
3. *$\mathscr{C}$ is monadically stable;*
4. *$\mathscr{C}$ is dependent;*
5. *$\mathscr{C}$ is monadically dependent.*

These two examples witness an intimate connection between graph theoretical and model theoretical dividing lines. This suggests a possible extension of the ideas and constructions introduced to deal with sparse graphs by using techniques borrowed from model theory, like interpretations and transductions. The hope, behind the search for an extension, is the possibility to define a dense analog of sparsity for hereditary classes of graphs, which would witness a relatively low complexity. In particular, we expect to cover the case of the *small* hereditary classes, which are classes with $O(n^c\,n!)$ labeled graphs with n vertices.

Another outcome of this connection between graph theory and model theory lies in the existence of totally Borel model theoretical limits for sequences of graphs in a nowhere dense graph. Here the notion of convergence consists in the convergence, for every first-order formula $\varphi(\overline{x})$ of the satisfaction probability of $\varphi(\overline{x})$ when considering a uniform and independent random assignment of the vertices to the free variables [27], which generalizes the existence of graphing limits for locally convergent sequence of graphs with bounded degrees (see [20]).

3 Transductions

A (first-order) *transduction* is a way to encode a structure within another structure by means of a coloring and a first-order formula. Precisely, a transduction T from graphs to graphs is defined by a first-order formula $\varphi(x, y)$ with two free-variable in the language of vertex-colored graphs. The atomic formulas of this language are of the form $x = y$, $E(x, y)$ (meaning x is adjacent to y) and $M_i(x)$ with $i \in \mathbb{N}$ (meaning x has color i). For a graph G, the set $\mathsf{T}(G)$ contains all the graphs H with vertex set $A \subseteq V(G)$, for which there is a vertex coloring G^+ of G such that $H \models E(u, v)$ if and only if $G^+ \models \varphi(u, v)$. It follows directly from the definition that $\mathsf{T}(G)$ is a hereditary class of graphs. A class $\mathscr{D}$ is a T-*transduction* of a class $\mathscr{C}$ if $\mathscr{D} \subseteq \mathsf{T}(\mathscr{C}) := \bigcup_{G\in\mathscr{C}} \mathsf{T}(G)$. The intuition here is that the graphs in the class $\mathscr{D}$ are non essentially more complex than the graphs in the class $\mathscr{C}$ as they can be "encoded" within them. It is easily checked that the existence of a transduction from a class to another defines a quasi-order. This quasi-order has a maximum, the class of all graphs. Classes that are not equivalent to this class – which are in some sense reasonably difficult – are exactly those graphs classes that are monadically dependent, in the model theoretical sense, as follows from [3]. It also follows from [3] that the so-called monadically stable classes of graphs are exactly those class that have no transduction to the class of all half-graphs.

Admittedly, checking if there exists a transduction from a class $\mathscr{C}$ to a class $\mathscr{D}$ may be a highly difficult task. However, the special case where $\mathscr{D}$ is the class of all graphs (that is checking if a class is not monadically dependent) is usually easier to handle. Moreover, if a class $\mathscr{C}$ is monadically dependent then checking the existence of a transduction from $\mathscr{C}$ to the class of all half-graphs (that is checking if $\mathscr{C}$ is not monadically stable) is much easier as it is surprisingly sufficient to check if arbitrarily large half-graphs are semi-induced subgraphs of graphs in $\mathscr{C}$ [28].

The structure of the transduction quasi-order seems to be difficult to establish [31]. As nowhere dense classes are monadically stable [2], every transduction of a nowhere dense class is also monadically stable. The converse statement is the object of the next conjecture.

▶ **Conjecture 3.1.** *Every monadically stable class of graphs is a transduction of a nowhere dense class.*

In general, when H is a transduction of a known graph G and that both the transduction and the vertex-coloring used by the transduction to get G are known, the problem of checking if H satisfies a first-order sentence can be easily transformed in the problem of checking if the (colored) graph G satisfies a derived formula. However, when only H and some basic information about G and the transduction are known, the problem may become much more difficult. For instance, if a graph G is a transduction of a graph with maximum degree d, computing such a pre-image and a transduction is provably hard. However, Gajarský et al. [12] proved the following (see [14] for some extension of this result).

▶ **Theorem 3.2.** *For every transduction T and every integer d there exist an integer d' (depending on T and d) and an interpretation I such that if G is a T-transduction of a graph H with maximum degree d, then there is a graph H' of maximum degree d', computable in polynomial time from G, such that $G = \mathsf{I}(H')$.*

This supports the next conjecture.

▶ **Conjecture 3.3.** *First-order model checking is fixed-parameter tractable on monadically stable classes of graphs.*

Some results have been obtained toward this conjecture, showing that first-order model checking is fixed-parameter tractable on transductions of bounded expansion classes, provided that a specific decomposition of the graphs in the graphs (a so-called depth-2 low shrub-depth cover) is given [15]. This result is proved by proving that first-order transductions transport low shrub-depth covers, which are a generalization of low tree-depth decompositions. A side consequence of this is that transductions of bounded expansion classes are linearly χ-bounded (see [15, 30, 28]). Some related results have been obtained for monadically stable classes with bounded linear rank-width [29] (showing that they are computable transductions of classes with bounded pathwidth) and for monadically stable classes with bounded rank-width (showing that they are computable transductions of classes with bounded treewidth) [28].

4 Partially ordered graphs

The use of transductions has been used to extend some properties of bounded expansion classes and nowhere dense classes within the monadically stable realm. To go further, it is necessary to introduce, at least locally, some order-like substructures. A way to it is to consider classes of *partially ordered graphs*, that is graphs with an additional partial order on the vertices, a special important case being *ordered graphs*, which are graphs with a total order on the vertices. Another example are *tree-ordered graphs*, which are graphs with a tree-order on the vertices. It appears that that the concept of ordered graphs particularly fits to the study of the recently introduced twin-width invariant [6], inspired by a width invariant defined on permutations by Guillemot and Marx [18]. Classes with bounded twin-width include several well studied classes of graphs, like classes of graphs excluding a minor, unit interval graphs, and classes with bounded clique-width. These classes are small [4] (contain at most $O(n^c\, n!)$ graphs with n vertices), and have fixed parameter tractable first-order model checking when a contraction sequence of the graphs is provided [6]. An essential property of twin-width is that its boundedness is preserved by transductions, as proved by Bonnet et al. [6].

▶ **Theorem 4.1.** *Every transduction of a class with bounded twin-width has bounded twin-width.*

Simon and Toruńczyk [38] recently announced the following characterization of bounded twin-width classes, which has been independently proved by Bonnet et al. [5]:

▶ **Theorem 4.2.** *A class of graphs has bounded twin-width if and only if it is the reduct of a monadically dependent class of ordered graphs.*

It is remarkable that for hereditary classes $\mathscr{C}$ of ordered graphs one can prove that, under the standard $\mathsf{FPT} \neq \mathsf{AW}[*]$ assumption from complexity theory, monadic dependence is equivalent to fixed parameter tractability of first-order model checking [5].

It might be possible that the characterization given by Theorem 4.2 could extend to the whole realm of monadically dependent classes, by considering tree-orders instead of linear orders.

▶ **Conjecture 4.3.** *Every monadically dependent class of graphs is a transduction of a monadically dependent class of tree-ordered graphs, whose reduct (obtained by forgetting the partial order) is monadically stable.*

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