3SUM and Related Problems in Fine-Grained Complexity

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Abstract

3SUM is a simple to state problem: given a set S of n numbers, determine whether S contains three a, b, c so that $a + b + c = 0$. The fastest algorithms for the problem run in $n^2 \text{poly}(\log\log n)/(\log n)^2$ time both when the input numbers are integers [1] (in the word RAM model with $O(\log n)$ bit words) and when they are real numbers [2] (in the real RAM model).

A hypothesis that is now central in Fine-Grained Complexity (FGC) states that 3SUM requires $n^{2-o(1)}$ time (on the real RAM for real inputs and on the word RAM with $O(\log n)$ bit numbers for integer inputs). This hypothesis was first used in Computational Geometry by Gajentaan and Overmars [4] [1] who built a web of reductions showing that many geometric problems are hard, assuming that 3SUM is hard. The web of reductions within computational geometry has grown considerably since then (see some citations in [11]).

A seminal paper by Pătraşcu [7] showed that the integer version of the 3SUM hypothesis can be used to prove polynomial conditional lower bounds for several problems in data structures and graph algorithms as well, extending the implications of the hypothesis to outside computational geometry. Pătraşcu proved an important tight equivalence between (integer) 3SUM and a problem called 3SUM-Convolution (see also [3]) that is easier to use in reductions: given an integer array a of length n, do there exist $i, j \in [n]$ so that $a[i] + a[j] = a[i + j]$. From 3SUM-Convolution, many 3SUM-based hardness results have been proven: e.g. to listing graphs in triangles, dynamically maintaining shortest paths or bipartite matching, subset intersection and many more.

It is interesting to consider more runtime-equivalent formulations of 3SUM, with the goal of uncovering more relationships to different problems. The talk will outline some such equivalences. For instance, 3SUM (over the reals or the integers) is equivalent to All-Numbers-3SUM: given a set S of n numbers, determine for every $a \in S$ whether there are $b, c \in S$ with $a + b + c = 0$ (e.g. [10]).

The equivalences between 3SUM, 3SUM-Convolution and All-Numbers 3SUM are (n^2, n^2)-fine-grained equivalences that imply that if there is an $O(n^{2-\varepsilon})$ time algorithm for one of the problems for $\varepsilon > 0$, then there is also an $O(n^{2-\varepsilon'})$ time algorithm for the other problems for some $\varepsilon' > 0$. More generally, for functions $a(n), b(n)$, there is an (a, b)-fine-grained reduction [11, 9, 10] from problem A to problem B if for every $\varepsilon > 0$ there is a $\delta > 0$ and an $O(a(n)^{1-\delta})$ time algorithm for A that does oracle calls to instances of B of sizes $n_1, \ldots, n_k$ (for some k) so that $\sum_{j=1}^{k} b(n_j)^{1-\varepsilon} \leq a(n)^{1-\delta}$. With such a reduction, an $O(b(n)^{1-\varepsilon})$ time algorithm for B can be converted into an $O(a(n)^{1-\delta})$ time algorithm for A by replacing the oracle calls by calls to the B algorithm. A and B are (a, b)-fine-grained equivalent if A (a, b)-reduces to B and B (b, a)-reduces to A.

One of the main open problems in FGC is to determine the relationship between 3SUM and the other central FGC problems, in particular All-Pairs Shortest Paths (APSP). A classical graph problem, APSP in n node graphs has been known to be solvable in $O(n^3)$ time since the 1950s. Its fastest known algorithm runs in $n^3/exp(\sqrt{\log n})$ time [14]. The APSP Hypothesis states that $n^{3-o(1)}$ time is needed to solve APSP in graphs with integer edge weights in the word-RAM model with $O(\log n)$ bit words. It is unknown whether APSP and 3SUM are fine-grained reducible to each other, in either direction. The two problems are very similar. Problems such as $(\min, +)$-convolution (believed to require $n^{2-o(1)}$ time) have tight fine-grained reductions to both APSP and 3SUM, and both 3SUM and APSP have tight fine-grained reductions to problems such as Exact Triangle [10, 8, 12] and (since very recently) Listing triangles in sparse graphs [7, 6, 13]. The talk will discuss these relationships and some of their implications, e.g. to dynamic algorithms.

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[1] They used a more stringent version of the hypothesis that said that (real) 3SUM requires $\Omega(n^2)$ time which we now know to be false [5].

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