A Stepping-Up Lemma for Topological Set Systems

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Abstract

Intersection patterns of convex sets in $\mathbb{R}^d$ have the remarkable property that for $d+1 \le k \le \ell$, in any sufficiently large family of convex sets in $\mathbb{R}^d$, if a constant fraction of the k-element subfamilies have nonempty intersection, then a constant fraction of the ℓ-element subfamilies must also have nonempty intersection. Here, we prove that a similar phenomenon holds for any topological set system $\mathcal{F}$ in $\mathbb{R}^d$. Quantitatively, our bounds depend on how complicated the intersection of ℓ elements of $\mathcal{F}$ can be, as measured by the maximum of the $\lceil \frac{d}{2} \rceil$ first Betti numbers. As an application, we improve the fractional Helly number of set systems with bounded topological complexity due to the third author, from a Ramsey number down to $d+1$. We also shed some light on a conjecture of Kalai and Meshulam on intersection patterns of sets with bounded homological VC dimension. A key ingredient in our proof is the use of the stair convexity of Bukh, Matoušek and Nivasch to recast a simplicial complex as a homological minor of a cubical complex.

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1 Introduction

In this paper, we investigate the intersection patterns of *topological set systems*, by which we mean families $\mathcal{F}$ of subsets of $\mathbb{R}^d$ such that for any $\mathcal{G} \subseteq \mathcal{F}$, the intersection of the elements of $\mathcal{G}$ has finite Betti numbers. Our main goal is to analyze how the intersection patterns of a set system $\mathcal{F}$ are influenced by the complexity of the intersection of subfamilies of $\mathcal{F}$. To measure this complexity, let us fix some integer h (set to $\lceil \frac{d}{2} \rceil$ for our purpose) and a ring to use for homology calculations ($\mathbb{Z}_2$ throughout this paper), and associate to $\mathcal{F}$ the function

$$\phi_{\mathcal{F}}^{(h)} : \begin{cases} \mathbb{N} & \to \quad \mathbb{N} \cup \{\infty\} \\ k & \mapsto \quad \sup\{\widetilde{\beta}_i\,(\cap \mathcal{G}) : \mathcal{G} \subseteq \mathcal{F}, |\mathcal{G}| \le k, 0 \le i < h\}. \end{cases}$$

We call it the (hth) *homological shatter function* of $\mathcal{F}$. Although here $\mathcal{F}$ is a set system in $\mathbb{R}^d$, the definition (and most of our methods) apply more generally (see Section 1.3).

A stepping-up phenomenon. For a set system $\mathcal{F}$, that is a family of subsets of some ground set, and an integer $k \ge 1$, let $\delta_{\mathcal{F}}(k) \in [0, 1]$ denote the proportion of the k-element subsets of $\mathcal{F}$ that have nonempty intersection. Technically, our main result is the following:

▶ **Theorem 1.** *For any $d+1 \leq k \leq \ell$ and $b \geq 0$, for any $\delta > 0$, there exists $\delta' > 0$ such that for any sufficiently large set system $\mathcal{F}$ in $\mathbb{R}^d$, if $\phi_{\mathcal{F}}^{(\lceil \frac{d}{2} \rceil)}(\ell) \leq b$ and $\delta_{\mathcal{F}}(k) \geq \delta$, then $\delta_{\mathcal{F}}(\ell) \geq \delta'$.*

Informally, Theorem 1 states that in nerves of topological set systems, positive densities tend to propagate towards *higher* dimension, in a form of "reverse Kruskal-Katona theorem". This generalizes earlier observations in combinatorial convexity and topological combinatorics, as we discuss in the next section. Moreover, the propagation rate can be bounded from below in terms of the homological shatter function.

Our result is existential and our proof does not aim at giving any reasonable estimate; the bound we obtain is an elementary recursive function, $\mathcal{E}^3$ in the Grzegorczyk hierarchy, of the parameters. Theorem 1 is, however, qualitatively sharp: without bounding the Betti numbers of intersections in *all* dimensions $0 \leq j < \lceil \frac{d}{2} \rceil$ we can obtain topological set systems in $\mathbb{R}^d$ with arbitrary intersection patterns. Indeed, fix an integer $k \geq 0$ and let K be the k-skeleton of the simplex on the vertex set V. Given nonempty subsets $S_1, \ldots, S_m$ of V, define induced subcomplexes $K_1, \ldots, K_m$ where $K_i = K[S_i]$. Note that for any subset $\emptyset \neq I \subset [m]$ we have $\bigcap_{i \in I} K_i = K[\bigcap_{i \in I} S_i]$, and so any nonempty intersection of the K_i has trivial j-dimensional homology for any $j \neq k$. Since K has a geometric realization in $\mathbb{R}^d$ for any $d \geq 2k+1$, it follows that the family $\mathcal{F} = \{K_1, \ldots, K_m\}$ forms a topological set system in $\mathbb{R}^d$ where intersections have Betti numbers equal to zero except possibly in dimension k. The subsets $S_1, \ldots, S_m$ can be chosen arbitrarily, so their intersection pattern is also arbitrary.

Lowering fractional Helly numbers. One use of Theorem 1 is the reduction of *fractional Helly numbers*. For instance, it easily improves a theorem of Patáková [29, Theorem 3] into:

▶ **Corollary 2.** *For every non-negative integers b and d, there is a function $\beta_{d,b} : (0,1) \to (0,1)$ such that for any $\alpha \in (0,1)$, for any sufficiently large set system $\mathcal{F}$ in $\mathbb{R}^d$ with $\phi_{\mathcal{F}}^{(\lceil d/2 \rceil)} \leq b$, if $\delta_{\mathcal{F}}(d+1) \geq \alpha$ then some $\beta_{d,b}(\alpha)|\mathcal{F}|$ members of $\mathcal{F}$ intersect.*

Specifically, [29, Theorem 3] required instead of "$\delta_{\mathcal{F}}(d+1) \geq \alpha$" that "$\delta_{\mathcal{F}}(m) \geq \alpha$" for some hypergraph Ramsey number m. The number m depends only on b and d, so we can first apply [29, Theorem 3], then follow up by Theorem 1 with $k = d+1$ and $\ell = m$. (Note that the implicit bound given by the proof of [29, Theorem 3] on the function $\beta_{d,b}$ also changes in the process.) This improvement in turn sharpens several other results, including a (p,q)-theorem, a weak ε-net theorem and a property testing algorithm. This systematic reduction of fractional Helly numbers also offers some evidence in support of some conjectures of Kalai and Meshulam [21] on a theory of *homological VC dimension*. We discuss these consequences in the next section.

1.1 Context and motivation

Let us briefly present some of the lines of research in discrete geometry, topological combinatorics and computational geometry that motivate Theorem 1.

Combinatorial convexity and beyond. The theorems of Carathéodory, Helly and Radon initiated a combinatorial theory of convexity that investigates the intersection patterns of families of convex sets, with a particular attention to the systems formed by the convex hulls of subsets of a finite point set. See the textbook of Matoušek [28, §8−10] and the surveys [8, 12, 9] for an introduction. One approach to extend combinatorial convexity is to think of a convex set as a hyperedge in an infinite hypergraph with vertex set $\mathbb{R}^d$. One can then reformulate results from combinatorial convexity as properties of this hypergraph, and investigate whether the dependencies between theorems found in the convex setting hold for more general hypergraphs. In this sense, Alon et al. [2] established that the (reformulation of the) fractional Helly theorem implies, among others, the (reformulations of the) (p, q)−theorem, the weak ε-net theorem and the selection lemma, three landmarks in combinatorial convexity. (Note that this pivotal role of the fractional Helly theorem is only surpassed by Radon's lemma [17].) The study of the homological shatter functions of topological set systems recasts several previous topological relaxations of convexity [27, 24, 23, 10, 14, 29] into a broader setting.

Fractional Helly, stepping up, and upper bound theorems. The original *fractional Helly theorem* of Katchalski and Liu [26] asserts that for any $d \geq 1$ there is a function $\beta_d : (0, 1) \to (0, 1)$ such that for any $\alpha \in (0, 1)$, any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ where a fraction α of the $(d + 1)$-element subsets have non-empty intersection must contain an intersecting subfamily of size at least $\beta_d(\alpha)|\mathcal{F}|$. In other words, if a positive fraction of the $(d+1)$-element subfamilies of $\mathcal{F}$ have nonempty intersection, then a positive fraction of $\mathcal{F}$ has nonempty intersection. The size of the subsets for which the "positive fraction" property is assumed, $d + 1$ here, is referred to as the *fractional Helly number*.

Katchalski and Liu [26] already observed that one can require that $\beta_d(\alpha) \to 1$ when $\alpha \to 1$. They derived it from the observation, which they dubbed the *stepping-up lemma*, that for any family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$,

$$\forall d + 1 \leq k \leq \ell, \quad \delta_{\mathcal{F}}(\ell) \geq 1 - (1 - \delta_{\mathcal{F}}(k)) \binom{\ell}{k}. \tag{1}$$

In particular, if $\delta_{\mathcal{F}}(k) > 1 - 1/\binom{\ell}{k}$, then $\delta_{\mathcal{F}}(\ell) > 0$. Kalai [20] later showed that one can take $\beta_d(\alpha) \stackrel{\text{def}}{=} 1 - (1 - \alpha)^{\frac{1}{d+1}}$, which is best possible. His proof is based on the *upper bound theorem* that he [20] and Eckhoff [11] established independently. The upper bound theorem asserts that for any family $\mathcal{F}$ of n convex sets in $\mathbb{R}^d$,

$$\forall d \leq k \leq d + r - 1, \quad f_k(\mathcal{F}) > \sum_{i=0}^{d} \binom{n - r}{i} \binom{r}{k - i + 1} \quad \Rightarrow \quad f_{d+r}(\mathcal{F}) > 0, \tag{2}$$

where $f_i(\mathcal{F}) = \binom{n}{i+1} \delta_{\mathcal{F}}(i + 1)$ denotes the number of $(i + 1)$-element subsets of $\mathcal{F}$ that have nonempty intersection. (That is, $f_i(\mathcal{F})$ is the number of i-dimensional faces of the nerve of $\mathcal{F}$.) It was recently shown [25] that the propagation phenomenon revealed by Equations (1) and (2) also holds for set systems in $\mathbb{R}^2$ (or on a surface) with bounded 1st homological shatter function, indicating that this phenomenon extends far beyond convexity. Our Theorem 1 gives further evidence of this by generalizing [25, Theorem 2.2] to arbitrary dimension.

Collapsibility, Leray number and homological VC dimension. The known proofs of the upper bound theorem (2) abstract away the geometry into some property shared by nerves of families of convex sets. The more elementary proofs apply to *d-collapsible* complexes [11, 20, 1], that is complexes that can be reduced by discrete homotopy moves (called *collapses*) to a d-dimensional complex [31, Lemma 1]. The more general proof, also due to Kalai (see Hell's PhD thesis [16, §5.2] for a presentation), applies to *d-Leray* complexes, that is complexes in which all induced subcomplexes have vanishing homology in dimension d and above.

Deeper connections between discrete geometry and topological combinatorics were suggested by Kalai and Meshulam in a program to develop a theory of *homological VC dimension*. The homological shatter function that we introduce here is already apparent in that program. Two of their conjectures, when combined, suggest that topological set systems with polynomial homological shatter function should enjoy a fractional Helly theorem:

▶ **Conjecture 3** (Following Kalai and Meshulam [21, Conjectures 6 and 7]). *For any integer $0 \leq k \leq d$ and any $A > 0$, there exists a function $\beta : (0,1) \to (0,1)$ such that the following holds. For any $\alpha > 0$ and any sufficiently large set system $\mathcal{F}$ in $\mathbb{R}^d$ such that $\forall m \geq 0$, $\phi_{\mathcal{F}}^{(d)}(m) \leq Am^k$, if $\delta_{\mathcal{F}}(d+1) \geq \alpha$ then some $\beta(\alpha)|\mathcal{F}|$ members of $\mathcal{F}$ have a point in common.*

This combination of Conjectures 6 and 7 from [21], also appeared in [22, Conjecture 17], except that we took upon ourselves to dissociate the dimension d of the space and the order k of the homological shatter function. Our Corollary 2 settles the case $k = 0$ of this conjecture. Moreover, our Theorem 1 reveals that for $k \geq 1$, if the assumptions of Conjecture 3 imply any fractional Helly theorem at all, then the fractional Helly number can be brought down to $d + 1$. (Note that Corollary 2 and Theorem 1 need only control $\phi_{\mathcal{F}}^{(\lceil \frac{d}{2} \rceil)}$, not $\phi_{\mathcal{F}}^{(d)}$.)

Property testing. The fact that fractional Helly theorems ensure the existence of small weak ε-nets already give them potential for computational applications. Let us stress another connection between fractional Helly theorems and algorithms, which comes from the area of *property testing*. Recall that Helly's theorem relates to the size of witness sets for convex programming, so that its generalizations, the so-called *Helly-type theorems*, correspond to the combinatorial dimension of *LP-type problems* [3] (see also [14, §1.3]). Similarly, generalizations of the fractional Helly theorem lead to *property testing* algorithms for optimization under constraints, by relating the probability that a random choice of k constraints is satisfiable to the size of the largest subset of constraints that can be simultaneously satisfied. (Here, k denotes the fractional Helly number.) This relation was spelled out by Chakraborty et al. [7] in the convex settings and holds more generally. Again, notice that reducing a fractional Helly number also improves, in principle, the efficiency of the related property testing algorithm.

1.2 Approach and further results

At a high-level, we prove Theorem 1 by a two-stage approach that we learned from Bárány and Matoušek [5], who use it for convex lattice sets. We first identify (or, in our case, prove) a Helly-type theorem that turns some intersection pattern on the k-element subsets of some family of *constant size* into *at least one* intersection of $k + 1$ sets. We then use the positive density assumption $\delta_{\mathcal{F}}(k) > 0$ to find many occasions to apply that constant-size theorem.

Supersaturation brings out colors... For this approach to work, we need the intersection pattern used in the first stage to be "massively unavoidable" when $\delta_{\mathcal{F}}(k) > 0$. Here, some extremal hypergraph theory comes into play, as in Matoušek's words [28, §9.2], "*Hypergraphs with many edges need not contain complete hypergraphs, but they have to contain complete multipartite hypergraphs*". So, our Helly theorem for constant size should not rely on a complete intersection pattern, as in Helly's original theorem, but rather on a complete k-partite intersection pattern, as in the *colorful Helly theorem* [4]. The fact that complete k-partite intersection patterns can be found in abundance as soon as $\delta_{\mathcal{F}}(k) > 0$ follows from the *supersaturation* theorem of Erdős and Simonovits; We postpone its presentation to Section 7, as we will only need it (and the related terminology) in the final step of our proof.

... and colors lead to stair convexity. Given set systems $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$, a subfamily $\mathcal{G} \subset \bigcup_{i=1}^m \mathcal{F}_i$ is called *colorful* (with respect to the families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$) if $\mathcal{G}$ contains at most one member from each $\mathcal{F}_i$. Here is our colorful Helly theorem:

▶ **Theorem 4.** *For any integers $b \geq 0$ and $m > d \geq 1$ there exists an integer $t = t(b, d, m)$ such that for any topological set systems $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ in $\mathbb{R}^d$, each of size t, if every colorful subfamily $\mathcal{G}$ satisfies $\phi_{\mathcal{G}}^{(\lceil d/2 \rceil)} \leq b$ and has nonempty intersection, then some $2m - d$ members of $\bigcup_{i=1}^m \mathcal{F}_i$ have nonempty intersection.*

We prove Theorem 4 using a technique developed in [14, 29] for studying intersection patterns via homological minors [30]. In short, a cellular chain complex C_1 is a minor of a cellular chain complex C_2 if there is a non-trivial chain map $C_1 \to C_2$ that sends disjoint faces to chains with disjoint supports. One novelty here is that, in order to account for the k-partite structure of the color classes, we work with minors in chain complexes built out of cubical cells rather than simplices. For this, we have to transfer some results from simplicial homology, like the homological Van Kampen-Flores Theorem (adapted in Proposition 8). We show that the stair convexity of Bukh et al. [6] offers a systematic way of building chain maps from simplicial complexes into grid-like complexes (Proposition 7). Along the way, we also establish a new Ramsey-type result (Lemma 9) which we use to construct grid-like minors inside the intersections of sets with bounded homological shatter function.

Further consequences. Let us say a word on some of the consequences of the fractional Helly theorem as spelled out by Alon et al. [2]. Our Corollary 2, via Theorems 8(i) and 9 and the discussion in §2.1 in [2], implies:[1]

▶ **Corollary 5.** *For every d, b and $p \geq d+1$, there exists $\tau = \tau(d, b, p)$ such that the following holds. Let $\mathcal{F}$ be a set system in $\mathbb{R}^d$ with $\phi_{\mathcal{F}}^{(\lceil d/2 \rceil)} \leq b$. If among any p members of $\mathcal{F}$, some $d+1$ intersect, then there exists a set of τ points of $\mathbb{R}^d$ that intersects every member of $\mathcal{F}$.*

Here, without our sharpening of Corollary 2 by Theorem 1, it would take any $p \geq m$ members of $\mathcal{F}$ to contain some m intersecting members, for some hypergraph Ramsey number m. Similarly, our Corollary 2 together with the Theorem 9 and the discussion in §2.1 from [2] imply that for any b and d, there are c_1 and c_2 such that any topological set system in $\mathbb{R}^d$ with $\lceil d/2 \rceil$th homological shatter function bounded from above by b admits a weak ε-net of size c_1/ε^{c_2}. Here, the effect of our sharpening is to reduce the constants c_1 and c_2.

1.3 Remarks and open questions

1. Do nerves of topological set systems with finite homological shatter functions have bounded Leray number? Indeed Theorem 1 reveals that these nerves enjoy some of the consequences of d-Lerayness. A first step in this direction was done by Holmsen, Kim and Lee [18], but the question remains open, already for a homological shatter function bounded by a *constant*.
2. In the proof of Theorem 1, the assumption that the sets are in $\mathbb{R}^d$ is used in exactly one place, namely in Section 4 when we invoke the homological version of the Van Kampen-Flores theorem [14, Corollary 14]. The proof therefore generalizes readily to, for instance, topological set systems in a manifold with some forbidden homological minor.

[1] More precisely, the statements follow from Corollary 2 applied to the family $\mathcal{F}^\cap \stackrel{\text{def}}{=} \{\cap_{S \in \mathcal{G}} S : \mathcal{G} \subset \mathcal{F}\}$. Note that $\phi_{\mathcal{F}^\cap}^{(\lceil d/2 \rceil)}$ is bounded from above by b if and only if $\phi_{\mathcal{F}}^{(\lceil d/2 \rceil)}$ is.

3. The idea behind stretched grids [6] suggest a general transfer principle between stair convex hulls and (limits of) simplices in $\mathbb{R}^d$. Perhaps this could offer a more conceptual approach to proving Proposition 7. Our efforts in that direction were not fruitful.

2 Background

We write $\mathbb{N} = \{1, 2, \ldots\}$ for the set of positive integers and $\mathbb{N}_0 = \{0, 1, \ldots\}$ for the set of non-negative integers. We write $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ and $\binom{[n]}{k}$ for the set of k-element subsets of $[n]$. All homology in this paper is computed with coefficients in $\mathbb{Z}_2$, and we write $C_\#\left(\mathbb{R}^d\right)$ for the singular chain complex of $\mathbb{R}^d$ with $\mathbb{Z}_2$ coefficients. Given a chain α in a chain complex, we write $\operatorname{supp}(\alpha)$ for its support. We use homology on a family of cubical complexes, in the sense, *e.g.*, of Kaczynski et al. [19], which we now define.

2.1 Grid complexes

Let $G[n]$ denote the 1-dimensional cell complex whose vertices (0-cells) are the singletons $\{1\}, \{2\}, \ldots, \{n\}$ and whose closed 1-cells are the closed intervals $[1, 2], [2, 3], \ldots, [n-1, n]$. For $m \geq 1$, define the *grid complex* $G[n]^m$ as the m-fold product $G[n]^m \stackrel{\text{def}}{=} \underbrace{G[n] \times \cdots \times G[n]}_{m\text{-fold}}$, equipped with the product topology.

Cells. For every integer $a \in [n]$ we use interchangeably the notations $[a, a] = \{a\}$ to denote the corresponding 0-cell in $G[n]$. For every integers $a, b \in [n]$ with $a < b$ we let $[a, b] = [b, a]$ denote the 1-chain with $\mathbb{Z}_2$ coefficients

$$\left.\begin{array}{l}[a,b]\\ [b,a]\end{array}\right\} \stackrel{\text{def}}{=} [a, a+1] + [a+1, a+2] + \cdots + [b-1, b].$$

For any *pairwise distinct* integers $a, b, c \in [n]$ we have $[a, c] = [a, b] + [b, c]$. Every (closed) k-cell σ in $G[n]^m$ can be written as the product of exactly $(m-k)$ 0-cells and k 1-cells

$$\sigma = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m],$$

where $1 \leq a_i \leq b_i \leq a_i + 1 \leq n$. A *$k$-chain* is a sum of k-cells in $G[n]^m$ with coefficients in $\mathbb{Z}_2$. We note that $G[n]^m$ is a regular cell complex of dimension m which can be realized geometrically as an m-dimensional axis-parallel cube in $\mathbb{R}^m$ with sidelength $n-1$. For $\ell \leq m$, the set of cells of dimension at most ℓ of $G[n]^m$ is a regular cell complex of dimension ℓ, called the *ℓ-dimensional skeleton* of $G[n]^m$ and denoted by $(G[n]^m)^{(\ell)}$. For X a subcomplex of a grid complex, we write $V(X)$ for the set of vertices of X.

Products and boundaries. The *product* $\times$ of a k_1-cell of $G[n]^{m_1}$ by a k_2-cell of $G[n]^{m_2}$ is a $(k_1 + k_2)$-cell of $G[n]^{m_1+m_2}$. We extend it to chains by putting

$$(\sigma_1 + \cdots + \sigma_{\ell_1}) \times (\tau_1 + \cdots + \tau_{\ell_2}) \stackrel{\text{def}}{=} \sum_{i=1}^{\ell_1} \sum_{j=1}^{\ell_2} \sigma_i \times \tau_j.$$

We denote the null chain (with empty support) by 0 and clarify that for any chain σ we have $\sigma \times 0 = 0 \times \sigma = 0$. We can now define the *boundary* of a cell of $G[n]^m$ recursively, as follows:

$$
\begin{array}{rrcll}
\textbf{(0-cells)} & \partial\{a\} & \stackrel{\text{def}}{=} & 0 & \text{(the trivial chain)} \\
\textbf{(1-cells)} & \partial[a,a+1] & \stackrel{\text{def}}{=} & \{a\}+\{a+1\} & \\
\textbf{($\geq$ 2-cells)} & \partial(\sigma\times\tau) & \stackrel{\text{def}}{=} & \partial\sigma\times\tau+\sigma\times\partial\tau &
\end{array}
$$

The definition of ∂ extends from k-cells to k-chains by linearity. For X a (skeleton of a) grid complex, we write $C_{\#}(X)$ for the chain complex defined by the chains of X together with ∂.

2.2 Stair convexity

The stair convex hull was introduced by Bukh, Matoušek, and Nivasch [6] as a tool for analyzing point configurations and extremal problems in discrete geometry such as lower bounds on the size of weak ε-nets. We now reformulate this notion in terms of chains of the grid complex $G[n]^m$; this resembles their recursive definition.

Stair convex chains. We fix some integer $n \geq 2$ and work, implicitly, in the grid complexes $G[n]^m$. For any $m \geq k \geq 0$ and any integers $1 \leq a_1 < \cdots < a_{k+1} \leq n$ we define a *stair convex* k-chain $\mathrm{sc}^m(a_1,\ldots,a_{k+1}) \in C_k(G[n]^m)$, which we also denote by $\mathrm{sc}^m_k(a_1,\ldots,a_{k+1})$ when we want to make its dimension explicit. The definition is recursive:

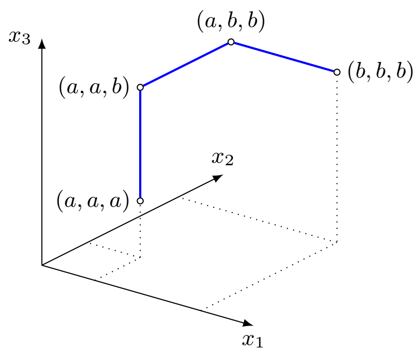
$$
\begin{array}{lrcl}
\mathbf{(k=0)} & \mathrm{sc}^m(a) & \stackrel{\text{def}}{=} & \overbrace{(a,\ldots,a)}^{m\text{-fold}} \\
\mathbf{(k>m)} & \mathrm{sc}^m(a_1,\ldots,a_{k+1}) & \stackrel{\text{def}}{=} & 0 \qquad \text{(the trivial chain)} \\
\mathbf{(0<k<m)} & \mathrm{sc}^m(a_1,\ldots,a_{k+1}) & \stackrel{\text{def}}{=} & \mathrm{sc}^{m-1}_{k-1}(a_1,\ldots,a_k)\times[a_k,a_{k+1}] \\
 & & & \quad +\ \mathrm{sc}^{m-1}_{k}(a_1,\ldots,a_{k+1})\times\{a_{k+1}\} \\
\mathbf{(0<k=m)} & \mathrm{sc}^m(a_1,\ldots,a_{m+1}) & \stackrel{\text{def}}{=} & [a_1,a_2]\times[a_2,a_3]\times\ldots\times[a_m,a_{m+1}]
\end{array}
$$

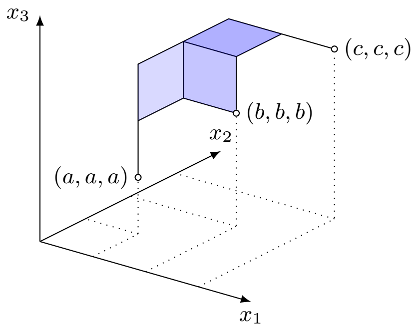
First examples. At one end, for $k=0$, $\mathrm{sc}^m(a)$ is a vertex on the diagonal of $G[n]^m$. At the other end, for $k=m$, $\mathrm{sc}^m(a_1,\ldots,a_{m+1})$ is a m-dimensional box. Let us examine some simple examples of what happens in-between. For $m=2$ and $k=1$ we have

$$\mathrm{sc}^2(a,b) = \mathrm{sc}^1(a)\times[a,b]+\mathrm{sc}^1(a,b)\times\{b\} = \{a\}\times[a,b]+[a,b]\times\{b\}$$

which is a rectilinear path from (a,a) to (b,b) with a bend at (a,b). Here are more examples:

$$
\begin{aligned}
&\mathrm{sc}^3(a,b)\\
&= \mathrm{sc}^2(a)\times[a,b]+\mathrm{sc}^2(a,b)\times\{b\}\\
&= (a,a)\times[a,b]+(\{a\}\times[a,b]+[a,b]\times\{b\})\times\{b\}\\
&= (a,a)\times[a,b]+\{a\}\times[a,b]\times\{b\}+[a,b]\times(b,b)
\end{aligned}
$$





$$\begin{aligned}
&\mathrm{sc}^3(a,b,c)\\
&= \mathrm{sc}^2(a,b) \times [b,c] + \mathrm{sc}^2(a,b,c) \times \{c\}\\
&= \{a\} \times [a,b] \times [b,c]\\
&\quad + [a,b] \times \{b\} \times [b,c]\\
&\qquad + [a,b] \times [b,c] \times \{c\},
\end{aligned}$$

A non-recursive definition. Let us extend the definition of stair convex hull to the case $m = k = 0$ by putting, for any integer a and any chain σ, $\mathrm{sc}_0^0(a) \times \sigma = \sigma \times \mathrm{sc}_0^0(a) = \sigma$. With this convention, we can "unwrap" the recursive definition of sc_k^m:

$$\mathrm{sc}_k^m(a_1,\ldots,a_{k+1}) = \sum_{\substack{t_1,t_2,\ldots,t_{k+1}\in\mathbb{N}_0\\ t_1+t_2+\ldots+t_{k+1}=m-k}} \mathrm{sc}_0^{t_1}(a_1) \times \prod_{i=1}^{k} \Big([a_i,a_{i+1}] \times \mathrm{sc}_0^{t_{i+1}}(a_{i+1})\Big) \tag{3}$$

So, for instance, $\mathrm{sc}^3(a,b) = \{a\}^2 \times [a,b] + \{a\} \times [a,b] \times \{b\} + [a,b] \times \{b\}^2$. From (3) we get:

▶ **Lemma 6.** *If an axis-aligned hyperplane $x_j = a$ contains a k-dimensional face of the support of $\mathrm{sc}_k^m(a_1,\ldots,a_{k+1})$, then $a \in \{a_1,\ldots,a_{k+1}\}$.*

Proof. The k-dimensional faces of the support of $\mathrm{sc}_k^m(a_1,\ldots,a_{k+1})$ are the summands of the right hand term of (3). A summand is contained in $x_j = a$ only if for some i we have $t_i \neq 0$ and $a_i = a$. ◀

3 Stair convex hulls and boundary operator

The key property of stair convex hull for our purpose is that under some conditions, it behaves like k-dimensional simplices with respect to the boundary operator on grid complexes. Figure 1 illustrates this phenomenon in 2 dimensions. To formalize this claim, let us write $(a_1,\ldots,\widehat{a_i},\ldots,a_{k+1}) \stackrel{\text{def}}{=} (a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{k+1})$, that is, $\hat{}$ denotes coordinates to be omitted. (Recall that all homology in this paper has coefficients in $\mathbb{Z}_2$.)

▶ **Proposition 7.** *For integers $m \geq k \geq 1$ and any sequence $a_1 < a_2 < \ldots < a_m$ of elements from $[n]$ we have $\partial\,\mathrm{sc}_k^m(a_1,\ldots,a_{k+1}) = \sum_{i=1}^{k+1} \mathrm{sc}_{k-1}^m(a_1,\ldots,\widehat{a_i},\ldots,a_{k+1})$.*

Sketch of proof. Our proof is a (not so short) calculation. We set up an induction on m by using the recursive definition of sc_k^m (for $k < m$) or by applying the product rule after singling out the factor $[a_m,a_{m+1}]$ (for $k = m$). One important idea is to handle the factors $[a_{i-1},a_{i+1}]$ arising from $\mathrm{sc}_{k-1}^m(a_1,\ldots,\widehat{a_i},\ldots,a_{k+1})$ using the identity $[a_{i-1},a_{i+1}] = [a_{i-1},a_i] + [a_i,a_{i+1}]$ between 1-chains. See the full version for the details. ◀

4 A homological van Kampen-Flores theorem for grid complexes

We now use Proposition 7 to prove a non-embeddability result, in homological terms, that is well-suited for grid complexes. Following [30], we call chain map *non-trivial* if the image of every vertex is a 0-chain supported on an odd number of vertices.

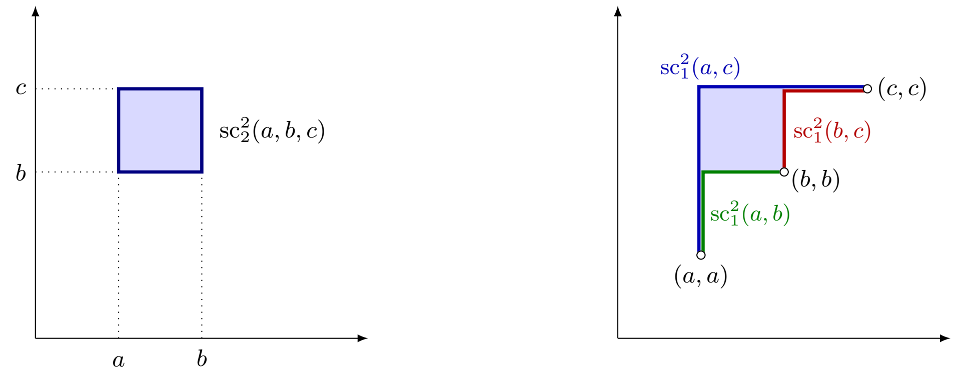


Figure 1 Left: An illustration of the 2-chain $\mathrm{sc}_2^2(a,b,c)$ with highlighted boundary. Right: An illustration of the boundary of $\mathrm{sc}_2^2(a,b,c)$ decomposed into the sum of $\mathrm{sc}_1^2(a,b), \mathrm{sc}_1^2(b,c)$ and $\mathrm{sc}_1^2(a,c)$, respectively. Note that both $\{a\} \times [a,b]$ and $[b,c] \times \{c\}$ cancel out since we work with $\mathbb{Z}_2$ coefficients.

▶ **Proposition 8.** *Let $m > d \geq 1$ be integers and let X be the $\lceil d/2 \rceil$-skeleton of the grid complex $G[d+3]^m$. For every nontrivial chain map $f_\# : C_\#(X) \to C_\#(\mathbb{R}^d)$, there exist cells σ and τ in X such that (i) $\dim \sigma + \dim \tau \leq d$, (ii) σ and τ are not contained in a common axis-parallel hyperplane, and (iii) the supports of $f_\#(\sigma)$ and $f_\#(\tau)$ intersect.*

Proof. For $p \geq 0$ even, let M_p be the $p/2$-skeleton of the $(p+2)$-dimensional simplex. For $p \geq 1$ odd, let M_p be the cone over M_{p-1}. The simplicial complex M_d satisfies:

- The dimensions of any two disjoint faces in M_d sum to at most d. Indeed, for d odd, every face of dimension $\lceil d/2 \rceil$ contains the coning vertex.
- For any non-trivial chain map $C_\#(M_d) \to C_\#(\mathbb{R}^d)$, there exist two disjoint simplices of M_d whose images have non-disjoint supports. In other words, M_d enjoys a homological van Kampen-Flores theorem. For d even, it follows from a standard proof of the van Kampen–Flores theorem through the *Van Kampen obstruction* [14, § 2]. The case where d is odd can be reduced to the even case $d-1$ using properties of this obstruction with respect to coning, see the proof of [14, Corollary 14].

Let us label the vertices of M_d by $v_1, v_2, \ldots, v_{d+3}$. For every simplex $\{v_{i_1}, \ldots, v_{i_{k+1}}\}$ in M_d with $i_1 < i_2 < \ldots < i_{k+1}$, we let $g(\{v_{i_1}, \ldots, v_{i_{k+1}}\}) \stackrel{\text{def}}{=} \mathrm{sc}^m(i_1, \ldots, i_{k+1})$. We extend g linearly into a map $g_\# : C_\#(M_d) \to C_\#(X)$ and note that $g_\#$ is a chain map, as Proposition 7 ensures that for any simplex $\sigma \in C_\#(M_d)$ we have $\partial(g_\#(\sigma)) = g_\#(\partial\sigma)$. Now, let us consider the chain map $f_\# \circ g_\# : M_d \to \mathbb{R}^d$. It is non-trivial, so by the homological van Kampen–Flores theorem there exist two disjoint simplices σ_M and τ_M of M_d whose images under $f_\# \circ g_\#$ have non-disjoint supports. So, there exists a cell σ in the support of $g(\sigma_M)$ with $\dim \sigma \leq \dim \sigma_M$ and a cell τ in the support of $g(\tau_M)$ with $\dim \tau \leq \dim \tau_M$ such that $f_\#(\sigma)$ and $f_\#(\tau)$ have non-disjoint support. Note that σ and τ satisfy condition (iii). The dimensions of σ and τ, like the dimensions of σ_M and τ_M, sum to at most d, so condition (i) is also satisfied. Finally, since σ_M an τ_M are disjoint, Lemma 6 ensures that σ and τ are not contained in a common axis-aligned hyperplane, and condition (ii) is satisfied as well. ◀

Note that a k-dimensional cell of $G[n]^m$ has $m-k$ coordinates that are constant. So, for any two simplices σ_1, σ_2 of X such that $\dim \sigma_1 + \dim \sigma_2 < m$, there is at least one coordinate that is constant for both. In particular, if two such simplices intersect they must be contained in a common axis-parallel hyperplane. So Conditions (i) and (ii) imply that σ and τ are vertex-disjoint, as in the usual van Kampen–Flores theorem. More generally, the chain map $g_\#$ maps disjoint simplices to chains with disjoint support: in the general sense of [30], it is a homological almost embedding that shows that M_d is a homological minor of X.

We have no reason to believe that the complex $X = (G[d+3]^m)^{(\lceil d/2 \rceil)}$ is a minimal one for which the conclusion of Proposition 8 holds. For instance, it is easy to see that the conclusion holds for chain maps $C_\#((G[2]^m)^{(1)}) \to C_\#(\mathbb{R}^1)$. Furthermore, we could show that the conclusion also holds for chain maps $C_\#((G[3]^3)^{(1)}) \to C_\#(\mathbb{R}^2)$ by computation of the van Kampen obstruction (but were unable to extend this to a general approach).

5 Filling holes: A Ramsey-type result for grid complexes

We now prove the last ingredient of our colorful Helly type Theorem 4: a Ramsey-type result.

5.1 Subgrids and the subgrid lemma

The structure that our Ramsey-type result identifies is a subgrid of $G[n]^m$. Formally, a *subgrid* of $G[n]^m$ is a map $\gamma : V(G[\ell]^m) \to V(G[n]^m)$, for some $1 \le \ell \le n$, given by $(x_1, \dots, x_m) \longmapsto (\gamma_1(x_1), \dots, \gamma_m(x_m))$, where each $\gamma_i : [\ell] \to [n]$ is a strictly increasing function. We write the fact that γ is a subgrid by $\gamma : G[\ell]^m \hookrightarrow G[n]^m$. For any $a, b \in [n]$ we let $\gamma_i(\{a\}) \stackrel{\text{def}}{=} \{\gamma_i(a)\}$ and $\gamma_i([a,b]) \stackrel{\text{def}}{=} [\gamma_i(a), \gamma_i(b)]$, and let

$$\gamma_\# : \begin{cases} C_\#(G[\ell]^m) & \to & C_\#(G[n]^m) \\ \sigma_1 \times \cdots \times \sigma_m & \mapsto & \gamma_1(\sigma_1) \times \cdots \times \gamma_m(\sigma_m). \end{cases}$$

As we spell out in the full version, $\gamma_\#$ is a chain map. Now, fix an integer $k \ge 1$ and consider a group homomorphism h from the group $(C_k(G[n]^m), +)$ of k-chains into $(\mathbb{Z}_2)^b$, where $b \in \mathbb{N}$. We say that a subgrid $\gamma : G[a]^m \hookrightarrow G[n]^m$ *lies in the kernel of* h if $h(\gamma_\#(c)) = 0$ for every $c \in C_k(G[a]^m)$. Here is our Ramsey-type statement:

▶ **Lemma 9** (subgrid lemma). *For any $b, k, m, \ell \in \mathbb{N}$, $\ell \ge 2$, there exists $N = N(b, k, m, \ell)$ such that for any group homomorphism $h : C_k(G[N]^m) \to (\mathbb{Z}_2)^b$, there exists a subgrid $\gamma : G[\ell]^m \hookrightarrow G[N]^m$ in the kernel of h.*

5.2 Proof of the subgrid lemma

The rest of this section is devoted to the proof of Lemma 9. It may help the reader to first read the rest of this section once with the sole case $k = m$ in mind.

Coloring vertices. We first establish a simple Ramsey-type property of subgrids:

▶ **Lemma 10.** *For any m, ℓ and q there exists $N = N(m, \ell, q)$ such that for every q-coloring of $V(G[N]^m)$, there exists a monochromatic subgrid $G[\ell]^m \hookrightarrow G[N]^m$.*

This follows for instance from the Gallai-Witt theorem [15, p. 40], but it is in fact much simpler as it dispenses of the "homothetic" constraint. See the full version for a direct proof.

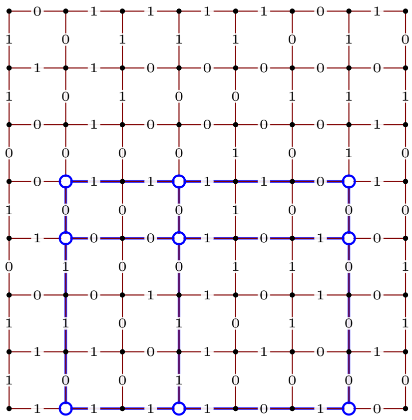


Figure 2 A homomorphism $h : C_1(G[8]^2) \to \mathbb{Z}_2$. The blue subgrid $\gamma : G[3]^2 \hookrightarrow G[8]^2$ lies in the kernel of h.

Boxes. Let $\mathbf{1} \stackrel{\text{def}}{=} (1, 1, \ldots, 1) \in [N]^m$. Given two grid points $x, y \in [N]^m$ we write $x \preceq y$ if $x_i \le y_i$ for $1 \le i \le m$. We also put $\operatorname{diff}(x, y) \stackrel{\text{def}}{=} \{i \in [m] : x_i \neq y_i\}$. For any two vertices $x \preceq y$ in $G[N]^m$ we put

$$\operatorname{box}_k(x, y) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } |\operatorname{diff}(x, y)| \neq k, \\ [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_m, y_m] & \text{otherwise.} \end{cases}$$

That is, $\operatorname{box}_k(x, y)$ is a k-chain. It is non-trivial if and only if x and y disagree on exactly k coordinates. In that case, it is a k-dimensional box, contained in the intersection of all coordinate hyperplanes where x and y agree.

Shuffles. For any $I \subset [m]$ and $x, y \in [N]^m$ we define the I-*shuffle* of x and y as

$$\lfloor x, y \rceil_I \stackrel{\text{def}}{=} (z_1, z_2, \ldots, z_m) \quad \text{where} \quad z_i = \begin{cases} x_i & \text{if } i \in I, \text{ and} \\ y_i & \text{otherwise.} \end{cases}$$

Notice that if $\operatorname{box}_k(x, y)$ is a non-trivial k-chain, then its support is the k-dimensional box with corners $\{\lfloor x, y \rceil_I\}_{I \subset \operatorname{diff}(x,y)}$. For $z \in [\ell]^m$, we let $G_k(z)$ denote the set of vertices of $G([\ell]^m)$ that can be reached from z by incrementing exactly k coordinates, that is

$$G_k(z) \stackrel{\text{def}}{=} \{w \in [\ell]^m \colon z \preceq w, \operatorname{diff}(z, w) = k, \forall i \in [m]\ z_i \le w_i \le z_i + 1\}.$$

Notice that the set $\{\operatorname{box}_k(z, w)\}_{z \in [\ell]^m, w \in G_k(z)}$ generates the vector space $C_k(G[\ell]^m)$. It follows that a subgrid $\gamma : G[\ell]^m \hookrightarrow G[N]^m$ lies in the kernel of h if

$$\forall z \in [\ell]^m, \forall w \in G_k(z), \quad h\left(\operatorname{box}_k(\gamma(z), \gamma(w))\right) = 0. \tag{4}$$

Inclusion-exclusion. We next define for any $x, y \in [N]^m$ a *base point*, also in $[N]^m$:

$$\operatorname{base}(x, y) \stackrel{\text{def}}{=} (z_1, z_2, \ldots, z_m) \quad \text{where} \quad z_i = \begin{cases} 1 & \text{if } i \in \operatorname{diff}(x, y), \\ x_i = y_i & \text{otherwise.} \end{cases}$$

Now, for any $x \preceq y$ in $[N]^m$ with $|\operatorname{diff}(x, y)| = k$, the inclusion-exclusion principle yields

$$\operatorname{box}_k(x, y) = \sum_{I \subseteq \operatorname{diff}(x,y)} (-1)^{|I|} \operatorname{box}_k\left(\operatorname{base}(x, y), \lfloor x, y \rceil_I\right). \tag{5}$$

Indeed, for $k = m$, this merely writes a full-dimensional, axis-parallel box B as the alternating sum of the boxes spanned by $\mathbf{1}$ and each of corners of B. The argument is the same for $k < m$ by simply dropping all coordinates where x and y agree. The factors $(-1)^{|I|}$, which would be necessary if the chains had coefficients in $\mathbb{Z}$, may be surprising over $\mathbb{Z}_2$. Their interest comes from considering the above identity through the homomorphism h:

$$h\left(\mathrm{box}_k(x,y)\right) = \sum_{I \subseteq \mathrm{diff}(x,y)} (-1)^{|I|} h\left(\mathrm{box}_k\left(\mathrm{base}(x,y), \lfloor x,y \rceil_I\right)\right). \tag{6}$$

Coloring. Let us associate to the group homomorphism $h : C_k(G[N]^m) \to (\mathbb{Z}_2)^b$ the coloring

$$\chi_h : \begin{cases} V(G[N]^m) & \to \quad (\mathbb{Z}_2)^{b\binom{m}{k}} \\ \qquad z & \mapsto \quad \left(h\left(\mathrm{box}_k\left(\lfloor \mathbf{1}, z \rceil_F, z\right)\right)\right)_{F \in \binom{[m]}{k}} \end{cases}$$

In plain english, $\chi_h(z)$ is a vector of $\binom{m}{k}$ elements of $(\mathbb{Z}_2)^b$, one per k-element subset $F \subseteq [m]$ (the order in which these subsets are considered is irrelevant). The element of $(\mathbb{Z}_2)^b$ associated to subset F is obtained by considering the k-dimensional axis parallel subspace through x where coordinates with index in F are fixed: $(\chi_h(z))_F$ is the image under h of the k-dimensional box spanned by the base point $\lfloor \mathbf{1}, z \rceil_F$ and z.

Wrapping up. Let $N_0 \stackrel{\text{def}}{=} N\left(m, \ell, 2^{b\binom{m}{k+1}}\right)$ for the function $N(\cdot,\cdot,\cdot)$ from Lemma 10. So, if $N \geq N_0$, then there exists a subgrid $\gamma : G[\ell]^m \hookrightarrow G[N]^m$ that is monochromatic for χ_h.

To argue that γ lies in the kernel of h, we use Condition (4): we consider some arbitrary $z \in [\ell]^m$ and $w \in G_k(z)$, let $x \stackrel{\text{def}}{=} \gamma(z)$ and $y \stackrel{\text{def}}{=} \gamma(w)$, and set out to prove that $h(\mathrm{box}_k(x,y)) = 0$. Note that the fact that γ is a subgrid and $w \in G_k(z)$ ensures that $|\,\mathrm{diff}(x,y)| = k$. For $I \subseteq \mathrm{diff}(x,y)$ let us write $c_I \stackrel{\text{def}}{=} \lfloor x,y \rceil_I$. Each coordinate of c_I comes from either $x = \gamma(z)$ or $y = \gamma(w)$, so c_I is a vertex of the subgrid γ. Moreover, for every $i \notin \mathrm{diff}(x,y)$ we have $x_i = y_i = (c_I)_i$ so $\mathrm{base}(x,y) = \lfloor \mathbf{1}, x \rceil_{\mathrm{diff}(x,y)} = \lfloor \mathbf{1}, y \rceil_{\mathrm{diff}(x,y)} = \lfloor \mathbf{1}, c_I \rceil_{\mathrm{diff}(x,y)}$. Equation (6) then rewrites as

$$h\left(\mathrm{box}_k(x,y)\right) = \sum_{I \subseteq \mathrm{diff}(x,y)} (-1)^{|I|} h\left(\mathrm{box}_k\left(\lfloor \mathbf{1}, c_I \rceil_{\mathrm{diff}(x,y)}, c_I\right)\right). \tag{7}$$

Notice that the value $h\left(\mathrm{box}_k\left(\lfloor \mathbf{1}, c_I \rceil_{\mathrm{diff}(x,y)}, c_I\right)\right)$ is independent of $I \subseteq \mathrm{diff}(x,y)$. This follows from the facts that (i) $|\,\mathrm{diff}(x,y)| = k$, (ii) c_I is a vertex of the subgrid γ, and (iii) γ is monochromatic for χ_h. Equation (7) therefore rewrites as

$$h\left(\mathrm{box}_k(x,y)\right) = h\left(\mathrm{box}_k\left(\lfloor \mathbf{1}, c_\emptyset \rceil_{\mathrm{diff}(x,y)}, c_\emptyset\right)\right) \left(\sum_{I \subseteq \mathrm{diff}(x,y)} (-1)^{|I|} \right) = 0.$$

This concludes the proof of the subgrid lemma.

6 A weak colorful Helly theorem

We now set out to prove Theorem 4. Recall that we are given arbitrary integers $b \geq 0$ and $m > d \geq 1$. Our task is to prove that there exists an integer $t = t(b,d,m)$ such that for any topological set systems $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ in $\mathbb{R}^d$, each of size t, if every colorful subfamily $\mathcal{G}$ satisfies $\phi_{\mathcal{G}}^{(\lceil d/2 \rceil)} \leq b$ and the members of $\mathcal{G}$ have nonempty intersection, then some $2m - d$ members of $\bigcup_{i=1}^m \mathcal{F}_i$ have nonempty intersection. Before we state the main technical step of our proof, Lemma 14, we need some definitions.

6.1 Setup

We define constants $t_0 > t_1 > \cdots > t_{\lceil d/2\rceil}$ starting with $t_{\lceil d/2\rceil} \stackrel{\text{def}}{=} d+3$ and, having defined t_{i+1}, setting $t_i \stackrel{\text{def}}{=} N(b,i,m,t_{i+1})$ where $N(\cdot,\cdot,\cdot,\cdot)$ is the function from the subgrid lemma (Lemma 9). We prove Theorem 4 for $t(b,d,m) \stackrel{\text{def}}{=} t_0$. We let $X_i \stackrel{\text{def}}{=} G[t_i]^m$ and note that the definition of the t_i's ensures:

▷ Claim 11. For any homomorphism $h : C_i(X_i) \to (\mathbb{Z}_2)^b$, there exists a subgrid $\gamma : X_{i+1} \hookrightarrow X_i$ in the kernel of h.

We label the members of each $\mathcal{F}_i$ arbitrarily as $\mathcal{F}_i = \{S_{(1,i)}, \dots, S_{(t_0,i)}\}$. For every vertex $v = (v_1, v_2, \dots, v_m)$ of X_0 we set $\mathcal{G}(v) \stackrel{\text{def}}{=} \{S_{(v_1,1)}, \dots, S_{(v_m,m)}\}$.

▷ Claim 12. $v \mapsto \mathcal{G}(v)$ is a bijection between the vertices of X_0 and the maximal colorful subfamilies of $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$.

Let A be an axis-parallel k-dimensional affine subspace A, or *axis parallel k-flat* for short. We put $\mathcal{G}(A) \stackrel{\text{def}}{=} \bigcap_{v\in V(X_0)\cap A} \mathcal{G}(v)$.

▷ Claim 13. The map $A \mapsto \mathcal{G}(A)$ induces a bijection between the axis-parallel k-flats that intersect $V(X_0)$ and the colorful subfamilies of size $m-k$.

Last, we further associate to any chain $\alpha \in C_k(X_i)$ a colorful family $\mathcal{G}(\alpha)$ set to be $\mathcal{G}(A)$, where A is the smallest axis-parallel flat of $\mathbb{R}^m$ that contains the support of α (that is, its affine span). Note that if σ is a k-face of X_i, then $|\mathcal{G}(\alpha)| = m-k$.

6.2 A constrained chain map

The main technical step in the proof of Theorem 4 is the following construction.

▶ **Lemma 14.** *Under the conditions of Theorem 4, there exists a subgrid $\gamma : X_{\lceil d/2\rceil} \hookrightarrow X_0$ and a nontrivial chain map $f_\# : C_\#\left((X_{\lceil d/2\rceil})^{(\lceil d/2\rceil)}\right) \to C_\#(\mathbb{R}^d)$ with the property that* $\operatorname{supp} f_\#(\sigma) \subset \bigcap \mathcal{G}\left(\gamma_\#(\sigma)\right)$ *for any cell $\sigma \in (X_{\lceil d/2\rceil})^{(\lceil d/2\rceil)}$,*

Before we describe the construction of γ and $f_\#$, let us see how our weak colorful Helly theorem follows from Lemma 14.

Proof of Theorem 4. Let γ and $f_\#$ be as given by Lemma 14. Since $X_{\lceil d/2\rceil} = G[d+3]^m$, we can apply Proposition 8 to find cells σ and τ in $(X_{\lceil d/2\rceil})^{(\lceil d/2\rceil)}$ such that:

1. $\dim\sigma + \dim\tau \le d$, from Proposition 8,
2. σ and τ are not contained in any axis parallel hyperplane, from Proposition 8,
3. the supports of $f_\#(\sigma)$ and $f_\#(\tau)$ intersect, again from Proposition 8, and
4. $\operatorname{supp} f_\#(\sigma) \subset \bigcap \mathcal{G}\left(\gamma_\#(\sigma)\right)$ and $\operatorname{supp} f_\#(\tau) \subset \bigcap \mathcal{G}\left(\gamma_\#(\tau)\right)$, from Lemma 14.

From (3) and (4) it comes that there is a point contained in every member of $\mathcal{G}\left(\gamma_\#(\sigma)\right) \cup \mathcal{G}\left(\gamma_\#(\tau)\right)$. From (2) and the definition of subgrids, it comes that the span of $\gamma_\#(\sigma)$ and the span of $\gamma_\#(\tau)$ are not contained in a common hyperplane. This in turn implies that $\mathcal{G}\left(\gamma_\#(\sigma)\right)$ and $\mathcal{G}\left(\gamma_\#(\tau)\right)$ are disjoint. Finally, we have

$$|\mathcal{G}\left(\gamma_\#(\sigma)\right) \cup \mathcal{G}\left(\gamma_\#(\tau)\right)| = |\mathcal{G}\left(\gamma_\#(\sigma)\right)| + |\mathcal{G}\left(\gamma_\#(\tau)\right)| = (m - \dim\sigma) + (m - \dim\tau)$$

which is at least $2m-d$ by (1). ◀

6.3 Proof of Lemma 14

It remains to construct the announced subgrid γ and constrained chain map $f_\#$.

Proof of Lemma 14. We construct the subgrid $\gamma : X_{\lceil d/2 \rceil} \hookrightarrow X_0$ and the chain map $f_\#$ inductively using the subgrid lemma. For each $i = 0, 1, \ldots, \lceil d/2 \rceil$ we claim there exists a subgrid $\gamma^{(i)} : X_i \hookrightarrow X_0$ and a nontrivial chain map $f_\#^{(i)} : C_\# \left((X_i)^{(i)}\right) \to C_\#(\mathbb{R}^d)$ such that

$$\forall \sigma \in (X_i)^{(i)}, \quad \operatorname{supp} f_\#^{(i)}(\sigma) \subset \cap \mathcal{G}\left(\gamma_\#^{(i)}(\sigma)\right). \tag{8}$$

Setting $\gamma = \gamma^{(\lceil d/2 \rceil)}$ and $f_\# = f_\#^{\lceil d/2 \rceil}$ will then complete the proof.

For $i = 0$, we let $\gamma^{(0)}$ be the trivial inclusion $X_0 \hookrightarrow X_0$. For each vertex $v \in X_0$ we fix a point p_v in the intersection $\cap \mathcal{G}(v)$ of the maximal colorful family $\mathcal{G}(v)$, which is nonempty by hypothesis. We define the chain map $f_\#^{(0)}$ by setting $f_\#^{(0)}(v) = p_v$ for every vertex v of X_0.

Before proceeding to the inductive step, for each colorful subfamily $\mathcal{G}$ of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ we fix a basis (arbitrarily) for $\widetilde{H}_i(\cap \mathcal{G})$, $0 \le i \le \lceil d/2 \rceil$. These bases remains fixed for remainder of the proof. The hypothesis $\phi_{\mathcal{G}}^{(\lceil d/2 \rceil)} \le b$ allows to consider each homology group $\widetilde{H}_i(\cap \mathcal{G})$ as a subgroup of $(\mathbb{Z}_2)^b$.

Let $0 \le i < \lceil d/2 \rceil$ and suppose we are given the subgrid $\gamma^{(i)} : X_i \hookrightarrow X_0$ and the chain map $f_\#^{(i)} : C_\# \left((X_i)^{(i)}\right) \to C_\#(\mathbb{R}^d)$ satisfying Condition (8). Let σ be an $(i+1)$-cell in X_i. The chain $\gamma_{\# i+1}^{(i)}(\sigma)$ is well-defined and has the same affine span as the chain $\gamma_{\# i}^{(i)}(\partial\sigma)$, so

$$\operatorname{supp} f_{\# i}^{(i)}(\partial\sigma) \subset \bigcap \mathcal{G}\left(\gamma_{\# i}^{(i)}(\partial\sigma)\right) = \bigcap \mathcal{G}\left(\gamma_{\# i+1}^{(i)}(\sigma)\right).$$

Define a homomorphism $h : C_{i+1}(X_i) \to (\mathbb{Z}_2)^b$ by setting

$$h(\sigma) \stackrel{\text{def}}{=} \left[f_{\# i}^{(i)}(\partial\sigma)\right] \in \widetilde{H}_i\left(\bigcap \mathcal{G}\left(\gamma_{\# i+1}^{(i)}(\sigma)\right)\right).$$

In other words, $h(\sigma)$ equals the homology class of the image $f_{\# i}^{(i)}(\partial\sigma)$ in the i-dimensional (reduced) homology group of $\cap \mathcal{G}\left(\gamma_{\# i+1}^{(i)}(\sigma)\right)$, which we can view as an element in $(\mathbb{Z}_2)^b$. By Claim 11, there exists a subgrid $\varphi : X_{i+1} \hookrightarrow X_i$ in the kernel of h. We set $\gamma^{(i+1)} \stackrel{\text{def}}{=} \gamma^{(i)} \circ \varphi$ and note that $\gamma^{(i+1)}$ is indeed a subgrid $X_{i+1} \hookrightarrow X_0$. Moreover, for every $(i+1)$-cell $\tau \in X_{i+1}$ we have $h(\varphi(\tau)) = 0$, that is $\left[f_{\# i}^{(i)}(\partial\varphi(\tau))\right] = 0 \in \widetilde{H}_i\left(\bigcap \mathcal{G}\left(\gamma_{\# i+1}^{(i)}(\varphi(\tau))\right)\right)$, which rewrites

$$\left[f_{\# i}^{(i)}(\varphi(\partial\tau))\right] = 0 \in \widetilde{H}_i\left(\bigcap \mathcal{G}\left(\gamma_{\# i+1}^{(i+1)}(\tau)\right)\right). \tag{9}$$

For a cell $\sigma \in X_{i+1}$ of dimension at most i, we set $f_\#^{(i+1)}(\sigma) \stackrel{\text{def}}{=} f_\#^{(i)}\left(\varphi_\#(\sigma)\right)$. For any $(i+1)$-cell $\tau \in X_{i+1}$, Equation (9) reveals that $f_\#^{(i+1)}(\partial\tau)$ is a boundary in $C_i\left(\bigcap \mathcal{G}\left(\gamma_{\# i+1}^{(i+1)}(\tau)\right)\right)$. We pick some arbitrary $\alpha \in C_{i+1}\left(\bigcap \mathcal{G}\left(\gamma_{\# i+1}^{(i+1)}(\tau)\right)\right)$ such that $\partial\alpha = f_\#^{(i+1)}(\partial\tau)$, and set $f_{\# i+1}^{(i+1)}(\tau) \stackrel{\text{def}}{=} \alpha$. Thus defined, $f_\#^{(i+1)}$ is indeed a chain map, from $C_\#\left((X_{i+1})^{(i+1)}\right)$ to $C_\#\left(\mathbb{R}^d\right)$, and it satisfies Condition (8). ◀

7 A stepping-up lemma for topological set systems

We can finally prove Theorem 1. Recall that we are given integers $d+1 \le k \le \ell$ and $b \ge 0$. Our task is to show that for any $\delta > 0$, there exists $\delta' > 0$ such that for any sufficiently large topological set system $\mathcal{F}$ in $\mathbb{R}^d$, if $\phi_{\mathcal{F}}^{(\lceil \frac{d}{2} \rceil)}(\ell) \le b$ and $\delta_{\mathcal{F}}(k) \ge \delta$, then $\delta_{\mathcal{F}}(\ell) \ge \delta'$.

Preparation. An m-uniform hypergraph is a pair $H = (V, E)$ where $V = V(H)$ is a finite set of vertices and $E = E(H) \subset \binom{V}{m}$ is the edge set. A hypergraph H *contains* a hypergraph H' if there is an injection $f : V(H') \to V(H)$ such that for every $e' \in E(H')$, $f(e) \in E(H)$. (In particular, we do *not* require that H' is an *induced* sub-hypergraph of H.) An m-uniform hypergraph is m-partite if the vertex set can be partitioned into disjoint sets (vertex classes) $V(H) = V_1 \cup \cdots \cup V_m$ such that every edge contains exactly one vertex from each V_i. Given integers $m \ge 2$ and $t \ge 1$, we let $K^m(t)$ denote the complete m-partite m-uniform hypergraph on vertex classes $V_1, \ldots, V_m$ where $|V_i| = t$. That is, the edge set of $K^m(t)$ consists of all m-tuples of $V_1 \cup \cdots \cup V_m$ that contain exactly one element from each V_i. We use the following "supersaturation" theorem of Erdős and Simonovits:

▶ **Theorem** ([13, Corollary 2]). *For any positive integers m and t and any $\varepsilon > 0$ there exists $\rho = \rho(\varepsilon, m, t) > 0$ such that any m-uniform hypergraph $H = (V, E)$ with $|E| \ge \varepsilon\binom{|V|}{m}$ contains at least $\rho|V|^{mt}$ copies of $K^m(t)$.*

Proof of Theorem 1. The general case follows from the special case where $\ell = k+1$ by stepping-up one dimension at a time.[2] Consider some topological set system $\mathcal{F}$ in $\mathbb{R}^d$. Let $t = t(b, d, k)$ be the constant from Theorem 4 where $b \overset{\text{def}}{=} \phi_{\mathcal{F}}^{(\lceil \frac{d}{2} \rceil)}(\ell)$ and the number m of colors is now k. For $\mathcal{F}' \subseteq \mathcal{F}$, let $H[\mathcal{F}']$ be the k-uniform hypergraph whose vertices are the members of $\mathcal{F}'$ and whose edges are the k-tuples of $\mathcal{F}'$ with nonempty intersection.

Now, our hypergraph $H[\mathcal{F}]$ contains at least $\delta\binom{|\mathcal{F}|}{k}$ edges. By the Erdős–Simonovits theorem it follows that for some constant $\rho > 0$ depending only on k, t, and δ, there are at least $\rho\binom{|\mathcal{F}|}{kt}$ distinct kt-element subfamilies $\mathcal{F}'$ of $\mathcal{F}$ such that $H[\mathcal{F}']$ contains a copy of $K^m(t)$. Our choice of t ensures that Theorem 4 applies to every such subfamily $\mathcal{F}'$, and therefore each $\mathcal{F}'$ contributes some $2k - d \ge k+1$ members with non-empty intersection. Each $(k+1)$-element subset of $\mathcal{F}$ with non-empty intersection is contained in $\binom{|\mathcal{F}|-(k+1)}{kt-(k+1)}$ distinct (kt)-tuples $\mathcal{F}'$. There are therefore at least

$$\frac{\rho\binom{|\mathcal{F}|}{kt}}{\binom{|\mathcal{F}|-(k+1)}{kt-(k+1)}} = \frac{\rho}{\binom{kt}{k+1}}\binom{|\mathcal{F}|}{k+1}$$

$(k+1)$-tuples of $\mathcal{F}$ with nonempty intersection. In other words, $\delta_{\mathcal{F}}(k+1)$ is at least $\delta' \overset{\text{def}}{=} \rho/\binom{kt}{k+1}$, where ρ depends only on k, t, and δ, that is on k, b, d and δ. ◀

[2] The careful reader may note that Theorem 4 allows to step up more than one dimension at a time, therefore weakening the assumption $\phi_{\mathcal{F}}^{(\lceil \frac{d}{2} \rceil)}(\ell) \le b$ to $\phi_{\mathcal{F}}^{(\lceil \frac{d}{2} \rceil)}(\ell') \le b$ with $\ell' \overset{\text{def}}{=} \max\left(\lceil \frac{d+\ell}{2} \rceil, k\right)$.

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