Stabbing Convex Bodies with Lines and Flats

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Abstract

We study the problem of constructing weak ε-nets where the stabbing elements are lines or k-flats instead of points. We study this problem in the simplest setting where it is still interesting – namely, the uniform measure of volume over the hypercube $[0,1]^d$. Specifically, a (k,ε)-net is a set of k-flats, such that any convex body in $[0,1]^d$ of volume larger than ε is stabbed by one of these k-flats. We show that for $k \geq 1$, one can construct (k,ε)-nets of size $O(1/\varepsilon^{1-k/d})$. We also prove that any such net must have size at least $\Omega(1/\varepsilon^{1-k/d})$. As a concrete example, in three dimensions all ε-heavy bodies in $[0,1]^3$ can be stabbed by $\Theta(1/\varepsilon^{2/3})$ lines. Note, that these bounds are *sublinear* in $1/\varepsilon$, and are thus somewhat surprising.

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1 Introduction

Range spaces and ε-nets. A **range space** is a pair $\mathsf{X} = (\mathcal{U}, \mathcal{R})$, where $\mathcal{U}$ is the **ground set** (finite or infinite) and $\mathcal{R}$ is a (finite or infinite) family of subsets of $\mathcal{U}$. The elements of $\mathcal{R}$ are **ranges**.

Suppose that $\mathcal{U}$ is a finite set. For a parameter $\varepsilon \in (0,1)$, a subset $\mathsf{S} \subseteq \mathcal{U}$ is an **ε-net** for the range space X, if for every range $\mathsf{r} \in \mathcal{R}$ with $|\mathsf{r} \cap \mathcal{U}| \geq \varepsilon|\mathcal{U}|$ has $\mathsf{r} \cap \mathsf{S} \neq \varnothing$. The ε-net theorem of Haussler and Welzl [5] implies the existence of ε-nets of size $O(\delta\varepsilon^{-1}\log\varepsilon^{-1})$, where δ is the VC dimension of the range space X. The use of ε-nets is widespread in computational geometry [7, 4].

Weak ε-nets. Consider the range space $(P, \mathcal{C})$, where $\mathcal{C}$ is the collection of all compact convex bodies in $\mathbb{R}^d$ and $P \subset \mathbb{R}^d$ is a point set of size n. This range space has infinite VC dimension – the standard ε-net constructions do not work for this range space. The notion of **weak ε-nets** bypasses this issue by allowing the net S to use points outside of P. Specifically, any convex body Ξ that contains at least εn points of P must contain a point of S. The first construction of weak ε-net is due to Bárány et al. [1]. There was quite a bit of work on this problem, culminating in the somewhat simpler construction of Matoušek and Wagner [8], who constructed weak ε-nets of size $O(\varepsilon^{-d}\log^{f(d)}\varepsilon^{-1})$, where $f(d) = O(d^2\log d)$. Recently, Rubin [10, 11] gave an improved bound, showing existence of weak ε-nets of size $O(\varepsilon^{-(d-0.5+\alpha)})$ for arbitrarily small $\alpha > 0$. For more detailed history of the problem, see the introduction of Rubin [10, 11]. As for a lower bound, Bukh et al. [2] gave constructions of point sets for which any weak ε-net must have size $\Omega(\varepsilon^{-1}\log^{d-1}\varepsilon^{-1})$. Closing this gap remains a major open problem.

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(k,ε)-nets and uniform measure. A natural extension of weak ε-nets is to allow the net S to contain other geometric objects. Given a collection of n points $P \subset \mathbb{R}^d$ and a parameter $0 \le k < d$, we define a (weak) (k,ε)-net to be a collection of k-flats S such that if Ξ is a convex body containing at least εn points of P, then there exists a k-flat in S intersecting Ξ. Note that $(0,\varepsilon)$-nets are exactly weak ε-nets.

In general, one would expect that as k increases, the size of the (k,ε)-net shrinks. For example, a $(1,\varepsilon)$-net for a collection of points in $\mathbb{R}^3$ can be constructed by projecting the points down onto the xy-plane and applying Rubin's construction in the plane to obtain a weak ε-net S of size $O(\varepsilon^{-(3/2+\alpha)})$ [10]. Lifting S up back into three dimensions results in a $(1,\varepsilon)$-net of the same size, which is smaller than the best known weak ε-net size in $\mathbb{R}^3$ [8, 10, 11]. However, one might expect that a $(1,\varepsilon)$-net of even smaller size is possible in $\mathbb{R}^3$, as this construction uses a set of parallel lines (i.e., one would expect the lines in an optimal net to be arbitrarily oriented).

Here, we study an even simpler version of the problem, where the ground set is the hypercube $B = [0,1]^d$. In particular, for $\varepsilon \in (0,1)$ and $0 \le k < d$, we are interested in computing the smallest set K of k-flats, such that if Ξ is a convex body with $\mathsf{vol}(\Xi \cap B) \ge \varepsilon$, then there is a k-flat in K which intersects Ξ. For sake of exposition, throughout the rest of the paper we refer to this set K as a **(k,ε)-net for volume measure**. We note that $[0,1]^d$ can be replaced with any arbitrary compact convex body in the definition (the size of the (k,ε)-net increases by a factor depending on d, see Appendix B).

1.1 Our results & paper organization

Notation. Throughout, the notation O_d, Ω_d, and Θ_d hides constants depending on the dimension d.

First, we show that any (k,ε)-net for volume measure must have size $\Omega_d(1/\varepsilon^{1-k/d})$ (Lemma 3). Perhaps surprisingly, we give a relatively simple construction of (k,ε)-nets for volume measure of size $O_d(1/\varepsilon^{1-k/d})$ for $k \ge 1$ (Theorem 6). For $k = 0$, we obtain nets of size $O_d((1/\varepsilon)\log^{d-1}(1/\varepsilon))$ (Theorem 11). Importantly, both constructions are deterministic and explicit (see the discussion below).

As far as the authors are aware, this particular problem we study has not been addressed before. The only related result known is the existence of explicit constructions of $(0,\varepsilon)$-nets for volume measure for axis parallel boxes in $\mathbb{R}^d$, and is briefly mentioned in [2]. In this case, one can construct a $(0,\varepsilon)$-net for volume measure of size $O_d(1/\varepsilon)$ using Van der Corput sets in two dimensions, and Halton-Hammersely sets in higher dimensions. For completeness, we describe these construction in Appendix A.

Deterministic vs. explicit constructions of ε-nets. For the regular concept of ε-nets, there are known deterministic constructions. They work by repeatedly halving the input point set, using deterministic discrepancy constructions, until the set is of the desired size [6, 3]. On the one hand, for our setting (i.e., the measure is uniform volume on the unit hypercube) it is not clear what the generated ε-net is without running this construction algorithm outright. On the other hand, we develop a construction of weak ε-nets – for uniform volume measure over the hypercube for ellipsoids – which are much simpler and are explicit; one can easily compute the ith point in this net using polylogarithmic space.

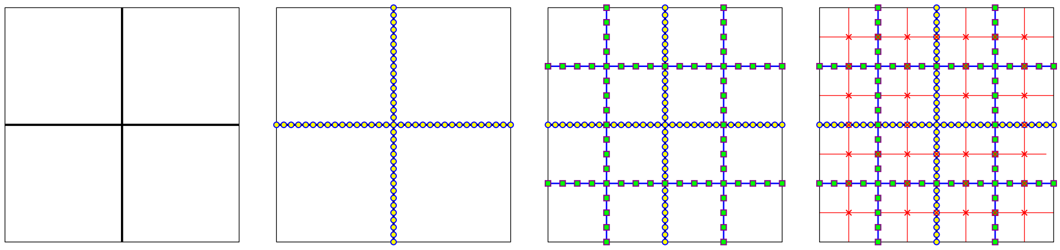

Figure 3.1 The multi-level grid, and its associated lines.

2 Lower bound

▶ **Definition 1.** *The affine hull of a point set* $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ *is the set*

$$\left\{ \sum_i \alpha_i p_i \;\middle|\; \forall i \quad \alpha_i \in \mathbb{R} \quad \text{and} \quad \sum_i \alpha_i = 1 \right\}.$$

For $0 \leq k < d$, *a* ***k*-flat** *is the affine hull of a set of* $k+1$ *(affinely independent) points.*

▶ **Definition 2.** *For parameters* $\varepsilon \in (0,1)$ *and* $k \in \{0, 1, \ldots, d-1\}$, *a set* K *of* k*-flats is a* $(\boldsymbol{k}, \boldsymbol{\varepsilon})$***-net for volume measure*** *if for any convex body* $\Xi \subseteq \mathbb{R}^d$ *with* $\mathsf{vol}\big(\Xi \cap [0,1]^d\big) \geq \varepsilon$, *there exists a flat* $\varphi \in K$ *such that* $\varphi \cap \Xi \neq \emptyset$.

▶ **Lemma 3.** *For a parameter* $\varepsilon \in (0,1)$, *any* (k, ε)*-net for volume measure must have size* $\Omega_d(1/\varepsilon^{1-k/d})$.

Proof. Let K be a (k, ε)-net for volume measure. For each k-flat $\varphi \in K$, let $H(\varphi, r)$ be the locus of points in $[0,1]^d$ within distance at most r from φ (for $k = 1$ in three dimensions, this is the intersection of $[0,1]^d$ and the cylinder with radius r centered at the line φ). Note that a ball b with center c and radius r intersects a k-flat φ if and only if $c \in H(\varphi, r)$.

Fix $r = (\varepsilon/\mu)^{1/d}$, where μ is a constant to be determined shortly. We claim that by choosing μ appropriately, if K is a (k, ε)-net for volume measure, then the collection of objects $\{H(\varphi, r) \mid \varphi \in K\}$ covers $[0,1]^d$. Indeed, suppose not. Then there exists a point $p \in [0,1]^d$ not covered by any of the objects $H(\varphi, r)$. This implies that a ball b centered at p with radius r does not intersect any k-flat of K, and its volume is $c_d r^d = c_d \varepsilon/\mu$, where c_d is a constant that depends on d. Choose $\mu = c_d$ so that b has volume at least ε, but does intersect any k-flat of K. A contradiction to the required net property.

Hence, by the choice of r, any (k, ε)-net for volume measure must satisfy the condition that $\{H(\varphi, r) \mid \varphi \in K\}$ covers $[0,1]^d$. For any k-flat φ, we have $\beta = \mathsf{vol}(H(\varphi, r)) = O_d(r^{d-k}) = O_d(\varepsilon^{1-k/d})$. Thus, to cover $[0,1]^d$, we have that $|K| \geq 1/\beta = \Omega_d(1/\varepsilon^{1-k/d})$. ◀

3 Constructing (k, ε)-nets for volume measure for $k \geq 1$

Here, we give a self-contained proof of a deterministic, explicit construction of (k, ε)-nets for volume measure of size $O_d(1/\varepsilon^{1-k/d})$ for $k \geq 1$ which matches the lower bound of Lemma 3 up to constant factors. The construction will be done recursively on the dimension d.

Base case: $\boldsymbol{k = d-1}$**.** Here a $(d-1, \varepsilon)$-net for volume measure of size $d/\varepsilon^{1/d} = O_d(1/\varepsilon^{1-k/d})$ follows readily by overlaying a d-dimensional grid of size length $\varepsilon^{1/d}$ and letting the net consist of the hyperplanes forming the grid. As such, we assume $k < d-1$.

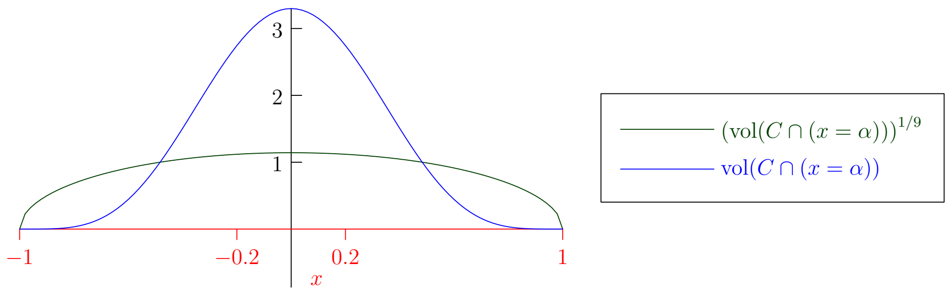

Figure 3.2 The slice volume, and its 1/9th power, for the unit radius ball Ξ in 10 dimensions. This is an example of the concavity implied by the Brunn-Minkowski inequality, which in turn implies that the slice function is unimodal.

3.1 Construction

The construction is based on quadtrees. Starting with the entire cube $[0,1]^d$, we construct d orthogonal hyperplanes which *split* the cube into 2^d cubes of side length $1/2$. We refer to such hyperplanes as **splitting hyperplanes**. This splitting process is continued recursively inside each cell, for $i = 0, \ldots, \tau$, where

$$\tau = \left\lceil \frac{1}{d} \lg \frac{1}{\varepsilon} \right\rceil + 3 \lceil \log(3d) \rceil + 1 \tag{3.1}$$

(and $\lg = \log_2$), so that cubes at the ith level of the construction has side length $1/2^i$. The number of such cubes at the ith level is 2^{di}. Naturally, these cubes together form a grid with side length $1/2^i$. See Figure 3.1 for an illustration of the construction in two dimensions.

For each splitting hyperplane h at level $i \geq 1$, which splits cells of side length $1/2^{i-1}$ into cells of side length $1/2^i$, we recursively construct a (k, ε_i)-net for volume measure on h (which lies in $d-1$ dimensions), where

$$\varepsilon_i = \frac{2^i \varepsilon}{4d}. \tag{3.2}$$

We collect all k-flats on all splitting hyperplanes at all levels into our (k, ε)-net for volume measure K.

3.2 Analysis

▶ **Lemma 4.** *The constructed (k, ε)-net for volume measure has size $O_d(1/\varepsilon^{1-k/d})$.*

Proof. Let $T(\varepsilon, d)$ denote the minimum size of a (k, ε)-net for volume measure over $[0,1]^d$. The proof is by induction on d. When $d = k+1$, we have $T(\varepsilon, k+1) \leq (k+1)/\varepsilon^{1/(k+1)}$, by the base case described above. So assume $d \geq k+2$ and $T(\delta, d') \leq \beta(d')/\delta^{1-k/d'}$ for all $d' < d$, where $\beta(d')$ is a constant to be determined. By the inductive hypothesis, the above construction produces a (k, ε)-net for volume measure of size

$$\begin{aligned} |K| &\leq d \sum_{i=1}^{\tau} 2^{i-1} T(\varepsilon_i, d-1) \leq d \sum_{i=1}^{\tau} \frac{2^{i-1} \beta(d-1)}{\varepsilon_i^{1-k/(d-1)}} \leq \frac{4d^2 \beta(d-1)}{\varepsilon^{1-k/d}} \sum_{i=1}^{\tau} \frac{2^{i-1}}{2^{i-ik/(d-1)}} \\ &\leq \frac{2d^2 \beta(d-1)}{\varepsilon^{1-k/d}} \sum_{i=1}^{\tau} 2^{ik/(d-1)} \leq \frac{4d^2 \beta(d-1)}{\varepsilon^{1-k/(d-1)}} \cdot 2^{\tau k/(d-1)} \leq \frac{16d^2 \beta(d-1)}{\varepsilon^{1-k/d}}. \end{aligned}$$

The last inequality follows since $\tau \leq \frac{1}{d} \lg \frac{1}{\varepsilon} + 2$. In particular, we obtain the recurrence $\beta(d) = 16d^2 \beta(d-1)$, which solves to $\beta(d) = d^{O(d)}$. As such, the size of K is $O_d(1/\varepsilon^{1-k/d})$. ◀

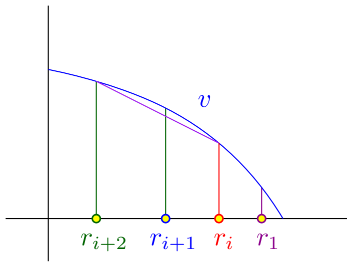

Figure 3.3 By the choice of $r_\tau \leq \ldots \leq r_1$, we have $v(r_\tau) \geq \ldots \geq v(r_1)$.

The Brunn-Minkowski inequality and unimodal functions. The Ξ be a convex body in $\mathbb{R}^d$. For a parameter $\alpha \in \mathbb{R}$, let $f(\alpha)$ denote the $(d-1)$-dimensional volume of Ξ intersected with the hyperplane $x = \alpha$. The Brunn-Minkowski inequality [7, 4] implies that the function $g(\alpha) = f(\alpha)^{1/(d-1)}$ is concave. In particular, g is **unimodal**. Namely, there exists a $\alpha \in \mathbb{R}$ such that g is non-decreasing on $(-\infty, \alpha]$ and non-increasing on $[\alpha, \infty)$. As such, the function f itself is unimodal. See Figure 3.2.

▶ **Lemma 5.** *The set K is a (k, ε)-net for volume measure.*

Proof. Let Ξ be a convex body contained in $[0,1]^d$ with volume at least ε. Assume, for the sake of contradiction, that Ξ is not stabbed by any of the k-flats of K.

Let $h(\alpha)$ be the hyperplane orthogonal to the first axis which intersects the first axis at $\alpha \in \mathbb{R}$. Define the function

$$f(\alpha) = \mathsf{vol}\big(\Xi \cap h(\alpha)\big).$$

By the Brunn-Minkowski inequality, the function $g(\alpha) = f(\alpha)^{1/(d-1)}$ is concave and unimodal. Define the point $x^* \in [0,1]$ so that $x^\star = \arg\max_\alpha f(\alpha)$.

Let $V(\Delta) = f(x^\star + \Delta)$, and let $v(\Delta) = (V(\Delta))^{1/(d-1)}$. The function v, being a translation of g, is concave and unimodal. Let $r_i \geq 0$ be the maximum number such that $V(r_i) = \varepsilon_i$, for $i = 1, \ldots, \tau$. Observe that if $r_i \geq 1/2^i$, then there is hyperplane orthogonal to the first axis that has a recursive construction of a net on it, for ε_i. This by induction would imply that the net intersects Ξ. We thus assume from this point on that

$$r_i < \frac{1}{2^i},$$

for all i. Observe that $r_1 \geq r_2 \geq \cdots \geq r_\tau$, as $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_\tau$ (more specifically, $\varepsilon_i = 2\varepsilon_{i-1}$ for all i).

The concavity of $v(\cdot)$, see Figure 3.3, implies that

$$\frac{v(r_{i+2}) - v(r_{i+1})}{r_{i+2} - r_{i+1}} \geq \frac{v(r_{i+1}) - v(r_i)}{r_{i+1} - r_i} \qquad \Longrightarrow \qquad \frac{r_{i+1} - r_i}{r_{i+2} - r_{i+1}} \leq \frac{v(r_{i+1}) - v(r_i)}{v(r_{i+2}) - v(r_{i+1})},$$

as $r_{i+1} - r_i < 0$ and $v(r_{i+2}) - v(r_{i+1}) > 0$. Since $V(r_{i+1}) = \varepsilon_{i+1} = 2\varepsilon_i = 2V(r_i)$, we have that $v(r_{i+1}) = 2^{1/(d-1)} v(r_i)$. For $i < \tau$, let $\ell_i = r_i - r_{i+1}$. Plugging this into the above, observe

$$\frac{\ell_i}{\ell_{i+1}} = \frac{r_i - r_{i+1}}{r_{i+1} - r_{i+2}} \leq \frac{v(r_{i+1}) - v(r_i)}{v(r_{i+2}) - v(r_{i+1})} = \frac{(2^{1/(d-1)} - 1)v(r_i)}{2^{1/(d-1)}(2^{1/(d-1)} - 1)v(r_i)} = \frac{1}{2^{1/(d-1)}}.$$

Since $\ell_{\tau-1} \leq r_{\tau-1} \leq 1/2^{\tau-1}$, we have

$$r_1 = r_\tau + \sum_{i=1}^{\tau-1} \ell_i \leq r_\tau + \ell_{\tau-1}\left(1 + \frac{1}{2^{1/(d-1)}} + \frac{1}{2^{2/(d-1)}} + \cdots\right)$$
$$\leq r_\tau + 2d\ell_{\tau-1} \leq (2d+1) r_{\tau-1} < \frac{2d+1}{2^{\tau-1}} < \frac{\varepsilon^{1/d}}{4d^2},$$

by the value of τ, see Eq. (3.1).

Let I_1 be the maximum interval, where the value of $V(x) \geq \varepsilon_1$, for any $x \in I_1$. By the above, we have that if the net does not intersect Ξ, then $\|I_1\| \leq 2r_1 \leq 2\varepsilon^{1/d}/(4d^2)$.

We define $I_2, \ldots, I_d$ in a similar fashion on the other axes, and the same argumentation would imply that $\|I_j\| \leq 2\varepsilon^{1/d}/(4d^2)$, for all j. Furthermore, any plane orthogonal to the axes that avoids the box $B = I_1 \times I_2 \cdots \times I_d$ has an intersection with Ξ of volume at most ε_1. We conclude that the total value of Ξ is at most

$$\mathsf{vol}(\Xi) \leq \mathsf{vol}(B) + \sum_{j=1}^{d} \int_{y \in [0,1] \setminus I_j} \mathsf{vol}\Big(\Xi \cap (x_j = y)\Big) dy \leq \prod_{j=1}^{d} \|I_j\| + d\varepsilon_1 \ll \varepsilon,$$

which is a contradiction to $\mathsf{vol}(\Xi) \geq \varepsilon$. ◀

▶ **Theorem 6.** *Given $\varepsilon \in (0,1)$ and $k \in \{1, \ldots, d-1\}$, the above is a deterministic and explicit construction of a (k,ε)-net for volume measure over $[0,1]^d$ of size $O_d(1/\varepsilon^{1-k/d})$.*

4 Constructing $(0,\varepsilon)$-nets for volume measure

4.1 Ellipsoids are enough

We now give constructions for $(0,\varepsilon)$-nets for volume measure. The following result shows that it suffices to build such nets when the convex bodies are restricted to be ellipsoids.

▶ **Lemma 7.** *Suppose there exists a $(0,\varepsilon)$-net for volume measure over $[0,1]^d$ for ellipsoids of size $T(\varepsilon,\tau)$, for $\tau = 1, \ldots, d$. Then one can construct a $(0,\varepsilon)$-net for volume measure over $[0,1]^d$ of size $T(\varepsilon/d^d, d)$.*

Proof. Consider any convex body Ξ, such that $\mathsf{vol}\big(\Xi \cap [0,1]^d\big) \geq \varepsilon$. Let $\mathcal{E}$ be the ellipsoid of largest volume contained inside $\Xi \cap [0,1]^d$. By John's ellipsoid theorem, we have that $\mathcal{E} \subseteq \Xi \subseteq d\mathcal{E}$. In particular,

$$\mathsf{vol}(\mathcal{E}) = \mathsf{vol}(d\mathcal{E})/d^d \geq \frac{\mathsf{vol}(\Xi)}{d^d} \geq \frac{\varepsilon}{d^d}.$$

As such, any $(0,\varepsilon/d^d)$-net for volume measure when the convex bodies are restricted to be ellipsoids is a $(0,\varepsilon)$-net for volume measure in the general setting. ◀

Hence, we focus on building $(0,\varepsilon)$-nets for volume measure (equivalently, these are also ε-nets for volume measure) for ellipsoids. Note that it is easy to obtain an ε-net of size $O_d(\varepsilon^{-1}\log\varepsilon^{-1})$ by random sampling [5]. Here, we give a deterministic, explicit construction of such a net.

4.2 Stabbing ellipsoids with points

4.2.1 Net construction in 2D

Let $\mathcal{E}$ be an ellipse contained in the unit square $[0,1]^2$ with $\mathsf{area}(\mathcal{E}) \geq \varepsilon$. The following construction is inspired by a construction of Pach and Tardos [9].

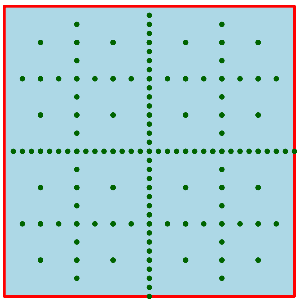

Figure 4.1 The net constructed.

Construction. Let $M = 3 + \lceil \lg \varepsilon^{-1} \rceil$. For $j = 1, \ldots, M-1$, consider the rectangle

$$R_j = [0, 1/2^{M-j}] \times [0, 1/2^j].$$

Consider the natural tiling of $[0,1]^2$ by the rectangle R_i, and let P_i be the set of vertices of the resulting grid G_i in the interior of the unit square. Let $N = \cup_i P_i$. See Figure 4.1.

Correctness. We need the following easy observation, whose proof is included for the sake of completeness.

▷ Claim 8. Let c be the center of an ellipse $\mathcal{E}$, and let $\hbar$ be the longest horizontal segment contained in $\mathcal{E}$. The segment $\hbar$ passes through c.

Proof. By the central symmetry of $\mathcal{E}$, if $\hbar$ does not pass through c, then it has a symmetric reflection $\hbar'$ through c, which is a horizontal segment of the same length. Let ℓ be the horizontal line through c, and observe that $|\ell \cap \mathcal{E}| \geq |\hbar|$ by convexity. By the smoothness of $\mathcal{E}$, it follows that $|\ell \cap \mathcal{E}| > |\hbar|$, which is a contradiction. ◁

▶ **Lemma 9.** *The set N constructed above is an ε-net for volume measure over $[0,1]^2$ for ellipses. Furthermore, $|N| = O(\varepsilon^{-1} \log \varepsilon^{-1})$.*

Proof. Observe that for any i, we have $\mathsf{area}(R_i) = 2^{-(M-j)-j} = 2^{-M} \geq \varepsilon/8$. As such, $|P_i| = O(1/\varepsilon)$, and $|N| = O(M/\varepsilon) = O(\varepsilon^{-1} \log \varepsilon^{-1})$.

Let $\mathcal{E} \subseteq [0,1]^2$ be any ellipse with $\mathsf{area}(\mathcal{E}) \geq \varepsilon$. Let Y denote the projection of $\mathcal{E}$ onto the y-axis. Observe that $|Y| \geq \varepsilon$. Let $\hbar$ be the longest horizontal segment contained in $\mathcal{E}$ (which passes through the center of $\mathcal{E}$ by Claim 8). The two extreme y-axis points in $\mathcal{E}$, and the segment $\hbar$ forms a quadrilateral in $\mathcal{E}$ of area $|\hbar| \, |Y| / 2$, see Figure 4.2. Let $Y = [y_-, y_+]$, and for $\alpha \in Y$, let $g(\alpha) = |\{y = \alpha\} \cap \mathcal{E}|$. We have that

$$|\hbar| \, |Y| / 2 \leq \mathsf{area}(\mathcal{E}) = \int_{\alpha = y_-}^{y_+} g(\alpha) \mathrm{d}\alpha \leq |\hbar| \, |Y| \, .$$

Since $\mathsf{area}(\mathcal{E}) \geq \varepsilon$, we conclude that $|\hbar| \geq \varepsilon / |Y|$.

We set $y_{1/4} = (3/4)y_- + (1/4)y_+$ and $y_{3/4} = (1/4)y_- + (3/4)y_+$. Consider the two horizontal segments $\hbar_{1/4} = \{y = y_{1/4}\} \cap \mathcal{E}$ and $\hbar_{3/4} = \{y = y_{3/4}\} \cap \mathcal{E}$. These two segments are of the same length and are parallel. Furthermore, $\gamma = |\hbar_{1/4}| = |\hbar_{3/4}| \geq |\hbar| / 2$, see Figure 4.2. Consider the parallelogram Z formed by the convex hull of $\hbar_{1/4}$ and $\hbar_{3/4}$. Observe, that for any $\alpha \in [y_{1/4}, y_{3/4}]$, we have that $|\{y = \alpha\} \cap Z| = \gamma$. As such, $\mathsf{area}(Z) = \gamma \cdot |Y| / 2 \geq |\hbar| / 2 \cdot |Y| / 2 \geq \varepsilon / 4$. Let k be the minimum integer such that $1/2^{k+1} \leq |Y| / 2$. Since $|Y| \geq \varepsilon$, it follows that $k < M - 2$.

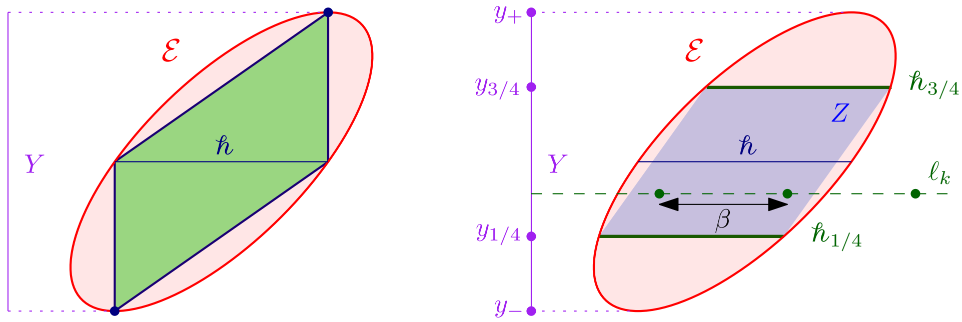

Figure 4.2 The setup for proof of correctness.

This implies that the grid G_{k+1} has a horizontal line ℓ_k that intersects Z. Furthermore, we have

$$|\ell_k \cap \mathcal{E}| \geq |\ell_k \cap Z| = \gamma \geq \frac{|\hbar|}{2} \geq \frac{\varepsilon}{2|Y|} \geq \varepsilon 2^k \geq \frac{8 \cdot 2^k}{2^M} = \frac{1}{2^{M-k-3}} > \frac{1}{2^{M-(k+1)}} = \beta,$$

since $M = 3 + \lceil \lg \varepsilon^{-1} \rceil$. Namely, the spacing of the points of G_{k+1} on the line ℓ_k (i.e., β) is shorter than the interval $\ell_k \cap \mathcal{E}$. It follows that a point of $P_{k+1} \subseteq N$ lies in $\mathcal{E}$, and thus establishing the claim. ◀

4.2.2 The construction in higher dimensions

We now extend the previous construction to higher dimensions. The construction is recursive. Namely, we assume that for all $d' < d$ we can construct an ε-net for volume measure over $[0,1]^{d'}$ for ellipsoids of size $(\beta(d')/\varepsilon)\lg^{d'-1}(1/\varepsilon)$, where $\beta(d')$ is a constant depending on the dimension d' (to be determined shortly). Lemma 9 proves the claim when $d = 2$.

Construction. Label the d axes $x_1, \ldots, x_d$. Let $\tau = \lceil (1/d)\lg(1/\varepsilon) \rceil$ and define the function $\Delta(i) = 2^i \varepsilon^{1/d}$. We repeat the following construction for each axis x_ℓ, where $\ell = 1, \ldots, d$. For each $i = 0, \ldots, \tau$, let $M_i = \lceil \lg(1/\Delta(i)) \rceil$. For each i, and for each $j = 0, \ldots, M_i$, form $2^j + 1$ evenly spaced hyperplanes which are orthogonal to the axis x_ℓ (thus consecutive hyperplanes are separated by distance 2^{-j}). For each hyperplane h, we recursively construct a $\varepsilon/\Delta(i+2)$-net $P_{\ell,i,j}$ for $[0,1]^{d-1}$ on $h \cap [0,1]^d$. Let $P_\ell = \cup_{i=1}^{\tau} \cup_{j=1}^{M_i} P_{\ell,i,j}$. Finally, we claim the point set $P = \cup_{\ell=1}^{d} P_\ell$ is the desired ε-net for volume measure.

▶ **Theorem 10.** *For $\varepsilon \in (0, 2^{-2d}]$, there exists a ε-net for volume measure over $[0,1]^d$ for ellipsoids, of size $2^{O(d^2)}\varepsilon^{-1}\lg^{d-1}\varepsilon^{-1}$.*

Proof. We first bound the size of the resulting net. Since $\varepsilon \leq 2^{-2d}$, by a direct calculation,

$$\begin{aligned}
|P| &\leq \sum_{\ell=1}^{d} |P_\ell| \leq d \sum_{i=0}^{\tau} \sum_{j=0}^{M_i} (2^j + 1) \cdot \beta(d-1) \cdot \left(\frac{\Delta(i+2)}{\varepsilon} \lg^{d-2}\left(\frac{\Delta(i+2)}{\varepsilon} \right) \right) \\
&\leq \frac{2d \cdot \beta(d-1)}{\varepsilon} \sum_{i=0}^{\tau} 2^{M_i+1} \cdot 2^2 \Delta(i) \lg^{d-2}\left(\frac{\Delta(i+2)}{\varepsilon} \right) \\
&\leq \frac{2^5 d \cdot \beta(d-1)}{\varepsilon} \sum_{i=0}^{\tau} \lg^{d-2}\left(\frac{2^{i+2}}{\varepsilon^{1-1/d}} \right)
\end{aligned}$$

$$\leq \frac{2^5 d \cdot \beta(d-1)}{\varepsilon} \sum_{i=0}^{\tau} \left((i+2) + \lg\left(\frac{1}{\varepsilon^{1-1/d}} \right) \right)^{d-2}.$$

Since $i+2 \leq \tau+2 \leq \lg(1/\varepsilon)$ for $\varepsilon \leq 2^{-2d}$, we have

$$|P| \leq \frac{2^5 d \cdot \beta(d-1)}{\varepsilon} \left[(\tau+1) \cdot 2^{d-2} \lg^{d-2}\left(\frac{1}{\varepsilon}\right) \right] \leq \frac{2^5 d \cdot \beta(d-1)}{\varepsilon} \left[\frac{4}{d} \lg \frac{1}{\varepsilon} \cdot 2^{d-2} \lg^{d-2} \frac{1}{\varepsilon} \right].$$

As such, $|P| \leq \frac{2^{d+5} \cdot \beta(d-1)}{\varepsilon} \lg^{d-1}\left(\frac{1}{\varepsilon}\right)$. In particular, we obtain the recurrence $\beta(d) = 2^{d+5}\beta(d-1)$, which solves to $\beta(d) = 2^{O(d^2)}$. Hence, $|P| = 2^{O(d^2)} \varepsilon^{-1} \lg^{d-1} \varepsilon^{-1}$.

We now argue correctness. Let $\mathcal{E}$ be an ellipsoid of volume at least ε. Let B be the smallest enclosing axis-aligned box for $\mathcal{E}$. Suppose that the longest edge of B is along the ℓth axis. In particular, along this ℓth axis B has side length $s \geq \varepsilon^{1/d}$, for otherwise $\mathsf{vol}(\mathcal{E}) \leq \mathsf{vol}(B) \leq s^d < \varepsilon$. We claim that $\mathcal{E}$ intersects a point in the set P_ℓ.

Let $L = [\ell_-, \ell_+]$ be the projection of $\mathcal{E}$ onto the ℓth axis, with $s = |L|$. For $x \in L$, define $H(x)$ to be the hyperplane orthogonal to the ℓth axis which intersects the ℓth axis at x. Finally, let K be the hyperplane through the center of $\mathcal{E}$ which is orthogonal to the ℓth axis and set $\mathcal{F} = \mathcal{E} \cap K$. We claim that $\mathsf{vol}(\mathcal{F}) \geq \varepsilon/s$. To prove the claim, suppose towards contradiction that $\mathsf{vol}(\mathcal{E} \cap K) < \varepsilon/s$. Then,

$$\mathsf{vol}(\mathcal{E}) = \int_{\ell_-}^{\ell_+} \mathsf{vol}(\mathcal{E} \cap H(x))\, dx < \frac{\varepsilon}{s} \int_{\ell_-}^{\ell_+} 1\, dx = \frac{\varepsilon}{s} |L| = \varepsilon,$$

a contradiction.

Choose an integer $i \geq 0$ such that $s \in [\Delta(i), \Delta(i+1))$. Let $z_{1/4} = (3/4)\ell_- + (1/4)\ell_+$ and $z_{3/4} = (1/4)\ell_- + (3/4)\ell_+$. Observe that for all $x \in [z_{1/4}, z_{3/4}]$, $\mathsf{vol}(\mathcal{E} \cap H(x)) \geq \varepsilon/(2s) \geq \varepsilon/\Delta(i+2)$. Next, let j be the minimum integer such that $1/2^{j+1} \leq s/2$. Note that such an integer exists, as we can choose $j = \lceil \lg(1/s) \rceil$. Since $s \geq \Delta(i)$, $j \leq \lceil \lg(1/\Delta(i)) \rceil \leq M_i$. Thus, for our choices of i and j, we have found a hyperplane h which intersects $\mathcal{E}$ with $\mathsf{vol}(\mathcal{E} \cap h) \geq \varepsilon/\Delta(i+2)$. By our recursive construction, there is a point in the net $P_{\ell,i,j}$ which intersects $\mathcal{E} \cap h$ and thus $\mathcal{E}$. ◀

▶ **Theorem 11.** *There is a deterministic, explicit construction of $(0,\varepsilon)$-nets for volume measure over $[0,1]^d$ of size*

$$O_d\left(\frac{1}{\varepsilon} \log^{d-1} \frac{1}{\varepsilon} \right).$$

Proof. Follows by plugging in the bound for Theorem 10 into Lemma 7. ◀

5 Conclusion

The main open problem left by our work is bounding the size of (k,ε)-nets in the general case. That is, the input is a set P of n points in $\mathbb{R}^d$, and we would like to compute a minimum set of k-flats which stab all convex bodies containing at least εn points of P. As noted earlier, there is a (k,ε)-net of asymptotically the same size as of a weak ε-net in $\mathbb{R}^{d-k}$. This follows by projecting the point set to a subspace of dimension $d-k$, constructing a regular weak ε-net, and lifting the net back to the original space. Can one do better than this somewhat naive construction?

Note that it is easy to show a lower bound of size $\Omega(1/\varepsilon)$ for $(1,\varepsilon)$-nets in the general case. Take a point set that consists of $\lceil 2/\varepsilon \rceil$ equally sized clusters of tightly packed points, such that no line passes through three clusters. Namely, our sublinear results in $1/\varepsilon$ are special for the uniform measure on the hypercube.

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A $(0, \varepsilon)$-nets for volume measure when the bodies are axis-aligned boxes

Here we show the existence of a $(0, \varepsilon)$-net for volume measure of size $O(1/\varepsilon)$ that intersects any axis-aligned box B with $\mathsf{vol}\big(B \cap [0,1]^2\big) \geq \varepsilon$. The following constructions are essentially described in [6] (in the context of low-discrepancy point sets), however the proofs use similar tools. We give the proofs for completeness.

▶ **Definition 12** (the Van der Corput set). *For an integer α, let $\mathsf{bin}(\alpha) \in \{0,1\}^\star$ denote the binary representation of α, and $\mathsf{rev}(\mathsf{bin}(\alpha))$ be the reversal of the string of digits in $\mathsf{bin}(\alpha)$. We define $\mathsf{br}(\alpha) \in [0,1]$ to be the **bit-reversal** of α, which is defined as the number obtained by concatenating "0." with the string $\mathsf{rev}(\mathsf{bin}(\alpha))$. For example, $\mathsf{br}(13) = 0.1011$. Formally, if $\alpha = \sum_{i=0}^{\infty} 2^i b_i$ with $b_i \in \{0,1\}$, then $\mathsf{br}(\alpha) = \sum_{i=0}^{\infty} b_i/2^{i+1}$.*

*For an integer n, the **Van der Corput set** is the collection of points $p_0, \ldots, p_{n-1}$, where $p_i = (i/n, \mathsf{br}(i))$. See Figure A.1.*

▶ **Lemma 13.** *For a parameter $\varepsilon \in (0,1)$,there is a collection of $O(1/\varepsilon)$ points $P \subset [0,1]^2$ such that any axis-aligned box B with $\mathsf{vol}\big(B \cap [0,1]^2\big) \geq \varepsilon$ contains a point of P.*

Proof. Let $n = \lceil 4/\varepsilon \rceil$. We claim that the Van der Corput set of size n is the desired point set P.

Let B be a box contained in $[0,1]^2$ of width w and height h, with $wh \geq \varepsilon$. Let $q \geq 2$ be the smallest integer such that $1/2^q < h/2 \leq 1/2^{q-1}$. By the choice of q, the projection of B onto the y-axis contains an interval of the form $I = [k/2^q, (k+1)/2^q)$ for some integer k. Let $B_I = B \cap \big\{(x,y) \in [0,1]^2 \,\big|\, y \in I\big\}$ be the box restricted to I along the y-axis. Observe that

$$\mathsf{vol}(B_I) = w/2^q = w/(4 \cdot 2^{q-2}) \geq wh/4 \geq \varepsilon/4 \iff w \geq 2^q \varepsilon/4.$$

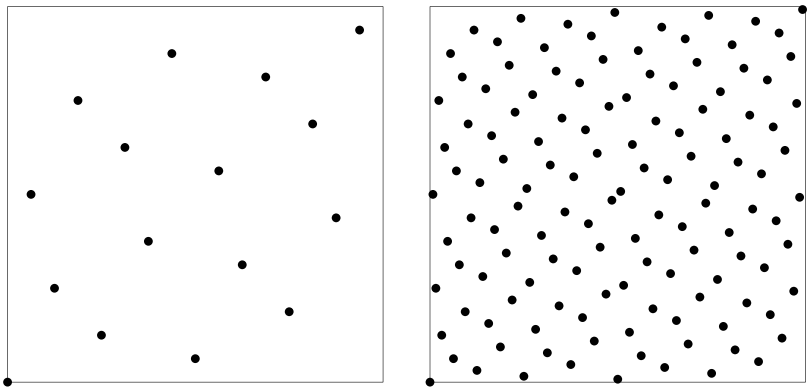

Figure A.1 The Van der Corput set with $n = 16$ (left) and $n = 128$ (right).

Let $S = [0,1] \times I$, so that each $p_j \in P \cap S$ has $\mathsf{br}(j) \in I$. In particular, the first q binary digits of $\mathsf{br}(j)$ are fixed. This implies that the q least significant binary digits of j are fixed. In other words, $P \cap S$ contains all points p_j such that $j \equiv \ell \pmod{2^q}$ for some integer ℓ – the x-coordinates of the points in P are regularly spaced in the strip S with distance $2^q/n$. If the width of B_I is at least $2^q/n$, then this implies that B contains a point of P in the strip S. Indeed, by the choice of n, $2^q/n \leq 2^q\varepsilon/4 \leq w$. ◀

By extending the definition of the Van der Corput set to higher dimensions, the above proof also generalizes.

▶ **Definition 14** (the Halton-Hammersely set). *For a prime number ρ and an integer $\alpha = \sum_{i=0}^{\infty} \rho^i b_i$, $b_i \in \{0, \ldots, \rho - 1\}$, written in base ρ, define $\mathsf{br}_\rho(\alpha) = \sum_{i=0}^{\infty} b_i/\rho^{i+1}$. Note that $\mathsf{br}_2 = \mathsf{br}$ from Definition 12.*

For integers n and d, the ***Halton-Hammersely set*** *is the collection of points*

$$p_1, \ldots, p_{n-1},$$

where $p_i = (\mathsf{br}_{\rho_1}(i), \mathsf{br}_{\rho_2}(i), \ldots, \mathsf{br}_{\rho_{d-1}}(i), i/n)$, and $\rho_1, \ldots, \rho_{d-1}$ are the first $d - 1$ prime numbers. (Making i/n the dth coordinate instead of the 1st coordinate simplifies future notation.)

▶ **Lemma 15.** *For a parameter $\varepsilon \in (0,1)$, there is a collection of $2^{O(d \log d)}/\varepsilon$ points $P \subset [0,1]^d$ such that any axis-aligned box B with $\mathsf{vol}\big(B \cap [0,1]^d\big) \geq \varepsilon$ contains a point of P.*

Proof. The proof is similar to Lemma 13, with the Chinese remainder theorem as the additional tool.

Let $n = \lceil (2^{d-1}/\varepsilon) \cdot (d-1)\sharp \rceil$, where $k\sharp$ is the **primorial** function, defined as the product of the first k prime numbers. It is known that $k\sharp \leq \exp((1 + o(1))k \log k)$, which implies $n = 2^{O(d \log d)}/\varepsilon$. We claim that the Halton-Hammersely set of size n is the desired point set P.

Denote the side lengths of the box B by $s_1, \ldots, s_d$, with $\prod_{i=1}^{d} s_i \geq \varepsilon$. For each $i = 1, \ldots, d-1$, let q_i be the smallest integer such that $1/\rho_i^{q_i} < s_i/2 \leq 1/\rho_i^{q_i - 1}$, where ρ_i is the ith prime number. By the choice of q_i, the projection of B onto the ith axis contains an interval of the form $I_i = [k_i/\rho_i^{q_i}, (k_i + 1)/\rho_i^{q_i}]$ for some integer k_i. Let S denote the box $I_1 \times \ldots \times I_{d-1} \times [0,1]$ and $B_S = B \cap S$. Observe that

$$\mathsf{vol}(B_S) = s_d \prod_{i=1}^{d-1} \frac{1}{\rho_i^{q_i}} \geq s_d \prod_{i=1}^{d-1} \frac{s_i}{2\rho_i} \geq \frac{\varepsilon}{2^{d-1}} \prod_{i=1}^{d-1} \frac{1}{\rho_i} \iff s_d \geq \frac{\varepsilon}{2^{d-1}} \prod_{i=1}^{d-1} \rho_i^{q_i - 1}.$$

Similar to Lemma 13, we observe that the point $p_j \in P$ falls into S when $j \equiv \ell_i \pmod{\rho_i^{q_i}}$ for some integers $\ell_1, \ldots, \ell_{d-1}$. By the Chinese remainder theorem, there is exactly one number in the set $\left\{0, 1, \ldots, \prod_{i=1}^{d-1} \rho_i^{q_i} - 1\right\}$ (the dth coordinate of p_j) which satisfies these $d-1$ equations. In particular, the points in $P \cap S$ are spaced regularly along the dth axis with distance $\delta = (1/n) \prod_{i=1}^{d-1} \rho_i^{q_i}$. Once again, we argue that the length of B along the dth axis is at least δ, which implies the result. Indeed, by our choice of n we have that,

$$\delta = \frac{1}{n} \prod_{i=1}^{d-1} \rho_i^{q_i} \leq \frac{\varepsilon}{2^{d-1}} \prod_{i=1}^{d-1} \rho_i^{q_i - 1} \leq s_d. \qquad \blacktriangleleft$$

B Extension: Replacing $[0,1]^d$ with other convex bodies

▶ **Lemma 16.** *Let $\mathcal{C}$ be an arbitrary compact convex body in $\mathbb{R}^d$ with non-empty interior. Suppose there is a (k, ε)-net for volume measure over $[0,1]^d$ of size $T(\varepsilon, k, d)$. For a given integer $k < d$ and $\varepsilon \in (0,1)$, there is a collection of k-flats K of size $T(\Omega_d(\varepsilon), k, d))$, such that any convex body Ξ with $\mathsf{vol}(\Xi \cap \mathcal{C}) \geq \varepsilon \, \mathsf{vol}(\mathcal{C})$ is intersected by a k-flat in K.*

Proof. Assume without loss of generality that $\Xi \subseteq \mathcal{C}$. John's ellipsoid theorem [7] implies that there exists a non-singular affine transformation $\mathbf{M}$, and a ball b of diameter 1, such that $\mathsf{b}/d \subseteq \mathbf{M}(\mathcal{C}) \subseteq \mathsf{b} \subseteq [0,1]^d$, where b/d is b scaled by a factor of $1/d$. We have that $\mathsf{vol}(\mathsf{b}) = c_d 2^{-d}$, where c_d is the volume of the unit ball in $\mathbb{R}^d$. Additionally,

$$\mathsf{vol}\big([0,1]^d\big) = 1 = \frac{2^d}{c_d} \mathsf{vol}(\mathsf{b}) = \frac{(2d)^d}{c_d} \mathsf{vol}(\mathsf{b}/d) \leq \frac{(2d)^d}{c_d} \mathsf{vol}(\mathbf{M}(\mathcal{C})).$$

Set $\delta = c_d/(2d)^d$. Compute a (k, ε')-net for volume measure K over $[0,1]^d$, where $\varepsilon' = \varepsilon/\delta$, which has size $T(\varepsilon', k, d)$. We claim that this is a (k, ε)-net for volume measure with respect to $\mathbf{M}(\Xi)$. Indeed, consider any convex body $\Xi \subseteq \mathcal{C}$ with $\mathsf{vol}(\Xi \cap \mathcal{C}) \geq \varepsilon \, \mathsf{vol}(\mathcal{C})$. Since $\mathbf{M}$ preserves the ratios of volumes, we have that

$$\mathsf{vol}\big(\mathbf{M}(\Xi) \cap [0,1]^d\big) \geq \mathsf{vol}(\mathbf{M}(\Xi) \cap \mathbf{M}(\mathcal{C})) \geq \varepsilon \, \mathsf{vol}(\mathbf{M}(\mathcal{C})) \geq \frac{\varepsilon}{\delta} \mathsf{vol}\big([0,1]^d\big) = \varepsilon' \, \mathsf{vol}\big([0,1]^d\big).$$

As such, one of the k-flats in K intersects $\mathbf{M}(\Xi)$. After applying the inverse transformation $\mathbf{M}^{-1}$ to each k-flat in K, one of the k-flats in $\mathbf{M}^{-1}(K)$ intersects Ξ. ◀