Asymptotically Optimal Welfare of Posted Pricing for Multiple Items with MHR Distributions

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Abstract

We consider the problem of posting prices for unit-demand buyers if all n buyers have identically distributed valuations drawn from a distribution with monotone hazard rate. We show that even with multiple items asymptotically optimal welfare can be guaranteed.

Our main results apply to the case that either a buyer's value for different items are independent or that they are perfectly correlated. We give mechanisms using dynamic prices that obtain a $1-\Theta\left(\frac{1}{\log n}\right)$-fraction of the optimal social welfare in expectation. Furthermore, we devise mechanisms that only use static item prices and are $1-\Theta\left(\frac{\log\log\log n}{\log n}\right)$-competitive compared to the optimal social welfare. As we show, both guarantees are asymptotically optimal, even for a single item and exponential distributions.

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1 Introduction

Posting prices is a very simple way to de-centralize markets. One assumes that buyers arrive sequentially. Whenever one of them arrives, a mechanism offers a menu of items at suitably defined prices. The buyer then decides to accept any offer, depending on what maximizes her own utility. Such a mechanism is incentive compatible by design, usually easy to explain and can be implemented online. For this reason, there is a large interest in understanding what social welfare and revenue can be guaranteed in comparison to mechanisms that optimize the respective objective.

Let us consider the following setting: There is a set of m heterogeneous items M, each of which we would like to be allocated to one of n buyers. Each buyer i has a private valuation function $v_i\colon 2^M \to \mathbb{R}_{\geq 0}$. We assume that valuation functions are unit-demand. That is, $v_i(S) = \max_{j\in S} v_i(\{j\})$, meaning that the value a buyer associates to a set is simply the one of the most valuable item in this set. Let $\mathrm{SW}_{\mathrm{opt}} = \max_{\text{allocations } (S_1,\ldots,S_n)} \sum_{i=1}^{n} v_i(S_i)$ denote the optimal (offline/ex-post) social welfare. Note that this optimal solution is nothing but the maximum-weight matching in a bipartite graph in which all buyers and items correspond to a vertex each and an edge between the vertices of buyer i and item j has weight $v_i(\{j\})$.

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To capture the pricing setting, we assume that the functions $v_1, \dots, v_n$ are unknown a priori; all of them are drawn independently from the same, publicly known distribution. For every item, one can either set a static item price or change the prices dynamically over time. Buyers arrive one-by-one and each of them chooses the set of items that maximizes her utility given the current prices among the remaining items. Static prices have the advantage that they are easier to explain and thus give easier mechanisms. However, dynamic prices can yield both higher welfare and revenue because they can be adapted to the remaining supply and the remaining number of buyers to appear.

Coming back to the interpretation of a bipartite matching problem, a posted-prices mechanism corresponds to an online algorithm, where the buyers correspond to online vertices and the items correspond to offline vertices. However, not every online algorithm necessarily corresponds to a posted-prices mechanism: There might not be item prices such that the choices of the algorithm correspond to the ones by a buyer maximizing their utility.

We would like to understand which fraction of the optimal (offline/ex-post) welfare posted-prices mechanisms can guarantee. The case of a single item is well understood via *prophet inequalities* from optimal stopping theory. Let us call a posted-prices mechanism β-competitive (with respect to social welfare) if its expected welfare $\mathbf{E}[\mathrm{SW}_{\mathrm{pp}}]$ is at least $\beta\mathbf{E}[\mathrm{SW}_{\mathrm{opt}}]$. For a static price and a single item, the best such guarantee is $\beta = 1 - \frac{1}{e} \approx 0.63$ [11, 15]; for dynamic pricing and a single item, it is $\beta \approx 0.745$ [1, 11]. There are a number of extensions of these results to multiple items (see Section 1.3 for details), also going beyond unit-demand valuations, many of which are $O(1)$-competitive.

The competitive ratios of $\beta = 1 - \frac{1}{e} \approx 0.63$ and $\beta \approx 0.745$ are optimal in the sense that there are distributions and choices of n such that no better guarantee can be obtained. Importantly, they are still tight when imposing a lower bound on n. That is, even for large n, there is a distribution such that if all values are drawn from this distribution the respective bound cannot be beaten.

1.1 Distributions with Monotone Hazard Rate

In this paper, we strengthen previous results by restricting the class of distributions to ones with monotone hazard rate. The single-item case is defined as follows. Consider a probability distribution on the reals with probability density function (PDF) f and cumulative distribution function (CDF) F, its hazard rate h is defined by $h(x) = \frac{f(x)}{1-F(x)}$ for x with $F(x) < 1$. It has a *monotone hazard rate* (MHR) – more precisely, increasing hazard rate – if h is a non-decreasing function. It has become a common and well-studied approach to model buyer preferences by MHR distributions. One of the reasons is that many standard distributions exhibit a monotone hazard rate such as, for example, uniform, normal, exponential and logistic distributions. (For a much more extensive list see [25].) Furthermore, the monotone hazard rate of distributions is also preserved under certain operations; for example, order statistics of MHR distributions also have an MHR distribution. Additionally, every MHR distribution is regular in the sense that its virtual value function [8, 24] is increasing.

We generalize results to multiple items and consider two fundamental settings. On the one hand, we consider *independent item valuations*, i.e. $v_{i,j} \sim \mathcal{D}_j$ is an independent draw from a distribution $\mathcal{D}_j$. In other words, the value of item j is independent of the value of item j' and both values are drawn from (possibly different) MHR distribution as defined above. On the other hand, we assume correlated values for items via the notion of *separable item valuations*, which are common in ad auctions [14, 27]: Each buyer has a type $t_i \geq 0$ and each item has an item-dependent multiplier α_j where now $v_{i,j} = \alpha_j \cdot t_i$. Again, $t_i \sim \mathcal{D}$ and $\mathcal{D}$ is a distribution with monotone hazard rate. We note that this case subsumes and extends the case of k identical items.

As we show, in these cases, asymptotically optimal welfare can be guaranteed. That is, if n grows large, the social welfare when suitably choosing prices is within a $1 - o(1)$ factor of the optimum, where the $o(1)$ term is independent of the distribution as long as its marginals satisfy the MHR property. Stated differently, there is a sequence $(\beta_n)_{n\in\mathbb{N}}$ with $\beta_n \to 1$ for $n \to \infty$ such that for every number of buyers n there exists a posted-prices mechanism that takes any distribution with MHR as input and guarantees $\mathbf{E}[\mathrm{SW}_{\mathrm{pp}}] \geq \beta_n \mathbf{E}[\mathrm{SW}_{\mathrm{opt}}]$. As pointed out before, such a result does not hold for arbitrary, non-MHR distributions. Even with a single item, the limit is then upper-bounded by ≈ 0.745.

A similar effect has already been observed by Giannakopoulos and Zhu [17]. They show that the revenue of static pricing for a single item with MHR distributions asymptotically reaches the optimal revenue. In contrast, our results concern welfare. Still, some of our results also have implications for revenue, either because we bound the revenue or because one could apply the results to virtual values.

1.2 Our Results and Techniques

We design mechanisms for both independent and separable item valuations. The ones using dynamic prices ensure a $\left(1 - O\left(\frac{1}{\log n}\right)\right)$-fraction of the expected optimal social welfare. The ones using static prices guarantee a $\left(1 - O\left(\frac{\log\log\log n}{\log n}\right)\right)$-fraction. We also show that these guarantees are best possible, even in the case of only a single item. Note that the bounds are independent of the number of items m, which may also grow in the number of buyers n.

Independent Valuations (Section 3)

The technically most interesting result is the one on dynamic pricing when values are independent across items. The idea is to set prices so that the offline optimum is mimicked. If item j is allocated in the optimal allocation with probability q_j, then we would like it to be sold in every step with an ex-ante probability of $\frac{q_j}{n}$. However, analyzing such a selling process is still difficult because items are incomparable and bounds for MHR distributions cannot be applied directly to draws from multiple distributions, which are not necessarily identical. To bypass this problem, we introduce a reduction that allows us to view item valuations not only as independent but also as identically distributed. To this end, we compare the selling process of our mechanism to a hypothetical setting, in which buyers do not make their decisions based on the actual utility but in quantile space. We observe that the revenue of both is identically distributed and utility is maximized in the former mechanism. As a consequence, the welfare obtained by the quantile allocation rule is a feasible lower bound on the welfare of the sequential posted-prices mechanism. Only afterwards, we can apply a concentration bound due to the MHR restriction.

The idea of our mechanism using static prices is to set prices suitably high in order to bound the revenue of our mechanism with a sufficient fraction of the optimum. While all other bounds apply for any number of items m, this bound unfortunately requires $m \leq \frac{n}{(\log\log n)^2}$. We leave it as an open problem to extend the result for larger number of items.

Separable Valuations (Section 4)

Our way of setting dynamic prices in the case of separable valuations is similar to the approach in independent valuations. This setting is even a little simpler because we can assume without loss of generality that there are as many items as buyers. Our pricing strategy ensures that in each step each item is sold equally likely as well as one item is sold for sure. In the analysis, we observe that we match a buyer and an item if the quantile of the buyer's value is in a specific range. Now, the MHR property comes into play which allows to bound quantiles of the distribution in a suitable way.

In the static case, to lower-bound the welfare of our mechanism, we compare it to the one of the VCG mechanism [10, 18, 28] which maximizes social welfare. To this end, we split social welfare in revenue and utility and bound each quantity separately. That is, we relate the revenue and the sum of buyer utilities of our posted-pricing mechanism to the ones of the VCG mechanism. For the revenue, we set prices fairly low to ensure that we sell all items with reasonably high probability. Still, these prices are high enough to use the MHR property and derive a suitable lower bound of the prices. The utility comparison is more complicated, we solve this issue by an unusual application of the equality of expected revenue and virtual welfare due to Myerson [24].

Optimality (Section 5)

The achieved bounds on the competitive ratio are optimal for both dynamic as well as static pricing. We show this by considering the single-item case with an exponential distribution, which is a special case of both independent and separable valuations. For dynamic prices, we use the correspondence to a Markov decision process showing that no online algorithm is better than $1-\Omega\left(\frac{1}{\log n}\right)$-competitive. Then we also show that the competitive ratio cannot be better than $1-\Omega\left(\frac{\log\log\log n}{\log n}\right)$ for any choice of a static price by writing out the expected social welfare explicitly.

Subadditive Valuations (Section 6)

We also demonstrate that our techniques are applicable beyond unit-demand settings by giving mechanisms for the more general class of subadditive valuation functions. Our dynamic pricing mechanism is $1-O\left(\frac{1+\log m}{\log n}\right)$-competitive for subadditive buyers. We complement this by a static pricing mechanism which is $1-O\left(\frac{\log\log\log n}{\log n}+\frac{\log m}{\log n}\right)$-competitive. Both guarantees can be derived by showing that the revenue of the posted pricing mechanism is at least as high as the respective fraction of the optimal social welfare. As a consequence, these bounds directly imply the competitive ratios for welfare and revenue. For small m, these bounds are again tight by our optimality results. Obtaining tight bounds for large m still remains an open problem.

1.3 Further Related Work

As mentioned already, our setup restricted to a single item is highly related to prophet inequalities. Prophet inequalities have their origin in optimal stopping theory, dating back to the 1970s [22]. Only much later they were considered as a tool to understand the loss by posting prices as opposed to using other mechanisms. In this context, Samuel-Cahn's result [26] then got the interesting interpretation that posting an appropriately chosen static price for an item is $\frac{1}{2}$-competitive for any buyer distributions; different buyers may even be drawn from different distributions. This guarantee is optimal, even for dynamic pricing.

Improvements for the single-item case are only possible by imposing further assumptions. Most importantly, this concerns the case that all buyer values are drawn from the same distribution. While already discussed by Hill et al. [20], this problem has been solved only very recently by devising an ≈ 0.745-competitive mechanism that relies on dynamic pricing [1, 11]. By using static pricing, one cannot be better than $1-\frac{1}{e}\approx 0.63$-competitive [11, 15].

Better guarantees can also be achieved by assuming that multiple, identical items are for sale. In this case, one can use concentration results. The respective competitive ratios tend to 1 for a growing number of item copies. Hajiaghayi et al. [19] gave the first guarantee for such a setting, Alaei [2] later improved it to tightness.

For identical regular distributions, all of the above results also apply to welfare as well as revenue maximization because the prices can be imposed in the space of the virtual valuation [24]; also see the work by Chawla et al. [9]. Also impossibility results transfer [12].

When it comes to multiple, heterogeneous items, there is a significant difference between welfare and revenue maximization because Myerson's characterization does not apply anymore. For welfare maximization, Feldman et al. [16] show that static item prices still yield a competitive ratio of $\frac{1}{2}$ even for XOS valuations and not necessarily identically distributed buyer valuations. Concerning subadditive valuation functions, Zhang [29] gives a $O(\log m/\log\log m)$-competitive prophet inequality, Dütting et al. [13] show how to obtain a competitive ratio of $O(\log\log m)$. The only improvement for identically distributed buyers is to $1-\frac{1}{e}$ for unit-demand buyers based on dynamic pricing [15]. Among others, Chawla et al. [7] considered a combinatorial generalization of such a setting with many item copies (see Lucier's survey [23] on a broader overview of combinatorial generalizations).

For revenue maximization, one usually imposes the additional assumption that items are independent. This makes it possible to also apply prophet inequalities on the sequence of items rather than buyers and thus maximize revenue for unit-demand buyers via posted prices [8, 9]. Cai and Zhao [6] consider more general XOS and subadditive valuations and apply a duality framework instead. They design a posted-prices mechanism with an entry fee that gives an $O(1)$ or $O(\log m)$ approximation to the optimal revenue. In Dütting et al. [13], the approximation of the optimal revenue for subadditive valuations is improved to $O(\log\log m)$.

There are surprisingly few results on pricing and prophet inequalities that derive better guarantees by imposing additional assumptions on the distribution. Babaioff et al. [4] consider the problem of maximizing revenue when selling a single item to one of n buyers drawn i.i.d. from an *unknown* MHR distribution with a bounded support $[1,h]$. If n is large enough compared to h, they get a constant-factor approximation to the optimal revenue using dynamic posted prices. Note that in contrast, in our paper, we assume to know the underlying distributions perfectly. Giannakopoulos and Zhu [17] consider revenue maximization in the single-item setting with valuations drawn independently from the same MHR distribution. They show that by offering the item for the same static price to all bidders one can achieve asymptotically optimal revenue. More precisely, one of their main results is that one gets within a factor of $1-O\left(\frac{\ln\ln n}{\ln n}\right)$. While they claim this result is "essentially tight", we show that the best factor is indeed $1-\Theta\left(\frac{\ln\ln\ln n}{\ln n}\right)$ because it is a special case of our results (see Section 6). It is not clear, how one could apply their result to welfare maximization as the MHR property is not preserved when moving between virtual and actual values. Furthermore, their results do not admit any apparent generalization to multiple items. Jin et al. [21] also consider revenue maximization in the single-item setting with identical and independent MHR values but in a non-asymptotic sense, providing a bound for every n.

2 Preliminaries

We consider a setting of n buyers N and a set M of m items. Every buyer has a valuation function $v_i\colon 2^M \to \mathbb{R}_{\geq 0}$ mapping each bundle of items to the buyer's valuation. We assume buyers to be unit-demand, that is $v_i(S) = \max_{j\in S} v_{i,j}$. The functions $v_1,\ldots,v_n$ are unknown a priori but all drawn independently from the same, publicly known distribution $\mathcal{D}$. Let $\mathcal{D}_j$ be the marginal distribution of $v_{i,j}$, which is the value of a buyer for being allocated item j. We assume that $\mathcal{D}_j$ is a continuous, real, non-negative distribution with monotone hazard rate. That is, let F_j be the cumulative distribution function of $\mathcal{D}_j$ and f_j its probability density function. The distribution's hazard rate is defined as $h_j(x) = f_j(x)/(1 - F_j(x))$ for all x such that $F_j(x) < 1$. We assume a *monotone hazard rate*, which means that h_j is a non-decreasing function. Equivalently, we can require $x \mapsto \log(1 - F_j(x))$ to be a concave function.

We design *posted-prices mechanisms*. That is, the buyers arrive one by one in order $1, \ldots, n$. In the i-th step, buyer i arrives and has the choice between all items which have not been allocated so far. Let $M^{(i)}$ denote this set of available items. The mechanism presents the i-th buyer a menu of prices $p_j^{(i)}$ for all items $j \in M^{(i)}$. The buyer then picks the item $j_i \in M^{(i)}$ which maximizes her *utility* $v_{i,j_i} - p_{j_i}^{(i)}$ if positive[1]. Buyer i and item j_i are matched immediately and irrevocably. If buyer i has negative utility for all items $j \in M^{(i)}$, then buyer i does not buy any item and remains unmatched. Generally, the prices for buyer i may depend arbitrarily on $M^{(i)}$ and the distribution $\mathcal{D}$. We call prices *static* if there are $p_1, \ldots, p_m$ such that $p_j^{(i)} = p_j$ for all i and all j.

Fix any posted-prices mechanism and let j_i denote the item allocated to buyer i (set $j_i = \perp$ if i remains unmatched in the mechanism). The *expected social welfare* of the mechanism is given by $\mathbf{E}\left[\sum_{i=1}^n v_{i,j_i}\right] =: \mathbf{E}\left[\mathrm{SW}_{\mathrm{pp}}\right]$. In comparison, let the social welfare maximizing allocation assign item j_i^* to buyer i. Its expected social welfare is therefore given by $\mathbf{E}\left[\sum_{i=1}^n v_{i,j_i^*}\right] =: \mathbf{E}\left[\mathrm{SW}_{\mathrm{opt}}\right]$.

We call a posted-prices mechanism *β-competitive* if it ensures that the expected social welfare of its allocation is at least a β-fraction of the expected optimal social welfare. That is, for any choice of distribution,

$$\mathbf{E}\left[\mathrm{SW}_{\mathrm{pp}}\right] = \mathbf{E}\left[\sum_{i=1}^n v_{i,j_i}\right] \geq \beta \mathbf{E}\left[\sum_{i=1}^n v_{i,j_i^*}\right] = \beta \mathbf{E}\left[\mathrm{SW}_{\mathrm{opt}}\right] \quad .$$

3 Asymptotically Tight Bounds for Independent Valuations

In this section, we show how to derive bounds if the buyers' values are independent across items. That is, each $v_{i,j} \sim \mathcal{D}_j$ is drawn independently from a distribution with monotone hazard rate. This is a standard assumption when considering multiple items [8, 9]. As a consequence, the distribution over valuations is a product distribution $v_i = (v_{i,1}, \ldots, v_{i,m}) \sim \mathcal{D} = \prod_{j=1}^n \mathcal{D}_j$ for any $i \in N$ and every $\mathcal{D}_j$ satisfies the MHR condition.

3.1 Dynamic prices

We first consider the case of dynamic pricing mechanisms. Without loss of generality, we can assume that $m \geq n$. If we have less items than buyers, i.e. $m < n$, we can add dummy items with value 0 to ensure $m = n$. Matching i to one of these dummy items in the mechanism then corresponds to leaving i unmatched. Observe that technically a point mass on 0 is not a MHR distribution. However, all relevant statements still apply.

[1] We can assume that any buyer is buying at most one item as buyers are unit-demand. Hence, no buyer can increase utility by buying a second (lower valued) item.

Our mechanism is based on a pricing rule which balances the probability of selling a specific item. To this end, let $M^{(i)}$ be the set of remaining items as buyer i arrives. We determine dynamic prices such that one item is allocated for sure in every step. Therefore, always $|M^{(i)}| = m - i + 1$. We can now define $q_j^{(i)}$ to be the probability that item j is allocated in the "remaining" offline optimum on $M^{(i)}$ and $n - i + 1$ buyers if $j \in M^{(i)}$ and 0 else. In other words, if $j \in M^{(i)}$, $q_j^{(i)}$ is the probability that item j is allocated in the offline optimum constrained to buyers $1, \dots, i-1$ receiving the items from $M \setminus M^{(i)}$. The prices $(p_j^{(i)})_{j \in M^{(i)}}$ are now chosen such that buyer i buys item j with probability $\frac{q_j^{(i)}}{n-i+1}$ and one item is allocated for sure. To see that such prices exist, observe the following: fix any price vector $\mathbf{x} = (x_j)_{j \in M^{(i)}}$ and denote by $r_j^{(i)}(\mathbf{x}) = \mathbf{Pr}\left[i \text{ buys item } j \text{ at prices } \mathbf{x} \;\middle|\; M^{(i)}\right]$. As the random variables $v_{i,j}$ are continuous and independent, the probability that buyer i buys item j at prices $\mathbf{x}$ given the current set of items $M^{(i)}$ is continuous in x_j. Hence, we can consider the mapping $\left(\phi^{(i)}(\mathbf{x})\right)_j = \frac{n-i+1}{q_j^{(i)}} \cdot r_j^{(i)}(\mathbf{x}) \cdot x_j$ for any $j \in M^{(i)}$ which is also continuous and hence, by the use of Brouwer's fixed point theorem[2] has our desired price vector $(p_j^{(i)})_{j \in M^{(i)}}$ as fixed point. This allows us to state the following theorem.

▶ **Theorem 1.** *The posted-prices mechanism with dynamic prices and independent item-valuations is $1 - O\left(\frac{1}{\log n}\right)$-competitive with respect to social welfare.*

Note that in the case $m \le n$ we will always have $q_j^{(i)} = 1$ for $j \in M^{(i)}$. This significantly simplifies the argument. The proof for the general case can be found in the full version of the paper. Here, we give a sketch with the major steps and key techniques.

In order to bound the social welfare obtained by the posted-prices mechanism, we consider the following *quantile allocation rule*. For any $j \in M^{(i)}$ with $q_j^{(i)} > 0$, compute $R_j^{(i)} := F_j(v_{i,j})^{\frac{1}{q_j^{(i)}}}$ and allocate buyer i the item j which maximizes $R_j^{(i)}$. Observe that by this definition for any i, any j and any $t \in [0,1]$,

$$\mathbf{Pr}\left[R_j^{(i)} \le t\right] = \mathbf{Pr}\left[F_j(v_{i,j}) \le t^{q_j^{(i)}}\right] = t^{q_j^{(i)}} \quad ,$$

as $F_j(v_{i,j}) \sim \text{Unif}[0,1]$. In particular, note that for $q_j^{(i)} = 1$, this is exactly the CDF of a random variable drawn from Unif$[0,1]$. Define indicator variables $X_{i,j}$ which are 1 if buyer i is allocated item j in the quantile allocation rule. Then, we can observe the following.

▶ **Observation 2.** *It holds*

$$\mathbf{Pr}\left[X_{i,j} = 1 \;\middle|\; M^{(i)}\right] = \frac{q_j^{(i)}}{n-i+1} \quad .$$

Note that by this, the probability of allocating item j in step i via the quantile allocation rule is $\frac{q_j^{(i)}}{n-i+1}$, exactly as in the posted-prices mechanism.

Proof. We allocate item j in the quantile allocation rule if $R_j^{(i)} \ge R_{j'}^{(i)}$ for any $j' \in M^{(i)}$. For fixed $M^{(i)}$, also the values of $q_j^{(i)}$ are fixed. Hence, we can use independence of the $v_{i,j}$

[2] In addition, we can use that prices $\mathbf{x} = (x_j)_{j \in M^{(i)}}$ are always bounded by $0 \le x_j \le F_j^{-1}\left(1 - \frac{q_j^{(i)}}{n-i+1}\right)$ to get a convex and compact set of price vectors.

variables to compute:

$$\begin{aligned}\mathbf{Pr}\left[X_{i,j}=1 \,\middle|\, M^{(i)}\right] &= \mathbf{Pr}\left[\max_{j'\neq j} R_{j'}^{(i)} \leq R_j^{(i)} \,\middle|\, M^{(i)}\right] \\ &= \int_0^1 \mathbf{Pr}\left[\max_{j'\neq j} R_{j'}^{(i)} \leq t \,\middle|\, M^{(i)}\right] q_j^{(i)} t^{q_j^{(i)}-1} dt \\ &= \int_0^1 \prod_{j'\neq j} \mathbf{Pr}\left[R_{j'}^{(i)} \leq t \,\middle|\, M^{(i)}\right] q_j^{(i)} t^{q_j^{(i)}-1} dt \\ &= \int_0^1 \left(\prod_{j'\neq j} t^{q_{j'}^{(i)}}\right) q_j^{(i)} t^{q_j^{(i)}-1} dt \\ &= q_j^{(i)} \int_0^1 t^{(n-i+1)-1} dt = \frac{q_j^{(i)}}{n-i+1} \quad ,\end{aligned}$$

where we use that $\sum_{j\in M^{(i)}} q_j^{(i)} = n-i+1$ for any value of i. ◀

Now, the crucial observation is that the expected contribution of any buyer to the social welfare in the posted-prices mechanism is at least as large as under the quantile allocation rule. To see this, fix buyer i and split buyer i's contribution to the social welfare into revenue and utility. Concerning revenue, note that in both cases the probability of selling any item j to buyer i is equal to $\frac{q_j^{(i)}}{n-i+1}$ and we allocate one item for sure. So, the expected revenue is identical. Further, since we maximize utility in the posted-prices mechanism, the achieved utility is always at least the utility of the quantile allocation rule. So, overall, we get $\mathbf{E}\left[\mathrm{SW}_{\mathrm{quantile}}\right] \leq \mathbf{E}\left[\mathrm{SW}_{\mathrm{pp}}\right]$.

Now, we aim to control the distribution of $v_{i,j}$ given that $X_{i,j}=1$ in order to get access to the value of an agent being allocated an item in the quantile allocation rule. To this end, we use the following lemma.

▶ **Lemma 3.** *For all i, j and $M^{(i)}$, we have*

$$\mathbf{Pr}\left[v_{i,j} \leq t \,\middle|\, X_{i,j}=1, M^{(i)}\right] = F_j(t)^{\frac{n-i+1}{q_j^{(i)}}}$$

Proof. Observe that in the vector $\left(R_j^{(i)}\right)_{j\in M^{(i)}}$, we choose j to maximize $R_j^{(i)}$. Now, for any value $v_{i,j'}$, we consider the following transform ψ_j: For any $j' \in M^{(i)}$, define

$$\psi_j(v_{i,j'}) := F_j^{-1}\left(\left(R_{j'}^{(i)}\right)^{q_j^{(i)}}\right) \quad .$$

Observe that for $j'=j$, we get that

$$\psi_j(v_{i,j}) = F_j^{-1}\left(\left(R_j^{(i)}\right)^{q_j^{(i)}}\right) = F_j^{-1}\left(\left(F_j(v_{i,j})^{\frac{1}{q_j^{(i)}}}\right)^{q_j^{(i)}}\right) = F_j^{-1}\left(F_j(v_{i,j})\right) = v_{i,j} \quad .$$

Further, we can compute the CDF as

$$\begin{aligned}\mathbf{Pr}\left[\psi_j(v_{i,j'}) \leq t\right] &= \mathbf{Pr}\left[F_j^{-1}\left(\left(R_{j'}^{(i)}\right)^{q_j^{(i)}}\right) \leq t\right] = \mathbf{Pr}\left[R_{j'}^{(i)} \leq F_j(t)^{\frac{1}{q_j^{(i)}}}\right] \\ &= \mathbf{Pr}\left[F_{j'}(v_{i,j'}) \leq F_j(t)^{\frac{q_{j'}^{(i)}}{q_j^{(i)}}}\right] = F_j(t)^{\frac{q_{j'}^{(i)}}{q_j^{(i)}}} \quad ,\end{aligned}$$

where in the last step, we used that $F_{j'}(v_{i,j'}) \sim \mathrm{Unif}[0,1]$. As a consequence,

$$\begin{aligned}\mathbf{Pr}\left[v_{i,j} \leq t \,\middle|\, X_{i,j} = 1, M^{(i)}\right] &= \mathbf{Pr}\left[\psi_j(v_{i,j}) \leq t \,\middle|\, \psi_j(v_{i,j}) > \psi_j(v_{i,j'}) \text{ for } j \neq j', M^{(i)}\right] \\ &= \mathbf{Pr}\left[\max_{j' \in M^{(i)}} (\psi_j(v_{i,j'})) \leq t \,\middle|\, M^{(i)}\right] \\ &= \prod_{j' \in M^{(i)}} \mathbf{Pr}\left[\psi_j(v_{i,j'}) \leq t\right] = \prod_{j' \in M^{(i)}} F_j(t)^{\frac{q_{j'}^{(i)}}{q_j^{(i)}}} \\ &= F_j(t)^{\frac{\sum_{j' \in M^{(i)}} q_{j'}^{(i)}}{q_j^{(i)}}} = F_j(t)^{\frac{n-i+1}{q_j^{(i)}}} . \end{aligned}$$ ◀

For integral values of $\frac{n-i+1}{q_j^{(i)}}$ (in particular $q_j^{(i)} = 1$), observe that this is exactly the CDF of the maximum of $\frac{n-i+1}{q_j^{(i)}}$ independent draws from distribution F_j.

For the remainder of the proof sketch, let us restrict to the case that $m = n$. Observe that in this special case, we have that all $q_j^{(i)} = 1$, so the probability in the quantile allocation rule of allocating any item $j \in M^{(i)}$ in Observation 2 simplifies to $\frac{1}{n-i+1}$. Therefore,

$$\mathbf{E}\left[v_{i,j} X_{i,j}\right] = \frac{n-i+1}{n} \cdot \frac{1}{n-i+1} \cdot \mathbf{E}\left[v_{i,j} \mid X_{i,j} = 1\right] = \frac{1}{n} \mathbf{E}\left[v_{i,j} \mid X_{i,j} = 1\right] \ .$$

Observe that this argument looks rather innocent in the special case of $m = n$, but requires a much more careful treatment in the general variant: The probabilities $q_j^{(i)}$ are random variables themselves depending on the set $M^{(i)}$. Hence, the calculation can not directly be extended and a more sophisticated argument needs to be applied. In addition, by the above considerations on the quantile allocation rule via Lemma 3, we have that $\mathbf{E}\left[v_{i,j} \mid X_{i,j} = 1\right] = \mathbf{E}\left[\max_{i' \in [n-i+1]} v_{i',j}\right]$ in the special case of $m = n$. Therefore, we can now simply apply Lemma 4 (see below) to get

$$\mathbf{E}\left[v_{i,j} X_{i,j}\right] = \frac{1}{n} \mathbf{E}\left[v_{i,j} \mid X_{i,j} = 1\right] = \frac{1}{n}\, \mathbf{E}\left[\max_{i' \in [n-i+1]} v_{i',j}\right] \geq \frac{1}{n} \cdot \frac{H_{n-i+1}}{H_n} \cdot \mathbf{E}\left[\max_{i' \in [n]} v_{i',j}\right]$$

Note that we take the maximum over i.i.d. random variables. As a consequence, we can conclude by basic calculations:

$$\begin{aligned}\mathbf{E}\left[\mathrm{SW}_{\mathrm{pp}}\right] \geq \mathbf{E}\left[\mathrm{SW}_{\mathrm{quantile}}\right] &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{E}\left[v_{i,j} X_{i,j}\right] \geq \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{H_{n-i+1}}{H_n} \mathbf{E}\left[\max_{i' \in [n]} v_{i',j}\right] \\ &= \frac{\sum_{i=1}^{n} H_{n-i+1}}{n H_n} \sum_{j=1}^{n} \mathbf{E}\left[\max_{i' \in [n]} v_{i',j}\right] = \left(1 - O\left(\frac{1}{\log n}\right)\right) \sum_{j=1}^{n} \mathbf{E}\left[\max_{i' \in [n]} v_{i',j}\right] \\ &\geq \left(1 - O\left(\frac{1}{\log n}\right)\right) \mathbf{E}\left[\mathrm{SW}_{\mathrm{opt}}\right]\end{aligned}$$

Observe that in the general version, comparing to $\sum_{j=1}^{n} \mathbf{E}\left[\max_{i' \in [n]} v_{i',j}\right]$ is a far too strong benchmark. Therefore, we consider an ex-ante relaxation of the offline optimum. As a new technical tool, we introduce in Lemma 5 (see below) an appropriate bound which allows to lower bound the expected maximum of k draws from an MHR distribution by a suitable fraction of $\mathbf{E}\left[v_{i,j} \,\middle|\, v_{i,j} \geq F^{-1}(1-q)\right]$ for any choice of $q \in [0,1]$. Applying this, we can lower bound the expected contribution of any item j to the quantile welfare by a suitable fraction of its contribution to the offline optimum.

We conclude by giving the lemmas which were used in the proof. First, let us restate a useful lemma from Babaioff et al. [4]. It allows to compare the expectation of the maximum of n and $n' \leq n$ draws from independent and identically distributed random variables, if the distribution has a monotone hazard rate.

▶ **Lemma 4** (Lemma 5.3 in [4]). *Consider a collection $(X_i)_i$ of independent and identically distributed random variables with a distribution with monotone hazard rate. Then, for any $n' \leq n$, we have*

$$\frac{\mathbf{E}\left[\max_{i \in [n']} X_i\right]}{\mathbf{E}\left[\max_{i \in [n]} X_i\right]} \geq \frac{H_{n'}}{H_n} \geq \frac{\log n'}{\log n} \quad .$$

In addition, we make use of the following lemma which is used in order to make a suitable comparison to the ex-ante relaxation.

▶ **Lemma 5.** *Let $z \in [0,1]$ and $k \in \mathbb{N}$. Further, let $\mathcal{D}$ be a distribution with monotone hazard rate with CDF F, let $X, (Y_i)_i \sim \mathcal{D}$ be independent and identically distributed. For $\alpha \geq \frac{1+\ln\left(\frac{1}{z}\right)}{H_k}$, $\alpha \geq 1$, and $\alpha k \leq \frac{1}{z}$, we have $\mathbf{E}\left[X \;\middle|\; X \geq F^{-1}(1-z)\right] \leq \alpha \mathbf{E}\left[\max_{i \in [k]} Y_i\right]$.*

The proof of this lemma can be found in the full version of the paper.

3.2 Static prices

Next, we would like to demonstrate how to use static prices. We consider the case that the number of items m is upper bounded by $\frac{n}{(\log\log n)^2}$. We set the price for item j to

$$p_j = F_j^{-1}(1-q) \ \text{, where } q = \frac{\log\log n}{n} \ ,$$

which allows us to state the following theorem.

▶ **Theorem 6.** *The posted-prices mechanism with static prices and independent item-valuations is $1 - O\left(\frac{\log\log\log n}{\log n}\right)$-competitive with respect to social welfare.*

As before, we defer the proof of this theorem to the full version of the paper and give a quick sketch here: First, observe that we can bound the probability of selling item j to buyer i by the probability of the event that buyer i has only non-negative utility for this item. This implies a bound on the probability of selling item j in our mechanism. Finally, we combine this with a lower bound on the price p_j and hence are able to bound the revenue (and thus the welfare) obtained by our mechanism. Observe that our guarantee only applies if the number of items m is bounded by $\frac{n}{(\log\log n)^2}$. We leave the extension to the general case as an open problem. As a first step, one could try to derive a suitable bound on the utility of agents in order to extend the result.

4 Asymptotically Tight Bounds for Separable Valuations

Let us now come to *separable valuations*, which are common in ad auctions [14, 27]. That is, in order to determine buyer i's value for item j, let each buyer i have a type $v_i \geq 0$ and let each item have an item-dependent multiplier α_j which can be interpreted as a click through rate in online advertising. Buyer i's value $v_{i,j}$ for being assigned item j is given by $\alpha_j \cdot v_i$. Without loss of generality, we assume that $\alpha_1 \geq \alpha_2 \geq \ldots$ and that $m = n$. The former can be ensured by reordering the items; the latter by adding items with $\alpha_j = 0$ or removing all items $j \in M$ with $j > n$ respectively. Observe that in the case of $m > n$, items of index larger than n are not matched in either the optimum, nor is it beneficial to match one of these items and leave an item $j \leq n$ unmatched. Note that this correlated setting also contains the single-item scenario as a special case since it can be modeled by $\alpha_1 = 1, \alpha_2 = \ldots = 0$. More generally, k identical items can be modeled by $\alpha_1 = \ldots = \alpha_k = 1, \alpha_{k+1} = \ldots = 0$.

The types $v_1, \ldots, v_n \geq 0$ are non-negative, independent and identically distributed random variables with a continuous distribution satisfying the MHR condition. Let j_i denote the item allocated to buyer i and $j_i = m + 1$ if buyer i is not allocated any item where $\alpha_{m+1} = 0$. We can specify the expected social welfare of the matching computed by the mechanism as $\mathbf{E}\left[\sum_{i=1}^n \alpha_{j_i} v_i\right] =: \mathbf{E}\left[\mathrm{SW_{pp}}\right]$.

Additionally, the structure of the optimal matching can be stated explicitly. Given any type profile $v = (v_1, \ldots, v_n)$, we let $v_{(k)}$ denote the k-th highest order statistics. That is, $v_{(k)}$ is the largest x such that there are at least k entries in v whose value is at least x. Denote its expectation by $\mathbf{E}\left[v_{(k)}\right] = \mu_k$. The allocation that maximizes social welfare assigns item 1 to a buyer of type $v_{(1)}$, item 2 to a buyer of type $v_{(2)}$ and so on. Hence, the expected optimal social welfare is given by $\mathbf{E}\left[\mathrm{SW_{opt}}\right] = \mathbf{E}\left[\sum_{j=1}^m \alpha_j v_{(j)}\right] = \sum_{j=1}^m \alpha_j \mu_j$.

4.1 Dynamic prices

First, we focus on posted-prices mechanisms with dynamic prices. Consider step i and buyer i arrives. Let $M^{(i)}$ be the set of remaining items at this time. Our choice of prices ensures that in each step one item is allocated. Therefore, always $|M^{(i)}| = n - i + 1$.

We choose prices $(p_j^{(i)})_{j \in M^{(i)}}$ with the goal that each item is allocated with probability $\frac{1}{n-i+1}$. To this end, let $M^{(i)} = \{\ell_1, \ldots, \ell_{n-i+1}\}$ with $\ell_1 < \ell_2 < \ldots < \ell_{n-i+1}$ and set

$$p_j^{(i)} = \sum_{k: j \leq \ell_k \leq n-i} (\alpha_{\ell_k} - \alpha_{\ell_{k+1}}) F^{-1}\left(1 - \frac{k}{n-i+1}\right) \quad .$$

Given this pricing scheme, we can state the following theorem.

▶ **Theorem 7.** *The posted-prices mechanism with dynamic prices and separable valuations is* $1 - O\left(\frac{1}{\log n}\right)$*-competitive with respect to social welfare.*

The proof can be found in the full version of the paper. First, observe that in principle, buyers will be indifferent between two items j and j' if $\alpha_j = \alpha_{j'}$. As these items are indistinguishable for later buyers anyway and new prices will be defined, we can assume that ties are broken in our favor. That is why we can assume that buyer i will prefer item ℓ_k if and only if $F^{-1}\left(1 - \frac{k}{n-i+1}\right) \leq v_i < F^{-1}\left(1 - \frac{k-1}{n-i+1}\right)$.

Using a suitable lower bound for $F^{-1}\left(1 - \frac{k}{n-i+1}\right)$ via the MHR property, we get a lower bound for the value $v_{i,j}$ if i is matched to j. To this end, we compare quantiles of MHR distributions to the respective order statistics. As stated before, by Section 5, the competitive ratio is optimal.

4.2 Static prices

When restricting to the case of static prices, we define probabilities q_j having the interpretation that a buyer drawn from the distribution has one of items $1, \ldots, j$ as their first choice. For technical reasons, we discard items $\widehat{m} + 1, \ldots, n$ for $\widehat{m} = n - n^{5/6}$ by setting $p_j = \infty$ for these items. For $j \leq \widehat{m}$, we set prices

$$p_j = \sum_{k=j}^{n} (\alpha'_k - \alpha'_{k+1}) F^{-1}(1 - q_k) \ , \text{ where } q_k = \min\left\{ \frac{k}{n} 2 \log\log n, \frac{k}{n} + \sqrt{\frac{\log n}{n}} \right\} \ ,$$

where $\alpha'_k = \alpha_k$ for $k \leq \widehat{m}$ and 0 otherwise.

Note the similarity of this price definition to the payments when applying the VCG mechanism. There, the buyer being assigned item j has to pay $\sum_{k=j}^{n} (\alpha'_k - \alpha'_{k+1}) v_{(k+1)}$.

▶ **Theorem 8.** *The posted-prices mechanism with static prices and separable valuations is* $1 - O\left(\frac{\log\log\log n}{\log n}\right)$*-competitive with respect to social welfare.*

The proof of this theorem can be found in the full version of the paper. The general steps are as follows. We first show that our prices are fairly low, meaning that the probability of selling all items $1, \dots, \widehat{m}$ is reasonably high. Having this, we decompose the social welfare into utility and revenue. The revenue of our mechanism is bounded in terms of the VCG revenue. To this end, we use that our pricing rule is quantile-based and exploit that the quantiles of any MHR distributions are lower-bounded by suitable fractions of expected order statistics. Talking about utility, we use a link to Myerson's theory and virtual values in order to achieve our desired bound. Again, by Section 5, the competitive ratio is asymptotically tight.

5 Asymptotically Upper Bounds on the Competitive Ratios

Our competitive ratios are asymptotically tight. In this section we provide matching upper bounds showing optimality. To this end, we consider the case of selling a single item with static and dynamic prices respectively. In any of the two cases, we can achieve asymptotic upper bounds on the competitive ratio of posted prices mechanisms which match our results from the previous sections. In particular, we prove that these bounds hold for any choice of pricing strategy.

5.1 Dynamic prices

We consider the guarantee of our dynamic-pricing mechanisms first. Even with a single item and types drawn from an exponential distribution, the best competitive ratio is $1 - \Omega\left(\frac{1}{\log n}\right)$. We simplify notation by omitting indices when possible.

▶ **Proposition 9.** *Let $v_1, \dots, v_n \in \mathbb{R}_{\geq 0}$ be random variables where each v_i is drawn i.i.d. from the exponential distribution with rate 1, i.e., $v_1, \dots, v_n \sim \mathrm{Exp}(1)$. For all dynamic prices, the competitive ratio of the mechanism picking the first v_i with $v_i \geq p^{(i)}$ is upper bounded by* $1 - \Omega\left(\frac{1}{\log n}\right)$.

In order to prove Proposition 9, we use that the expected value of the optimal offline solution (the best value in hindsight) is given by $\mathbf{E}\left[\max_{i \in [n]} v_i\right] = H_n$ [3]. Therefore, it suffices to show that the expected value of any dynamic pricing rule is upper bounded by $H_n - c$ for some constant $c > 0$.

To upper-bound the expected social welfare of any dynamic pricing rule, we use the fact that this problem corresponds to a Markov decision process and the optimal dynamic prices are given by[3]

$$p^{(n)} = 0 \quad \text{ and } \quad p^{(i)} = \mathbf{E}\left[\max\{v_{i+1}, p^{(i+1)}\}\right] \quad \text{ for } i < n \ .$$

[3] To the best of our knowledge, this is a folklore result.

Furthermore, $p^{(0)}$ is exactly the expected social welfare of this mechanism. Therefore, the following lemma with $k = n$ directly proves our claim.

▶ **Lemma 10.** *Let $v_1, \dots, v_n \in \mathbb{R}_{\geq 0}$ be random variables where each v_i is drawn i.i.d. from the exponential distribution* $\mathrm{Exp}(1)$*. Moreover, let $p^{(n)} = 0$ and $p^{(i)} = \mathbf{E}\left[\max\{v_{i+1}, p^{(i+1)}\}\right]$ for $i < n$. Then, we have $p^{(n-k)} \leq H_k - \frac{1}{8}$ for all $2 \leq k \leq n$.*

The proof via induction over k can be found in the full version of the paper.

5.2 Static prices

For static pricing rules, we show that any mechanism is $1 - \Omega\left(\frac{\log\log\log n}{\log n}\right)$-competitive. Again, this bound even holds for a single item and the valuations being drawn from an exponential distribution.

▶ **Proposition 11.** *Let $v_1, \dots, v_n \in \mathbb{R}_{\geq 0}$ be random variables where each v_i is drawn i.i.d. from the exponential distribution with rate 1, i.e., $v_1, \dots, v_n \sim \mathrm{Exp}(1)$. For all static prices $p \in \mathbb{R}_{\geq 0}$ the competitive ratio of the mechanism picking the first v_i with $v_i \geq p$ is upper bounded by $1 - \Omega\left(\frac{\log\log\log n}{\log n}\right)$.*

The proof of Proposition 11 can be found in the full version of the paper. The idea is as follows. The expected welfare obtained by the static-price mechanism using price p is given by $\mathbf{E}\left[\mathrm{SW_{pp}}\right] = \mathbf{E}\left[v \mid v \geq p\right] \cdot \mathbf{Pr}\left[\exists i : v_i \geq p\right] = (p+1) \cdot \left(1 - (1 - \mathrm{e}^{-p})^n\right)$. This has to be compared to the expected value of the optimal offline solution (the best value in hindsight), which is given by $\mathbf{E}\left[\max_{i \in [n]} v_i\right] = H_n$ [3].

6 Extensions to Subadditive Buyers and Revenue Considerations

In this section, we illustrate that the same style of mechanisms used for unit-demand buyers in principle is also applicable for subadditive buyers. A valuation function v_i is subadditive if $v_i(S \cup T) \leq v_i(S) + v_i(T)$ for any $S, T \subseteq M$. This generalizes unit-demand functions considered so far in this paper. Instead of being interested in only a single-item, each buyer now has a subadditive valuation function over item bundles and can thus be interested in multiple items.

To generalize the MHR property, we assume that the subadditive valuation functions are drawn from distributions with MHR marginals. That is, $v_i \sim \mathcal{D}$ and we assume that $v_i(\{j\})$ has a marginal distribution with monotone hazard rate. Buyers arrive online and purchase the bundle of items which maximizes the buyer's utility.

We can construct a dynamic-pricing mechanism which is $1 - O\left(\frac{1+\log m}{\log n}\right)$-competitive. For a detailed explanation, we refer to the full version of the paper. The general approach is to split the set of buyers in subgroups of size $\left\lfloor \frac{n}{m} \right\rfloor$ and sell each item to one of these groups. For the k-th buyer in every group, the price for the item in question is set to $p_j^{(k)} = F_j^{-1}\left(1 - \frac{1}{\left\lfloor \frac{n}{m} \right\rfloor - k + 1}\right)$, where F_j^{-1} denotes the quantile function of the marginal distribution of $v_i(\{j\})$. Using techniques similar to the ones in the unit-demand case allows to bound the revenue of the posted-prices mechanism by the desired fraction of the optimal social welfare. Hence, the argument directly implies the respective bounds for welfare and revenue.

In the static pricing environment, our results can be extended to a mechanism which is $1 - O\left(\frac{\log\log\log n}{\log n} + \frac{\log m}{\log n}\right)$-competitive for subadditive buyers. Details can be found in the full version. As before, let F_j^{-1} be the quantile function of the marginal distribution of

$v_i(\{j\})$. Setting fairly low prices of $p_j = F_j^{-1}(1-q)$ for $q = \frac{m \log\log n}{n}$ ensures that we sell all items with a suitably high probability. Afterwards, we can apply the same bounds for MHR distributions as in the previous sections in order to bound the revenue of the mechanism with the respective fraction of the optimal social welfare. Again, this directly implies the mentioned competitive ratio for welfare as well as revenue, as the revenue of any individually rational mechanism is upper-bounded by the corresponding social welfare.

Note that the guarantees now depend on the number of items m. To make them meaningful, we need $m = o(n)$. This makes them significantly worse than the ones we obtain for unit-demand functions with a much more careful treatment. However, they are stronger in one aspect, namely that in both cases we bound the revenue of the mechanism in terms of the optimal social welfare. In particular, this means that they are also approximations of the optimal revenue. Interestingly, the optimality results from Section 5 also transfer.

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