Approximation Schemes for Bounded Distance Problems on Fractionally Treewidth-Fragile Graphs

Zdeněk Dvořák
Charles University, Prague, Czech Republic

Abhiruk Lahiri
Ariel University, Israel

Abstract

We give polynomial-time approximation schemes for monotone maximization problems expressible in terms of distances (up to a fixed upper bound) and efficiently solvable on graphs of bounded treewidth. These schemes apply in all fractionally treewidth-fragile graph classes, a property which is true for many natural graph classes with sublinear separators. We also provide quasipolynomial-time approximation schemes for these problems in all classes with sublinear separators.

2012 ACM Subject Classification Theory of computation → Approximation algorithms analysis

Keywords and phrases approximation, sublinear separators

Digital Object Identifier 10.4230/LIPIcs.ESA.2021.40

Related Version *Full Version*: `https://arxiv.org/abs/2105.01780`

Funding *Zdeněk Dvořák*: Supported by the ERC-CZ project LL2005 (Algorithms and complexity within and beyond bounded expansion) of the Ministry of Education of Czech Republic.
Abhiruk Lahiri: Supported by ISF grant 822/18 and Ariel University Post-doctoral fellowship.

Acknowledgements The second author wish to thank the Caesarea Rothschild Institute and the Department of Computer Science, University of Haifa, for providing its facilities for his research activities.

1 Introduction

In this paper, we consider optimization problems such as:

- Maximum r-Independent Set, $r \in \mathbb{Z}^+$: Given a graph G, the objective is to find a largest subset $X \subseteq V(G)$ such that distance in G between any two vertices in X is at least r.
- Maximum weight induced forest: Given a graph G and an assignment $w : V(G) \to \mathbb{Z}_0^+$ of non-negative weights to vertices, the objective is to find a subset $X \subseteq V(G)$ such that $G[X]$ does not contain a cycle and subject to that, $w(X) := \sum_{v \in X} w(v)$ is maximized.
- Maximum (F, r)-Matching, for a fixed connected graph F and $r \in \mathbb{Z}^+$: Given a graph G, the objective is to find a largest subset $X \subseteq V(G)$ such that $G[X]$ can be partitioned into vertex-disjoint copies of F such that distance in G between any two vertices belonging to different copies is at least r.

To be precise, to fall into the scope of our work, the problem must satisfy the following conditions:

- It must be a **maximization problem on certain subsets of vertices** of an input graph, possibly with non-negative weights. That is, the problem specifies which subsets of vertices of the input graph are *admissible*, and the goal is to find an admissible subset of largest size or weight.

29th Annual European Symposium on Algorithms (ESA 2021).
Editors: Petra Mutzel, Rasmus Pagh, and Grzegorz Herman; Article No. 40; pp. 40:1–40:10
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

- The problem must be **defined in terms of distances between the vertices, up to some fixed bound**. That is, there exists a parameter $r \in \mathbb{Z}^+$ such that for any graphs G and G', sets $X \subseteq V(G)$ and $X' \subseteq V(G')$, and a bijection $f : X \to X'$, if $\min(r, d_G(u,v)) = \min(r, d_{G'}(f(u), f(v)))$ holds for all $u, v \in X$, then X is admissible in G if and only if X' is admissible in G'.
- The problem must be **monotone** (i.e., all subsets of an admissible set must be admissible), or at least **near-monotone** (as happens for example for Maximum (F,r)-Matching) in the following sense: There exists a parameter $c \in \mathbb{Z}^+$ such that for any admissible set A in a graph G, there exists a system $\{R_v \subseteq A : v \in A\}$ of subsets of A such that
  - each vertex belongs to R_v for at most c vertices $v \in A$,
  - $v \in R_v$ for each $v \in A$, and
  - for every $Z \subseteq A$, the subset $A \setminus \bigcup_{v \in Z} R_v$ is admissible in G.
- The problem must be **tractable in graphs of bounded treewidth**, that is, there must exist a function g and a polynomial p such that given any graph G, its tree decomposition of width t, an assignment w of non-negative weights to the vertices of G, and a set $X_0 \subseteq X$, it is possible to find a maximum-weight admissible subset of X_0 in time $g(t)p(|V(G)|)$.

Let us call such problems $(\leq r)$*-distance determined* c*-near-monotone* (g,p)*-tw-tractable.* Note that a convenient way to verify these assumptions is to show that the problem is expressible in *solution-restricted Monadic Second-Order Logic* (MSOL) *with bounded-distance predicates*, i.e., by a MSOL formula with one free variable X such that the quantification is restricted to subsets and elements of X, and using binary predicates $d_1, \ldots, d_r$, where $d_i(u,v)$ is interpreted as testing whether the distance between u and v in the whole graph is at most i. For example, r-Independent Set is expressed by the formula $(\forall u, v \in X)\ u = v \vee \neg d_r(u,v)$. This ensures that the problem is $(\leq r)$-distance determined, and $(g, O(n))$-tw-tractable for some function g by Courcelle's meta-algorithmic result [5].

Of course, the problems satisfying the assumptions outlined above are typically hard to solve optimally, even in rather restrictive circumstances. For example, Maximum Independent Set is NP-hard even in planar graphs of maximum degree at most 3 and arbitrarily large (fixed) girth [1]. Moreover, it is hard to approximate it within factor of 0.995 in graphs of maximum degree at most three [4]. Hence, to obtain polynomial-time approximation schemes (PTASes), i.e., polynomial-time algorithms for approximating within any fixed precision, a restriction other than just bounding the maximum degree is needed.

A natural restriction that has been considered in this context is the requirement that the graphs have sublinear separators (a set S of vertices of a graph G is a *balanced separator* if every component of $G \setminus S$ has at most $|V(G)|/2$ vertices, and a hereditary class $\mathcal{G}$ of graphs has *sublinear separators* if for some $c < 1$, every graph $G \in \mathcal{G}$ has a balanced separator of size $O(|V(G)|^c)$). This restriction still lets us speak about many interesting graph classes (planar graphs [19] and more generally proper minor-closed classes [2], many geometric graph classes [21], ...). Moreover, the problems discussed above admit PTASes in all classes with sublinear separators or at least in substantial subclasses of these graphs:

- Maximum Independent Set has been shown to admit a PTAS in graphs with sublinear separators already in the foundational paper of Lipton and Tarjan [20].
- For any positive integer, Maximum r-Independent Set and several other problems are known to admit PTASes in graphs with sublinear separators by a straightforward local search algorithm [16].
- All of the problems mentioned above (and more) are known to admit PTASes in planar graphs by a layering argument of Baker [3]; this approach can be extended to some related graph classes, including all proper minor-closed classes [6, 12].

- The problems also admit PTASes in graph classes that admit thin systems of overlays [11], a technical property satisfied by all proper minor-closed classes and by all hereditary classes with sublinear separators and bounded maximum degree.
- Bidimensionality arguments [7] apply to a wide range of problems in proper minor-closed graph classes.

However, each of the outlined approaches has drawbacks. On one side, the local search approach only applies to specific problems and does not work at all in the weighted setting. On the other side of the spectrum, Baker's approach is quite general as far as the problems go, but there are many hereditary graph classes with sublinear separators to which it does not seem to apply. The approach through thin systems of overlays tries to balance these concerns, but it is rather technical and establishing this property is difficult.

Another option that has been explored is via *fractional treewidth-fragility.* For a function $f\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ and a polynomial p, a class of graphs $\mathcal{G}$ is *p-efficiently fractionally treewidth-f-fragile* if there exists an algorithm that for every $k \in \mathbb{Z}^+$ and a graph $G \in \mathcal{G}$ returns in time $p(|V(G)|)$ a collection of subsets $X_1, X_2, \ldots, X_m \subseteq V(G)$ such that each vertex of G belongs to at most m/k of the subsets, and moreover, for $i = 1, \ldots, m$, the algorithm also returns a tree decomposition of $G \setminus X_i$ of width at most $f(k, |V(G)|)$. We say a class is *p-efficiently fractionally treewidth-fragile* if f does not depend on its second argument (the number of vertices of G). This property turns out to hold for basically all known natural graph classes with sublinear separators. In particular, a hereditary class $\mathcal{G}$ of graphs is efficiently fractionally treewidth-fragile if

- $\mathcal{G}$ has sublinear separators and bounded maximum degree [9],
- $\mathcal{G}$ is proper minor-closed [8, 12], or
- $\mathcal{G}$ consists of intersection graphs of convex objects with bounded aspect ratio in a finite-dimensional Euclidean space and the graphs have bounded clique number, as can be seen by a modification of the argument of Erlebach et al. [15]. This includes all graph classes with polynomial growth [18].

In fact, Dvořák conjectured that every hereditary class with sublinear separators is fractionally treewidth-fragile, and gave the following result towards this conjecture.

▶ **Theorem 1** (Dvořák [10]). *There exists a polynomial p so that the following claim holds. For every hereditary class $\mathcal{G}$ of graphs with sublinear separators, there exists a polynomial q such that $\mathcal{G}$ is p-efficiently fractionally treewidth-f-fragile for the function $f(k, n) = q(k \log n)$.*

Moreover, Dvořák [9] observed that weighted Maximum Independent Set admits a PTAS in any efficiently fractionally treewidth-fragile class of graphs. Indeed, the algorithm is quite simple, based on the observation that for the sets $X_1, \ldots, X_m$ from the definition of fractional treewidth-fragility, at least one of the graphs $G \setminus X_1, \ldots, G \setminus X_m$ (of bounded treewidth) contains an independent set whose weight is within the factor of $1 - 1/k$ from the optimal solution. A problem with this approach is that it does not extend to more general problems; even for the Maximum 2-Independent Set problem, the approach fails, since a 2-independent set in $G \setminus X_i$ is not necessarily 2-independent in G. Indeed, this observation served as one of the motivations behind more restrictive (and more technical) concepts employed in [11, 12].

As our main result, we show that this intuition is in fact false: There is a simple way how to extend the approach outlined in the previous paragraph to all bounded distance determined near-monotone tw-tractable problems.

▶ **Theorem 2.** *For every class $\mathcal{G}$ of graphs with bounded expansion, there exists a function $h : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following claim holds. Let c and r be positive integers, $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ functions and p and q polynomials. If $\mathcal{G}$ is q-efficiently fractionally treewidth-f-fragile, then for every $(\leq r)$-distance determined c-near-monotone (g, p)-tw-tractable problem, there exists an algorithm that given a graph $G \in \mathcal{G}$, an assignment of non-negative weights to vertices, and a positive integer k, returns in time $h(r, c)|V(G)| + q(|V(G)|) \cdot p(|V(G)|) \cdot g(f(h(r, c)k, |V(G)|))$ an admissible subset of $V(G)$ whose weight is within the factor of $1 - 1/k$ from the optimal one.*

Note that the assumption that $\mathcal{G}$ has bounded expansion is of little consequence – it is satisfied for any hereditary class with sublinear separators [14] as well as for any fractionally treewidth-fragile class [9]; see Section 2 for more details. The time complexity of the algorithm from Theorem 2 is polynomial if f does not depend on its second argument, and quasipolynomial (exponential in a polylogarithmic function) if f is logarithmic in the second argument and g is single-exponential, i.e., if $g(n) = \exp\bigl(n^{O(1)}\bigr)$. Hence, we obtain the following corollaries.

▶ **Corollary 3.** *Let c and r be positive integers, $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ a function and p a polynomial. Every $(\leq r)$-distance determined c-near-monotone (g, p)-tw-tractable problem admits a* PTAS *in any efficiently fractionally treewidth-fragile class of graphs.*

We say a problem admits a quasipolynomial-time approximation schemes (QPTAS) if there exist quasipolynomial-time algorithms for approximating the problem within any fixed precision. Combining Theorems 1 and 2, we obtain the following result.

▶ **Corollary 4.** *Let c and r be positive integers, $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ a single-exponential function, and p a polynomial. Every $(\leq r)$-distance determined c-near-monotone (g, p)-tw-tractable problem admits a* QPTAS *in any hereditary class of graphs with sublinear separators.*

The idea of the algorithm from Theorem 2 is quite simple: We consider the sets $X_1, \ldots, X_m$ from the definition of fractional treewidth-f-fragility, extend them to suitable supersets Y_1, $\ldots$, Y_m, and argue that for $i = 1, \ldots, m$, any admissible set in $G \setminus X_i$ disjoint from Y_i is also admissible in G, and that for some i, the weight of the heaviest admissible set in $G \setminus X_i$ disjoint from Y_i is within the factor of $1 - 1/k$ from the optimal one. The construction of the sets Y_1, $\ldots$, Y_m is based on the existence of orientations with bounded outdegrees that represent all short paths, a generalization of a result Kowalik and Kurowski [17] that we present in Section 2.

Let us remark one can develop the idea of this paper in further directions. Dvořák proved in [13](via a substantially more involved argument) that every monotone maximization problem expressible in first-order logic admits a PTAS in any efficiently fractionally treewidth-fragile class of graphs. Note that this class of problems is incomparable with the one considered in this paper (e.g., Maximum Induced Forest is not expressible in first-order logic, while Maximum Independent Set consisting of vertices belonging to triangles is expressible in first-order logic but does not fall into the scope of the current paper).

Finally, it is worth mentioning that our results only apply to maximization problems. We were able to extend the previous uses of fractional treewidth-fragility by giving a way to handle dependencies over any bounded distance. However, for the minimization problems, we do not know whether fractional treewidth-fragility is sufficient even for the distance-1 problems. For a simple example, consider the Minimum Vertex Cover problem in fractionally treewidth-fragile graphs, or more generally in hereditary classes with sublinear separators. While the unweighted version can be dealt with by the local search method [16], we do not know whether there exists a PTAS for the weighted version of this problem.

2 Paths and orientations in graphs with bounded expansion

For $r \in \mathbb{Z}_0^+$, a graph H is an *r-shallow minor* of a graph G if H can be obtained from a subgraph of G by contracting pairwise vertex-disjoint connected subgraphs, each of radius at most r. For a function $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$, a class $\mathcal{G}$ of graphs has *expansion bounded* by f if for all non-negative integers r, all r-shallow minors of graphs from $\mathcal{G}$ have average degree at most $f(r)$. A class has *bounded expansion* if its expansion is bounded by some function f. The theory of graph classes with bounded expansion has been developed in the last 15 years, and the concept has found many algorithmic and structural applications; see [23] for an overview. Crucially for us, this theory includes a number of tools for dealing with short paths. Moreover, as we have pointed out before, all hereditary graph classes with sublinear separators [14] as well as all fractionally treewidth-fragile classes [9] have bounded expansion.

Let $\vec{G}$ be an orientation of a graph G, i.e, uv is an edge of G if and only if the directed graph $\vec{G}$ contains at least one of the directed edges (u,v) and (v,u); note that we allow $\vec{G}$ to contain both of them at the same time, and thus for the edge uv to be oriented in both directions. We say that a directed graph $\vec{H}$ with the same vertex set is a *1-step fraternal augmentation of* $\vec{G}$ if $\vec{G} \subseteq \vec{H}$, for all distinct edges $(x,y),(x,z) \in E(\vec{G})$, either (y,z) or (z,y) is an edge of $\vec{H}$, and for each edge $(y,z) \in E(\vec{H}) \setminus E(\vec{G})$, there exists a vertex $x \in V(\vec{G}) \setminus \{y,z\}$ such that $(x,y),(x,z) \in E(\vec{G})$. That is, to obtain $\vec{H}$ from $\vec{G}$, for each pair of edges $(x,y),(x,z) \in E(\vec{G})$ we add an edge between y and z in one of the two possible directions (we do not specify the direction, but in practice we would choose directions of the added edges that minimize the maximum outdegree of the resulting directed graph). For an integer $a \ge 0$, we say $\vec{F}$ is an *a-step fraternal augmentation of* $\vec{G}$ if there exists a sequence $\vec{G} = \vec{G}_0, \vec{G}_1, \ldots, \vec{G}_a = \vec{F}$ where for $i = 1, \ldots, a$, $\vec{G}_i$ is a 1-step fraternal augmentation of $\vec{G}_{i-1}$. We say $\vec{F}$ is an a-step fraternal augmentation of an undirected graph G if $\vec{F}$ is an a-step fraternal augmentation of some orientation of G.

A key property of graph classes with bounded expansion is the existence of fraternal augmentations with bounded outdegrees. Let us remark that whenever we speak about an algorithm returning an a-step fraternal augmentation $\vec{H}$ or taking one as an input, this implicitly includes outputting or taking as an input the whole sequence of 1-step fraternal augmentations ending in $\vec{H}$.

▶ **Lemma 5** (Nešetřil and Ossona de Mendez [22]). *For every class $\mathcal{G}$ with bounded expansion, there exists a function $d : \mathbb{Z}_0^+ \to \mathbb{Z}^+$ such that for each $G \in \mathcal{G}$ and each non-negative integer a, the graph G has an a-step fraternal augmentation of maximum outdegree at most $d(a)$. Moreover, such an augmentation can be found in time $O(d(a)|V(G)|)$.*

As shown already in [22], fraternal augmentations can be used to succinctly represent distances between vertices of the graph. For the purposes of this paper, we need a more explicit representation by an orientation of the original graph without the additional augmentation edges, as we only assume that the original (rather than the augmented) graph is fractionally treewidth-fragile. Let us remark that the existence of such a representation was shown by Kowalik and Kurowski [17] in a more restrictive setting of graph classes closed under topological minors.

By a *walk* in a directed graph $\vec{G}$, we mean a sequence $W = v_0 v_1 v_2 \ldots v_b$ such that for $i = 1, \ldots, b$, $(v_{i-1}, v_i) \in E(\vec{G})$ or $(v_i, v_{i-1}) \in E(\vec{G})$; that is, the walk does not have to respect the orientation of the edges. The walk W is *inward directed* if for some $c \in \{0, \ldots, b\}$, we have $(v_i, v_{i+1}) \in E(\vec{G})$ for $i = 0, \ldots, c-1$ and $(v_i, v_{i-1}) \in E(\vec{G})$ for $i = c+1, \ldots, b$. For a positive integer r, an orientation $\vec{G}$ of a graph G *represents $(\le r)$-distances* if for each $u, v \in V(G)$ and each $b \in \{0, \ldots, r\}$, the distance between u and v in G is at most b if and only if $\vec{G}$ contains an inward-directed walk of length at most b between u and v. Note that given such an orientation with bounded maximum outdegree for a fixed r, we can determine the distance between u and v (up to distance r) by enumerating all (constantly many) walks of length at most r directed away from u and away from v and inspecting their intersections.

Our goal now is to show that graphs from classes with bounded expansion admit orientations with bounded maximum outdegree that represent $(\leq r)$-distances. Let us define a more general notion used in the proof of this claim, adding to the fraternal augmentations the information about the lengths of the walks in the original graph represented by the added edges. A *directed graph with* $(\leq r)$*-length sets* is a pair $(\vec{H}, \ell)$, where $\vec{H}$ is a directed graph and ℓ is a function assigning a subset of $\{1, \ldots, r\}$ to each *unordered* pair $\{u, v\}$ of vertices of $\vec{H}$, such that if neither (u, v) nor (v, u) is an edge of $\vec{H}$, then $\ell(\{u, v\}) = \emptyset$. We say that $(\vec{H}, \ell)$ is an *orientation* of a graph G if G is the underlying undirected graph of $\vec{H}$ and $\ell(\{u, v\}) = \{1\}$ for each $uv \in E(G)$. We say that $(\vec{H}, \ell)$ is an $(\leq r)$*-augmentation* of G if $V(\vec{H}) = V(G)$, for each $uv \in E(G)$ we have $1 \in \ell(\{u, v\})$, and for each $u, v \in V(G)$ and $b \in \ell(\{u, v\})$ there exists a walk of length b from u to v in G. Let $(\vec{H}_1, \ell_1)$ be another directed graph with $(\leq r)$-length sets. We say $(\vec{H}_1, \ell_1)$ is a *1-step fraternal augmentation* of $(\vec{H}, \ell)$ if $\vec{H}_1$ is a 1-step fraternal augmentation of $\vec{H}$ and for all distinct $u, v \in V(\vec{H})$ and $b \in \{1, \ldots, r\}$, we have $b \in \ell_1(\{u, v\})$ if and only if $b \in \ell(\{u, v\})$ or there exist $x \in V(\vec{H}) \setminus \{u, v\}$, $b_1 \in \ell(\{x, u\})$, and $b_2 \in \ell(\{x, v\})$ such that $(x, u), (x, v) \in E(\vec{H})$ and $b = b_1 + b_2$. Note that a 1-step fraternal augmentation of an $(\leq r)$-augmentation of a graph G is again an $(\leq r)$-augmentation of G. The notion of an a-step fraternal augmentation of a graph G is then defined in the natural way, by starting with an orientation of G and performing the 1-step fraternal augmentation operation a times. Let us now restate Lemma 5 in these terms (we just need to maintain the edge length sets, which can be done with $O(a^2)$ overhead per operation).

▶ **Lemma 6.** *Let $\mathcal{G}$ be a class of graphs with bounded expansion, and let $d : \mathbb{Z}_0^+ \to \mathbb{Z}^+$ be the function from Lemma 5. For each $G \in \mathcal{G}$ and each non-negative integer a, we can in time $O(a^2 d(a)|V(G)|)$ construct a directed graph with $(\leq a+1)$-length sets $(\vec{H}, \ell)$ of maximum outdegree at most $d(a)$ such that $(\vec{H}, \ell)$ is an a-step fraternal augmentation of G.*

Let $(\vec{H}, \ell)$ be an $(\leq r)$-augmentation of a graph G. For $b \leq r$, a *walk of span* b in $(\vec{H}, \ell)$ is a tuple $(v_0 v_1 \ldots v_t, b_1, \ldots, b_t)$, where $v_0 v_1 \ldots v_t$ is a walk in $\vec{H}$, $b_i \in \ell(\{v_{i-1}, v_i\})$ for $i = 1, \ldots, t$, and $b = b_1 + \ldots + b_t$. Note that if there exists a walk of span b from u to v in $(\vec{H}, \ell)$, then there also exists a walk of length b from u to v in G. We say that $(\vec{H}, \ell)$ *represents* $(\leq r)$*-distances* in G if for all vertices $u, v \in V(G)$ at distance $b \leq r$ from one another, $(\vec{H}, \ell)$ contains an inward-directed walk of span b between u and v. Next, we show that this property always holds for sufficient fraternal augmentations.

▶ **Lemma 7.** *Let G be a graph and r a positive integer and let $(\vec{H}, \ell)$ be a directed graph with $(\leq r)$-length sets. If $(\vec{H}, \ell)$ is obtained as an $(r-1)$-step fraternal augmentation of G, then it represents $(\leq r)$-distances in G.*

Proof. For $b \leq r$, consider any walk $W = (v_0 v_1 \ldots v_t, b_1, \ldots, b_t)$ of span b in an $(\leq r)$-augmentation $(\vec{H}_1, \ell_1)$ of G, and let $(\vec{H}_2, \ell_2)$ be a 1-step augmentation of $(\vec{H}_1, \ell_1)$. Note that W is also a walk of span b between v_0 and v_t in $(\vec{H}_2, \ell_2)$. Suppose that W is not inward-directed in $(\vec{H}_1, \ell_1)$, and thus there exists $i \in \{1, \ldots, t-1\}$ such that $(v_i, v_{i-1}), (v_i, v_{i+1}) \in E(\vec{H}_1)$. By the definition of 1-step fraternal augmentation, this implies $b_i + b_{i+1} \in \ell_2(\{v_{i-1}, v_{i+1}\})$, and thus $(v_0 \ldots v_{i-1} v_{i+1} \ldots v_t, b_1, \ldots, b_i + b_{i+1}, \ldots b_t)$ is a walk of span b from v_0 to v_t in $(\vec{H}_2, \ell_2)$.

Let $(\vec{G}_0, \ell_0)$, …, $(\vec{G}_{r-1}, \ell_{r-1})$ be a sequence of $(\leq r)$-augmentations of G, where $(\vec{G}, \ell_0)$ is an orientation of G, $(\vec{G}_{r-1}, \ell_{r-1}) = (\vec{H}, \ell)$, and for $i = 1, \ldots, r-1$, $(\vec{G}_i, \ell_i)$ is a 1-step fraternal augmentation of $(\vec{G}_{i-1}, \ell_{i-1})$. Let u and v be any vertices at distance $b \leq r$ in G, and let P be a shortest path between them. Then P naturally corresponds to a walk P_0 of span b in $(\vec{G}_0, \ell_0)$. For $i = 1, \ldots, r-1$, if P_{i-1} is inward-directed, then let $P_i = P_{i-1}$, otherwise let P_i be a walk of span b in $(\vec{G}_i, \ell_i)$ obtained from P_{i-1} as described in the previous paragraph. Since each application of the operation decreases the number of vertices of the walk, we conclude that P_{r-1} is an inward-directed walk of span b between u and v in $(\vec{H}, \ell)$. Hence, $(\vec{H}, \ell)$ represents $(\le r)$-distances in G. ◀

Next, let us propagate this property back through the fraternal augmentations by orienting some of the edges in both directions. We say that $(\vec{H}, \ell)$ is an *a-step fraternal superaugmentation* of a graph G if there exists an a-step fraternal augmentation $(\vec{F}, \ell)$ of G such that $V(\vec{F}) = V(\vec{H})$, $E(\vec{F}) \subseteq E(\vec{H})$ and for each $(u, v) \in E(\vec{H}) \setminus E(\vec{F})$, we have $(v, u) \in E(\vec{F})$. We say that $(\vec{F}, \ell)$ is a *support* of $(\vec{H}, \ell)$.

▶ **Lemma 8.** *Let G be a graph and r a positive integer and let $(\vec{H}, \ell)$ be an $(\le r)$-augmentation of G of maximum outdegree Δ representing $(\le r)$-distances. For $a \ge 1$, suppose that $(\vec{H}, \ell)$ is an a-step fraternal superaugmentation of G. Then we can in time $O(r^2 \Delta |V(G)|)$ obtain an $(a-1)$-step fraternal superaugmentation of G representing $(\le r)$-distances, of maximum outdegree at most $(r+1)\Delta$.*

Proof. Let $(\vec{F}, \ell)$ be an a-step fraternal augmentation of G forming a support of $(\vec{H}, \ell)$, obtained as a 1-step fraternal augmentation of an $(a-1)$-step fraternal augmentation $(\vec{F}_1, \ell_1)$ of G. Let $(\vec{H}_1, \ell_1)$ be the $(a-1)$-step fraternal superaugmentation of G obtained from $(\vec{F}_1, \ell_1)$ as follows:

- For all distinct vertices $y, z \in V(G)$ such that $(y, z), (z, y) \in E(\vec{H})$, $(y, z) \in E(\vec{F}_1)$, and $(z, y) \notin E(\vec{F}_1)$, we add the edge (z, y).
- For each edge $(y, z) \in E(\vec{H})$ and integer $b \in \ell(\{y, z\}) \setminus \ell_1(\{y, z\})$, we choose a vertex $x \in V(G) \setminus \{y, z\}$ such that $(x, y), (x, z) \in E(\vec{F}_1)$ and $b = b_1 + b_2$ for some $b_1 \in \ell_1(\{x, y\})$ and $b_2 \in \ell_1(\{x, z\})$, and add the edge (y, x). Note that such a vertex x and integers b_1 and b_2 exist, since b was added to $\ell(\{y, z\})$ when $(\vec{F}, \ell)$ was obtained from $(\vec{F}_1, \ell_1)$ as a 1-step fraternal augmentation.

Each edge $(y, x) \in E(\vec{H}_1) \setminus E(\vec{H})$ arises from an edge $(y, z) \in E(\vec{H})$ leaving y and an element $b \in \ell(\{y, z\}) \setminus \ell_1(\{y, z\})$, and each such pair contributes at most one edge leaving y. Hence, the maximum outdegree of $\vec{H}_1$ is at most $(r+1)\Delta$.

Consider an inwards-directed walk $(v_0 v_1 \ldots v_t, b_1, \ldots, b_t)$ of span b in $\vec{H}$, for any $b \le r$. Then $\vec{H}$ contains an inwards-directed walk of span b from v_0 to v_t obtained by natural edge replacements: For any edge $(y, z) \in E(\vec{H})$ of this walk and $b' \in \ell_i(\{y, z\})$, the construction described above ensures that if $(y, z) \notin E(\vec{H}_1)$ or $b' \notin \ell_1(\{y, z\})$, then there exists $x \in V(G) \setminus \{y, z\}$ such that $(y, x), (x, z) \in E(\vec{H}_1)$ and $b' = b'' + b'''$ for some $b'' \in \ell_1(\{x, y\})$ and $b''' \in \ell_1(\{x, z\})$, and we can replace the edge (y, z) in the walk by the edges (y, x) and (x, z) of $E(\vec{H}_1)$. Since $\vec{H}$ represents $(\le r)$-distances in G, this transformation shows that so does $\vec{H}_1$. ◀

We are now ready to prove the main result of this section.

▶ **Lemma 9.** *For any class $\mathcal{G}$ with bounded expansion, there exists a function $d' : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for each $G \in \mathcal{G}$ and each positive integer r, the graph G has an orientation with maximum outdegree at most $d'(r)$ that represents $(\le r)$-distances in G. Moreover, such an orientation can be found in time $O(r^2 d'(r) |V(G)|)$.*

Proof. Let d be the function from Lemma 5, and let $d'(r) = (r+1)^{r-1} d(r-1)$. By Lemma 6, we obtain an $(r-1)$-step fraternal augmentation $(\vec{H}, \ell)$ of G of maximum outdegree at most $d(r-1)$. By Lemma 7, $(\vec{H}, \ell)$ represents $(\le r)$-distances in G. Repeatedly applying Lemma 8, we obtain a 0-step fraternal superaugmentation $(\vec{G}, \ell_0)$ of G of maximum outdegree at most $d'(r)$ representing $(\leq r)$-distances. Clearly, $\vec{G}$ is an orientation of G of maximum outdegree at most $d'(r)$ representing $(\leq r)$-distances. ◀

3 Approximation schemes

Let us now prove Theorem 2. To this end, let us start with a lemma to be applied to the sets arising from fractional treewidth-fragility.

▶ **Lemma 10.** *Let $\vec{G}$ be an orientation of a graph G with maximum outdegree Δ. Let A be a set of vertices of G and for a positive integer c, let $\{R_v : v \in A\}$ be a system of subsets of A such that each vertex belongs to at most c of the subsets. For $X \subseteq V(G)$ and a positive integer r, let $D_{\vec{G},r}(X) \subseteq A$ be the union of the sets R_v for all vertices $v \in A$ such that $\vec{G}$ contains a walk from v to X of length at most r directed away from v. For a positive integer k, let $X_1, \ldots, X_m$ be a system of subsets of $V(G)$ such that each vertex belongs to at most $\frac{m}{c(\Delta+1)^r k}$ of the subsets. For any assignment w of non-negative weights to vertices of G, there exists $i \in \{1, \ldots, m\}$ such that $w(A \setminus D_{\vec{G},r}(X_i)) \geq (1 - 1/k)w(A)$.*

Proof. For a vertex $z \in A$, let $B(z)$ be the set of vertices reachable in $\vec{G}$ from vertices $v \in A$ such that $z \in R_v$ by walks of length at most r directed away from v. Note that $|B(z)| \leq c(\Delta + 1)^r$ and that for each $X \subseteq V(G)$, we have $z \in D_{\vec{G},r}(X)$ if and only if $B(z) \cap X \neq \emptyset$.

Suppose for a contradiction that for each i we have $w(A \setminus D_{\vec{G},r}(X_i)) < (1 - 1/k)w(A)$, and thus $w(D_{\vec{G},r}(X_i)) > w(A)/k$. Then

$$\begin{aligned}\frac{m}{k}w(A) &< \sum_{i=1}^{m} w(D_{\vec{G},r}(X_i)) = \sum_{i=1}^{m} \sum_{z \in D_{\vec{G},r}(X_i)} w(z) = \sum_{i=1}^{m} \sum_{z \in A: B(z) \cap X_i \neq \emptyset} w(z) \\ &\leq \sum_{i=1}^{m} \sum_{z \in A} w(z)|B(z) \cap X_i| = \sum_{z \in A} w(z) \sum_{i=1}^{m} |B(z) \cap X_i| \\ &= \sum_{z \in A} w(z) \sum_{x \in B(z)} |\{i \in \{1, \ldots, m\} : x \in X_i\}| \leq \sum_{z \in A} w(z) \sum_{x \in B(z)} \frac{m}{c(\Delta+1)^r k} \\ &= \sum_{z \in A} w(z)|B(z)| \frac{m}{c(\Delta+1)^r k} \leq \sum_{z \in A} w(z)\frac{m}{k} = \frac{m}{k}w(A),\end{aligned}$$

which is a contradiction. ◀

Next, let us derive a lemma on admissibility for $(\leq r)$-distance determined problems.

▶ **Lemma 11.** *For a positive integer r, let $\vec{G}$ be an orientation of a graph G representing $(\leq r)$-distances. For a set $X \subseteq V(G)$, let $Y_{\vec{G},r}(X)$ be the set of vertices y such that $\vec{G}$ contains a walk from y to X of length at most r directed away from y. For any $(\leq r)$-distance determined problem, a set $B \subseteq V(G) \setminus Y_{\vec{G},r}(X)$ is admissible in G if and only if it is admissible in $G - X$.*

Proof. Since the problem is $(\leq r)$-distance determined, it suffices to show that $\min(r, d_G(u, v)) = \min(r, d_{G-X}(u, v))$ holds for all $u, v \in B$. Clearly, $d_G(u, v) \leq d_{G-X}(u, v)$, and thus it suffices to show that if the distance between u and v in G is $b \leq r$, then $G - X$ contains a walk of length b between u and v. Since $\vec{G}$ represents $(\leq r)$-distances, there exists an inward-directed walk P of length b between u and v in $\vec{G}$. Since $u, v \notin Y_{\vec{G},r}(X)$, we have $V(P) \cap X = \emptyset$, and thus P is also a walk of length b between u and v in $G - X$. ◀

We are now ready to prove the main result.

Proof of Theorem 2. Let d' be the function from Lemma 9 for the class $\mathcal{G}$. Let us define $h(r,c) = c(d'(r)+1)^r$. The algorithm is as follows. Since $\mathcal{G}$ is q-efficiently fractionally treewidth-f-fragile, in time $q(|V(G)|)$ we can find sets $X_1, \ldots, X_m \subseteq V(G)$ such that each vertex belongs to at most $\frac{m}{h(r,c)k}$ of them, and for each i, a tree decomposition of $G - X_i$ of width at most $f(h(r,c)k, |V(G)|)$. Clearly, $m \le q(|V(G)|)$. Next, using Lemma 9, we find an orientation $\vec{G}$ of G that represents $(\le r)$-distances. Let $Y_{\vec{G},r}$ be defined as in the statement of Lemma 11. Since the problem is (g,p)-tw-tractable problem, for each i we can in time $p(|V(G)|) \cdot g(f(h(r,c)k, |V(G)|))$ find a subset A_i of $V(G) \setminus Y_{\vec{G},r}(X_i)$ admissible in $G - X_i$ of largest weight. By Lemma 11, each of these sets is admissible in G; the algorithm return the heaviest of the sets $A_1, \ldots, A_m$.

As the returned set is admissible in G, it suffices to argue about its weight. Let A be a heaviest admissible set in G. Let $\{R_v \subseteq A : v \in A\}$ be the system of subsets from the definition of c-near-monotonicity, and let $D_{\vec{G},r}$ be defined as in the statement of Lemma 10. By the definition of c-near-monotonicity, for each i the set $A \setminus D_{\vec{G},r}(X_i)$ is admissible in G. Since $v \in R_v$ for each $v \in A$, we have $Y_{\vec{G},r}(X_i) \cap A \subseteq D_{\vec{G},r}(X_i)$, and thus $A \setminus D_{\vec{G},r}(X_i) \subseteq V(G) \setminus Y_{\vec{G},r}(X_i)$. By Lemma 11, $A \setminus D_{\vec{G},r}(X_i)$ is also admissible in $G - X_i$, and by the choice of A_i, we have $w(A_i) \ge w(A \setminus D_{\vec{G},r}(X_i))$. By Lemma 10, we conclude that for at least one i, we have $w(A_i) \ge (1 - 1/k)w(A)$, as required. ◀

References

1. Vladimir E. Alekseev, Vadim V. Lozin, Dmitriy S. Malyshev, and Martin Milanic. The maximum independent set problem in planar graphs. In *Mathematical Foundations of Computer Science 2008, 33rd International Symposium, MFCS 2008, Torun, Poland, August 25-29, 2008, Proceedings*, volume 5162 of *Lecture Notes in Computer Science*, pages 96–107. Springer, 2008. doi:10.1007/978-3-540-85238-4_7.
2. Noga Alon, Paul D. Seymour, and Robin Thomas. A separator theorem for graphs with an excluded minor and its applications. In *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing, May 13-17, 1990, Baltimore, Maryland, USA*, pages 293–299. ACM, 1990. doi:10.1145/100216.100254.
3. Brenda S. Baker. Approximation algorithms for np-complete problems on planar graphs. *J. ACM*, 41(1):153–180, 1994. doi:10.1145/174644.174650.
4. Piotr Berman and Marek Karpinski. On some tighter inapproximability results (extended abstract). In *Automata, Languages and Programming, 26th International Colloquium, ICALP'99, Prague, Czech Republic, July 11-15, 1999, Proceedings*, volume 1644 of *Lecture Notes in Computer Science*, pages 200–209. Springer, 1999. doi:10.1007/3-540-48523-6_17.
5. Bruno Courcelle. The monadic second-order logic of graphs. i. recognizable sets of finite graphs. *Inf. Comput.*, 85(1):12–75, 1990. doi:10.1016/0890-5401(90)90043-H.
6. Anuj Dawar, Martin Grohe, Stephan Kreutzer, and Nicole Schweikardt. Approximation schemes for first-order definable optimisation problems. In *21th IEEE Symposium on Logic in Computer Science (LICS 2006), 12-15 August 2006, Seattle, WA, USA, Proceedings*, pages 411–420. IEEE Computer Society, 2006. doi:10.1109/LICS.2006.13.
7. Erik D. Demaine and Mohammad Taghi Hajiaghayi. Bidimensionality: new connections between FPT algorithms and ptass. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2005, Vancouver, British Columbia, Canada, January 23-25, 2005*, pages 590–601. SIAM, 2005. URL: http://dl.acm.org/citation.cfm?id=1070432.1070514.

8 Matt DeVos, Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, Bruce A. Reed, Paul D. Seymour, and Dirk Vertigan. Excluding any graph as a minor allows a low tree-width 2-coloring. *J. Comb. Theory, Ser. B*, 91(1):25–41, 2004. doi:10.1016/j.jctb.2003.09.001.

9 Zdenek Dvorák. Sublinear separators, fragility and subexponential expansion. *Eur. J. Comb.*, 52:103–119, 2016. doi:10.1016/j.ejc.2015.09.001.

10 Zdenek Dvorák. On classes of graphs with strongly sublinear separators. *Eur. J. Comb.*, 71:1–11, 2018. doi:10.1016/j.ejc.2018.02.032.

11 Zdenek Dvorák. Thin graph classes and polynomial-time approximation schemes. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 1685–1701. SIAM, 2018. doi:10.1137/1.9781611975031.110.

12 Zdenek Dvorák. Baker game and polynomial-time approximation schemes. In *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 2227–2240. SIAM, 2020. doi:10.1137/1.9781611975994.137.

13 Zdenek Dvorák. Approximation metatheorem for fractionally treewidth-fragile graphs. *CoRR*, abs/2103.08698, 2021. arXiv:2103.08698.

14 Zdenek Dvorák and Sergey Norin. Strongly sublinear separators and polynomial expansion. *SIAM J. Discret. Math.*, 30(2):1095–1101, 2016. doi:10.1137/15M1017569.

15 Thomas Erlebach, Klaus Jansen, and Eike Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM J. Comput.*, 34(6):1302–1323, 2005. doi:10.1137/S0097539702402676.

16 Sariel Har-Peled and Kent Quanrud. Approximation algorithms for polynomial-expansion and low-density graphs. *SIAM J. Comput.*, 46(6):1712–1744, 2017. doi:10.1137/16M1079336.

17 Lukasz Kowalik and Maciej Kurowski. Oracles for bounded-length shortest paths in planar graphs. *ACM Trans. Algorithms*, 2(3):335–363, 2006. doi:10.1145/1159892.1159895.

18 Robert Krauthgamer and James R. Lee. The intrinsic dimensionality of graphs. *Comb.*, 27(5):551–585, 2007. doi:10.1007/s00493-007-2183-y.

19 Richard J. Lipton and Robert Endre Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.*, 36(2):177–189, 1979. doi:10.1137/0136016.

20 Richard J. Lipton and Robert Endre Tarjan. Applications of a planar separator theorem. *SIAM J. Comput.*, 9(3):615–627, 1980. doi:10.1137/0209046.

21 Gary L. Miller, Shang-Hua Teng, William P. Thurston, and Stephen A. Vavasis. Separators for sphere-packings and nearest neighbor graphs. *J. ACM*, 44(1):1–29, 1997. doi:10.1145/256292.256294.

22 Jaroslav Nesetril and Patrice Ossona de Mendez. Grad and classes with bounded expansion II. algorithmic aspects. *Eur. J. Comb.*, 29(3):777–791, 2008. doi:10.1016/j.ejc.2006.07.014.

23 Jaroslav Nesetril and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012. doi:10.1007/978-3-642-27875-4.