Hitting Weighted Even Cycles in Planar Graphs

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Abstract

A classical branch of graph algorithms is graph transversals, where one seeks a minimum-weight subset of nodes in a node-weighted graph G which intersects all copies of subgraphs F from a fixed family $\mathcal{F}$. Many such graph transversal problems have been shown to admit polynomial-time approximation schemes (PTAS) for *planar* input graphs G, using a variety of techniques like the shifting technique (Baker, J. ACM 1994), bidimensionality (Fomin et al., SODA 2011), or connectivity domination (Cohen-Addad et al., STOC 2016). These techniques do not seem to apply to graph transversals with parity constraints, which have recently received significant attention, but for which no PTASs are known.

In the *even-cycle transversal* (ECT) problem, the goal is to find a minimum-weight hitting set for the set of even cycles in an undirected graph. For ECT, Fiorini et al. (IPCO 2010) showed that the integrality gap of the standard covering LP relaxation is $\Theta(\log n)$, and that adding sparsity inequalities reduces the integrality gap to 10.

Our main result is a primal-dual algorithm that yields a $47/7 \approx 6.71$-approximation for ECT on node-weighted planar graphs, and an integrality gap of the same value for the standard LP relaxation on node-weighted planar graphs.

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1 Introduction

Transversal problems in graphs have received a significant amount of attention from the perspective of algorithm design. Such problems take as input a node-weighted graph G, and seek a minimum-weight subset S of nodes which intersect all graphs F from a fixed graph family $\mathcal{F}$ that appears as subgraph in G. A prominent example in this direction is the fundamental Feedback Vertex Set (FVS) problem, where $\mathcal{F}$ is the class of all cycles. FVS is one of Karp's 21 NP-complete problems [17]. It admits a 2-approximation in polynomial time [2, 5], which cannot be improved to a $(2-\varepsilon)$-approximation for any $\varepsilon > 0$ assuming the Unique Games Conjecture [18].

Recently, several graph transversal problems have been revisited in the presence of additional parity constraints [19, 21, 20, 23]. The natural parity variants of FVS are Odd Cycle Transversal (OCT) and Even Cycle Transversal (ECT), where one wishes to intersect the odd-length and even-length cycles of the input graph G, respectively. The

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approximability of these problems is much less understood than that of FVS: for OCT, only an $\mathcal{O}(\sqrt{\log n})$-approximation is known [1], and for ECT, only a 10-approximation is known [21].

Planar graphs are a natural subclass of graphs in which to consider graph transversal problems. The interest goes back to Baker's shifting technique [3], which yielded a PTAS for Vertex Cover in planar graphs (where $\mathcal{F}$ is the single graph consisting of an edge). The technique was generalized by Demaine et al. [8], who gave EPTASs for graph transversal problems satisfying a certain bidimensionality criterion, including FVS in *unweighted* planar graphs. That result was later extended to yield an EPTAS for FVS in unweighted H-minor free graphs [13], for any fixed graph H. In *edge-weighted* planar graphs, PTAS are known for edge-weighted Steiner Forest and OCT [4, 16, 10].

On *node-weighted* planar graphs, the situation appears to be more complex. First, the existence of a PTAS for FVS on node-weighted planar graphs was a long-standing open question which was resolved only recently in a paper of Cohen-Addad et al. [7]. The authors presented a PTAS for FVS in node-weighted planar graphs, crucially exploiting the fact that the treewidth of $G - S$ is bounded for feasible solutions S. The existence of an EPTAS for FVS in node-weighted planar graphs is still open.

To deal with cycle transversal problems (in node-weighted planar graphs) which are more complex than FVS, Goemans and Williamson [14] first proposed a primal-dual based framework. Their framework requires the cycle family $\mathcal{F}$ to satisfy a certain uncrossing property. The latter property can be seen to be satisfied by OCT, Directed FVS in directed planar graphs, and Subset FVS, which seeks a minimum-cost node set hitting all cycles containing a node from a given node set T. For those problems, the authors obtained 3-approximations[1]. The framework by Berman and Yaroslavtsev [14] also yields a 3-approximation for Steiner Forest in node-weighted planar graphs [9, 22]. Berman and Yaroslavtsev [6] later improved the approximation factor for the same class of uncrossable cycle transversal problems from 3 to 2.4. For none of those problems, though, the existence of a PTAS is known.

The main question driving our work is whether the framework of Goemans and Williamson [14] (and its improvement by Berman and Yaroslavtsev [6]) can be extended to cycle transversal problems that do not satisfy uncrossability. In this paper we focus on ECT in node-weighted planar graphs as a natural such problem: even cycles are not uncrossable, and hence the frameworks by Goemans and Williamson [14] does not apply. Furthermore, the framework of Cohen-Addad et al. [7] requires that contracting edges only reduces the solution value, which is not the case for even cycles either. For *unweighted* planar graphs, it is still possible to obtain an EPTAS for ECT, by building on the work of Fomin et al. [12]. Their main result are EPTASs for bidimensional problems, which ECT is not (as contracting edges can change the parity of cycles). Yet, to obtain their result, they show that any transversal problem that satisfies the "ν-transversability" and "reducibility" conditions has an EPTAS on H-minor free graphs (cf. [12, Theorem 1]). Both conditions are met by unweighted ECT[2], which thus admits an EPTAS on H-minor free graphs. For ECT on node-weighted planar graphs, though, reducibility fails, and the existence of a PTAS is unknown. The currently best result for ECT is a 10-approximation, which was given by Fiorini et al. [11] for general graphs. They showed that the integrality gap of the standard covering LP relaxation for ECT is $\Theta(\log n)$, but that adding sparsity inequalities reduces the integrality gap to 10. No better than 10-approximation is known for ECT in node-weighted planar graphs.

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[1] 18/7-approximations were claimed but later found to be incorrect [6].

[2] ν-transversability follows from as graphs without even cycles have treewidth 2, and reducibility from unit weights and connectedness of the to-be-hit subgraphs F.

1.1 Our results

We prove an improved approximation algorithm for ECT in node-weighted planar graphs.

▶ **Theorem 1.** *ECT has an efficient* $47/7 \approx 6.71$*-approximation on node-weighted planar graphs.*

This improves the previously best 10-approximation by Fiorini et al. [11] for planar graphs.

Our algorithm takes as input a node-weighted planar graph G with node weights $c_v \in \mathbb{N}$ for each $v \in V(G)$. We then employ a primal-dual algorithm that is based on the following natural covering LP for ECT and its dual, where $\mathcal{C}$ denotes the set of even cycles in G:

$$(\mathrm{P_{ECT}}) \quad \begin{aligned} \min\ & c^T x \\ \text{s.t. } & x(C) \geq 1 \quad \forall\, C \in \mathcal{C} \\ & x \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \max\ & \mathbb{1}^T y \\ \text{s.t. } & \sum_{C \in \mathcal{C}, v \in C} y_C \leq c_v \quad \forall v \in V(G) \\ & y \geq \mathbb{0} \end{aligned} \quad (\mathrm{D_{ECT}})$$

Fiorini et al. [11] proved that the integrality gap of this LP is $\Theta(\log n)$. Our main result is an improved integrality gap of this LP for ECT in planar graphs:

▶ **Theorem 2.** *The integrality gap of the LP* $(\mathrm{P_{ECT}})$ *is at most* $47/7 \approx 6.71$ *in planar graphs.*

1.2 Our approach

Designing a primal-dual algorithm is far from trivial, as the imposed parity constraints rule out a direct application of the framework proposed by Goemans and Williamson [14]. Unlike in their work, *face-minimal even cycles* (even cycles containing a minimal set of faces in their interior) are not necessarily faces, and may thus overlap. Indeed, increasing the dual variables of face-minimal even cycles does not yield a constant-factor approximation in general.

Consider Figure 1, and let F be the inner face that is only incident to blue and black nodes. For an even number of 5-cycles surrounding F, F is the only face-minimal even cycle

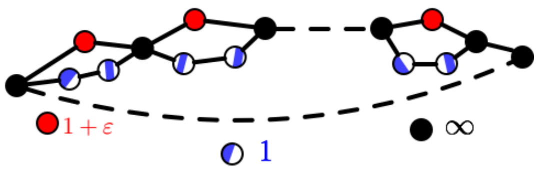

Figure 1 The bottom path has odd length, and the number of length-5 faces at the top is even.

in the graph. Using only F for the dual increase, even including a reverse-delete step, leaves one blue node of each 5-cycle. Yet, an optimal solution would take a single red and blue node from one 5-cycle.

To circumvent this impediment, we establish strong structural properties of planar graphs related to ECT. Those properties along with results from matching theory allow us to algorithmically find a large set of pairwise face-disjoint even cycles whose dual variables we can then increment. Even with this set of cycles found, it remains technically challenging to bound the integrality gap. For this purpose, we first use the structure of minimal hitting sets of our graph to associate each such set with a hitting set in a subdivision of the so called 2-compression of our graph; the latter is a certain minor that we define in detail shortly. We then show that faces that are contained in even cycles we increment are incident to few nodes on average. Crucial in this step is a technical result that is implicit in the work of Berman and Yaroslavtsev [6]. Eventually, this approach leads to an integrality gap of 47/7, and an algorithm with the same approximation guarantee.

Due to space constraints, we defer proofs of statements marked by ($\star$) to the full version of the paper [15].

2 Primal-dual algorithm for ECT on node-weighted planar graphs

We describe a primal-dual, constant-factor approximation for ECT on node-weighted planar graphs. Our algorithm borrows some ideas from Fiorini et al. [11] for the Diamond Hitting Set (DHS) problem, which seeks a minimum-cost set of nodes in a node-weighted graph G that hits all *diamonds* (sub-divisions of the graph consisting of three parallel edges). For DHS, Fiorini et al. [11] employ a primal-dual algorithm to prove that the natural covering LP ($\mathrm{P}_{\mathrm{ECT}}$) (where $\mathcal{C}$ is replaced by the set of diamonds) has integrality gap $\Theta(\log n)$. We develop several new ideas to obtain a constant integrality gap.

We now outline the ideas of our primal-dual approach. Consider a planar input graph G with node costs $c_v \in \mathbb{N}$ for each $v \in V(G)$. Given feasible dual solution y to ($\mathrm{D}_{\mathrm{ECT}}$), let the *residual cost* of node $v \in V(G)$ be $c_v - \sum_{C \in \mathcal{C}, v \in C} y_C$. Our primal-dual method begins with a trivial feasible dual solution $y = \mathbb{0}$, and the empty, infeasible hitting set $S = \emptyset$.

Then, in each iteration, we increase y_C for all C in some carefully chosen subset $\mathcal{C}' \subseteq \mathcal{C}$ of even cycles, while maintaining dual feasibility, and until some *primary condition* is achieved. A common such primary condition is that some dual node-constraint becomes *tight* in the increase process, and hence the corresponding node ends up having residual cost 0.

When this happens, we add the node to S. Once S is a feasible ECT, our algorithm ends its first phase, and executes a problem-specific *reverse-delete* procedure. Here, we consider all nodes in S in reverse order of addition to S, and we delete such a node if the feasibility of S is maintained. We will later describe a subtle and crucial refinement of this reverse-delete procedure. Call the resulting final output of the algorithm S'.

During our algorithm, we will use the term *hitting set* to refer to S, and during the analysis we will use the term *hitting set* to refer to S'. We will say a hitting set is *feasible* if it is a feasible ECT, and refer to nodes of the hitting set as *hit nodes*.

In the next subsections, we will fill in the details of the algorithm, and analyze the cost of S' compared to the value of an optimal solution. We begin by defining the concept of "blended inequalities" and necessary graph compression operations. Blended inequalities were used by Fiorini et al. [11], and our definitions follow their's closely.

2.1 Blended inequalities and compression

A *block* of G is an inclusion-maximal 2-connected subgraph of G. The *block graph* of G is the bipartite graph B_G with bipartition $V(B_G) = B_1 \cup B_2$, where B_1 are the blocks of G, B_2 are the cut nodes of G, and $(b_1, b_2) \in B_1 \times B_2$ is an edge if b_2 is a node of b_1.

Let S be a partial solution to the given ECT instance at some point during the execution of our algorithm. Let G^S be the corresponding residual graph that we obtain from $G - S$ by deleting all nodes that do not lie on even cycles. Our primal-dual algorithm first looks for an even cycle C in G^S such that at most two nodes of C have neighbours outside C. If such a cycle C is found, we increment its dual variable y_C until a node becomes tight. The reason for doing this is that such C will pay for at most two hit nodes, which we will show later.

If there is no even cycle C in G^S such that at most two nodes of C have neighbours outside C, we successively compress the residual graph G^S using two types of graph compression. To this end, first note that any minimal solution will only contain one node in the interior of any induced path in G^S. Suppose we contract some path P of G^S of length at least 2 down to an edge e. Choosing a node in the interior of P is "equivalent" to choosing the edge e. This is the motivation for the 1-*compression*.

Suppose we contract two u-v paths P_1, P_2 with lengths of different parity down to edges e_1, e_2, respectively. We will find it helpful to think of these edges as a single *twin* edge between u and v whose parity is *flexible*. This is the motivation for the 2-*compression*.

Formally, we will successively compress G^S as follows:

- Obtain the 1-*compression* G_1^S of G^S by repeatedly *folding* degree-2 nodes v, as long as they exist, which means to delete v and adding the edge uw between its neighbors u, w.
- Note that no pair of nodes in G_1^S is connected by more than two edges. Obtain $\bar{G}_1^S$ from G_1^S by replacing each pair of parallel edges by a *twin* edge. In $\bar{G}_1^S$, we now once again fold degree-2 nodes as long as those exist. The resulting graph is the 2-*compression* G_2^S of G^S.

See Figure 2 for examples of 1- and 2-compression of a graph. In the following, we will omit the superscript S from G_1^S, $\bar{G}_1^S$, and G_2^S if this is clear from the context. Let G_3 be obtained from G_2 by replacing every edge of G_2 with a path of length two. If a twin edge was replaced, call the two edges of the path added *twin edges*. By an abuse of notation, we call a cycle of G_1, G_2 or G_3 even if it contains a twin edge, or if its preimage in G is even.

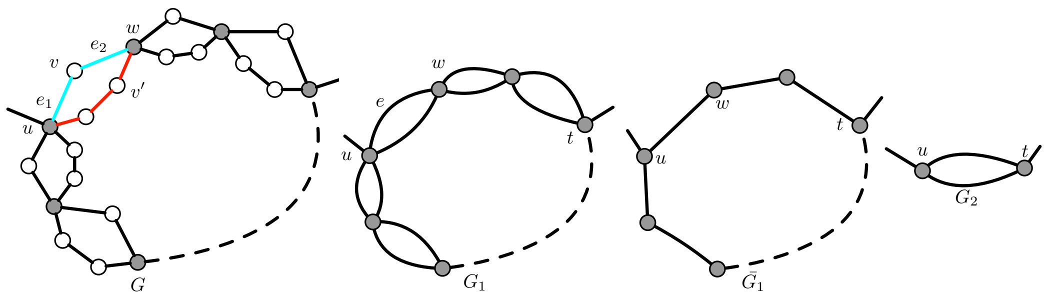

Figure 2 The graph G and its 1- and 2-compression G_1 and G_2.

In the following, we will sometimes call the subgraph Q of G whose contraction yields a subgraph R of G_2 the *preimage* of R. If R is an edge, call Q a *piece*, and say Q *corresponds* to R. Furthermore, call u, v *ends* of Q and other nodes of Q *internal nodes*. If the edge was twin, call the piece *twin*, otherwise, call the piece *single*. The blocks of a piece are cycles and paths, and the block graph of a piece is a path. Each cycle of a piece is called an *elementary cycle*. For an elementary cycle C, call its two nodes u_C and v_C with neighbours outside C *branch nodes*. Call the two $u_C - v_C$-paths P_1, P_2 in C the *handles* of C, which form the *handle pair* (P_1, P_2). For an illustration, see the red and light blue edges in Figure 2.

The reason for defining G_3 is that intuitively selecting a node inside a piece corresponds to selecting the edge corresponding to the piece in G_2. It will be simpler for us if our hitting set consists of only nodes, so we subdivide each edge of G_2. Suppose that S is the partial (and infeasible) hitting set for the cycles in $\mathcal{C}$ at some point during the algorithm. Further, assume that G^S has even cycles, but none with at most two outside neighbours. In this case, if an even cycle C' in G^S contains an internal node of some piece Q, then $C' \cap Q$ is a path between the ends of Q; see Figure 3. It follows that C' has the form $v_1 P_1 v_2 P_2 \ldots v_k P_k v_1$, where for $i = 1, \ldots, k$ nodes $v_i, v_{i+1 \bmod k}$ are ends of some piece Q_i, and P_i is a v_i-v_{i+1} path in Q_i. For $i = 1, \ldots, k$, pieces Q_i, Q_j for $i \neq j$ are disjoint except for their ends.

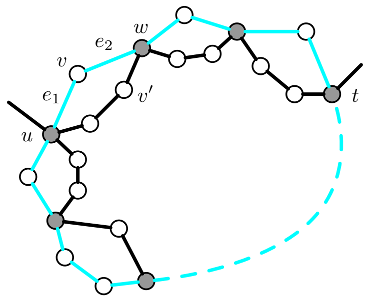

Figure 3 The light blue cycle in G has two u-t paths lying in different pieces of G; the dashed path has odd length.

We say that C' in G^S *corresponds* to cycle $C = (v_1, \ldots, v_k)$ in G^S_2. For such C, its *blended inequality* is

$$\sum_v a^C_v x_v \geq 1, \tag{$\circledast$}$$

where $a^C_v \in \{0, 1/2, 1\}$ for all nodes v, and where the support of a^C is contained in the node set of the preimage of C. We next provide a precise definition of the coefficients of ($\circledast$). With those, one can show that ($\circledast$) is dominated by a convex combination of inequalities $x(C) \geq 1$ in ($\mathrm{P}_{\mathrm{ECT}}$).

Consider an elementary cycle of the preimage of C and let h_1, h_2 be its two handles. For each of these handles, we define its residual cost as the smallest residual cost of any of its internal nodes. Suppose that the residual cost of h_2 is at most that of h_1. We will also call h_1 the *dominant*, and h_2 the *non-dominant* handle of this cycle. As an invariant, our algorithm maintains that the designation of dominant and non-dominant handles of an elementary cycle does not change throughout the algorithm's execution.

Suppose first that the residual cost of h_1 is strictly larger than that of h_2. In this case, let $a^C_v = 1$ for all internal nodes of handle h_1, and let $a^C_v = 0$ of the internal nodes of h_2. If the residual cost of both handles is the same, we let $a^C_v = 1/2$ on internal nodes of both handles.

In certain cases, we need to *correct the parity* of the constructed inequality. This is necessary if a^C as defined above is $0, 1$ (i.e., if all elementary cycles of C have a strictly dominant handle), and if the cycle formed by all dominant handles is odd. In this case, we pick an arbitrary elementary cycle on C, and declare it *special*. For this special cycle, we then set $a^C_v = 1$ for the internal nodes on *both* handles. Following the same reasoning as Fiorini et al. [11] for DHS, we can show the following for ECT:

▶ **Lemma 3.** *Each feasible point of our LP* ($\mathrm{P}_{\mathrm{ECT}}$) *satisfies any blended inequality.*

In our algorithm, we assume that inequalities ($\circledast$) are part of ($\mathrm{P}_{\mathrm{ECT}}$). Throughout the algorithm, we increase dual variables $y_\circledast$ of such inequalities.

We will sometimes say that variable $y_\circledast$ (or cycle C) *pays for* $\sum_{v \in S'} a^C_v$ hit nodes. It is well-known (see, e.g., Goemans and Williamson [14]) that if during any iteration dual variables for a family of blended inequalities are incremented uniformly, and the dual variables pay for α hit nodes (of S') on average, then the final solution produced by the algorithm is α-approximate.

The motivation for blended inequalities is to pay for no more than one node in each piece. Consider the example in Figure 1. Here, the bottom black dashed path is odd, there are an even number of handle pairs in the top part, and ε is small. Suppose we set $a^C_v = 1/2$

on internal nodes of each handle. If we were to increment the inequality (⊛), all the blue nodes of weight 1 would become tight, and after reverse-delete, the algorithm would keep one blue node for each handle pair. However, selecting a red node and a blue node would be a cheaper solution. This could be achieved by setting $a_v^C = 1$ for red and black nodes, and $a_v^C = 0$ on blue nodes, until the residual costs of the red nodes become 1, and afterwards setting $a_v^C = 1/2$ on internal nodes of each handle.

During its execution, the algorithm carefully chooses a family of even cycles $\mathcal{C}$ in G_2^S and increments the dual variables of certain blended inequalities for each $C \in \mathcal{C}$ until a node becomes tight, or the blended inequality changes; i.e. the residual costs of two handles of a handle pair, which were previously not equal, become equal.

In their primal-dual algorithms for cycle transversal problems with uncrossing property, Goemans and Williamson [14] started with the infeasible "hitting set" $S = \emptyset$. While S is infeasible, the dual variables for faces of the residual digraph that are cycles are incremented. A reverse-delete step is applied at the end. The authors show that tight examples for their algorithm feature so called *pocket* subgraphs. Not surprisingly, the improved algorithm of Berman and Yaroslavtsev [6] has to pay special attention to these pockets to obtain the improvement in performance guarantee.

2.2 Pockets and their variants

The following definition of crossing cycles was elementary to the approach by Goemans and Williamson [14] for cycle transversal problems in planar graphs.

▶ **Definition 4.** *In an embedded planar graph, two cycles C_1, C_2* cross *if C_i contains an edge intersecting the interior of the region bounded by C_{3-i}, for $i = 1, 2$. That is, the plane curve corresponding to the embedding of the edge in the plane intersects the interior of the region of the plane bounded by C_{3-i}. A set of cycles $\mathcal{C}$ is* laminar *if no two elements of $\mathcal{C}$ cross.*

Next, we formally define pockets, and we also introduce the new notion of "pseudo-pockets", the lack of which will help us "cover" our graph with even cycles.

▶ **Definition 5.** *Let G be a graph and let $\mathcal{C}$ be a collection of cycles in G. A* pseudo-pocket *of $(G, \mathcal{C})$ is a connected subgraph G' of G which contains a cycle such that at most two nodes of G' have neighbours outside G'. A* pocket *of $(G, \mathcal{C})$ is a pseudo-pocket that contains a cycle of $\mathcal{C}$. A pocket is* minimal *if it contains no pocket as a proper induced subgraph.*

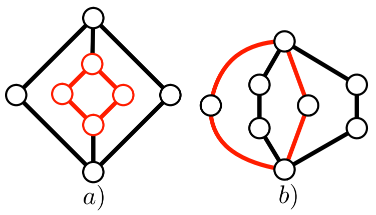

Figure 4 (a) Graph formed by red nodes is a pocket. (b) Crossing cycles in red and black.

2.3 Identifying families of even cycles via tilings

The 12/5-approximation algorithm of Berman and Yaroslavtsev [6] for Directed FVS in node-weighted planar digraphs G proceeds roughly as follows.

It starts with the empty hitting set $S = \emptyset$. As long as S is not a hitting set for the directed cycles of G, it first looks for a pocket H of the residual digraph G^S, that is the digraph obtained from $G - S$ by deleting all nodes not on a directed cycle. It then increments the dual variables for the set of face minimal directed cycles of H, which happen to be faces. It then adds any nodes that become tight to S. Once S is feasible, the algorithm performs a reverse deletion step.

As pointed out, in our setting, face-minimal even cycles may not be faces, and may cross. Following Berman and Yaroslavtsev [6], we wish to "cover" our residual graph with face-minimal even cycles which do not cross, we call this a "tiling"; see Figure 5 iii). As we will see, this tiling allows us to identify the dual variables to increase. Let us formalize the correspondence between edges of the dual between odd faces and even faces.

▶ **Definition 6.** *Let H be a plane graph without pseudo-pockets. For each face f of H, let v_f be the corresponding node of the planar dual H^*. A* tile *of H is an even cycle C of H bounding one or two faces. If C is a single face f, we say that C* corresponds *to the node v_f. If C bounds two faces f and g, we say that C corresponds to the edge $v_f v_g \in E(H^*)$. We say that nodes v_f, v_g and the faces f, g are* covered *by the tile.*

For a node v of H^*, let $f_v \subset E(H)$ be the edges on the boundary of the corresponding face of H. Denote by h_∞ the node of H^* corresponding to the infinite face.

Given $wh_\infty \in E(H^*)$, a cycle $C^1 \subset E(H)$ *corresponds* to wv_∞ if C^1 is a cycle of $f_w \Delta f_{h_\infty}$, or $C^1 = C' \Delta f_w$ and C' is a cycle of $f_w \Delta f_{h_\infty}$. We also call such a cycle C^1 a *tile* and say that C^1 covers h_∞, w, and the corresponding faces.

Given a matching $E' \subset E(H^*)$ and $V' \subset V(H^*)$, with $E' = \{e_1, \ldots, e_\ell\}$ and $V' = \{v_1, \ldots, v_t\}$, a set of tiles $\mathcal{T} = \{C_1, \ldots, C_{\ell+t}\}$ *corresponds* to $E' \cup V'$ if C_i corresponds to e_i for $i = 1, \ldots, \ell$ and $C_{j+\ell}$ corresponds to v_j for $j = 1, \ldots, t$.

In Figure 5 i), cycle C bounds two faces f and g; see also Figure 5 ii).

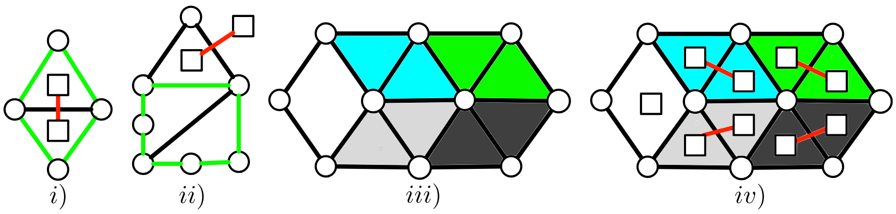

Figure 5 Diagrams i) and ii) show cycles in green and corresponding edges of the dual graph in red. (i) The red edge corresponds to the symmetric difference of two finite faces. (ii) The red edge corresponds to the symmetric difference of a finite and infinite face. Diagrams iii) and iv) show a tiling indicated by the boundaries of the various finite regions in white, light grey, etc and the corresponding matching.

▶ **Definition 7.** *For a plane graph H, a set $\mathcal{T}$ of tiles is a* pseudo-tiling *if no face of H is covered by more than one tile. If the node v_{h_∞} corresponding to the infinite face of H is not covered by $\mathcal{T}$, we call $\mathcal{T}$ a* tiling*.*

Certain tilings are particularly desirable; we will define these the next.

▶ **Definition 8.** *Let $\alpha \in (0, 1)$. A tiling is* α-quasi-perfect *if it covers all even finite faces, a β-fraction of odd finite faces of G^S, and a ψ-fraction of the finite faces of G^S are even, where $\beta(1 - \psi) + 2\psi \geq \alpha$.*

Let C be an even cycle in G_2^S, and recall that we say that C pays for $\sum_{v\in S} a_v^C$ hit nodes. For an even cycle in a tiling consisting of two faces, we bound the number of hit nodes it pays for by the number of hit nodes each face pays for.

We will show that a finite face of our graph intersects at most 18/7 hit nodes on average (over all finite faces). Ideally, we would want to cover all faces by a tiling. Then an even cycle of our tiling is incident to at most 36/7 hit nodes on average, twice the amount a face of our graph intersects on average. Alas, tilings covering all faces need not always exist. Thus, we try to find a tiling that covers as many finite faces as possible. Suppose that we find a tiling $\mathcal{T}$ that covers a set $\mathcal{T}_{\mathsf{Faces}}$ of finite faces consisting of α-fraction of the finite faces of our graph. It follows that a face of $\mathcal{T}_{\mathsf{Faces}}$ will be incident to at most $18/7\alpha$ hit nodes on average, and so an even cycle of the tiling $\mathcal{T}$ is incident to at most $36/7\alpha$ hit nodes on average. Intuitively, even faces pay for fewer hit nodes than even cycles containing two faces, so it is good if a tiling contains many even faces. The motivation for quasi-perfect tilings is that it is good if a large fraction of faces are covered by the tiling and if the tiling contains a lot of even faces. We prove the following key result in Appendix A.

▶ **Theorem 9.** *Let H be a 2-compression of some planar graph G, that has an even cycle and contains no pockets. Then H has a 2/3-quasi-perfect tiling.*

2.4 The algorithm in detail

We formally state our algorithm. It takes as input a planar graph G with cost function $c : V(G) \to \mathbb{N}$. Let $\mathcal{C}(G)$ be the set of even cycles of G, and let $\mathsf{opt}(G, c)$ be the minimum cost of an *even cycle transversal* of G, which is a set of nodes intersecting each cycle in $\mathcal{C}(G)$.

As we will see, the algorithm returns an even cycle transversal S of G whose cost is at most $(47/7)\mathsf{opt}(G, c)$. We start with the empty candidate $S := \emptyset$. In each iteration, the algorithm looks for an even cycle C in the residual graph G^S such that at most two nodes of C have outside neighbours. If we find such C, increment the variable y_C until a node becomes tight. If no such cycle exists, the algorithm computes the 2-compression of G^S, and in it, we find an inclusion-minimal pocket H of G_2^S. Using Theorem 9, we find a 2/3-quasi-perfect tiling $\mathcal{T}_H$ of H and increments the dual variables for the blended inequalities for each $C \in \mathcal{T}_H$. The algorithm then adds all nodes X that became tight to our candidate hitting set S.

During an iteration, for each handle pair (Q_1, Q_2) for which the set X of nodes that became tight contains a node in the interior of each handle, our algorithm will choose two nodes $a, b \in X$ with a in the interior of Q_1 and b in the interior of Q_2 and define (a, b) to be a *node pair*. For instance, in Figure 2 if v and v' are the only nodes added during some iteration then the algorithm would define (v, v') to be a node pair. For a set of nodes X added during the same iteration, nodes in a pair are considered to be added *before* any node not in a pair.

At the end of the algorithm, we perform a non-trivial reverse-delete procedure. Formally, let $w_1, \ldots, w_\ell$ be the nodes of S in the order they were added to S by the algorithm, where for nodes w_i, w_j that were added during the same iteration if w_i is in a pair and w_j is not, then $i < j$. That is, for reverse-delete purposes, nodes not in a pair are considered for deletion first. For $p = \ell, \ell - 1, \ldots, 1$, if w_p is not in a node pair, then if $S \backslash \{w_p\}$ is a feasible ECT, the algorithm deletes w_p from S; otherwise, it does not. If w_p is in a node pair (w_p, w'), then if $S \backslash \{w_p, w'\}$ is a feasible hitting set, then delete both w_p, w' from S; else, keep both w_p, w'.

The intuition behind the caveat in our reverse-delete step is that node pairs are often very useful to keep, because they disconnect a piece. Consider the example in Figure 6. There is a piece with green nodes of cost 2, and an odd number of length-5 faces with red and blue

Algorithm 2.1 EvenCycleTransversal(G, c).

Input : A planar graph G with node costs $c : V(G) \rightarrow \mathbb{N}$.
Output : An even cycle transversal S of G of cost at most $\frac{47}{7}\mathsf{opt}(G, c)$.

$S \leftarrow \emptyset$
while *residual graph* G^S *contains an even cycle* **do**
  if G^S *contains a cycle* C *with at most* 2 *outside neighbours* **then**
    increase the dual variable y_C for C until a node v becomes tight.
  else
    compute the 2-compression G_2^S of G^S.
  $H \leftarrow$ minimal pocket of G_2^S.
  $\mathcal{T}_H \leftarrow$ a 2/3-quasi-perfect tiling of H.
  Increment dual variables of blended inequalities of all $C \in \mathcal{T}_H$ until a node v becomes tight or the blended inequality changes.
  Denote by X the set of nodes that became tight, and add X to S.
  for *each handle pair* (Q_1, Q_2) **do**
    if X *contains a node in the interior of each handle* **then**
      choose two nodes $a, b \in X$ with a in the interior of Q_1 and b in the interior of Q_2 and define (a, b) to be a node pair.
$w_1, \ldots, w_\ell \leftarrow$ nodes of S in the order they were added, where for nodes X added during the same iteration, any node of X in a pair appears before others node of X not in pairs.
for $i = \ell$ **downto** 1 **do**
  if w_i *is not part of a pair* **then**
    if $S \backslash \{w_i\}$ *is feasible* **then**
      $S \leftarrow S \backslash \{w_i\}$.
  else
    Let (w_i, w_j) be the pair containing w_i. **if** $S \backslash \{w_i, w_j\}$ *is feasible* **then**
      $S \leftarrow S \backslash \{w_i, w_j\}$.
return S

striped nodes of cost 1. The black nodes have cost infinity. The bottom dashed path has odd length. In the 2-compression shown on the right, all length-5 faces in the figure belong to one piece. Suppose the blended inequality chooses the length-5 face with the green nodes as the special cycle, and increments the blended inequality for this graph. One sees that the red, blue striped and green nodes become tight simultaneously.

To see that reverse delete orders need to be chosen carefully, consider the following adversarial ordering: in reverse delete, consider the two green nodes other than v first, then consider the red nodes, and then consider one blue striped node on each handle. Finally, consider the remaining blue striped nodes. One can see that the algorithm would end up with v and one blue striped node per handle, which is significantly more costly than the optimum which selects the solution consisting of one red and one blue striped node on a handle pair. This completes the description of our approximation algorithm for ECT, whose complete pseudo-code is given as Algorithm 2.1.

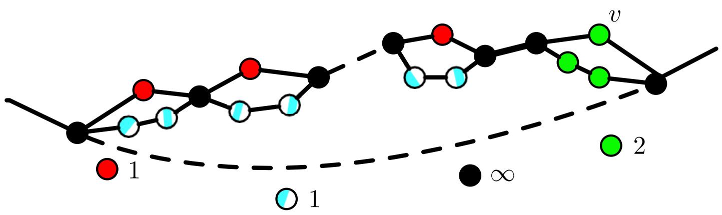

Figure 6 The red and blue striped nodes have weight 1, black nodes have infinite weight and green nodes have cost 2. The bottom dashed black path has odd length. The number of length-5 faces at the top is assumed to be even.

2.5 Analysis of approximation ratio

We claim the algorithm to be a 47/7-approximation for ECT on node-weighted planar graphs.

Fix an input planar graph G with node costs $c_v \in \mathbb{N}$. Consider a set $S \subseteq V(G)$ of nodes and a node $v \in S$. A cycle C is a *pseudo-witness cycle* for v with respect to S if $C \cap S = \{v\}$. If C is additionally even, then C is a *witness cycle* for v. Note that if S is an inclusion-minimal ECT for G, then there is a set W_v of witness cycles for each node in $v \in S$. If the reverse-delete procedure does not delete any node of S, then each node not in a pair has a witness cycle and for each pair, at least one of the nodes in the pair has a witness cycle.

The analyses of the algorithms by Goemans and Williamson [14] and by Berman and Yaroslavtsev [6] for Subset FVS on planar graphs rely crucially on the fact that, each node of an inclusion-wise minimal solution has a witness cycle. Goemans and Williamson [14] showed that one can find a laminar collection $\mathcal{A}$ of witness cycles. Laminar families are well-known to have a natural tree representation. The key argument of both algorithms is that for each *leaf* cycle C of the laminar family, one can increment the dual variable of at least one face contained in the region defined by C. Further, this dual variable pays only for the hit node that C is a witness of. This is used to argue that a large portion of the dual variables they incremented pay for a single hit node. An additional bound on how many nodes the other dual variables pay for is proven exploiting the sparsity of planar graphs.

For ECT, however, we do not have laminar witness cycles. Instead, we must extend the analysis of Berman and Yaroslavtsev [6] to find a set of laminar pseudo-witness cycles.

Consider some time $\bar{t}$ during the algorithm when applied to (G, c). Let $S_{\bar{t}}$ be the current hitting set and $G^{S_{\bar{t}}}$ the residual graph. Let $\{\sum_{v \in V(G)} a_v^C \geq 1\}_{C \in \mathcal{L}}$ be the set of inequalities of the increased dual variables. Here, $\mathcal{L}$ will be either a single cycle of $G^{S_{\bar{t}}}$, or a tiling of $G_2^{S_{\bar{t}}}$. We wish to show that the primal increase rate towards the final set S' at time $\bar{t}$, $\sum_{C \in \mathcal{L}} \sum_{v \in S'} a_v^C$ is at most 47/7 times the dual increase rate $|\mathcal{L}|$.

If the algorithm incremented y_C, where C was a cycle of G for which at most two nodes have outside neighbours, then the inequality we increase is $\sum_{v \in C} x_v \geq 1$. As S' is minimal under reverse-delete, $|C \cap S'| \leq 2$, and hence the primal increase rate $\sum_{v \in S'} a_v^C = |C \cap S'|$ is at most twice the dual increase rate 1.

Otherwise, if the algorithm did not increment y_C, then there is no cycle C of $G^{S_{\bar{t}}}$ such that at most two nodes of C have neighbours outside C. Hence, the set of increased inequalities are the blended inequalities of a tiling $\mathcal{T}_H$ of an inclusion-minimal pocket H of $G_2^{S_{\bar{t}}}$. For a cycle C of $G_2^{S_{\bar{t}}}$, let $\sum_{v \in V(G^{S_{\bar{t}}})} a_v^C \geq 1$ be the blended inequality C (see Equation ⊛).

Recall that informally speaking, we wish to pay for at most one hit node inside a piece. To do this, we need the following theorem which generalizes a result by Fiorini et al. [11, Theorem 5.7] and tells us the structure of a minimal solution within a piece.

▶ **Theorem 10.** *Let S' be the output of Algorithm 2.1 on input (G, c). Consider an edge $uw \in E(G_2^{S_{\bar{t}}})$ on the even cycle whose dual variable we increase, and let Q be the piece corresponding to uw in G. Then exactly one of the following occurs:*

(C1) *S' contains no internal node of Q,*

(C2) *S' contains exactly one node of Q, and this node is a cut-node of Q,*

(C3) *S' contains exactly two nodes of Q, and they belong to opposite handles of a cycle of Q,*

(C4) *S' contains exactly one node per elementary cycle of Q, each belonging to the interior of some handle of the corresponding cycle.*

Proof. If S' contains two nodes a and b in the interiors of different handles of a pair, then since removing both a and b disconnects u from w in Q, our algorithm would delete all other nodes of $V(Q)\backslash\{u, w\}$ from S'. If u or w were in S', then our algorithm would delete both a and b. Thus, $u, w \notin S'$, and case (C3) holds.

Similarly, if S' contains a cut node z, then since removing z disconnects from u from v in Q, our algorithm would delete all other nodes of $V(Q)\backslash\{u, v\}$ from S'. If u or w were in S', then our algorithm would delete z. Thus, $u, w \notin S'$, and case (C2) holds.

If u or w is in S', then for any $r \in S' \cap (V(Q)\backslash\{u, w\})$ there cannot be an even cycle of G which intersects S' only at r as such a cycle would have to go through u or w, and thus S' contains no internal node of Q and case (C1) holds.

Assume that cases (C1), (C2) and (C3) do not hold, so $u, w \notin S'$. Let (P_1, P_2) be a handle pair on Q such that P_1 contains a hit node t in its interior and P_2 does not. Suppose that Y_1, Y_2 was another handle pair with no hit node on either of Y_1 or Y_2. By our deletion procedure, there must be an even cycle C which intersects S' at t only. Such a cycle C uses the handle P_1 and one handle Y_i of the pair Y_1, Y_2. Let C' be the cycle obtained from C by replacing the paths P_1 and Y_i in C by the paths P_2 and Y_{3-i}. Since the lengths of different handles of a pair have different parity, C' is even. Since P_2, Y_1 and Y_2 contain no nodes of S', C' contains no nodes of S', which is a contradiction. Since a handle can only contain one hit node of S', this implies that case (C4) holds. ◀

Given a hitting set S' output by Algorithm 2.1, we wish to construct a corresponding hitting set for $G_3^{S_{\bar{t}}}$ such that the primal increase rate of any particular blended inequality (with respect to S') is equals the number of nodes of S'_3 on the corresponding cycle of $G_3^{S_{\bar{t}}}$.

▶ **Definition 11.** *Let S' be a hitting set output by Algorithm 2.1. The* corresponding hitting set for $G_3^{S_{\bar{t}}}$ *is the set $S'_3 \subset V(G_3^{S_{\bar{t}}})$ obtained by first taking the nodes of $S' \cap V(G_3^{S_{\bar{t}}})$. Now, consider an edge uv of $G_2^{S_{\bar{t}}}$ with corresponding piece P. Replace uv by the path uw_pv in $G_3^{S_{\bar{t}}}$, and add w_p to S'_3 if $P - S'$ has two components.*[3]

▷ Claim 12. Let C be the preimage of an even cycle in $G_2^{S_{\bar{t}}}$, and C_3 the corresponding cycle in $G_3^{S_{\bar{t}}}$. We claim $\sum_{v \in S'} a_v^C \leq |C_3 \cap S'_3| + 1$. Further, if C does not contain a twin edge, then $\sum_{v \in S'} a_v^C \leq |C_3 \cap S'_3|$.

Proof. Define b^C as follows: For a handle pair, while one handle has greater residual cost than the other set $b_v^C = 1$ for v on the handle of greater residual cost $b_v^C = 0$ on internal nodes of the other handle (change b^C whenever residual costs become equal). Otherwise, $b_v^C = 1/2$ on internal nodes of both handles. In short, b_v^C are the coefficients a_v^C if we had not redefined $a_v^C = 1$ for nodes on the special cycle.

[3] Note that the minimality of S' implies that removing S' from P yields at most two connected components.

Let $uw \in E(G_2^{S_{\bar{t}}})$, Q be the preimage of uw in $G^{S_{\bar{t}}}$ and $uw_Q w$ be the subdivision of uw in $G_3^{S_{\bar{t}}}$. Let S_3' be the corresponding hitting set of S' for $G_3^{S_{\bar{t}}}$. We claim $\sum_{v \in S' \cap (Q \setminus \{u,w\})} b_v^C = |S_3' \cap \{w_Q\}|$. We decide which case of Theorem 10 is satisfied by uw and S'.

- If uw and S' satisfy (C1), then $\sum_{v \in S' \cap (Q \setminus \{u,w\}} b_v^C = 0$. Since S' contains no internal node of Q, $Q \setminus S$ is connected, and hence S_3' does not contain w_Q. Hence $\sum_{v \in S' \cap (Q \setminus \{u,w\})} b_v^C = |S_3' \cap \{w_Q\}|$.
- If uw and S' satisfy (C2) or (C3), then S' does not contain either end node of Q, and contains either a single cut node of Q, or exactly two nodes of Q in the interiors of two handles of a handle pair of Q. Thus, $S' \cap Q$ consists either of a single node v for which $b_v^C = 1$, or two nodes j, k for which $b_j^C = b_k^C = 1/2$, and so $\sum_{v \in S' \cap Q} b_v^C = 1$.

In case (C2) or (C3), $Q \setminus S'$ is disconnected, so $|S_3' \cap \{w_Q\}| = 1$. Hence, $\sum_{v \in S' \cap (Q \setminus \{u,w\})} b_v^C = |S_3' \cap \{w_Q\}|$.

- Suppose S' satisfies (C4). Suppose, for sake of contradiction that, Algorithm 2.1 added a node pair (ℓ', m) on some handle pair (P_1, P_2) of Q. It then follows from the reverse-delete step that the final solution S' contains both ℓ' and m, or none of them. Since we do not contain a node pair, the deletion procedure of Algorithm 2.1 implies the algorithm did not add a node pair with nodes in Q. Hence, throughout the algorithm, for each handle pair (P_1, P_2) of Q, the handle P_i, which contains a hit node in its interior must have strictly less residual cost than the other. Hence $b_v^C = 0$ on handle P_i. This implies

$$\sum_{v \in (V(Q) \setminus \{u,w\})} b_v^C = 0 \ . \tag{1}$$

Thus $\sum_{v \in S' \cap (Q \setminus \{u,w\}} b_v^C = |S_3' \cap \{w_Q\}|$.

Let $C = v_1 v_2 \ldots v_\ell v_1$. Let Q_i be the piece corresponding to $v_i v_{i+1 \mod \ell}$. Let q_i be the node resulting from subdividing $v_i v_{i+1 \mod \ell}$ in $G_2^{S_{\bar{t}}}$ to obtain $G_3^{S_{\bar{t}}}$. Let $C_3 := v_1 q_1 v_2, q_2, \ldots, v_\ell q_\ell$ the cycle corresponding to C in $G_3^{S_{\bar{t}}}$. We showed

$$\sum_{v \in S' \cap (Q_i \setminus \{u,w\})} b_v^C = |S_3' \cap \{q_i\}| \ . \tag{2}$$

Summing (2) for $i - 1, .., l$ yields $\sum_{v \in S' \cap (\cup_{i=1}^{\ell} Q_i \setminus \{v_1, v_2, \ldots, v_\ell\})} b_v^C = |\{q_1, q_2, \ldots, q_\ell\} \cap C_3|$.

Noting $b_{v_i}^C = 1$ for each i and $b_v^C = 0$ for $v \notin \cup_{j=1}^{l} Q_j$, yields

$$\sum_{v \in S'} b_v^C = |C_3 \cap S_3'| \ . \tag{3}$$

Let us now relate a_v^C to b_v^C. If C has no twin edge, then the blended inequality coefficients a_v^C are equal to b_v^C, therefore $\sum_{v \in S} a_v^C = |C_3 \cap S_3'|$.

In general, C may contain a twin edge. In this case, a_v^C differs from b_v^C only in the interior of the handles H_1, H_2 of the special cycle: then either $b_v^C = \frac{1}{2}$ in the interior of H_1 and H_2, or $b_v^C = 0$ in the interior of the dominant handle, and $b_v^C = a_v^C$ everywhere else.

If $b_v^C = \frac{1}{2}$ in the interior of H_1 and H_2, then note from Theorem 10 there are at most two nodes of S' on $H_1 \cup H_2$. Thus, $\sum_{v \in S} a_v^C \le \sum_{v \in S} b_v^C + 1$.

Otherwise, $b_v^C = 0$ in the interior of the dominant handle, and $b_v^C = a_v^C$ everywhere else. Since S contains at most one node from the dominant handle $\sum_{v \in S} a_v^C \le \sum_{v \in S} b_v^C + 1$. Thus, $\sum_{v \in S} a_v^C \le |C_3 \cap S_3'| + 1$ completing the proof. $\triangleleft$

To show that $|C_3 \cap S_3'| + 1$ is small on average we need the fact that S_3' is a minimal ECT, which is stated in the following remark.

▶ Remark 13. Let S' be the output of Algorithm 2.1 on input (G, c). Let S'_3 be the corresponding hitting set for $G_3^{S_{\bar{t}}}$ in Definition 11. Then each $v \in S'_3$ has a witness cycle.

For a node h and cycle C, denote by $C \circ h$ that h lies on C.

▶ **Definition 14.** *Let $\mathcal{R}$ be a set of cycles of a graph G, and let $S \subset V(G)$. The* debit graph *for $\mathcal{R}$ and S is the bipartite graph $\mathcal{D}_G = (\mathcal{R} \cup S, E)$ with edges $E_{\mathcal{R}} = \{(C, s) \in \mathcal{R} \times S \mid C \circ s\}$.*

Given an embedding of G and a set $\mathcal{R}$ of faces of G, we can obtain an embedding of $\mathcal{D}_G$ by placing a node v_M inside the face R for each $R \in \mathcal{R}$. This shows the following observation.

▶ **Observation 15** ([14, 6]). *If $\mathcal{R}$ is a set of faces of G, then the debit graph is planar.*

Note that for $\mathcal{R}$ a set of cycles, a cycle $R \in \mathcal{R}$, the number of nodes $|R \cap S|$ that R pays for in the hitting set is the degree of R in the debit graph.

Recall the definition of the Subset FVS problem, which seeks a minimum-weight node set X which intersects all cycles from $\mathcal{C}_T$, the collection of cycles in G which contain some node from a given set $T \subseteq V(G)$. Observe that each node of S'_3 has a witness cycle in $G_3^{S_{\bar{t}}}$; therefore, it is an inclusion-minimal hitting set for the collection $\mathcal{C}_T$ with $T = S'_3$. Goemans and Williamson [14, Lemma 4.2] showed that any inclusion-minimal hitting set for $\mathcal{C}_T$ has a laminar set of witness cycles, which implies that there is a laminar set of pseudo-witness cycles $\mathcal{A}$ for hitting set S'_3.

▶ **Proposition 16** ([14, Lemma 4.2 specialized for Subset FVS]). *Let G' be a planar graph and let $T \subseteq V(G')$. Let $\mathcal{C}_T$ be the set of cycles of G' containing at least one node of T, and let X be an inclusion-minimal hitting set for $\mathcal{C}_T$. Then there is a laminar set of cycles $\mathcal{A} = \{A_x \mid x \in X\}$, satisfying $A_x \in \mathcal{C}_T$ and $A_x \cap X = \{x\}$.*

Applying Proposition 16 to $G' = G_3$ and $X = T = S'_3$ implies there is a laminar set $\mathcal{A} = \{A_x \mid x \in S'_3\}$ of cycles satisfying $A_x \cap S'_3 = \{x\}$. In other words, $\mathcal{A}$ is a laminar set of pseudo-witness cycles for S'_3. Note that cycles of $\mathcal{A}$ may not be even, hence they may be pseudo-witness cycles for S'_3, but not necessarily witness cycles for nodes of S'_3.

Recall that, during the current iteration, our algorithm incremented the blended inequalities of the cycles in a 2/3-quasi-perfect tiling $\mathcal{T}_H$ of H. Recall H is an inclusion-minimal pocket of $G_2^{S_{\bar{t}}}$. By abuse of notation, let $\mathcal{T}_H$ be the corresponding cycles of $G_3^{S_{\bar{t}}}$. Let $\mathcal{D}$ be the debit graph formed using $G_3^{S_{\bar{t}}}$, the cycle set $\mathcal{T}_H$ and hitting set S'_3.

Obtain graph $\mathcal{D}'$ from $\mathcal{D}$ by replacing each even cycle C containing two faces with the two faces that compose it. To be precise, construct $\mathcal{D}'$ by first taking all nodes of S'_3 and all faces of H that lie inside some even cycle of $\mathcal{T}_H$ as the vertex set. For each edge $(C, v) \in E(\mathcal{D})$, if the cycle C consist of two faces f_1, f_2 add the edges (f_1, v) and (f_2, v) to $\mathcal{D}'$, otherwise add the edge (C, v) to $\mathcal{D}'$ (see Figure 7). Delete isolated vertices from $\mathcal{D}'$. If f_i is not incident to any hit nodes v, we remove f_i from $\mathcal{D}'$. Let $\mathcal{T}_{\mathsf{Faces}(H)}$ be the "face nodes" of $\mathcal{D}'$. Let $\mathcal{F}_{\mathsf{all}(H)}$ denote the finite faces of H. Let $\mathcal{F}_H$ denote the set of finite faces of H that contain a hit node. Observe that $M \cap S'_3 = \emptyset$ for each $M \in \mathcal{F}_{\mathsf{all}(H)} \backslash \mathcal{F}_H$. Now

$$\begin{aligned}\sum_{M \in \mathcal{T}_H} |M \cap S'_3| &\leq \sum_{M \in \mathcal{T}_{\mathsf{Faces}(H)}} |M \cap S'_3| \\ &\leq \sum_{M \in \mathcal{F}_{\mathsf{all}(H)}} |M \cap S'_3| - |\mathcal{F}_H \backslash \mathcal{T}_{\mathsf{Faces}(H)}| = \sum_{M \in \mathcal{F}} |M \cap S'_3| - |\mathcal{F}_H \backslash \mathcal{T}_{\mathsf{Faces}(H)}| \ . \quad (4)\end{aligned}$$

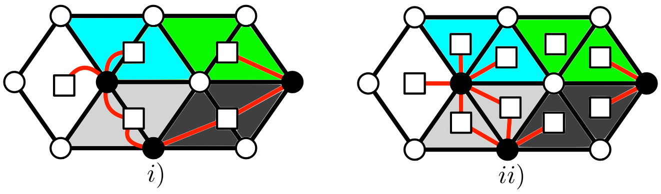

Figure 7 Left: A possible debit graph $\mathcal{D}$ with the cycles of the tiling in Figure 5. Right: the graph $\mathcal{D}'$ obtained by replacing each cycle with the faces that compose it.

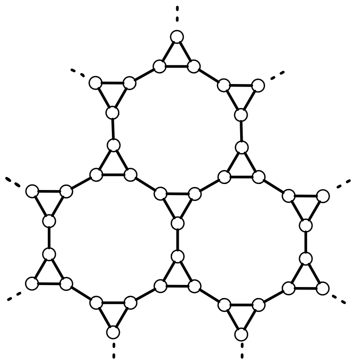

Figure 8 A graph consisting of a tessellation of the plane with twice as many triangles as dodecagons. None of the triangles are adjacent, so a maximum tiling covers only the even dodecagons.

The first inequality holds, because for each cycle C consisting of two faces f_1 and f_2 we have $|C \cap S_3'| \leq |f_1 \cap S_3'| + |f_2 \cap S_3'|$. The second inequality holds, because each face of $\mathcal{F}_H$ contains a hit node, and so $|C \cap S_3'| \geq 1$ for each $C \in \mathcal{F}_H$. The last inequality holds, because by definition $|M \cap S_3'| = 0$ for all $M \in \mathcal{F}_{\mathsf{all}(H)} \backslash \mathcal{F}_H$.

If our tiling covers 2/3 of all finite faces, then $|\mathcal{T}_{\mathsf{Faces}(H)}| \leq 2|\mathcal{T}_H|$ and $(2/3)|\mathcal{F}_H| \leq |\mathcal{T}_{\mathsf{Faces}(H)}|$, so $|\mathcal{F}_H| \leq 3|\mathcal{T}_H|$. Alas, one can show that a tiling that covers 2/3 of all finite faces does not always exist; see Figure 8. To overcome this impediment, we will show that $|\mathcal{F}_H| \leq 3|\mathcal{T}_H|$ holds for a 2/3-quasi-perfect tiling. Suppose that our 2/3-quasi-perfect tiling covers a b-fraction of the odd faces in $\mathcal{F}_H$, and a c-fraction of the faces in $\mathcal{F}_H$ which are even. Let $\mathcal{F}_{\mathsf{even}(H)}$ be the even finite faces of $\mathcal{F}_H$. Then, as $\mathcal{F}_H \backslash \mathcal{F}_{\mathsf{even}(H)}$ are the odd faces of $\mathcal{F}_H$, and $\mathcal{T}_{\mathsf{Faces}(H)} \backslash \mathcal{F}_{\mathsf{even}(H)}$ are the odd faces covered by our tiling, it holds that $b|\mathcal{F}_H \backslash \mathcal{F}_{\mathsf{even}(H)}| = |\mathcal{T}_{\mathsf{Faces}(H)} \backslash \mathcal{F}_{\mathsf{even}(H)}|$. Simplifying, we get

$$b|\mathcal{F}_H| + (1-b)|\mathcal{F}_{\mathsf{even}(H)}| \leq |\mathcal{T}_{\mathsf{Faces}(H)}| \leq 2|\mathcal{T}_H| - |\mathcal{F}_{\mathsf{even}(H)}| \ .$$

By rearranging, we get $b|\mathcal{F}_H \backslash \mathcal{F}_{\mathsf{even}(H)}| + 2|\mathcal{F}_{\mathsf{even}(H)}| \leq 2|\mathcal{T}_H|$. Noting that $b(1-c)+2c \geq 2/3$, and rearranging once more, yields

$$\frac{2}{3}|\mathcal{F}_H| \leq b|\mathcal{F}_H \backslash \mathcal{F}_{\mathsf{even}(H)}| + 2|\mathcal{F}_{\mathsf{even}(H)}| \leq |\mathcal{T}_{\mathsf{Faces}(H)}| \leq 2|\mathcal{T}_H| \ .$$

Noting that $|\mathcal{F}_{\mathsf{even}(H)}|/|\mathcal{F}_H| = c$ and $b(1-c) + 2c \geq 2/3$, we get

$$3|\mathcal{T}_H| \geq \frac{3}{2}(b(1-c) + 2c)|\mathcal{F}_H| \geq |\mathcal{F}_H| \ . \tag{5}$$

By (4), in order to bound $\sum_{M \in \mathcal{T}_H} |M \cap S_3'|$, it suffices to bound $\sum_{M \in \mathcal{F}} |M \cap S_3'|$. To do this, we prove the following extension of the work by Berman and Yaroslavtsev [6, Theorem 4.1].

▶ **Theorem 17.** *Let H be an inclusion-wise minimal pocket of G. Let $S \subset V(G)$ be a set of nodes with some set $\mathcal{A}$ of laminar pseudo-witness cycles. Let $\mathcal{R}$ be a set of finite faces of H such that each cycle of $\mathcal{A}$ contains a face of $\mathcal{R}$ in its interior. Then $\sum_{M \in \mathcal{R}} |M \cap S| \leq \frac{18}{7}|\mathcal{R}|$.*

We defer the proof of Theorem 17 to Subsection A.1.

Let $\mathcal{A}$ be a set of laminar witness cycles for S_3'. If we were to set $\mathcal{R} = \mathcal{F}_H$ (the set of finite faces of H incident to a hit node), then each cycle $A \in \mathcal{A}$ contains a face of $\mathcal{R}$ in its interior, namely any face inside A that is incident to the hit node of S_3' on A. Thus, $S_3', \mathcal{A}$ and $\mathcal{R}$ meet the conditions of Theorem 17.

To recap, we wish to bound the primal increase rate $\sum_{M \in \mathcal{T}_H} \sum_{v \in S} a_v^M$, so we analyze the expression $\sum_{M \in \mathcal{T}_H} |M \cap S_3'|$. Recall from Claim 12 that $\sum_{v \in S} a_v^M$ is at most one more than $|M \cap S_3'|$ and $\sum_{v \in S} a_v^M = |M \cap S_3'|$ if M contains no twin edge. We bound $\sum_{M \in \mathcal{T}_H} |M \cap S_3'|$ by looking at the quantity $\sum_{M \in \mathcal{F}_H} |M \cap S_3'|$, because $\mathcal{F}_H$ fits the conditions of Theorem 17. One could then use $|\mathcal{F}_H| \leq 3|\mathcal{T}_H|$ (by (5)), to bound $\sum_{M \in \mathcal{T}_H} \sum_{v \in S} a_v^M$ in terms of the dual increase rate $|\mathcal{T}_H|$. We will use $3|\mathcal{T}_H| \geq \frac{3}{2}(b(1-c)+2c)|\mathcal{F}_H|$ to obtain a stronger bound.

Let $\mathcal{T}$ be our 2/3-quasi-perfect tiling from Theorem 9. Recall from Definition 8 that the fraction β of odd finite faces that are covered by the tiling, and the fraction ψ of finite faces of H, that are even satisfy $\beta(1-\psi)+2\psi \geq \alpha$. Let $\mathcal{A}$ be a set of pseudo-witness cycles in H for S_3', the corresponding set for the hitting set S' returned by our algorithm. Define $\mathcal{R} = \mathcal{F}_H$. We have that every cycle of $\mathcal{A}$ contains a face of $\mathcal{R}$ in its interior. Thus, $\mathcal{R}, \mathcal{A}$ and S_3' satisfy the conditions of Theorem 17. Therefore,

$$\sum_{M \in \mathcal{T}_H} |M \cap S_3'| \leq \left(\sum_{M \in \mathcal{F}_H} |M \cap S_3'| \right) - |\mathcal{F}_H \backslash \mathcal{T}_{\mathsf{Faces}(H)}| \leq \frac{18}{7}|\mathcal{F}_H| - |\mathcal{F}_H \backslash \mathcal{T}_{\mathsf{Faces}(H)}| \ . \tag{6}$$

Note that $\sum_{v \in S} a_v^M \leq |M \cap S|$, unless M contains a twin edge. If $M \in \mathcal{T}$ is the disjoint union of two odd faces which share an edge, then M will not contain a twin edge. That is, M can only contain a twin edge if $M \in \mathcal{F}_{\mathsf{even}(H)}$, so M is an even face then. So

$$\sum_{M \in \mathcal{T}_H} \sum_{v \in S} a_v^M \leq \sum_{M \in \mathcal{T}_H} |M \cap S| + |\mathcal{F}_{\mathsf{even}(H)}| \leq \frac{18}{7}|\mathcal{F}_H| - |\mathcal{F}_H \backslash \mathcal{T}_{\mathsf{Faces}(H)}| + |\mathcal{F}_{\mathsf{even}(H)}| \ . \tag{7}$$

Recall that $c = |\mathcal{F}_{\mathsf{even}(H)}|/|\mathcal{F}_H|$ is the fraction of finite faces of $\mathcal{F}_H$ which are even, and that $b = |\mathcal{T}_{\mathsf{Faces}(H)} \backslash \mathcal{F}_{\mathsf{even}(H)}|/|\mathcal{F}_H \backslash \mathcal{F}_{\mathsf{even}(H)}|$ is the fraction of odd finite faces of $\mathcal{F}_H$ covered by our tiling. Note that

$$\begin{aligned}|\mathcal{F} \backslash \mathcal{T}_{\mathsf{Faces}(H)}| &= |\mathcal{F}_H \ \mathcal{F}_{\mathsf{even}(H)}| - |\mathcal{T}_{\mathsf{Faces}(H)} \backslash \mathcal{F}_{\mathsf{even}(H)}| \\ &= |\mathcal{F} \backslash \mathcal{F}_{\mathsf{even}(H)}| - b|\mathcal{F}_H \backslash \mathcal{F}_{\mathsf{even}(H)}| = (1-b)(1-c)|\mathcal{F}_H| \ .\end{aligned}$$

We now recall (5), by which $3|\mathcal{T}_H| \geq \frac{3}{2}(b(1-c)+2c)|\mathcal{F}_H|$.
Substituting these bounds for $|\mathcal{F}_H|$ and $|\mathcal{F}_H \backslash \mathcal{T}_{\mathsf{Faces}(H)}|$ into (7), we obtain

$$\begin{aligned}\sum_{M \in \mathcal{T}_H} \sum_{v \in S} a_v^M &\leq c|\mathcal{F}_H| + \frac{18}{7}\left(\frac{2}{b(1-c)+2c}|\mathcal{T}_H|\right) - (1-b)(1-c)|\mathcal{F}_H| \\ &= \frac{2c}{b(1-c)+2c}|\mathcal{T}_H| + \frac{18}{7}\left(\frac{2}{b(1-c)+2c}|\mathcal{T}_H|\right) - \frac{2(1-b)(1-c)}{b(1-c)+2c}|\mathcal{T}_H| \ .\end{aligned}$$

If we maximize the right-hand side factor $\frac{2c}{(b(1-c)+2c)} + \frac{36}{7(b(1-c)+2c)} - \frac{2(1-b)(1-c)}{(b(1-c)+2c)}$ subject to $b(1-c)+2c \geq 2/3$, we obtain that the right-hand side is bounded by $\frac{47}{7}|\mathcal{T}_H|$.

This completes the proof of Theorem 1 modulo the proof of Theorem 9; i.e., the fact that large quasi-perfect tilings can be computed efficiently. We deal with this in the appendix.

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A Obtaining a 2/3-quasi-perfect tiling

We now show how to find the 2/3-quasi perfect tiling in line 8 of Algorithm 2.1. The following result states that the minimal pockets picked by the algorithm have such tilings.

▶ **Theorem 9.** *Let H be a 2-compression of some planar graph G, that has an even cycle and contains no pockets. Then H has a 2/3-quasi-perfect tiling.*

To prove this theorem we will use the following lemma.

▶ **Lemma 18.** *For any set S, any pseudo-pocket contained in G_2^S contains an even cycle.*

Proof. Informally speaking, the proof will show that any pseudo-pocket without even cycles contains an odd cycle for which only two nodes have outside neighbours; this, however, cannot appear in the 2-compression, as we would have replaced this cycle by an edge in G_2^S.

Suppose, for sake of contradiction, that G_2^S contained a pseudo-pocket Q without even cycles. Since each node of Q is in an even cycle of G_2 and Q contains no even cycle, Q contains exactly two nodes u and v with neighbours outside Q, and each node of Q lies on a u-v path of Q. Let B_u and B_v be the blocks of Q containing u and v in the block graph $\mathcal{B}$ of Q, respectively (see Figure 9).

If $\mathcal{B}$ is not a path, then there would be some block B_1 that does not lie on a B_u-B_v path in $\mathcal{B}$, and thus there would be a node of B_1 that would not lie on a u-v path in Q – a contradiction. Hence, $\mathcal{B}$ is a path. Let B be a block of Q. Suppose for a contradiction

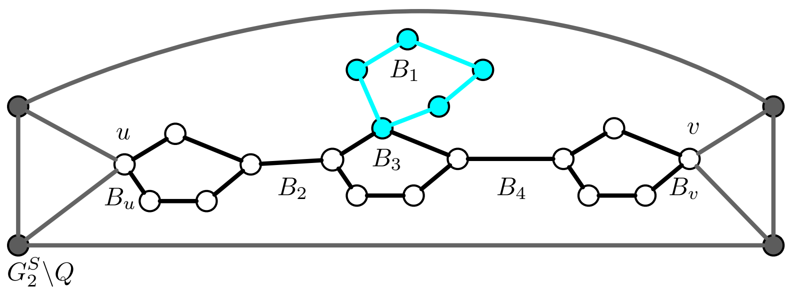

Figure 9 Graph Q consisting of blocks labelled $B_1, B_2, B_3, B_4, B_u, B_v$. Block B_1 depicted in blue contains nodes not on any u-v path, which is a contradiction.

that B contains a cycle C and a node v' of C with a neighbour $u' \in V(B)$ outside C. Since v' is not a cut node, there is a path P from u' to $C \backslash v'$. Construct the u'-v' path P' from P by traversing P from u' to the first node w' of $C \backslash v'$ and appending to that a w'-v' path in C. Since Q contains no even cycles, the cycles $P' \cup v'u'$ and C are odd. Then the cycle formed by the edges $E(C) \Delta E(P' \cup v'u')$, that is edges of C or $P' \cup v'u'$, but not both, has length $|E(C)| + |E(P' \cup v'u')| - 2|E(C) \cap E(P' \cup v'u')|$ which is even, and hence a contradiction. Thus if B contains a cycle then it does not contain nodes outside the cycle, or put simply B is a cycle. Since we assume B contains no even cycles, B is an odd cycle. Thus, the blocks of Q are odd cycles or edges. Since Q contains at least one cycle, there is an odd cycle C'. Since $\mathcal{B}$ is a path, C' contains 2 nodes a and b with neighbours outside C'. However, G_2^S

cannot contain such an odd cycle, as that we would have contracted the two a-b paths of C' to parallel edges and then replaced them by a twin edge; see Figure 10. This completes the proof. ◀

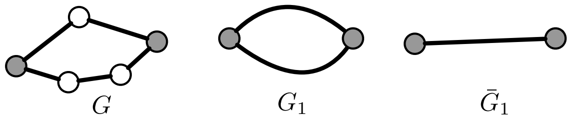

Figure 10 Cycle is replaced by an edge in 2-compression.

For any set S, if G_3^S contained a pseudo-pocket Q without even cycles, then Q was obtained from a subgraph Q' of G_2^S by subdividing edges. Then Q' would be a pseudo-pocket of G_2^S without even cycles. This contradicts Lemma 18. This shows the following corollary.

▶ **Corollary 19.** *For any set S, any pseudo-pocket of G_3^S contains an even cycle.*

Recall from Definition 6 and the paragraph afterwards, that a pseudo-tiling of our graph corresponds to the union of a matching of the dual graph and a set of even faces. A tiling corresponds to the union of a matching of the dual graph not containing any edge incident to the infinite face and a set of even finite faces. Under this correspondence, the existence of large pseudo-tilings is a much more natural thing to prove. Let us first formally define a large pseudo-tiling.

▶ **Definition 20.** *Let $\alpha \in (0, 1)$. A pseudo-tiling $\mathcal{T}$ is* α-pseudo-perfect *if it covers all even faces (including the infinite face if it is even) and a β-fraction of the odd faces, and a ψ-fraction of the faces of H are even, where $\beta(1 - \psi) + 2\psi \geq \alpha$.*

We will first prove the existence of large pseudo-perfect pseudo-tilings. We fix an embedding of H. For any multigraph W, let $\mathsf{oc}(W)$ be the number of odd components of W. Recall pseudo-tilings correspond to matchings. Our proof will use Tutte's Theorem stated below, which informally speaking, says that the absence of a large matching implies the existence of a small set of vertices whose removal results in a graph with a large number of connected components of odd size.

▶ **Theorem 21** (Tutte's Theorem). *For any graph G, the number of nodes of G which are not covered by a maximum size matching of G is at most*

$$\mathsf{oc}(G \backslash X) - |X| \ . \tag{8}$$

for some $X \subset V(G)$. Further, if some node $v \in V(G)$ is covered by every maximum matching of G, then (8) holds for some $X \subset V(G)$ containing v.

The main idea of why such large pseudo-perfect pseudo-tilings should exist is that by Tutte's Theorem, the absence of a large pseudo-tiling implies that for some set X of nodes of the dual graph H^*, the set of odd components of $H^* \backslash X$ is large relative to $|X|$.

Construct a new graph H^1 as follows. Start with the graph H^* and add as many edges as possible between nodes of X while preserving planarity and not creating any faces of length two (see Figure 11). We will show that each odd component of $H^1 \backslash X$ lies in a different face of $H^1[X]$ and that H^1 contains at most two faces of length two. Thus using Euler's formula, $|E(H^1[X])| \leq 3|V(H^1[X])| - 4$, $H^1[X]$ does not have too many edges. The crucial observation is that since each odd component of $H^1 \backslash X$ lies in a different face of $H^1[X]$, each

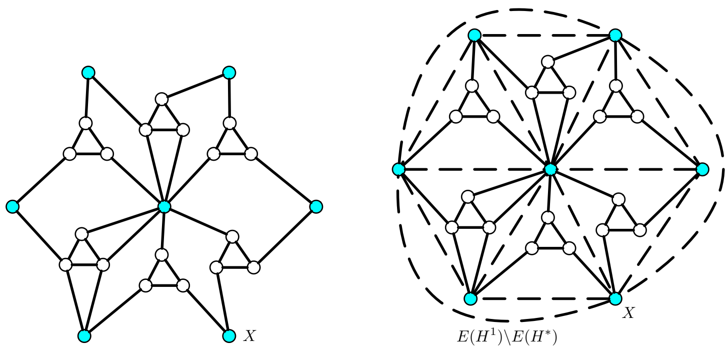

Figure 11 The graph H^* with set $X \subset V(H^*)$ (depicted in blue) on the left. On the right, the graph H^1 obtained from H^* by adding edges (dashed) between X.

node $x \in X$ is adjacent to more other nodes of X in H^1 than there are odd components of $H^1\backslash X$ which contain a neighbour of x. By facial region, we mean the region of the plane bounded by a face. We will also show there are at most two odd components J_1, J_2 for which at most two nodes of X have neighbours in J_i. We can then show that the number of odd components is at most 2/3 the number of edges of $H^1[X]$ plus $\frac{2}{3}$, which will contradict that the set of odd components is large.

▶ **Lemma 22** (⋆). *Let H be as in Algorithm 2.1, that is, H is a minimal pocket of G_2^S. Then H has a 2/3-pseudo-perfect pseudo-tiling.*

So let $\mathcal{T}$ be a 2/3-pseudo-perfect pseudo-tiling of H. Let β' be the fraction of odd faces of H which are covered by $\mathcal{T}$, and let ψ' be the fraction of even faces of H. Next, we will show that if $\mathcal{T}$ covers more faces than a maximum tiling of H, then $\mathcal{T}$ satisfies a slightly stronger condition than 2/3-pseudo-perfect, namely, $\beta'(1-\psi')|V(H^*)| + 2\psi'|V(H^*)| \geq \frac{2}{3}|V(H^*)| + \frac{4}{3}$. Formally, this means:

▶ **Lemma 23** (⋆). *Let H be as in Algorithm 2.1, that is, H is a minimal pocket of G_2^S. Suppose that any maximum size pseudo-tiling of H covers the infinite face. Then H has a pseudo-tiling covering a β'-fraction of all odd faces such that*

$$\beta'(1-\psi')|V(H^*)| + 2\psi'|V(H^*)| \geq \frac{2}{3}|V(H^*)| + \frac{4}{3} \quad . \tag{9}$$

▶ **Theorem 24.** *Let H be an inclusion-minimal pocket of G_2^S. Then we can obtain 2/3-quasi-perfect tiling of H in polynomial time.*

Proof. We first show that H admits a 2/3-quasi-perfect tiling. Let us show that if some tiling $\mathcal{T}$ is 2/3-pseudo-perfect, then it is 2/3-quasi-perfect. Let β' be the fraction of odd faces of H that are covered by $\mathcal{T}$ and ψ' the fraction of faces of H, that are even. As $\mathcal{T}$ is 2/3-pseudo-perfect, it covers all even faces. Since $\mathcal{T}$ is a tiling, the infinite face is odd. As the number of even finite faces is $\psi'|V(H^*)|$, so $\frac{\psi'|V(H^*)|}{|V(H^*)|-1}$ is the fraction of finite faces of H that are even. $(1-\psi')|V(H^*)|$ is the number of odd faces of H, so $\beta'(1-\psi')|V(H^*)|$ is the number of odd faces of H covered by $\mathcal{T}$. Since the infinite face is odd, $(1-\psi')|V(H^*)|-1$ is the number of odd finite faces. Thus $\frac{\beta'(1-\psi')|V(H^*)|}{(1-\psi')|V(H^*)|-1}$ is the fraction of odd finite faces of H covered by $\mathcal{T}$. Since

$$
\begin{aligned}
&\frac{\beta'(1-\psi')|V(H^*)|}{(1-\psi')|V(H^*)|-1}(1-\frac{\psi'|V(H^*)|}{|V(H^*)|-1})+\frac{2\psi'|V(H^*)|}{|V(H^*)|-1} \\
&=\frac{\beta'(1-\psi')|V(H^*)|}{(1-\psi')|V(H^*)|-1}(1-\psi')+2\psi'+\left(2-\frac{\beta'(1-\psi')|V(H^*)|}{(1-\psi')|V(H^*)|-1}\right)\left(\psi'-\frac{\psi'|V(H^*)|}{|V(H^*)|-1}\right) \\
&\leq \frac{\beta'(1-\psi')|V(H^*)|}{(1-\psi')|V(H^*)|-1}(1-\psi')+2\psi' \leq \frac{2}{3},
\end{aligned}
$$

it holds that $\mathcal{T}$ is 2/3-quasi-perfect.

If there is a maximum size pseudo-tiling that is also a tiling, then it follows from Lemma 22 that such a tiling is 2/3-quasi-perfect.

Otherwise, if no pseudo-tiling exists, the largest pseudo-tiling is larger than the largest tiling. Let $\mathcal{T}$ be a maximum size pseudo-tiling.

If the infinite face of $\mathcal{T}$ is even, consider the tiling $\mathcal{T}'$ obtained by removing the infinite face from $\mathcal{T}$. Let $\psi^{(1)} := (\psi'|V(H^*)|-1)/(|V(H^*)|-1)$ be the fraction of finite faces of H which are even. As the infinite face is even, β' is the fraction of odd finite faces of H which are covered by $\mathcal{T}'$. It holds that

$$
\begin{aligned}
\beta'(1-\psi^{(1)})(|V(H^*)|-1)+2\psi^{(1)}(|V(H^*)|-1) &= \beta'|V(H^*)|(1-\psi')+\psi'|V(H^*)|-1 \\
&\geq \frac{2}{3}|V(H^*)|+\frac{4}{3}-1=\frac{2}{3}(|V(H^*)|-1)\ .
\end{aligned}
$$

So $\mathcal{T}'$ is 2/3-quasi-perfect.

If the infinite face is odd, consider the tiling $\mathcal{T}'$ obtained by removing the even cycle covering the infinite face from $\mathcal{T}$. Let $\psi^{(2)} := \psi'|V(H^*)|/(|V(H^*)|-1)$ be the fraction of finite faces of H that are even. At least $\beta'|V(H^*)|-2$ of the finite faces of H are covered by $\mathcal{T}'$ so the fraction β''' of finite odd faces of H that are covered satisfies $b'' \geq (\beta'|V(H^*)|-1)/(1-\psi^{(2)})(|V(H^*)|-1)$. Therefore,

$$
\begin{aligned}
b''(1-\psi^{(2)})(|V(H^*)|-1)+2\psi^{(2)}(|V(H^*)|-1) &\geq (\beta'|V(H^*)|-1)+2c|V(H^*)| \\
&\geq \frac{2}{3}|V(H^*)|+\frac{4}{3}-1=\frac{2}{3}(|V(H^*)|-1)\ .
\end{aligned}
$$

Hence also in this case, $\mathcal{T}'$ is 2/3-quasi-perfect.

Finally, since a tiling corresponds to the union of a matching and a set of even faces, finding a maximum tiling of H corresponds to finding a maximum matching of the odd finite faces of H. Computing such a maximum matching can be done in polynomial time. ◀

A.1 Proof of Theorem 17

In this section we will prove Theorem 17. Let $G, H, \mathcal{R}, S, \mathcal{A}$ be as in the statement of Theorem 17. Let $\mathcal{D}_G$ be the debit graph of G with respect to S.

We introduce the notion of "balance", which captures for subsets $\mathcal{R}' \subseteq \mathcal{R}$ of cycles are incident to more or less than 18/7 nodes of S on average.

▶ **Definition 25.** *For each subset $\mathcal{R}' \subseteq \mathcal{R}$, its* balance $\mathsf{bal}(\mathcal{R}')$ *is the quantity* $|\mathcal{R}'| - \frac{7}{18}|E'_{\mathcal{R}}|$.

Our proof follows the same methodology as Berman and Yaroslavtsev [6]. First, it shows a pseudo-witness cycle that is not a face and is minimally so, that is any pseudo-witness cycle lying in the finite region bounded by it is a face, has balance at least $1 - \frac{7}{18}$. Then it uses this to apply a reduction on G. We will use the following result of theirs.

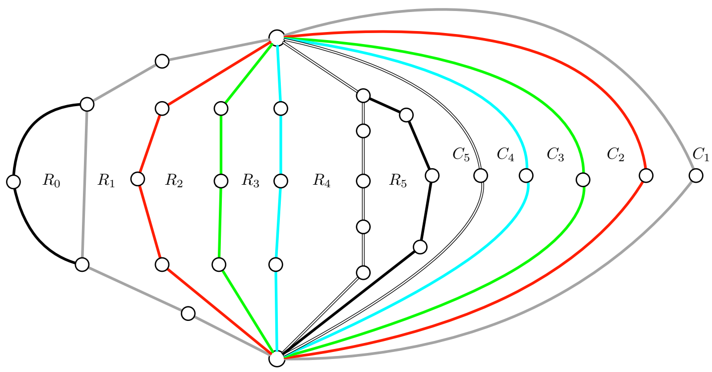

Figure 12 Pseudo-witness cycles $C_1, \ldots, C_5$ divide H into regions $R_0, R_1, \ldots, R_5$.

▶ **Proposition 26** ([6, Lemma 4.3]). *Let W be a planar graph, $\hat{S}$ be a set of nodes of W and $Q \subset \hat{S}$ be a set of nodes of W that we call* outer nodes*. Let $\mathcal{R}_W$ be a set of faces of W such that each non-outer node of $\hat{S} \cap W$ has a pseudo-witness cycle in $\mathcal{R}_W$. If W contains $a \leq 2$ outer nodes, then $\mathsf{bal}(\mathcal{R}_W) \geq 1 - \frac{7}{18}a$.*

If all nodes of a pseudo-witness cycle A are contained in H, call A a *hierarchical* pseudo-witness cycle. Otherwise, call A a *crossing* pseudo-witness cycle. Denote the set of crossing pseudo-witness cycles by $\hat{\mathcal{A}}$. We are now ready to complete the proof of Theorem 17. We begin by reductions on our instance $(G, H, \mathcal{R}, \mathcal{A}, S)$ which simplify our instance and do not increase the balance. If after applying this reduction our instance has positive balance, then our instance had positive balance before the reduction. We define the reduction below.

▶ **Definition 27.** *We define the following reduction on our instance $(G, H, \mathcal{R}, \mathcal{A}, S)$. If H contains a hierarchical pseudo-witness cycle A that is not a face of $\mathcal{R}$, delete all nodes, edges and faces of $\mathcal{R}$ inside A from H and add A to $\mathcal{R}$. If H does not contain a hierarchical witness cycle, we call the instance $(G, H, \mathcal{R}, \mathcal{A}, S)$* reduced*.*

Let $\mathcal{R}_C$ be the faces in $\mathcal{R}$ contained in the region bounded by C. Let $H^1, \mathcal{R}^1$ be the result of applying the reduction in Definition 27 on $H, \mathcal{R}$. The balance of $H^1, \mathcal{R}^1$ is equal to

$$|(\mathcal{R} \backslash \mathcal{R}_C) \cup \{C\}| - \sum_{M \in (\mathcal{R} \backslash \mathcal{R}_C) \cup \{C\}} |M \cap S|$$
$$= |\mathcal{R}| - \sum_{M \in \mathcal{R}} |M \cap S| - (|\mathcal{R}_C| + 1 - (\sum_{M \in \mathcal{R}_C} |M \cap S|) + 1) = \mathsf{bal}(H) + 1 - \mathsf{bal}(\mathcal{R}_C) - \frac{7}{18} \ .$$

That is to say, the reduction changes the balance by $1 - \mathsf{bal}(\mathcal{R}_C) - \frac{7}{18}$, which by Proposition 26 is non-positive. Thus, if after applying the reduction in Definition 27, our instance has positive balance then it initially had positive balance. We know apply the reduction in Definition 27 until our instance is reduced, for simplicity we will continue to call this graph H.

The crossing pseudo-witness cycles $\hat{\mathcal{A}}$ partition H into *regions*, see Figure 12. That is, consider the subgraph $K \subset H$ consisting of nodes and edges lying on a witness cycle of $\hat{\mathcal{A}}$ or on the outside face of H. The regions are defined as the portions of the plane bounded by the finite faces of K. Define a *subpocket* [6] as the subgraph of H consisting of the nodes and edges lying in or on the boundary of a region.

▶ **Proposition 28** ([6]). *The regions that the set of crossing cycles $\hat{\mathcal{A}}$ partition the plane into satisfy the following. For each region, there is a set $\tilde{\mathcal{A}}$ of at most two pseudo-witness cycles of $\hat{\mathcal{A}}$ such that each node bounding the region either does not lie on a pseudo-witness cycle in $\hat{\mathcal{A}}$ or lies on a cycle of $\tilde{\mathcal{A}}$.*

By the reduction described in Definition 27 each non-crossing cycle of $\mathcal{A}$ is a face. Since by Proposition 28, the outside face of each subpocket W contains nodes from at most two crossing pseudo-witness cycles, and contains all nodes that belong to pseudo-witness cycles lie on the outside face, there are at most two hit nodes of W whose pseudo-witness is not a face and they must lie on the outside face of W. Hence, each subpocket satisfies the conditions of Proposition 26 and hence has positive balance. Thus, H has positive balance, that is, $0 \leq |\mathcal{R}| - \frac{7}{18}|E_{\mathcal{R}}| = |R| - \sum_{M \in R} |M \cap S|$. Rearranging, $\sum_{M \in R} |M \cap S| \leq \frac{18}{7}|\mathcal{R}|$, which completes the proof of Theorem 17. ◀