

# Correlation-Intractable Hash Functions via Shift-Hiding

Alex Lombardi ✉

Massachusetts Institute of Technology, Cambridge, MA, USA

Vinod Vaikuntanathan ✉

Massachusetts Institute of Technology, Cambridge, MA, USA

---

## Abstract

A hash function family  $\mathcal{H}$  is correlation intractable for a  $t$ -input relation  $\mathcal{R}$  if, given a random function  $h$  chosen from  $\mathcal{H}$ , it is hard to find  $x_1, \dots, x_t$  such that  $\mathcal{R}(x_1, \dots, x_t, h(x_1), \dots, h(x_t))$  is true. Among other applications, such hash functions are a crucial tool for instantiating the Fiat-Shamir heuristic in the plain model, including the only known NIZK for NP based on the learning with errors (LWE) problem (Peikert and Shiehian, CRYPTO 2019).

We give a conceptually simple and generic construction of single-input CI hash functions from shift-hiding shiftable functions (Peikert and Shiehian, PKC 2018) satisfying an additional one-wayness property. This results in a clean abstract framework for instantiating CI, and also shows that a previously existing function family (PKC 2018) was already CI under the LWE assumption.

In addition, our framework transparently generalizes to other settings, yielding new results:

- We show how to instantiate certain forms of *multi-input* CI under the LWE assumption. Prior constructions either relied on a very strong “brute-force-is-best” type of hardness assumption (Holmgren and Lombardi, FOCS 2018) or were restricted to “output-only” relations (Zhandry, CRYPTO 2016).
- We construct single-input CI hash functions from indistinguishability obfuscation (iO) and one-way permutations. Prior constructions relied essentially on variants of fully homomorphic encryption that are impossible to construct from such primitives. This result also generalizes to more expressive variants of multi-input CI under iO and additional standard assumptions.

**2012 ACM Subject Classification** Theory of computation → Cryptographic primitives

**Keywords and phrases** Cryptographic hash functions, correlation intractability

**Digital Object Identifier** 10.4230/LIPIcs.ITCS.2022.102

**Related Version** *Full Version*: <https://eprint.iacr.org/2020/1378>

**Funding** *Alex Lombardi*: Research supported in part by an NDSEG Fellowship, by a Charles M. Vest Fellowship, and by the grants of the second author.

*Vinod Vaikuntanathan*: Supported in part by NSF Grants CNS-1350619, CNS-1414119 and CNS-1718161, by DARPA under Agreement No. HR00112020023, by a Microsoft Faculty Fellowship, and by an MIT/IBM grant.

**Acknowledgements** We thank an anonymous reviewer for pointing out that the [34] hash function can likely also be shown to satisfy multi-input CI for shifted sum relations.

## 1 Introduction

The random oracle model [4] is a powerful but controversial paradigm in cryptography in which the proof of security of a cryptographic scheme assumes that a certain publicly computable function  $H$  that is used in the scheme behaves like a random function to the adversary. The random oracle model is hugely influential in designing concretely efficient cryptosystems, but is inherently problematic theoretically: how could a *public*, and therefore completely predictable, function behave in all aspects like a random function? Indeed,



© Alex Lombardi and Vinod Vaikuntanathan;

licensed under Creative Commons License CC-BY 4.0

13th Innovations in Theoretical Computer Science Conference (ITCS 2022).

Editor: Mark Braverman; Article No. 102; pp. 102:1–102:16

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Canetti, Goldreich and Halevi [15] demonstrated cryptographic schemes that one could prove secure in the random oracle model, but which are insecure no matter how one tries to instantiate the oracle with a concrete function (or even a function chosen at random from an exponential-size family). Nevertheless, this negative result and the notions introduced therein led to a long line of research that asked *what concrete properties* of a random oracle are instantiable in the standard model (see, e.g., [16] for an early work in this direction), and opened the door to groundbreaking positive results two decades later [12–14, 26, 28, 34].

The key notion introduced in [15] is that of correlation intractability (CI), which captures a general and powerful form of cryptographic hardness for a hash family  $\mathcal{H}$ . For any binary relation  $R(x, y)$ , a hash family  $\mathcal{H}$  is correlation-intractable for  $R$  if it is computationally hard (given a hash function  $h \leftarrow \mathcal{H}$ ) to find an input  $x$  such that  $R(x, h(x))$  is true. For this definition to make sense, we require that the relation  $R$  is sparse: for any  $x$ , all but a negligible fraction of  $y$  do not satisfy the relation with  $x$ .

For decades, there was little progress on building correlation-intractable hash functions in the standard model outside of a few extremely simple cases (such as one-way functions). However, there has been much recent work [10, 12–14, 26, 28, 30, 34] on instantiating restricted but expressive variants of CI. Namely, these works made the following simplifications:

- Starting with [13, 26], additional *efficiency* requirements were placed on the relation  $R$ . For example, one can require that  $R(x, y)$  is decidable in (bounded) polynomial time.
- Starting with [12], the relation  $R$  was further specialized to represent an *efficiently computable function*  $f$ . A hash family  $\mathcal{H}$  is CI for  $f$  if it is hard, given  $h$ , to find an input  $x$  such that  $h(x) = f(x)$ .

While these restrictions may seem extreme, these limited forms of CI remain expressive and powerful. In particular, even CI for efficiently computable functions has implications for the instantiability of the Fiat-Shamir transform [20] in the standard model [3, 13, 19] for constant-round public-coin interactive proof systems. Most notably, [12, 34] construct hash families  $\mathcal{H}$  that are CI for efficiently computable functions under standard cryptographic assumptions related to the learning with errors (LWE) problem, and use these hash families to build the first lattice-based non-interactive zero-knowledge (NIZK) proof systems for NP.

Let us recall the [12, 34] constructions at a high level. [12] gives a *generic* construction using fully homomorphic encryption (FHE) [11, 22]. The construction is simple: a hash function  $h \leftarrow \mathcal{H}$  is parameterized by a FHE ciphertext  $\text{Enc}(g)$  for some (dummy) function  $g$ . To evaluate  $h(x)$ , simply homomorphically evaluate  $g$  on  $x$  to obtain some ciphertext of the form  $\text{Enc}(g(x))$ . One can show that this hash family is CI for a function  $f$  if the FHE scheme is *circular secure*: since  $g$  is computationally hidden, we can replace it in the security proof with a function  $g^*(x) = \text{Dec}_{\text{sk}}(f(x)) + 1$  specifically designed to avoid  $f(x)$  at the ciphertext level.

While this construction is both simple and generic, it has the significant drawback that it relies on the circular security (rather than semantic security) of the FHE, and therefore cannot be proven secure under the plain LWE assumption. Peikert and Shiehian [34] then gave an ingenious construction of CI based on plain LWE. Their construction uses the algebra of the [23] FHE scheme to give a special-purpose variant of the [12] approach that avoids reliance on circular security. However, this requires making a number of changes to the hash function: at a high level, they “downgrade” plain LWE-based GSW ciphertexts after evaluation to Regev “ciphertexts” (where the plaintext space is  $\mathbb{Z}_q$  and decryption correctness is only approximate) with circular dependencies. This results in a LWE-based CI hash family, but loses the conceptual simplicity of the [12] construction.

## 1.1 Our Results and Techniques

Our main result is a new framework for constructing CI hash functions using a cryptographic primitive called *shift-hiding shiftable functions* (SHSFs) [33], a twist on private constrained pseudorandom functions [7,8,29]. A SHSF family is a function family  $\{F_{\text{msk}}\}$  that additionally supports the ability to *delegate* a constrained key  $\text{sk}_f$  that enables computation of the map  $x \mapsto F_{\text{msk}}(x) + f(x)$ , without revealing the “shift function”  $f$ . Shift-hiding shiftable functions were originally introduced for the purpose of constructing private constrained PRFs, but have since found several other applications [18,35].

In a nutshell, we show that SHSFs are intimately tied to correlation intractability via an extremely short proof. We further develop this framework in three directions.

1. We obtain a conceptually simple construction of CI for functions based on LWE. This construction can replace the FHE-based approach of [12,34] and shows that the prior function family of [33] (constructed for an entirely different purpose) was *already* a good CI hash family.
2. We show that our construction transparently generalizes to new variants of *multi-input* CI, which is currently poorly understood.
3. We give additional instantiations of our framework (which are new, in both the single- and multi-input settings) using indistinguishability obfuscation and other standard assumptions.

Moreover, we believe that our framework and new approach to constructing CI hash functions may be useful for future progress on and understanding of this primitive.

### Lifting CI

We begin with a description of (1). Our main technique is a *lifting theorem* (Theorem 17) that allows us to construct CI hash functions for complex relations starting from CI hash functions for simpler relations. In the single-input setting, it states that any SHSF family (for a function class  $\mathcal{F}$ ) satisfying a *very weak* form of correlation intractability is essentially already a CI hash family for  $\mathcal{F}$ .

► **Theorem 1 (Informal).** *Suppose that  $\text{SHSF} = \{F_{\text{msk}}\}$  is a family of SHSFs for a function class  $\mathcal{F}$ , and suppose that  $F_{\text{msk}}$  satisfies either of the following two one-wayness properties:*

- *Given  $\text{msk}$ , it is hard to find an element in  $F_{\text{msk}}^{-1}(0)$ , or*
- *Given  $\text{msk}$  and a uniformly random target  $r$ , it is hard to find an element in  $F_{\text{msk}}^{-1}(r)$ .*

*Then, the shifted evaluation algorithm of SHSF describes a hash family  $\mathcal{H}$  that is correlation-intractable for all functions  $f \in \mathcal{F}$ .*

The CI hash function is extremely simple to describe. Hash keys are shifted keys  $\text{sk}_{\mathcal{Z}}$  for the all-zero function  $\mathcal{Z}$ , and hash function evaluation is simply the shifted evaluation using  $\text{sk}_{\mathcal{Z}}$  which computes exactly the function  $F_{\text{msk}}$ . (Philosophically, the CI hash family constructed in this theorem is a form of “obfuscated PRF evaluation” although shift-hiding functions are decidedly more complex to construct than PRFs.) The proof of Theorem 1 is also simple.

**Proof Sketch.** If an adversary  $\mathcal{A}$ , given a hash key  $\text{sk}_{\mathcal{Z}}$ , finds an input  $x$  such that

$$\text{Hash}(x) := F_{\text{sk}_{\mathcal{Z}}}(x) = f(x) ,$$

then by the shift-hiding property of SHSF,  $\mathcal{A}$  also produces such an  $x$  when given  $\text{sk}_f$  instead of  $\text{sk}_{\mathcal{Z}}$ . In that case,  $\mathcal{A}$  solves the equation

$$f(x) = F_{\text{sk}_f}(x) = F_{\text{msk}}(x) + f(x),$$

which is equivalent to the equation  $F_{\text{msk}}(x) = 0$ . This yields a 0-inversion attack on  $F_{\text{msk}}$ . The “random target” version of the theorem holds by the same argument, using a shifted key  $\text{sk}_{f_r}$  for the function  $f_r(x) = f(x) - r$ . ◀

We note that Theorem 1 could be proved under a weaker one-wayness assumption, namely, that *it is hard to find an input  $x$  such that  $F_{\text{msk}}(x) = 0$ , given a shifted key  $\text{sk}_f$  for any pre-specified  $f$*  (as opposed to being given  $\text{msk}$  in the clear). However, we phrase Theorem 1 under the assumption that  $F_{\text{msk}}$  is one-way (given  $\text{msk}$  in the clear) because this is a clean,  $f$ -independent security property, which also makes it more amenable to instantiation/proof. In our constructions below, we prove the stronger one-wayness property of  $F_{\text{msk}}$ .

### Instantiation from LWE

Given Theorem 1, it remains to construct an SHSF family satisfying this one-wayness property. We show that a variant of the Peikert-Shiehian SHSF [33] satisfies this.

► **Theorem 2** (Informal, see the full version [31]). *Assuming the hardness of standard lattice problems (LWE and 1-dimensional SIS variants), the [33] SHSF<sup>1</sup> is one-way.*

We now sketch our proof assuming some knowledge of LWE-based cryptography.

**Proof Sketch.** In the Peikert-Shiehian SHSF construction,  $\text{msk} = \mathbf{s} \in \mathbb{Z}_q^n$  is an LWE secret, and

$$F_{\text{msk}}(x) = \lfloor \mathbf{s}\mathbf{A}_x + \mathbf{u} \cdot \mathbf{G}^{-1}(\mathbf{A}_x) \rfloor_p \in \mathbb{Z}_p^\mu$$

where  $\mathbf{G} \in \mathbb{Z}_q^{n \times m}$  is the gadget matrix,  $\mathbf{u} \in \mathbb{Z}_q^m$  is a uniformly random row vector,  $\mathbf{A}_x \in \mathbb{Z}_q^{n \times \mu}$  is a matrix constructed out of (uniformly random) matrices  $\mathbf{A}_1, \dots, \mathbf{A}_\ell$  using the gadget homomorphisms from [6], and  $\lfloor \cdot \rfloor_p$  denotes the rounding operation that (roughly speaking) keeps the top  $\log p$  bits of the argument and discards the rest. By [33], this family is shift-hiding under the LWE assumption and (computationally) correct under the 1D-SIS assumption.

If the adversary finds an  $x$  such that  $F_{\text{msk}}(x) = 0$ , there are two cases; the first case is when  $\mathbf{G}^{-1}(\mathbf{A}_x)$  is non-zero. This gives an approximate subset sum solution for the instance  $\mathbf{s}\mathbf{G} + \mathbf{u}$ , that is,

$$(\mathbf{s}\mathbf{G} + \mathbf{u})\mathbf{G}^{-1}(\mathbf{A}_x) \in q\mathbb{Z}^\mu + \left[-\frac{q}{p}, \frac{q}{p}\right]^\mu.$$

This violates (on whichever column of  $\mathbf{G}^{-1}(\mathbf{A}_x)$  is nonzero) a natural one-dimensional variant of SIS that we show is as hard as worst-case lattice problems provided that  $p$  is large enough<sup>2</sup> (see the full version [31]).

The second case is when the adversary finds an  $x$  such that  $\mathbf{G}^{-1}(\mathbf{A}_x) = 0$ , which implies that  $\mathbf{A}_x = 0$ . We show that the adversary cannot make this happen without violating SIS (again!) Roughly speaking, we use the fact that if we *program* the matrices  $\mathbf{A}_i = \mathbf{A}\mathbf{R}_i + h_i\mathbf{G}$  where  $\mathbf{R}_i$  are matrices with small entries and  $h$  is the description of a constant function with

<sup>1</sup> Compared to [33], (1) our construction is slightly modified for ease of proof, and (2) particular parameter settings are required.

<sup>2</sup> Some care must be taken to set parameters so that the SHSF security reductions still hold for this choice of  $p$ .

image  $y \neq 0 \in \mathbb{Z}_q^\mu$ , the following equation holds for each column  $\mathbf{a}_x^{(j)}$  of  $\mathbf{A}_x$  due to the gadget homomorphisms of Boneh et al. [6]:

$$\mathbf{a}_x^{(j)} = \mathbf{A}\mathbf{r}_x^{(j)} + y_j\mathbf{u}_1$$

(where  $\mathbf{u}_1$  is the first standard basis vector) for some  $\mathbf{r}_x^{(j)}$  that is a function of  $\mathbf{R}_1, \dots, \mathbf{R}_\ell$ . We know by assumption that  $\mathbf{A}_x = 0$ . Since  $y \neq 0$ , this means that the adversary found a valid solution  $\mathbf{R}_x = \begin{bmatrix} \mathbf{r}_x^{(1)} & \dots & \mathbf{r}_x^{(\mu)} \end{bmatrix}$  to the (inhomogenous) SIS problem  $\mathbf{A}\mathbf{R}_x = -\mathbf{u}_1 y^\top \in \mathbb{Z}_q^{n \times \mu}$ , which is hard assuming that worst-case lattice problems are hard. This finishes the proof of one-wayness.  $\blacktriangleleft$

Combining Theorem 2 with Theorem 1, we already recover a similar result to [34]. That is, assuming the hardness of standard lattice problems, there exists a hash family that is correlation-intractable for all bounded-size functions. By appealing to [12], this also gives a lattice-based NIZK argument system for NP. However, our approach leverages this new, conceptually simple connection to SHSFs and shows that [33] were “most of the way” to LWE-based CI. Besides the extremely simple bootstrapping theorem, the missing piece was whether a natural PRF construction [33] satisfies a one-wayness property given  $\text{msk}$  in the clear. A similar question was previously studied for the GGM PRF family [17], but does not appear to have been addressed for other concrete PRF families.

Next, we describe how our techniques extend to give new feasibility results in two different directions:

- They immediately generalize to setting of *multi-input* CI, and
- They allow for new generic instantiations based on indistinguishability obfuscation.

We remark that constructing (single- or multi-input) CI hash functions even assuming indistinguishability obfuscation is far from straightforward. Indeed, the initial works [13, 26, 28] in this line all made non-standard assumptions *in addition to iO*. Non-standard assumptions were required until the work of [12] which constructed single-input CI hash functions under circular-secure LWE. However, they only managed to do this for a tiny subset of relations that [13, 28] achieved. In particular, replicating the results of [28] or even [13] assuming *only iO* (plus standard assumptions) is a challenging open problem.

## 1.2 Applications: Multi-Input CI from LWE and CI from iO

So far, we have only discussed *single-input* CI; that is, we considered CI for relations with a single input  $x$  and single corresponding output  $y$ . However, there is a natural generalization of CI to relations with many input-output pairs: a hash family  $\mathcal{H}$  is defined to be CI for a relation  $R(x_1, \dots, x_t, y_1, \dots, y_t)$  if it is computationally hard (given a hash function  $h \leftarrow \mathcal{H}$ ) to find inputs  $x_1, \dots, x_t$  such that  $(x_1, \dots, x_t, h(x_1), \dots, h(x_t)) \in R$ . In contrast to the single-input case, *multi-input* correlation intractability (for any  $t \geq 2$ ) is a far less well-understood primitive. Perhaps the simplest nontrivial example of multi-input CI is for the relation  $R$  where  $R(x_1, x_2, y_1, y_2) = 1$  if and only if  $y_1 = y_2$  but  $x_1 \neq x_2$ . A CI hash family for  $R$  is precisely a collision-resistant hash family. However, most multi-input relations do not correspond to security notions that are simple-to-understand or previously studied. CI for more general multi-input relations also has interesting applications, including:

1. As a useful tool for the *untrusted setup* of public parameters [13, 38]: Multi-input CI hash functions allow  $n$  parties  $P_1, \dots, P_n$  with inputs  $x_1, \dots, x_n$  to compute public outputs  $y_i = H(x_i)$  that can be used to generate public parameters for a multi-party protocol. Correlation intractability of  $H$  is necessary to ensure that a “bad CRS” is not accidentally (or maliciously) agreed on.

2. As a hash function in proof-of-work protocols [13,14]: In the bitcoin protocol [32], a miner succeeds in adding a block to the blockchain when she finds an  $x$  such that  $y = H(x||B_i)$  starts with a specified number of zeroes (here,  $B_i$  is the  $i$ -th block and once found,  $y$  is placed in the next block  $B_{i+1}$ ). A very desirable property in this setting is that a single miner (or collection of colluding miners) cannot find *multiple consecutive blocks* with significantly less effort than finding them sequentially. This property can be formalized as a quantitatively precise<sup>3</sup> variant of multi-input CI. For example, in the case of two consecutive blocks, simplifying the setting a little, we require a 2-input CI for the relation  $R$  where  $R(x_1, x_2, y_1, y_2) = 1$  iff  $y_1$  and  $y_2$  start with a pre-specified number  $\ell$  of zeroes, and  $y_1$  is a suffix of  $x_2$ .

Unfortunately, multi-input CI has so far proved hard to achieve. In particular, the constructions of [10,12–14,28,34] are only known to achieve single-input CI. Holmgren and Lombardi [26] do achieve multi-input CI for a large class of relations that they call *locally sampleable* relations. However, they require both an indistinguishability obfuscation (iO) scheme [2] as well as an “optimally-secure” one-way product function [26]. While iO can now be achieved under relatively standard assumptions [9,21,27,37], the latter is a very strong “brute force is optimal”-type assumption. Zhandry [38] constructed a hash family satisfying a very special form of multi-input CI called “output intractability”. Output intractability is a form of CI for relations  $R(x_1, \dots, x_t, y_1, \dots, y_t)$  that depend only on the  $y_i$ , which captures some variants of application (1) above. On the plus side, the construction is based on the exponential hardness of the Diffie-Hellman problem.<sup>4</sup> To summarize, multi-input CI is either known for a small class of relations under standard assumptions, or for a larger class of relations under very strong assumptions. We refer the reader to Section 1.3 for more details and further comparisons.

### Multi-Input CI via Shift-Hiding

One consequence of our shift-hiding technique is a collection of feasibility results for multi-input correlation intractability based on standard assumptions. We obtain two flavors of results: constructions from standard (lattice) assumptions, and constructions from indistinguishability obfuscation.

Our results are obtained via a generalization of our lifting theorem (Theorem 1) to multi-input relations. This gives us three new constructions of multi-CI hash functions under different assumptions:

- Our first construction considers the shifted linear relation

$$\mathcal{R}_{\text{lin}} = \{(x_1, \dots, x_t, y_1, \dots, y_t) : \sum w_i y_i = \sum w_i f(x_i) \pmod{p}\}$$

where  $p$  is some large integer (roughly  $2^\lambda$ ),  $w_i$  are small weights and  $f$  is an arbitrary polynomial-time computable function. We construct a multi-input CI hash function for  $\mathcal{R}_{\text{lin}}$  under the same lattice assumptions as in the single-input case (all approximation ratios are larger by a factor of  $t$ ).

<sup>3</sup> As noted in [13], CI following the (poly, negl) security definition framework is insufficient for this application. Instead, these protocols desire a concrete “moderately small” probability of breaking CI and a tight gap between honest and adversarial parties’ probabilities of doing so in a fixed runtime. We do not attempt to address this subtlety in this work.

<sup>4</sup> Moreover, given an inverse-subexponential lower bound on the sparsity of the relation, Zhandry’s construction is secure under (the more standard) sub-exponential DDH.

- Our second and third constructions consider the shifted general relation

$$\mathcal{R} = \{(x_1, \dots, x_t, y_1, \dots, y_t) : \mathcal{R}_0(y_1 - f(x_1), \dots, y_t - f(x_t)) = 1\}$$

where  $\mathcal{R}_0$  is any polynomial-time decidable relation. In particular, our second construction achieves a multi-input CI hash function for  $\mathcal{R}$  under subexponential iO, subexponential OWFs, and (sufficiently) lossy functions.

### Our Generalized Lifting Theorem

Given any output-only relation  $\mathcal{R}_0$ , we say that a hash family  $\mathcal{H}$  is  $\mathcal{R}_0$ -output intractable if it is hard (given  $h$ ) to find distinct<sup>5</sup> inputs  $x_1, \dots, x_t$  such that  $(y_1, \dots, y_t) \in \mathcal{R}_0$  for  $y_i = h(x_i)$ . Output intractability as a standalone property (like collision-resistance) is known to be instantiable based on standard cryptographic assumptions (e.g., lossy functions [36]) as we discuss in Section 1.3. Our generalization of Theorem 1 states that *SHSFs that are output-intractable* lead to interesting new CI constructions.

► **Theorem 3** (Also see Theorem 17). *Suppose that SHSF is a shift-hiding shiftable function family. Assume that it is hard, given  $\text{msk}$ , to find distinct  $x_1, \dots, x_t$  such that  $\mathcal{R}_0(y_1, \dots, y_t) = 1$  where  $y_i = F_{\text{msk}}(x_i)$  and  $\mathcal{R}_0$  is some polynomial-time computable relation. Then, there is a CI hash family for the shifted output relation*

$$\mathcal{R} = \{(x_1, \dots, x_t, y_1, \dots, y_t) : \mathcal{R}_0(y_1 - f(x_1), \dots, y_t - f(x_t)) = 1\}$$

The proof of Theorem 3 follows from that of the single-input CI case *mutatis mutandis*. Thus, all that remains is to construct SHSFs that are *output-intractable*. We show three constructions.

### Instantiation from LWE

To obtain a form of multi-input CI from LWE, we combine Theorem 3 with a generalization of Theorem 2:

► **Theorem 4.** *Under standard lattice assumptions, there exists a SHSF family SHSF satisfying the following form of correlation intractability: for every nonzero vector  $w \in \{-1, 0, 1\}^t$ , it is hard (given  $\text{msk}$ ) to find  $t$  distinct inputs  $x_1, \dots, x_t$  such that*

$$\sum_i w_i \cdot F_{\text{msk}}(x_i) = 0,$$

where the sum is computed modulo some (large enough) integer  $p$ .

Our modification of the Peikert-Shiehian [33] construction satisfies this more general form of output intractability (for small linear equations), although the proof (in “Case 2” above) is more complicated (see the full version [31]). Note that this is a strict generalization of both single-input CI for functions (where  $t = 1, w = 1$ ) and collision-resistance (where  $t = 2, w = (-1, 1)$  and  $f$  is the constant function). Previously, this form of correlation intractability was only known assuming iO and (extremely hard) one-way product functions [26].

<sup>5</sup> For the relation  $\sum_i w_i y_i = 0$  implicitly described above, it is enough to assume that the inputs  $x_i$  are not all equal for the relation to be sparse. We elaborate on this weakening of output intractability as compared to [26, 38] in Section 2.

**Instantiation from IO + lossiness**

Our second construction achieves correlation intractability for shifted  $\mathcal{R}_0$ -output relations for a large class of  $\mathcal{R}_0$  simultaneously (as opposed to linear  $\mathcal{R}_0$  as in the LWE case above). It can be thought of as a (non-black-box) combination of our approach with a construction due to Zhandry [38] of output-intractable hash functions.

► **Theorem 5.** *Assume the existence of subexponential iO, subexponential OWFs, and lossy functions with input domain  $\{0,1\}^n$  with a range of size  $\leq 2^\ell$  in lossy mode. Then, there exists a hash family  $\mathcal{H}$  that is CI for all (efficiently decidable) shifted  $t$ -ary output relations with sparsity at most  $2^{-t\ell}$ .*

As a corollary, we conclude that additionally assuming the existence of *extremely lossy functions* [38], there is a hash family  $\mathcal{H}$  that is CI for all (efficiently decidable) shifted  $t$ -ary output relations with sparsity  $2^{-\omega(t)}$ . As another corollary, we note that by combining Theorem 5 with [12], we obtain a construction of dual-mode NIZKs for NP based on iO, (injective) lossy functions, and lossy encryption. This closely matches the assumptions used in the work [25] but with a simpler construction. The corollary follows because the hash family from Theorem 5 satisfies “somewhere statistical correlation intractability.”

**A Separation between Single-Input and Multi-Input CI**

Finally, we show that single-input and multi-input CI hash functions are fundamentally different primitives by demonstrating a separation between them. This follows from our third new CI instantiation, which is interesting even in the single-input setting.

► **Theorem 6.** *Assume the existence of subexponentially secure indistinguishability obfuscation, subexponentially secure one-way functions, and a hash family  $\mathcal{H}$  such that  $\mathcal{H}$  is  $\mathcal{R}_0$ -output intractable, and for a random input  $X$ ,  $h_k(X)$  is  $2^{-n}$ -indistinguishable from uniform (even given  $k$ ). Then, there exists a hash family that is CI for shifted  $\mathcal{R}_0$ -relations.*

This theorem says that assuming subexponential iO and one-way functions, shifted-CI for  $\mathcal{R}_0$  can be constructed (semi-)generically from output intractability for  $\mathcal{R}_0$ . Theorem 6 is proved by combining Theorem 3 with a construction of an  $\mathcal{R}_0$ -output intractable SHSF using iO, puncturable PRFs, and an output-intractable hash function satisfying the above statistical requirement.

We note that as a corollary to Theorem 6, we obtain a construction of single-input CI for all efficient functions from iO and one-way permutations.<sup>6</sup>

► **Corollary 7.** *If subexponential iO, subexponential OWFs, and (polynomially-secure) OWPs exist, then there exists a hash family that is CI for all efficient functions, that is, relations  $\mathcal{R}(x,y)$  which is true iff  $y = f(x)$ .*

Corollary 7 follows from Theorem 6 by setting the output-intractable hash function  $\mathcal{H}$  to be  $h_k(x) := f(x) + k$ , where  $f$  is a one-way permutation<sup>7</sup> and  $k$  is a uniformly random key. This construction is notable in that it separates *single-input* correlation intractability

<sup>6</sup> As is common [24], one must be careful about which definitions of “one-way permutation” suffice for this result. In our proof (which suffices for the separation), we assume that the one-way permutation has domain  $\{0,1\}^n$ . It turns out that the proof can be made to work for discrete log-based one-way permutations, but does *not* appear to work for the (trapdoor) permutations constructed based on iO [5].

<sup>7</sup> It suffices for  $f$  to be a OWF whose output distribution is close to uniform, e.g., a surjective regular OWF.



(theoretically) from *two-input* correlation intractability: due to an impossibility result of Asharov-Segev [1], it is known that there is no (black-box) construction of CRHFs from iO and one-way permutations (even with exponential security). A similar separation was shown in [26], but the “positive result” required assuming *optimally hard* one-way functions along with iO to obtain CI for all efficient functions (and more). In contrast, our construction is based on assumptions in the quantitatively standard regime.

### 1.3 Additional Related Work Discussion

#### Multi-Input Correlation Intractability

We summarize what was previously known regarding multi-input correlation intractability:

- For subexponentially sparse output relations  $\mathcal{R}_0$ , output intractability for  $\mathcal{R}_0$  can be constructed based on lossy functions (following [38], but relying on less extreme forms of lossiness). Based on “extremely lossy functions”, Zhandry [38] constructs a hash family that is CI for all sparse (efficiently decidable) output relations.<sup>8</sup>
- Similarly to Zhandry [38], the construction  $x \mapsto p(H_k(x))$  (where  $H_k$  is a sufficiently shrinking collision-resistant hash function and  $p$  is sampled from a  $t$ -wise independent hash family) also yields output intractability for subexponentially sparse (and efficiently decidable) output relations.
- Holmgren and Lombardi [26] construct output-intractable hash functions for all sparse (even inefficient)  $R$  based on “one-way product functions” (OWPFs), OWFs satisfying a quantitatively extreme assumption about the hardness of inverting many one-way function challenges in parallel. OWPFs (in different parameter regimes) are existentially incomparable to lossy functions and CHRFs. Under sufficiently strong assumptions, these hash families achieve quantitatively better security than is possible for the previous two constructions.
- Holmgren and Lombardi [26] also construct correlation-intractable hash families for relations  $R(\mathbf{x}, \mathbf{y})$  that include all shifted output relations. However, they rely on both indistinguishability obfuscation and OWPFs (as above).

#### Comparison with Peikert-Shiehian [34]

[34] constructs single-input CI based on the LWE (or SIS) assumption. Their construction improves upon the construction of [12] based on circular-secure FHE: by making use of special properties of the [23] (and related) FHE schemes, they can remove the need for a circular ciphertext  $\text{Enc}(\text{sk}, \text{sk})$  in a specific GSW-based construction. By comparison, we show that any SHSF that is one-way is also CI for bounded functions, and that (essentially) the [33] SHSF is one-way. It does not seem easy to abstract out a simple, generic property of the [34] hash function that implies multi-input correlation intractability.

Given our generalization to multi-input CI, it is also reasonable to ask whether the [34] hash function also satisfies a form of multi-input CI. In fact, it appears likely that it satisfies CI for shifted-sum relations (just like our construction). However, a proof of this fact requires some of our analysis in the security proof of our multi-input CI construction (Theorem 4).

<sup>8</sup> This is a special case of Zhandry’s actual result; we refer the reader to [38] for more details.

### Comparison with Brakerski-Koppula-Mour [10]

We also note that our construction shares some conceptual similarity to the recent CI construction of [10]. We highlight the similarity here:

- In [10], they show that a hash function  $x \mapsto h_k(x) - r$  (for a random  $r$ ) is CI for a (low-degree) function  $f$  by writing down an indistinguishable key distribution  $k_f$  so that  $h_{k_f}(x) - f(x)$  lies in some sparse set  $S_f$ . Then,  $h_{k_f}(x) - f(x) = r$  typically has no (information theoretic) solution.
- In our construction, we show that a hash function  $x \mapsto h_k(x) - r$  is CI for  $f$  by writing down an indistinguishable key distribution  $k_f$  so that  $h_{k_f}(x) - f(x)$  is the evaluation of a PRF  $\text{PRF}_s(x)$ . Then, as long as it is computationally hard to find a PRF inverse  $F_s^{-1}(r)$  (i.e. as long as  $F_s$  is one-way), we can conclude that the equation  $h_{k_f}(x) - f(x) = r$  is computationally hard to solve.

## 2 Preliminaries

Some of the preliminaries below are adapted from [12, 26].

### 2.1 Hash Functions and Correlation Intractability

► **Definition 8.** For a pair of efficiently computable functions  $(\nu(\cdot), \mu(\cdot))$ , a hash family with input length  $\nu$  and output length  $\mu$  is a collection  $\mathcal{H} = \{h_\lambda : \{0, 1\}^{\kappa(\lambda)} \times \{0, 1\}^{\nu(\lambda)} \rightarrow \{0, 1\}^{\mu(\lambda)}\}_{\lambda \in \mathbb{N}}$  of keyed hash functions, along with a pair of p.p.t. algorithms:

- $\mathcal{H}.\text{Gen}(1^\lambda)$  outputs a hash key  $k \in \{0, 1\}^{\kappa(\lambda)}$  describing a hash function  $h$ .
- $\mathcal{H}.\text{Hash}(k, x)$  computes the function  $h_\lambda(k, x) = h(x)$ . We may use the notation  $h(x)$  to denote hash evaluation when the hash family is clear from context.

Following [12, 26], we consider the security notion of correlation intractability [15] for multi-input relations.

► **Definition 9 (Multi-Input Correlation Intractability).** For a given relation ensemble  $R = \{R_\lambda \subseteq (\{0, 1\}^{\nu(\lambda)})^{t(\lambda)} \times (\{0, 1\}^{\mu(\lambda)})^{t(\lambda)}\}$ , a hash family  $\mathcal{H} = \{h_\lambda : \{0, 1\}^{\kappa(\lambda)} \times \{0, 1\}^{\nu(\lambda)} \rightarrow \{0, 1\}^{\mu(\lambda)}\}$  is said to be  $R$ -correlation intractable with security  $(s, \delta)$  if for every  $s$ -size adversary  $\mathcal{A} = \{\mathcal{A}_\lambda\}$ ,

$$\Pr_{\substack{k \leftarrow \mathcal{H}.\text{Gen}(1^\lambda) \\ \mathbf{x} = (x_1, \dots, x_t) \leftarrow \mathcal{A}(k)}}} \left[ (\mathbf{x}, \mathbf{y} = (h(x_1), \dots, h(x_t))) \in R \right] = O(\delta(\lambda)).$$

We say that  $\mathcal{H}$  is  $R$ -correlation intractable with security  $\delta$  if it is  $(\lambda^c, \delta)$ -correlation intractable for all  $c > 1$ . Finally, we say that  $\mathcal{H}$  is  $R$ -correlation intractable if it is  $(\lambda^c, \frac{1}{\lambda^c})$ -correlation intractable for all  $c > 1$ .

A random oracle is correlation intractable for relations that are *sparse*, defined as follows:

► **Definition 10 (Sparsity).** A relation  $R = \{R_\lambda \subseteq (\{0, 1\}^{\nu(\lambda)})^{t(\lambda)} \times (\{0, 1\}^{\mu(\lambda)})^{t(\lambda)}\}$  is  $\rho(\lambda)$ -sparse if for every  $\mathbf{x} \in (\{0, 1\}^{\nu(\lambda)})^{t(\lambda)}$ ,

$$\Pr_{\mathbf{y} \leftarrow (\{0, 1\}^{\mu(\lambda)})^{t(\lambda)}} [(\mathbf{x}, \mathbf{y}) \in R] \leq \rho(\lambda).$$

We say that  $R$  is sparse if it is  $\text{negl}(\lambda)$ -sparse.

In this work, we focus on *distinct input relations*, i.e., relations  $R$  such that for any  $(\mathbf{x}, \mathbf{y}) \in R$ , we have that  $x_i \neq x_j$  for any pair  $(i, j)$ .

We now describe some special cases of the above definition. Two of them (CI for efficient functions and Output Intractability) have been discussed in prior works [12, 26, 34, 38], while a third – which we call “CI for shifted relations” – we introduce in this work.

► **Definition 11** (Correlation Intractability for Functions). *For a given function ensemble  $\mathcal{F} = \{f_\lambda : \{0, 1\}^{\nu(\lambda)} \rightarrow \{0, 1\}^{\mu(\lambda)}\}$ , a hash family  $\mathcal{H} = \{h_\lambda : \{0, 1\}^{\kappa(\lambda)} \times \{0, 1\}^{\nu(\lambda)} \rightarrow \{0, 1\}^{\mu(\lambda)}\}$  is said to be  $f$ -correlation intractable if it is  $R$ -correlation intractable for the single-input relation*

$$R = \left\{ (x, f(x)) : x \in \{0, 1\}^* \right\}.$$

Formally, the requirement is that for every poly-size  $\mathcal{A} = \{\mathcal{A}_\lambda\}$ ,

$$\Pr_{\substack{k \leftarrow \mathcal{H}.\text{Gen}(1^\lambda) \\ x \leftarrow \mathcal{A}(k)}}} \left[ h(k, x) = f(x) \right] = \text{negl}(\lambda).$$

► **Definition 12** (Output Intractability). *For a given relation ensemble  $R_{\text{out}} = \{R_{\text{out}, \lambda} \subseteq (\{0, 1\}^{\mu(\lambda)})^{t(\lambda)}\}$ , a hash family  $\mathcal{H} = \{h_\lambda : \{0, 1\}^{\kappa(\lambda)} \times \{0, 1\}^{\nu(\lambda)} \rightarrow \{0, 1\}^{\mu(\lambda)}\}$  is said to be  $R_{\text{out}}$ -output intractable if it is  $R$ -correlation intractable for the relation*

$$R = \left\{ (\mathbf{x}, \mathbf{y}) : \mathbf{y} \in R_{\text{out}} \text{ and } x_i \neq x_j \text{ for all } i \neq j \right\}.$$

Formally, the requirement is that for every poly-size  $\mathcal{A} = \{\mathcal{A}_\lambda\}$ ,

$$\Pr_{\substack{k \leftarrow \mathcal{H}.\text{Gen}(1^\lambda) \\ \mathbf{x} = (x_1, \dots, x_t) \leftarrow \mathcal{A}(k)}}} \left[ x_i \neq x_j \text{ for all } i \neq j \text{ and } (\mathbf{y} = (h(x_1), \dots, h(x_t)) \in R_{\text{out}}) \right] = \text{negl}(\lambda).$$

In this work, we also consider a strengthening of  $R_{\text{out}}$ -output intractability (as defined above) in which the inputs  $x_1, \dots, x_t$  are not required to be distinct; of course, this larger relation must still be sparse in order for correlation intractability to be feasible.

► **Definition 13** (Not-All-Equal (NAE) Output Intractability). *For a given relation ensemble  $R_{\text{out}} = \{R_{\text{out}, \lambda} \subseteq (\{0, 1\}^{\mu(\lambda)})^{t(\lambda)}\}$ , a hash family  $\mathcal{H} = \{h_\lambda : \{0, 1\}^{\kappa(\lambda)} \times \{0, 1\}^{\nu(\lambda)} \rightarrow \{0, 1\}^{\mu(\lambda)}\}$  is said to be not-all-equal  $R_{\text{out}}$ -output intractable if it is  $R$ -correlation intractable for the relation*

$$R = \left\{ (\mathbf{x}, \mathbf{y}) : \mathbf{y} \in R_{\text{out}} \text{ and } x_1, \dots, x_t \text{ are not all equal} \right\}.$$

When  $t$  is a constant, not-all-equal output intractability for a  $t$ -output relation  $R_{\text{out}}$  follows from standard output intractability for  $\leq t^t$  different relations defined based on  $R_{\text{out}}$  (there is one distinct-input relation for each partition of  $[t]$ ). When  $t$  is superconstant it becomes better to prove the security property directly (without incurring a  $t^t$  security loss).

► **Definition 14** ((Not-All-Equal) Multi-Input CI for  $\mathbb{Z}_p$ -Shifted Relations). *Let  $p = p(\lambda)$  be an efficiently computable function of  $\lambda$ .*

*For a given function ensemble  $\mathcal{F} = \{f_\lambda : \{0, 1\}^{\nu(\lambda)} \rightarrow \mathbb{Z}_p^{\mu(\lambda)}\}$  and relation ensemble  $R_{\text{out}} = \{R_{\text{out}, \lambda} \subseteq (\mathbb{Z}_p^{\mu(\lambda)})^{t(\lambda)}\}$ , a hash family  $\mathcal{H} = \{h_\lambda : \{0, 1\}^{\kappa(\lambda)} \times \{0, 1\}^{\nu(\lambda)} \rightarrow \mathbb{Z}_p^{\mu(\lambda)}\}$  is said to be  $(R_{\text{out}}, f)$ -correlation intractable (respectively, not-all-equal  $(R_{\text{out}}, f)$ -correlation intractable) if it is correlation intractable for the shifted relation*

$$R = \left\{ (\mathbf{x}, \mathbf{y}) : x_i \neq x_j \text{ for all } i \neq j \text{ and } (y_1 - f(x_1), \dots, y_t - f(x_t)) \in R_{\text{out}} \right\},$$

## 102:12 Correlation-Intractable Hash Functions via Shift-Hiding

respectively,

$$R_{\text{NAE}} = \left\{ (\mathbf{x}, \mathbf{y}) : x_1, \dots, x_t \text{ are not all equal } (y_1 - f(x_1), \dots, y_t - f(x_t)) \in R_{\text{out}} \right\}$$

We note that Definition 14 generalizes both Definition 11 and Definition 12/Definition 13. In particular, when  $p(\lambda)$  is a power-of-two, Definitions 12 and 13 can be recovered (identifying  $\mathbb{Z}_p^\mu = \{0, 1\}^{\mu \log p}$ ) by setting  $f$  to be the all-zero function, while Definition 11 can be recovered by setting  $R_{\text{out}} = \{\mathbf{0}^\mu \in \mathbb{Z}_p^\mu = \{0, 1\}^{\mu \log p}\}$ .

Finally, we describe an interesting special case of Definition 14 that we securely instantiate under LWE.

► **Definition 15** (Weighted Sum Resistance mod  $p$ ). *Let  $t = t(\lambda)$ . A hash function family  $\mathcal{H}$  with output space  $\mathbb{Z}_p^\mu$  is weighted sum resistant mod  $p$  with weights  $w \in \{-1, 0, 1\}^t$  if it is output intractable for the  $t$ -output relation*

$$R_{\text{out}} = \left\{ \mathbf{y} : \sum_{i=1}^t w_i y_i = 0^\mu \pmod{p} \right\}.$$

*Similarly, it is not-all-equal weighted sum resistant mod  $p$  with weights  $w$  if it is NAE output intractable for  $R_{\text{out}}$ .*

We say that  $\mathcal{H}$  is weighted sum resistant if it is sum resistant for all nonzero weight vectors  $w$ , and NAE-weighted sum resistant if it is NAE-sum resistant for all weight vectors  $w$  such that  $\sum_i w_i \neq 0$ . As shown in the full version [31], our LWE-based hash family satisfies (NAE) multi-input CI for (both variants of) *shifted* weighted sum resistance mod  $p$  with  $p \approx 2^\lambda$ .

## 2.2 Shift-Hiding Shiftable Functions

We consider a weakening of the original definition of Peikert and Shiehian [33] that does not give the adversary oracle access to the SHSF. We also consider a modified definition with exact correctness rather than approximate correctness (this corresponds to the “rounded version” of the [33] construction).

► **Definition 16** (Shift-Hiding Shiftable Functions [33]). *Let  $p = p(\lambda)$  be an efficiently computable function of  $\lambda$ . We define a family of shift-hiding shiftable functions with input space  $\{0, 1\}^{\nu(\lambda)}$  and output space  $\mathbb{Z}_p^{\mu(\lambda)} = \{0, 1\}^{\mu(\lambda) \log p(\lambda)}$  for arbitrary polynomial functions  $(\nu(\lambda), \mu(\lambda))$ .*

*For a given class  $\mathcal{C}$  of function ensembles  $\mathcal{F} = \{f_\lambda : \{0, 1\}^{\nu(\lambda)} \rightarrow \mathbb{Z}_p^{\mu(\lambda)}\}$ , a shift-hiding shiftable function family  $\text{SHSF} = (\text{Gen}, \text{Shift}, \text{Eval}, \text{SEval})$  consists of four PPT algorithms:*

- $\text{Gen}(1^\lambda)$  outputs a master secret key  $\text{msk}$  and public parameters  $\text{pp}$ .
- $\text{Shift}(\text{msk}, f)$  takes as input a secret key  $\text{msk}$  and a function  $f \in \mathcal{F}$ . It outputs a shifted key  $\text{sk}_f$ .
- $\text{Eval}(\text{pp}, \text{msk}, x)$ , given a secret key  $\text{msk}$  and input  $x \in \{0, 1\}^{\nu(\lambda)}$ , outputs an evaluation  $y \in \mathbb{Z}_p^{\mu(\lambda)}$ .
- $\text{SEval}(\text{pp}, \text{sk}_f, x)$ , given a shifted key  $\text{sk}_f$  and input  $x \in \{0, 1\}^{\nu(\lambda)}$ , outputs an evaluation  $y \in \mathbb{Z}_p^{\mu(\lambda)}$ .

*We will sometimes use the notation  $F_{\text{sk}}(x)$  to mean either  $\text{Eval}(\text{sk}, x)$  or  $\text{SEval}(\text{sk}, x)$  when the context is clear.*

We require that SHSF satisfies the following two properties:

- **Computational Correctness:** for any function  $f \in \mathcal{C}$ , given public parameters  $\text{pp}$  and a shifted key  $\text{sk}_f \leftarrow \text{Shift}(\text{msk}, f)$  (for  $(\text{pp}, \text{msk}) \leftarrow \text{Gen}(1^\lambda)$ ), it is computationally hard to find an input  $x \in \{0, 1\}^{\nu(\lambda)}$  such that  $\text{Eval}(\text{sk}_f, x) \neq \text{Eval}(\text{msk}, x) + f(x) \pmod{p}$ . In other words, the equation

$$F_{\text{sk}_f}(x) = F_{\text{msk}}(x) + f(x)$$

holds computationally (mod  $p$ ).

- **Shift Hiding:** for any pair of functions  $f, g \in \mathcal{C}$ ,

$$\text{sk}_f \approx_c \text{sk}_g,$$

where  $\text{sk}_f \leftarrow \text{Shift}(\text{msk}, f)$ ,  $\text{sk}_g \leftarrow \text{Shift}(\text{msk}, g)$ , and  $\text{msk} \leftarrow \text{Gen}(1^\lambda)$ .

### 3 Correlation Intractability from Shift-Hiding Shiftable Functions

In this section, we show that shift-hiding shiftable functions (Definition 16) that are *output intractable* (Definitions 12 and 13) can be used to construct correlation-intractable hash functions for shifted relations (Definition 14). As a special case, this shows that SHSFs that are *hard to invert* yield correlation-intractable hash functions for all circuits (Definition 11) supported by the SHSF function class  $\mathcal{C}$ . In other words, SHSFs allow us to *lift* a form of output intractability to a more general form of correlation intractability.

Formally, let  $\text{SHSF} = (\text{Gen}, \text{Shift}, \text{Eval})$  be a SHSF family that represents functions of the form  $F_{\text{sk}} : \{0, 1\}^{\nu(\lambda)} \rightarrow \mathbb{Z}_p^{\mu(\lambda)}$  and supports shifts for functions  $f \in \mathcal{C}$ , where  $\mathcal{C}$  is some class that contains the all zero function ensemble. We then consider two hash functions  $\mathcal{H}_{\text{plain}}, \mathcal{H}_{\text{shift}}$ :

- $\mathcal{H}_{\text{plain}}$  uses  $\text{msk}$  as a hash key, and computes the function  $h(\text{msk}, x) = F_{\text{msk}}(x)$ .
- $\mathcal{H}_{\text{shift}}$  uses  $\text{sk}_Z$  as a hash key, where  $Z : \{0, 1\}^\nu \rightarrow \mathbb{Z}_p^\mu$  is an identically zero function. It computes the function  $h(\text{sk}_Z, x) = F_{\text{sk}_Z}(x)$ .

► **Theorem 17.** *Let  $R_{\text{out}}$  be an efficiently decidable output relation. If SHSF is a shift-hiding shiftable function family for  $\mathcal{C}$  and  $\mathcal{H}_{\text{plain}}$  is  $R_{\text{out}}$ -output intractable, then  $\mathcal{H}_{\text{shift}}$  is  $(R, f)$ -correlation intractable for any  $f \in \mathcal{C}$ .*

*Moreover, if  $\mathcal{H}_{\text{plain}}$  is NAE- $R_{\text{out}}$ -output intractable, then  $\mathcal{H}_{\text{shift}}$  is NAE- $(R, f)$ -CI for any  $f \in \mathcal{C}$ .*

**Proof.** Suppose that a PPT adversary  $\mathcal{A}$  breaks the  $(R, f)$ -correlation intractability of  $\mathcal{H}_{\text{shift}}$ , which means that  $\mathcal{A}$  wins the following challenger-based security game with non-negligible probability:

1. The challenger samples  $\text{msk} \leftarrow \text{Gen}(1^\lambda)$ .
2. The challenger samples  $\text{sk} = \text{sk}_Z \leftarrow \text{Shift}(\text{msk}, Z)$  and sends  $\text{sk}$  to  $\mathcal{A}$ .
3.  $\mathcal{A}(\text{sk})$  outputs  $\mathbf{x} = (x_1, \dots, x_t)$ .
4.  $\mathcal{A}$  wins if (i) the inputs  $x_i$  are distinct, and (ii) for  $y_i = F_{\text{sk}}(x_i) - f(x_i)$ , the relation  $R_{\text{out}}(\mathbf{y})$  holds.

Then,  $\mathcal{A}$  also wins each of the following **modified** security games with non-negligible probability.

- Hybrid  $\text{Hyb}_1$ : same as the honest security game, except that in step (2), we sample  $\text{sk}_f \leftarrow \text{Shift}(\text{msk}, f)$

This is indistinguishable from the original security game by the shift-hiding of SHSF.

- Hybrid  $\text{Hyb}_2$ : same as  $\text{Hyb}_1$ , except that in step (4), we change the win condition (ii) so that  $\mathcal{A}$  wins if for  $y_i = F_{\text{msk}}(x_i)$ , the relation  $R_{\text{out}}(\mathbf{y})$  holds.

This is indistinguishable from  $\text{Hyb}_1$  by the computational correctness of SHSF.

Finally, we show that  $\mathcal{A}$ 's success in  $\text{Hyb}_2$  leads to an attack  $\mathcal{A}'$  on the  $R_{\text{out}}$ -output intractability of  $\mathcal{H}_{\text{plain}}$ . The attack works as follows:

1. The challenger samples  $\text{msk} \leftarrow \text{Gen}(1^\lambda)$  and sends  $\text{msk}$  to  $\mathcal{A}'$ .
2.  $\mathcal{A}'(\text{msk})$  samples  $\text{sk} = \text{sk}_f \leftarrow \text{Shift}(\text{msk}, f)$ .
3.  $\mathcal{A}'$  then calls  $\mathcal{A}(\text{sk}_f)$  and outputs  $\mathbf{x} = (x_1, \dots, x_\ell)$ .
4. By definition,  $\mathcal{A}'$  wins if (i) the  $x_i$  are distinct, and (ii) for  $y_i = F_{\text{msk}}(x_i)$ , the relation  $R_{\text{out}}(\mathbf{y})$  holds.

By construction,  $\mathcal{A}'$  above wins with the same probability that  $\mathcal{A}$  wins in  $\text{Hyb}_2$ , contradicting the  $R_{\text{out}}$ -output intractability of  $\mathcal{H}_{\text{plain}}$ .

The same argument as above applies to NAE-CI, with the condition (i) replaced by “the inputs  $x_i$  are not all equal.” This completes the proof of Theorem 17. ◀

---

## References

- 1 Gilad Asharov and Gil Segev. Limits on the power of indistinguishability obfuscation and functional encryption. In Venkatesan Guruswami, editor, *56th FOCS*, pages 191–209. IEEE Computer Society Press, October 2015. doi:10.1109/FOCS.2015.21.
- 2 Boaz Barak, Oded Goldreich, Russell Impagliazzo, Steven Rudich, Amit Sahai, Salil P. Vadhan, and Ke Yang. On the (im)possibility of obfuscating programs. In Joe Kilian, editor, *CRYPTO 2001*, volume 2139 of *LNCS*, pages 1–18. Springer, Heidelberg, August 2001. doi:10.1007/3-540-44647-8\_1.
- 3 Boaz Barak, Yehuda Lindell, and Salil P. Vadhan. Lower bounds for non-black-box zero knowledge. In *44th FOCS*, pages 384–393. IEEE Computer Society Press, October 2003. doi:10.1109/SFCS.2003.1238212.
- 4 Mihir Bellare and Phillip Rogaway. Entity authentication and key distribution. In Douglas R. Stinson, editor, *CRYPTO'93*, volume 773 of *LNCS*, pages 232–249. Springer, Heidelberg, August 1994. doi:10.1007/3-540-48329-2\_21.
- 5 Nir Bitansky, Omer Paneth, and Daniel Wichs. Perfect structure on the edge of chaos - trapdoor permutations from indistinguishability obfuscation. In Eyal Kushilevitz and Tal Malkin, editors, *TCC 2016-A, Part I*, volume 9562 of *LNCS*, pages 474–502. Springer, Heidelberg, January 2016. doi:10.1007/978-3-662-49096-9\_20.
- 6 Dan Boneh, Craig Gentry, Sergey Gorbunov, Shai Halevi, Valeria Nikolaenko, Gil Segev, Vinod Vaikuntanathan, and Dhinakaran Vinayagamurthy. Fully key-homomorphic encryption, arithmetic circuit ABE and compact garbled circuits. In Phong Q. Nguyen and Elisabeth Oswald, editors, *EUROCRYPT 2014*, volume 8441 of *LNCS*, pages 533–556. Springer, Heidelberg, May 2014. doi:10.1007/978-3-642-55220-5\_30.
- 7 Dan Boneh and Brent Waters. Constrained pseudorandom functions and their applications. In Kazue Sako and Palash Sarkar, editors, *ASIACRYPT 2013, Part II*, volume 8270 of *LNCS*, pages 280–300. Springer, Heidelberg, December 2013. doi:10.1007/978-3-642-42045-0\_15.
- 8 Elette Boyle, Shafi Goldwasser, and Ioana Ivan. Functional signatures and pseudorandom functions. In Hugo Krawczyk, editor, *PKC 2014*, volume 8383 of *LNCS*, pages 501–519. Springer, Heidelberg, March 2014. doi:10.1007/978-3-642-54631-0\_29.
- 9 Zvika Brakerski, Nico Döttling, Sanjam Garg, and Giulio Malavolta. Factoring and pairings are not necessary for io: Circular-secure lwe suffices. *IACR Cryptology ePrint Archive*, 2020:1024, 2020.
- 10 Zvika Brakerski, Venkata Koppula, and Tamer Mour. NIZK from LPN and trapdoor hash via correlation intractability for approximable relations. In Daniele Micciancio and Thomas

- Ristenpart, editors, *CRYPTO 2020, Part III*, volume 12172 of *LNCS*, pages 738–767. Springer, Heidelberg, August 2020. doi:10.1007/978-3-030-56877-1\_26.
- 11 Zvika Brakerski and Vinod Vaikuntanathan. Efficient fully homomorphic encryption from (standard) LWE. In Rafail Ostrovsky, editor, *52nd FOCS*, pages 97–106. IEEE Computer Society Press, October 2011. doi:10.1109/FOCS.2011.12.
  - 12 Ran Canetti, Yilei Chen, Justin Holmgren, Alex Lombardi, Guy N. Rothblum, Ron D. Rothblum, and Daniel Wichs. Fiat-Shamir: from practice to theory. In Moses Charikar and Edith Cohen, editors, *51st ACM STOC*, pages 1082–1090. ACM Press, June 2019. doi:10.1145/3313276.3316380.
  - 13 Ran Canetti, Yilei Chen, and Leonid Reyzin. On the correlation intractability of obfuscated pseudorandom functions. In Eyal Kushilevitz and Tal Malkin, editors, *TCC 2016-A, Part I*, volume 9562 of *LNCS*, pages 389–415. Springer, Heidelberg, January 2016. doi:10.1007/978-3-662-49096-9\_17.
  - 14 Ran Canetti, Yilei Chen, Leonid Reyzin, and Ron D. Rothblum. Fiat-Shamir and correlation intractability from strong KDM-secure encryption. In Jesper Buus Nielsen and Vincent Rijmen, editors, *EUROCRYPT 2018, Part I*, volume 10820 of *LNCS*, pages 91–122. Springer, Heidelberg, April / May 2018. doi:10.1007/978-3-319-78381-9\_4.
  - 15 Ran Canetti, Oded Goldreich, and Shai Halevi. The random oracle methodology, revisited (preliminary version). In *30th ACM STOC*, pages 209–218. ACM Press, May 1998. doi:10.1145/276698.276741.
  - 16 Ran Canetti, Daniele Micciancio, and Omer Reingold. Perfectly one-way probabilistic hash functions (preliminary version). In *30th ACM STOC*, pages 131–140. ACM Press, May 1998. doi:10.1145/276698.276721.
  - 17 Aloni Cohen and Saleet Klein. The GGM function family is a weakly one-way family of functions. In Martin Hirt and Adam D. Smith, editors, *TCC 2016-B, Part I*, volume 9985 of *LNCS*, pages 84–107. Springer, Heidelberg, October / November 2016. doi:10.1007/978-3-662-53641-4\_4.
  - 18 Yevgeniy Dodis, Vinod Vaikuntanathan, and Daniel Wichs. Extracting randomness from extractor-dependent sources. In Anne Canteaut and Yuval Ishai, editors, *EUROCRYPT 2020, Part I*, volume 12105 of *LNCS*, pages 313–342. Springer, Heidelberg, May 2020. doi:10.1007/978-3-030-45721-1\_12.
  - 19 Cynthia Dwork, Moni Naor, Omer Reingold, and Larry J. Stockmeyer. Magic functions. In *40th FOCS*, pages 523–534. IEEE Computer Society Press, October 1999. doi:10.1109/SFFCS.1999.814626.
  - 20 Amos Fiat and Adi Shamir. How to prove yourself: Practical solutions to identification and signature problems. In Andrew M. Odlyzko, editor, *CRYPTO'86*, volume 263 of *LNCS*, pages 186–194. Springer, Heidelberg, August 1987. doi:10.1007/3-540-47721-7\_12.
  - 21 Romain Gay and Rafael Pass. Indistinguishability obfuscation from circular security. Proceedings of STOC 2021, 2021. URL: <https://eprint.iacr.org/2020/1010>.
  - 22 Craig Gentry. Fully homomorphic encryption using ideal lattices. In Michael Mitzenmacher, editor, *41st ACM STOC*, pages 169–178. ACM Press, May / June 2009. doi:10.1145/1536414.1536440.
  - 23 Craig Gentry, Amit Sahai, and Brent Waters. Homomorphic encryption from learning with errors: Conceptually-simpler, asymptotically-faster, attribute-based. In Ran Canetti and Juan A. Garay, editors, *CRYPTO 2013, Part I*, volume 8042 of *LNCS*, pages 75–92. Springer, Heidelberg, August 2013. doi:10.1007/978-3-642-40041-4\_5.
  - 24 Oded Goldreich and Ron D. Rothblum. Enhancements of trapdoor permutations. *Journal of Cryptology*, 26(3):484–512, July 2013. doi:10.1007/s00145-012-9131-8.
  - 25 Dennis Hofheinz and Bogdan Ursu. Dual-mode NIZKs from obfuscation. In Steven D. Galbraith and Shihō Moriai, editors, *ASIACRYPT 2019, Part I*, volume 11921 of *LNCS*, pages 311–341. Springer, Heidelberg, December 2019. doi:10.1007/978-3-030-34578-5\_12.

## 102:16 Correlation-Intractable Hash Functions via Shift-Hiding

- 26 Justin Holmgren and Alex Lombardi. Cryptographic hashing from strong one-way functions (or: One-way product functions and their applications). In Mikkel Thorup, editor, *59th FOCS*, pages 850–858. IEEE Computer Society Press, October 2018. doi:10.1109/FOCS.2018.00085.
- 27 Aayush Jain, Huijia Lin, and Amit Sahai. Indistinguishability obfuscation from well-founded assumptions. Proceedings of STOC 2021, 2021. URL: <https://eprint.iacr.org/2020/1003>.
- 28 Yael Tauman Kalai, Guy N. Rothblum, and Ron D. Rothblum. From obfuscation to the security of Fiat-Shamir for proofs. In Jonathan Katz and Hovav Shacham, editors, *CRYPTO 2017, Part II*, volume 10402 of *LNCS*, pages 224–251. Springer, Heidelberg, August 2017. doi:10.1007/978-3-319-63715-0\_8.
- 29 Aggelos Kiayias, Stavros Papadopoulos, Nikos Triandopoulos, and Thomas Zacharias. Delegatable pseudorandom functions and applications. In Ahmad-Reza Sadeghi, Virgil D. Gligor, and Moti Yung, editors, *ACM CCS 2013*, pages 669–684. ACM Press, November 2013. doi:10.1145/2508859.2516668.
- 30 Alex Lombardi and Vinod Vaikuntanathan. Fiat-shamir for repeated squaring with applications to PPAD-hardness and VDFs. In Daniele Micciancio and Thomas Ristenpart, editors, *CRYPTO 2020, Part III*, volume 12172 of *LNCS*, pages 632–651. Springer, Heidelberg, August 2020. doi:10.1007/978-3-030-56877-1\_22.
- 31 Alex Lombardi and Vinod Vaikuntanathan. Multi-input correlation-intractable hash functions via shift-hiding. *IACR Cryptol. ePrint Arch.*, 2020:1378, 2020.
- 32 Satoshi Nakamoto. Bitcoin: A peer-to-peer electronic cash system. Technical report, Manubot, 2008. URL: <https://git.dhimmel.com/bitcoin-whitepaper/>.
- 33 Chris Peikert and Sina Shiehian. Privately constraining and programming PRFs, the LWE way. In Michel Abdalla and Ricardo Dahab, editors, *PKC 2018, Part II*, volume 10770 of *LNCS*, pages 675–701. Springer, Heidelberg, March 2018. doi:10.1007/978-3-319-76581-5\_23.
- 34 Chris Peikert and Sina Shiehian. Noninteractive zero knowledge for NP from (plain) learning with errors. In Alexandra Boldyreva and Daniele Micciancio, editors, *CRYPTO 2019, Part I*, volume 11692 of *LNCS*, pages 89–114. Springer, Heidelberg, August 2019. doi:10.1007/978-3-030-26948-7\_4.
- 35 Chris Peikert and Sina Shiehian. Constraining and watermarking PRFs from milder assumptions. In Aggelos Kiayias, Markulf Kohlweiss, Petros Wallden, and Vassilis Zikas, editors, *PKC 2020, Part I*, volume 12110 of *LNCS*, pages 431–461. Springer, Heidelberg, May 2020. doi:10.1007/978-3-030-45374-9\_15.
- 36 Chris Peikert and Brent Waters. Lossy trapdoor functions and their applications. In Richard E. Ladner and Cynthia Dwork, editors, *40th ACM STOC*, pages 187–196. ACM Press, May 2008. doi:10.1145/1374376.1374406.
- 37 Hoeteck Wee and Daniel Wichs. Candidate obfuscation via oblivious LWE sampling. In Anne Canteaut and François-Xavier Standaert, editors, *EUROCRYPT 2021, Part III*, volume 12698 of *LNCS*, pages 127–156. Springer, Heidelberg, October 2021. doi:10.1007/978-3-030-77883-5\_5.
- 38 Mark Zhandry. The magic of ELFs. In Matthew Robshaw and Jonathan Katz, editors, *CRYPTO 2016, Part I*, volume 9814 of *LNCS*, pages 479–508. Springer, Heidelberg, August 2016. doi:10.1007/978-3-662-53018-4\_18.