# On the Existence of Competitive Equilibrium with Chores

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## Abstract

We study the chore division problem in the classic Arrow-Debreu exchange setting, where a set of agents want to divide their divisible chores (bads) to minimize their disutilities (costs). We assume that agents have linear disutility functions. Like the setting with goods, a division based on competitive equilibrium is regarded as one of the best mechanisms for bads. Equilibrium existence for goods has been extensively studied, resulting in a simple, polynomial-time verifiable, necessary and sufficient condition. However, dividing bads has not received a similar extensive study even though it is as relevant as dividing goods in day-to-day life.

In this paper, we show that the problem of checking whether an equilibrium exists in chore division is NP-complete, which is in sharp contrast to the case of goods. Further, we derive a simple, polynomial-time verifiable, sufficient condition for existence. Our fixed-point formulation to show existence makes novel use of both Kakutani and Brouwer fixed-point theorems, the latter nested inside the former, to avoid the *undefined demand* issue specific to bads.

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## 1 Introduction

Fair division has developed into a fundamental branch in mathematical economics, computational social choice theory and computer science over the last several decades. In a classical fair division problem, the goal is to divide a set of items among agents in a *fair* and *efficient* manner. Such problems have been extensively studied when the items to be divided are all *goods*. The problem of dividing *chores* (items creating disutility) has not received a similar extensive investigation even though it is as relevant as dividing goods in day-to-day life; for instance division of daily household chores among tenants, teaching load among faculty, job shifts among workers, and so on. A division based on *competitive equilibrium* (CE) has emerged as one of the best mechanisms for this problem due to its remarkable fairness and efficiency guarantees [27, 3].

In this paper, we consider the problem of computing a CE with *divisible* chores in the fundamental Arrow-Debreu *exchange* model. The exchange model is like a barter system, where each agent brings a set of chores that needs to be completed and exchanges them with others to optimize their (dis)utility. For example, a set of university students teaching each other in a group study, to optimize the time and effort required. At a larger scale, *timebanks*[1] are such reciprocal service exchange platforms which have around 30,000 to 40,000 users from the United States. In a timebank, individuals from a certain community give services to one another and earn *time credit*. Thereafter, each individual uses their time credit to receive services. CE provides a systematic way to do the exchange: it constitutes of prices (payment)[2] for chores and an allocation such that all chores are completely assigned and each agent gets her most preferred bundle (*optimal bundle*) subject to her budget constraint[3].

We assume that agents have linear disutility (cost) functions, i.e., the disutility of an agent is $\sum_j d_{ij} X_{ij}$, where $d_{ij}$ is the disutility agent $i$ gets from doing a unit amount of chore $j$, and $X_{ij}$ indicates the amount of chore $j$ that agent $i$ does. Clearly, an agent can do a chore within a reasonable amount of time only if she has the skill set required for it. For example, a professor trained in computer science (CS) can teach a CS course in the upcoming semester, but may not have skill set to teach a course in music. This essentially boils down to not allocating certain chores to certain agents. In the case of goods, this is achieved by specifying *zero* utility values to some items, and its analogue for chores is specifying *infinite* disutility.

The existence of CE is well understood for goods. In particular, when agents have (quasi-)concave utility functions, Arrow and Debreu [1], and Mckenzie [18, 19] had shown the existence of CE under some mild conditions. Both the theorems make use of Kakutani's fixed point formulations. When the utility functions are further restricted to be linear, there are well known convex programs that capture the competitive equilibrium in the exchange model [23, 15, 11]. Such convex programs have been instrumental in designing polynomial time algorithms for finding a competitive equilibrium when agents have linear utility functions [15, 29].

Interestingly, CE with chores behaves significantly differently than CE with goods. Bogomolnaia et al. [3] considered the CEEI (CE with equal income) model, a special case of the exchange where every agent owns one unit of every chore, with finite and homogeneous disutilities. They gave an involved characterization of CE, and through this showed that with chores, the set of CE is non-convex and disconnected even when disutility functions are restricted to linear. While, in the case of linear CEEI model with goods, there are even simpler convex programs [12, 10] that capture CE.

In this paper, we analyze existence of CE with chores in the exchange model where agents may have infinite disutility for certain chores. Although infinite disutilities seem natural, they create more challenges. For example, we observe that CE in the CEEI model may not exist in contrast to guaranteed existence with finite disutilities [3]. Furthermore, it is NP-hard to determine the existence in both exchange and CEEI. This is in sharp contrast to the goods case, where there is a polynomial time verifiable necessary and sufficient condition for existence of CE in the exchange setting [13, 11]. Our NP-hardness result rules out the possibility of obtaining such conditions for chores case unless P=NP! Furthermore, we strengthen our NP-hardness result to hold for 11/12-approximate CE.

1 `https://timebanks.org/`
2 Equivalent of time credit in time banks.
3 Here the budget constraint of an agent is that she has to earn enough to pay for her initial set of chores.

The next best hope is to obtain weakest possible sufficient conditions that also capture interesting instances, leading to our main question: *Are there polynomial-time verifiable, natural, sufficient conditions that guarantee the existence of a CE with chores?*

Our result address the above question. First, we show the existence of a CE under two conditions. The first condition, known as *strong connectivity of the exchange graph*, is an artifact of the exchange model, and is required in the case of goods as well [17, 28]. Intuitively, it ensures that no set of agents can consume only a strict subset of the chores they cumulatively own, as otherwise no prices can ensure demand equals supply. Our second condition depends on the disutility values. While this condition is specific to only bads, it is simple, polynomial-time verifiable, and unavoidable (see Example 6).

The proof of existence of a CE under these two conditions makes use of Kakutani's as well as Brouwer's fixed-point theorems, with the latter nested inside the former. The fixed point formulations for the goods case define a *correspondence* (or equivalently a *set valued function*) on the *simplex* domain of prices [1, 25, 17]. The correspondence maps each price vector to a set of price vectors in the simplex obtained by adjusting the price of each good depending on its *excess demand*.[4] Thereafter, by Kakutani's fixed point theorem the correspondence admits a fixed point, which is mapped to a CE by showing no excess demand at a fixed-point.

With chores, the simplex domain of prices pose the issue of *undefined optimal bundles* of the agents: If an agent owns chores that have positive prices but all the chores she can do (has finite disutility towards) have zero prices, then there is no way she can earn the money needed for her endowment, thereby making her optimal bundle undefined. We fix this issue by adding a set of linear constraints to our price domain, which ensures that if the total prices of the chores an agent is interested in is zero, then her total endowment money is also zero, implying that she does not need to earn anything and the *doing-no-chores* is an admissible optimal bundle. However, such a fix makes it harder to define an appropriate correspondence: be mindful that given a price vector, we need our correspondence to adjust prices depending on excess demand as before, but now map it back to a more involved domain (earlier it was a simplex). Additionally, it should satisfy the continuity-like property. It is unclear whether a correspondence with all the desired properties exist. This is where we use Brouwer's fixed point theorem to show the existence of such a correspondence. An overview of this technique can be found in Section 1.2.1.

## 1.1 Model and Notations

A chore division problem consists of a set of $m$ divisible chores (bads), namely $B = \{b_1, \ldots, b_m\}$, and a set of $n$ agents $A = \{a_1, \ldots, a_n\}$. Each agent $a_i$ has $d(a_i, b_j) \in (0, \infty]$ disutility (pain) for doing unit amount of chore $b_j$.[5] Here, infinite disutility implies that the agent does not have required skill set to do the chore in a reasonable amount of time. If agent $a_i$ is assigned bundle $X_i = \langle X_{i1}, \ldots, X_{im} \rangle \in \mathbb{R}^m_{\geq 0}$ where $X_{ij}$ is the amount of chore $b_j$ she gets, then her total disutility is $d_i(X_i) = \sum_{j \in [m]} d(a_i, b_j) \cdot X_{ij}$. We study the problem under exchange model, where agent $a_i$ brings $w(a_i, b_j)$ amount of chore $b_j$ to be done (by herself or other agents).

[4] At any given price, an agent can be content with several allocations, i.e., there are multiple optimal bundles at a given price. As different optimal bundles can lead to different excess demands for the goods, such a correspondence maps a price vector to several price vectors in the simplex.

[5] If $d(a_i, b_j)$ is zero, then chore $b_j$ can be safely assigned to agent $a_i$ and can be removed from the instance.

Given prices $p = \langle p(b_1), p(b_2), \ldots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ for chores, where $p(b_j)$ denotes the payment for doing unit amount of chore $b_j$, agent $a_i$ needs to earn $\sum_{j\in[m]} w(a_i, b_j)\cdot p(b_j)$ in order to pay to get her own chores done. Let $F_i(p)$ be the feasible set of bundles, i.e., the bundles with which an agent can earn her required money:

$$F_i(p) = \left\{ X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij}\cdot p(b_j) \geq \sum_{j\in[m]} w(a_i, b_j)\cdot p(b_j) \right\}.$$

Clearly $a_i$ would like to choose the feasible bundle that minimizes her disutility – this defines her *optimal bundle* (or optimal chore set).

$$OB_i(p) = \underset{X_i\in F_i(p)}{\arg\min}\ d_i(X_i). \tag{1}$$

It is easy to see that in her optimal bundle agent $a_i$ gets assigned only those chores that minimizes her disutility per dollar earned and agent $i$ earns money exactly equal to the total price of her endowments. Formally, if $X_i \in OB_i(p)$, then,

$$\forall j \in [m],\quad X_{ij} > 0 \quad \Rightarrow \quad \tfrac{d(a_i,b_j)}{p(b_j)} \leq \tfrac{d(a_i,b_{j'})}{p(b_{j'})} \quad \forall j' \in [m],$$

and

$$\sum_{j\in[m]} X_{ij}\cdot p(b_j) = \sum_{j\in[m]} w(a_i, b_j)\cdot p(b_j).$$

In the above ratios, to deal with zero prices and infinite disutilities we assume $\infty/a > b/0$ for any $a, b \in [0, \infty)$. Clearly, an optimal bundle of an agent contains only those chores for which she has finite disutility.

Price vector $p$ is said to be at a *Competitive Equilibrium (CE)* if all chores are completely assigned when every agent gets one of her optimal bundles, *i.e.,* $X_i \in OB_i(p)$ and $\sum_{i\in[n]} X_{ij} = \sum_{i\in[n]} w(a_i, b_j),\ \forall j \in [m]$. It is without loss of generality to assume that each chore is available in one unit total, i.e. for each $b_j \in B$, $\sum_{i\in[n]} w(a_i, b_j) = 1$ (through appropriate scaling of the disutility values). We now formally describe our problem.

▶ **Definition 1** (Chore Division in the Exchange Model)**.** *Given a set of agents $A = \{a_1, a_2, \ldots, a_n\}$, chores $B = \{b_1, b_2, \ldots, b_m\}$, disutilities $d(\cdot,\cdot)$ and endowments $w(\cdot,\cdot)$, our goal is to find a price vector $p = \langle p(b_1), p(b_2), \ldots, p(b_m)\rangle \in \mathbb{R}^m_{\geq 0}$ and allocation $X = \langle X_1, X_2, \ldots, X_n\rangle$, such that*

- *Every agent gets their optimal bundle: $X_i \in OB_i(p)$.*
- *All chores are completely allocated: $\sum_{i\in[n]} X_{ij} = \sum_{i\in[n]} w(a_i, b_j) = 1$, for all $b_j \in B$.*

Observe that the equilibrium prices are scale invariant: if $p$ is an equilibrium price vector then so is $\alpha\cdot p$ for any positive scalar $\alpha$. Furthermore, at equilibrium $p(b_j) > 0$ for each chore $j$, otherwise no agent would be willing to do it. A CE $\langle p, X\rangle$ has many desirable properties like envy-freeness and Pareto optimality in the chore division with equal income [3]. Similarly, CE for the exchange model too satisfies Pareto optimality and *weighted* envy-freeness[6].

[6] Weight of an agent at given prices is the total monetary cost of the chores she brings. Naturally, higher the cost of her chores (more money she has to earn), larger is her share of disutility.

**Fisher Model and CEEI.** The Fisher model is a special case of exchange model, where instead of the endowment of chores, each agent $a_i$ has a requirement of earning a fixed amount of money $e(a_i) \geq 0$, i.e., the only change is in the definition of the feasible set of chores that can be allocated to an agent at a given price vector $p$, $F_i(p) = \left\{X_i \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} X_{ij} \cdot p(b_j) \geq e(a_i)\right\}$. If $e(a_i) = 1$ for all $a_i \in A$ then resulting equilibrium is called *Competitive Equilibrium with Equal Income* (CEEI). Clearly, CEEI is a special case of Fisher. Observe that determining CE in the Fisher model, can be modeled as determining CE in the exchange model, by setting $w(a_i, b_j) = e(a_i)$ for each $a_i \in A$ and $b_j \in B$, while keeping the disutility values as is.

## 1.2 Overview of Our Results and Techniques

In this section we discuss the high-level ideas and techniques used to prove our main results. We first note that in general, a chore division instance may not admit a CE as demonstrated by the following example.

▶ **Example 2.** There are two agents $a_1$ and $a_2$, and two chores $b_1$ and $b_2$. We have $w(a_i, b_j) = 1$ for all $i, j \in [2]$, and $d(a_1, b_1) = d(a_2, b_1) = 1$, and $d(a_1, b_2) = \infty$ and $d(a_2, b_2) = 2$. Let $p(b_1)$ and $p(b_2)$ be the prices of the chores at a CE.

Observe that since $d(a_1, b_2) = \infty$, $a_1$ earns her entire money of $w(a_1, b_1) \cdot p(b_1) + w(a_1, b_2) \cdot p(b_2)$ from $b_1$. Therefore, at a CE, the total price of the chore $b_1$ is at least the total money earned by $a_1$: $(w(a_1, b_1) + w(a_2, b_1)) \cdot p(b_1) \geq (w(a_1, b_1) \cdot p(b_1) + w(a_1, b_2) \cdot p(b_2))$. This implies that $2 \cdot p(b_1) \geq p(b_1) + p(b_2)$, further implying that $p(b_1) \geq p(b_2)$. In that case observe that the disutility to price ratio of $b_2$ is strictly greater than that of $b_1$ for $a_2$: $d(a_2, b_1)/p(b_1) = 1/p(b_1) < 2/p(b_1) \leq 2/p(b_2) = d(a_2, b_2)/p(b_2)$. Thus, none of the agents are willing to do chore $b_2$, and therefore it remains unassigned, a contradiction.

It is well known that the a CE may not exist while dividing goods as well under the exchange model. And, there are *polynomial time checkable* necessary and sufficient conditions for the existence of CE. The next natural question is to obtain similar conditions for the chore division as well. However, in this paper, we prove the following theorem.

▶ **Theorem 3.** *Determining whether an instance of chore division in the Fisher model admits a CE is strongly NP-hard, even for the case of equal incomes (CEEI). This also holds for the constant-approximate CE.*

The above theorem rules out obtaining polynomial time checkable necessary and sufficient conditions for existence of a CE unless P=NP. The next best hope is to design weakest possible conditions that ensures a CE and captures an interesting class of instances. Towards this we derive two conditions.

The first condition is an artifact of the exchange setting, and is required for dividing goods as well [17]: if a set of agents are interested to consume only a strict subset of the endowment that they cumulatively own, then no prices can ensure demand equals supply (we elaborate this shortly in Example 9). We now define a condition that helps us resolve this issue.[7] To define the condition, we first define the *economy graph* of a given instance of chore division.

[7] In fact, Condition 1 is the analogue of the necessary and sufficient condition required for a CE to exist in exchange markets with goods.

▶ **Definition 4** (Economy Graph [17]). *Given an instance $I = \langle A, B, d(\cdot,\cdot), w(\cdot,\cdot)\rangle$, an Economy Graph $G = (A, E)$ is a graph, with vertices corresponding to the agents and there exists an edge from $a_i$ to $a_j$ if and only if there exist a chore $c \in B$, such that $w(a_i, c) > 0$ and $d(a_j, c) \neq \infty$.*

Now we define the first condition.

▶ **Definition 5** (Condition 1 [17]). *The economy graph of the instance is strongly connected.*

Observe that our instance in Example 2 does satisfy Condition 1, yet does not admit a CE. The primary reason for non-existence of CE in Example 2 is that sets $\{b \in B \mid d(a_1, b) \neq \infty\}$ and $\{b \in B \mid d(a_2, b) \neq \infty\}$ are neither same nor disjoint. Next by generalizing this example we demonstrate that unless finite disutility chore sets of any two agents are either same or disjoint, the equilibrium may not exist. In particular, given any integer $n > 1$ and $m > 1$, we create a chore division instance with $n$ agents and $m$ chores that satisfies Condition 1, has exactly one agent-chore pair with infinite disutility, and does not admit a CE.

▶ **Example 6.** There are $n$ agents $a_1, a_2, \dots, a_n$, and $m$ chores $b_1, b_2, \dots, b_m$. We set $w(a_i, b_j) = 1$ for all $i \in [n]$ and $j \in [m]$. So there is a total of $n$ units of each chore $b_j$, for all $j \in [m]$. Now, we set $d(a_i, b_j) = 1$ for all $i \in [n]$ and $j \in [m-1]$. We set $d(a_i, b_m) = nm$ for all $i \in [n-1]$ and $d(a_n, b_m) = \infty$.

Since $w(a_i, b_j) = 1$, for all $i \in [n]$ and $j \in [m]$, the instance in Example 6 does satisfy Condition 1 (the economy graph of the instance is a clique). Observe that since all the agents have the same disutility for the chores $\cup_{j\in[m-1]} b_j$, the prices of all these chores will be the same at a CE (otherwise some of the chores will remain unassigned). Therefore, let $p$ be the price of a chore $b_j$ for $j \in [m-1]$, and $p'$ be the price of the chore $b_m$ at a CE. Since $a_n$ has infinite disutility for $b_m$, she will earn her entire money of $\sum_{j\in[m]} w(a_n, b_j) \cdot p(b_j) = (m-1) \cdot p + p'$ from the chores in $\cup_{j\in[m-1]} b_j$. Therefore, at a CE, the total price of the chores in $\cup_{j\in[m-1]} b_j$ is at least the total money earned by $a_n$, i.e., total prices of the chores owned by agent $a_n$, implying that $\sum_{j\in[m-1]} \sum_{i\in[n]} w(a_i, b_j) \cdot p(b_j) \geq \sum_{j\in[m]} w(a_n, b_j) \cdot p(b_j)$. This implies that $(m-1)\cdot n\cdot p \geq (m-1)\cdot p + p'$, further implying that $(m-1)\cdot(n-1)\cdot p \geq p'$. In that case observe that the disutility to price ratio of $b_m$ is strictly less than that of $b_1$ for any agent $a_i$, for $i \in [n-1]$: $d(a_i, b_1)/p(b_1) = 1/p \leq ((n-1)\cdot(m-1))/p' < nm/p' = d(a_i, b_m)/p(b_m)$. Thus, none of the agents are willing to do chore $b_m$, and it remains unassigned, a contradiction.

Our next condition is to circumvent the primary issue in Example 6 that renders CE to not exist. To this end, we define the disutility graph $D = (A \cup B, E_D)$ as the bipartite graph with the set of agents $A$ and the set of chores $B$ forming the vertex sets on two sides and there is an edge from an $a \in A$ to a $b \in B$ when $d(a, b) \neq \infty$. Examples 2 and 6 demonstrate that whenever there is a connected component $D'$ of $D$ which is not a biclique, there exists disutility values for which the instance will not admit a CE. This brings us to our second condition.

▶ **Definition 7** (Condition 2). *The disutility graph is a disjoint union of bicliques.*

The second main result of our paper shows that Conditions 1 and 2 guarantee the existence of a CE.

▶ **Theorem 8.** *A chore division instance satisfying Conditions 1 and 2 admits a CE.*

We now quickly show that even if one of the two conditions is not satisfied, the instance may not admit a CE. Examples 2 and 6 already outline instances that satisfy Condition 1, but do not satisfy Condition 2 and as a result do not admit a CE. We next give an example that satisfies Condition 2, but not Condition 1, and does not admit a CE.

▶ **Example 9.** There are three agents $a_1$, $a_2$, $a_3$ and three chores $b_1$, $b_2$, $b_3$. Agents $a_1$ and $a_2$ own $1/2$ units of chores $b_1$ and $b_2$ each, i.e., $w(a_i, b_j) = 1/2$ for all $i, j \in [2]$. Agent $a_3$ owns one unit of $b_3$, i.e., $w(a_3, b_3) = 1$. We set $d(a_1, b_1) = d(a_2, b_1) = 1$, and $d(a_3, b_2) = d(a_3, b_3) = 1$. The disutility value of all other agent chore pair is infinity.

Observe that the disutility graph is a disjoint union of bicliques – one biclique comprising of agents $a_1$, $a_2$ and the chore $b_1$, and the second biclique comprising of the agent $a_3$ and chores $b_2$ and $b_3$. Therefore the instance satisfies Condition 2. We now show that the instance does not admit a CE. Let $p(b_1)$, $p(b_2)$ and $p(b_3)$ denote the prices of chores $b_1$, $b_2$ and $b_3$ at a CE. Since agents $a_1$ and $a_2$ earn their entire endowment money from the chore $a_1$, we have that $\sum_{i\in[2]} \sum_{j\in[3]} w(a_i, b_j) \cdot p(b_j) = \sum_{i\in[3]} w(a_i, b_1) \cdot p(b_1)$, implying that $p(b_1) + p(b_2) = p(b_1)$, further implying that $p(b_2) = 0$. Therefore, at any CE $b_2$ will remain unassigned as it will not be a part of the optimal bundle set of the agent $a_3$ when $p(b_2) = 0$, which is contradiction.

In the subsections that follow, we briefly elaborate our techniques and novel ideas used to prove Theorems 3 and 8.

### 1.2.1 Existence of a CE under the Sufficient Condition

In this section, we sketch the approach and main ideas to show existence of a CE assuming the instance satisfies two sufficient conditions, that is proof of Theorem 8. Most equilibrium existence results [22, 1] are based on either Brouwer's or Kakutani's fixed-point theorems. The Brouwer's (Kakutani's) fixed-point theorem says that given a function (correspondence) $\phi$ from $D$ to itself, there exists an $x \in D$ such that $f(x) = x$ ($x \in f(x)$), if $f$ is continuous (has closed graph) and $D$ is convex and compact [8, 16]. Our proof invokes both Brouwer's and Kakutani's fixed-point theorems, the former nested inside the latter. This approach may be of independent interest to prove existence in other settings.

We first briefly discuss why existence proofs for determining a CE with goods do not easily extend to chores, and this will eventually lead us to the new approach. Most existence proofs for a CE with goods define a fixed-point formulation on the domain of prices that forms a simplex [1, 17], i.e., if there are $m$ goods, then the domain is the simplex $\Delta_m = \{p \in \mathbb{R}^m_{\geq 0} \mid \sum_{j=1}^m p_j = 1\}$. Given the prices, it computes the optimal bundles of agents and adjusts prices based on *excess demand*. At a fixed-point, no change in prices will imply no excess demand, leading to a CE.

This approach immediately fails for the chore division problem due to the issue of *infeasible optimal bundle*: Given a price vector from the simplex domain, if agent $a_i$'s chore endowment has positive total monetary cost, while the chores she is able to do have zero prices, then there is no way she can earn enough money to pay for her chores, in turn making the set $F_i(p)$ in (1) empty. The reason why this issue does not arise in case of goods is that, there, agents are allowed to spend *at most* the total price of their endowments (for bads it is *at least*), thereby reversing the inequality in the definition of the set $F_i(p)$, which ensures that the all zero vector in $\mathbb{R}^m_{\geq 0}$ is always a feasible vector.

To circumvent the above issue, first we need to work with a more involved price domain that ensures that total monetary cost of the chores and endowments is the same inside every component of the disutility graph. Recall the bipartite disutility graph $D = (A \cup B, E_D)$ where there is an edge $(a, b) \in E_D$ if and only if $d(a, b) \neq \infty$. Let $D_1 = (A_1 \cup B_1, E_{D_1}), D_2 = (A_2 \cup B_2, E_{D_2}), \ldots, D_d = (A_d \cup B_d, E_{D_d})$ be the connected components of $D$. Then, our new price domain is,

$$\mathbf{P} = \left\{ p \in \mathbb{R}^m_{\geq 0} \mid \sum_{j\in[m]} p(b_j) = 1 \text{ and } \sum_{b\in B_k} p(b) = \sum_{a\in A_k} \sum_{j\in[m]} w(a, b_j) p(b_j) \;\; \forall k \in [d] \right\} \tag{2}$$

Now observe that if for any agent $a \in A_k$, for some $k \in [d]$, the chores she is interested in (the set $B_k$), have zero prices, then the total price of her endowment is also zero as $p \in \mathbf{P}$. In this case, agent $a$ need not earn anything. As a result, she does not need to do any chore and the all zero vector in $\mathbb{R}^m_{\geq 0}$ is a feasible optimal chore set for agent $a$. Therefore, for any price vector $p \in \mathbf{P}$, for any agent $i$, we have that the set $F_i(p)$ is not empty and neither is the optimal bundle set in (1). However, there is still an issue with zero prices, a different one: It can be the case that for some component $(A_k \cup B_k, E_k)$, the prices of all the chores in $B_k$ are zero, and prices of all the chores that agents in $A_k$ bring are also zero. In that case, the optimal bundle of any agent $a \in A_k$ consists of only the all zero vector because none of them have to earn anything! However, this will make the optimal bundle set change *non-continuously* with respect to prices, which is a major roadblock in proving continuity like property (the closed graph property) for the fixed-point formulation: for instance consider a simple scenario where there is a component $D_k$ in the disutility graph comprising of just one agent $a$ and one chore $b$. Agent $a$ has some positive endowment of only one chore $b' \neq b$, say $w(a, b') = 1$ and $w(a, j) = 0$ for all other $j \in B$. Now, consider a sequence of price-vectors $(p_n)_{n \in \mathbb{N}}$ in $\mathbf{P}$, such that $p_n(b') = p_n(b) = 1/n$. Observe that for every $n \in \mathbb{N}$, the optimal bundle of agent $a$ is $X_{ab} = 1$ and $X_{at} = 0$ for all other $t \in B$, as the only chore $a$ is interested in is $b$, and she has to do one unit of $b$, to earn her money of $w(a, b) \cdot p(b') = 1 \cdot (1/n) = 1/n$. However, at the limit of the sequence $(p_n)_{n \in \mathbb{N}}$, say $p_*$, we have $p_*(b) = p_*(b') = 0$ and the only unique optimal bundle for agent $a$ is the all zero vector in $\mathbb{R}^m_{\geq 0}$. Thus, the optimal bundle may not change continuously with the price-vectors in $\mathbf{P}$.

To fix the above issue, we define the *extended optimal bundle* set, which is same as the optimal bundle set of an agent $a_i \in A_k$, if the total price of the chores in $B_k$ is strictly positive, otherwise it is the set of all feasible allocations of chores in $B_k$. This will help us ensure *continuity* of the final correspondence. However, we will have to make sure that at the fixed-point, the *extended optimal bundle* is the *optimal bundle* for every agent (one way to do this is to ensure that there are no zero prices at the fixed point). For the allocations, we will work with the following domain: for some sufficiently large constant $C$, we define

$$\mathbf{X} = \{X \in \mathbb{R}^{mn}_{\geq 0} \;\mid\; 0 \leq X_{ij} \leq C, \;\forall a_i \in A, \forall b_j \in B\} \tag{3}$$

Then the set of extended optimal bundles of an agent $a_i \in A_k$ is:

$$EOB_i(p) = \begin{cases} \{X_i \in \mathbf{X} \mid X_{ij} > 0 \text{ only if } d(a_i, b_j) \neq \infty\} & \text{if } \sum_{b \in B_k} p(b) = 0, \\ OB_i(p) & \text{otherwise.} \end{cases} \tag{4}$$

*Fixed-point formulation.* The domain of our fixed point formulation is $\mathbf{S} = \mathbf{P} \times \mathbf{X}$. Next, we define a correspondence $\phi : \mathbf{S} \to 2^{\mathbf{S}}$ that is product of two correspondences $\phi_1 : \mathbf{S} \to 2^{\mathbf{P}}$ and $\phi_2 : \mathbf{S} \to 2^{\mathbf{X}}$. For a given $(p, X) \in \mathbf{S}$, $\phi(p, X) = \phi_1(p, X) \times \phi_2(p, X)$. Out of these, $\phi_2(p, X)$ is the set of extended optimal bundles at prices $p$. Formally,

$$\phi_2(p, X) = \{X \in \mathbf{X} \mid X_i \in EOB_i(p), \;\forall a_i \in A\}$$

The exact formulation of $\phi_1$ is involved and requires to invoke Brouwer's fixed-point theorem. Therefore, let us first state the properties of $\phi_1$ that we need to ensure, and discuss how they help us map fixed-points of $\phi$ to the competitive equilibria of the chore division instance. For a given $(p, X) \in \mathbf{S}$, if $p' \in \phi_1(p, X)$, then it must be that

- $p' \in P$ and for all components $D_k = (A_k \cup B_k, E_k)$ of the disutility graph, and chores $b_j$ and $b_{j'}$ in $B_k$, where $p(b_{j'}) > 0$, we have

$$\frac{p'(b_j)}{p'(b_{j'})} = \frac{p(b_j) + max(\sum_{i \in [n]} w(a_i, b_j) - \sum_{i \in [n]} X_{ij}, 0)}{p(b_{j'}) + max(\sum_{i \in [n]} w(a_i, b_{j'}) - \sum_{i \in [n]} X_{ij'}, 0)}. \tag{5}$$

*Fixed-points to CE.* Let $(p, X)$ be a fixed-point of $\phi$, i.e., $(p, X) \in \phi(p, X)$. We first show that at any fixed-point, the prices of all the chores are strictly positive. To the contrary, suppose $p(b_j) = 0$ for some $b_j \in B$, and let $b_j$ belong to component $D_k = (A_k \cup B_k, E_{D_k})$ of the disutility graph $D$. Then, some component of $D$ has chores with both zero and positive prices. Either it is $D_k$ itself, or if all the chores in $D_k$ have zero prices, then using the fact that $p \in \mathbf{P}$, we have $\sum_{a_i \in A_k} \sum_{j\in[m]} w(a_i, b_j) \cdot p(b_j) = \sum_{b_j \in B_k} p(b_j) = 0$. This implies that the prices of all the chores owned by agents in $D_k$ are zero, and some of them must belong to other components due to the *strong connectivity* of the economy graph (Condition 1). Recursing this argument, and also using the fact that sum of all the prices is 1, there must be a component with a zero priced chore, but the sum of prices of the chores in the component is positive, say component $D_\ell = (A_\ell \cup B_\ell, E_{D_\ell})$.

Let $b^0$ and $b^+$ be the chores in $D_\ell$ with $p(b^0) = 0$ and $p(b^+) > 0$. For every agent in $a_i \in A_\ell$, their $EOB_i(p) = OB_i(p)$, since total price of the chores in $B_\ell$ is positive (by (4)). Since every $a_i \in D_\ell$ has finite disutility for both $b^0$ and $b^+$ (due to Condition 2), her disutility-per-buck for $b^0$ is strictly more than that for $b^1$. Due to (1), if $X_i \in OB_i(p)$ then $X_{ib^0} = 0$ for all $i \in A_\ell$. Furthermore, every agent $a \notin A_\ell$ has infinite disutility for $b^0$, we have that $X_{ib^0} = 0$ for all $i \in [n]$. Now given that our correspondence $\phi$ satisfies (5), and $p(b^0) = 0$ and $p(b^+) > 0$, we have,

$$\begin{aligned} 0 = \frac{p(b^0)}{p(b^+)} &= \frac{p(b^0) + max(\sum_{i\in[n]} w(a_i, b^0) - \sum_{i\in[n]} X_{ib^0}, 0)}{p(b^+) + max(\sum_{i\in[n]} w(a_i, b^+) - \sum_{i\in[n]} X_{ib^+}, 0)} \\ &= \frac{0 + \sum_{i\in[n]} w(a_i, b^0)}{p(b^+) + max(\sum_{i\in[n]} w(a_i, b^+) - \sum_{i\in[n]} X_{ib^+}, 0)} \\ &> 0, \text{ a contradiction.} \end{aligned}$$

Therefore, at a fixed point, there is no chore with a zero price. Now, we briefly describe why fixed-point $(p, X)$ correspond to the prices and allocation at a CE. Let $r_j(X)$ denote the amount of the chore $b_j$ left undone under $X$, i.e.,

$$r_j(X) = max(\sum_{i\in[n]} w(a_i, b_j) - \sum_{i\in[n]} X_{ij}, 0).$$

Since all chores have positive price at $p$, extended optimal bundle set of every agent is her optimal bundle set (by (4)) and thereby $X \in \phi_2(p, X)$ ensures that $X_i \in OB_i(p)$ for every agent $a_i \in A$. Now we only need to ensure demand meets supply for every chore. If not, then some chore $b_j$ in component $D_k$, which is not completed, i.e., $r_j(X) > 0$. Since $p \in \mathbf{P}$, we have that the cumulative price of the endowments of the agents in a component of the disutility graph equals the total price of the chores in the same component. Since every agent spends on their optimal bundle, the cumulative price of the endowments of the agents equals the total earning of that agents in $A_k$ from $B_k$. Therefore, if one chore $b_j$ is underdone, i.e., $r_j(X) > 0$, then there exists some other chore $b_{j'}$, which is overdone, i.e., $r_{j'}(X) = 0$. Again using (5), we have $\frac{p(b_j)}{p(b_{j'})} = \frac{p(b_j)+r_j(X)}{p(b_{j'})+r_{j'}(X)} > \frac{p(b_j)}{p(b_{j'})+r_{j'}(X)} = \frac{p(b_j)}{p(b_{j'})}$, a contradiction.

*Mapping to P and Ensuring Condition (5).* Our next task is to define the correspondence $\phi_1$, so that for any given $(p, X) \in \mathbf{S}$, (5) holds for every $p' \in \phi_1(p, X)$, and $p' \in \mathbf{P}$. This in fact is the trickiest part of our proof and constitutes the main bulk of our efforts.

To get $p' \in \mathbf{P}$, we need to make sure that the $p' \in \Delta_m$, and for every component $D_k$ of the disutility graph $D$, total prices of the chores in $D_k$ equals total cost of endowments of agents in $D_k$. To this end, for every chore $b_j$ in component $D_k$, let $q(b_j) = p(b_j) + r_j(X)$, where $r_j(X)$ is the non-negative excess supply as defined above, and $\beta_j = \frac{q(b_j)}{\sum_{b \in D_k} q(b)}$. Note that for (5), we want that for any $b_j, b_{j'} \in D_k$ with $p(b_{j'}) > 0$, $\frac{p'(b_j)}{p'(b_{j'})} = \frac{q(b_j)}{q(b_{j'})} = \frac{\beta_j}{\beta_{j'}}$. Thus, if $\tilde{p}_k = \sum_{b \in D_k} p'(b)$ then $p'(b_j)$ must be $\beta_j \tilde{p}_k$. This reduces to one unknown per component of $D$, namely $\tilde{p}_k$ for each $k \in [d]$.

Next, we write a system of linear equations to compute $\tilde{p}_k$'s such that all the constraints of domain $\mathbf{P}$ are satisfied. The simplex constraints for the prices in $\mathbf{P}$ can be encoded by ensuring $\tilde{p} \in \Delta_d$. Next, for each component $D_k$, the following constraint imposes total endowment costs of agents in $D_k$ equals total prices of chores in $D_k$.

$$\sum_{a_i \in A_k} \sum_{k' \in [d]} \sum_{b_{j'} \in B_{k'}} w(a_i, b_{j'}) \cdot (\beta_{j'} \tilde{p}_{k'}) = \sum_{b_j \in B_k} (\beta_j \tilde{p}_k)$$

Let $M(\beta) \in \mathbb{R}^{d \times d}$ denote the matrix of this linear system. Then, our goal becomes to find a vector $v \in \Delta_d$, in the null space of $M(\beta)$. It is not obvious why such a vector should exist. Our high-level approach to show the same is as follows: We can equivalently express the linear system of equations $M(\beta) \cdot v = 0$ as $M'(\beta) \cdot v = v$, where $M'(\beta) = M(\beta) + I$, where $I$ is the identity matrix. *We show that if we define a function $f : \mathbb{R}^d \to \mathbb{R}^d$ as $f(v) = M'(\beta) \cdot v$, then $f$ maps the d-dimensional simplex $\Delta_d$ to itself (this is non-trivial).* Restricting $f$ to only the simplex, we get a continuous map $f : \Delta_d \to \Delta_d$ and therefore it has a fixed-point by the Brouwer's fixed-point theorem. At every fixed-point $v$ we have $M'(\beta) \cdot v = v$ implying $M(\beta) \cdot v = 0$. Since $v \in \Delta_d$ we get the vector we needed.

The above scheme will work if the $\beta_j$s are well defined. However, for a component $D_k$ if $\sum_{b \in D_k} q(b)$ turns out to be zero, then $\beta_j$s are ill-defined and cause issues with proving continuity like properties of $\phi$. To handle this, we define a set of permissible $\beta$s, namely,

$$\mathcal{B} = \left\{ \beta \in \mathbb{R}^m_{\geq 0} \mid \forall k \in [d], \quad \begin{array}{ll} \sum_{b_j \in D_k} \beta_j = 1 & \text{if } \sum_{b \in D_k} q(b) = 0 \\ \forall b_j \in D_k,\ \beta_j = \frac{q(b_j)}{\sum_{b \in D_k} q(b)} & \text{otherwise} \end{array} \right\}.$$

And for each $\beta \in \mathcal{B}$, the above process will compute a $p' \in \phi_1(p, X)$. By construction, each of these $p'$'s will satisfy, $p' \in \mathbf{P}$ and equation (5), as needed. However, it is not immediate why such a set of $p'$'s will form a convex set, as required to apply the Kakutani's fixed point theorem.

In fact, to apply the Kakutani's fixed-point theorem, we need to show that the above complex process creates a $\phi$, that has closed graph (continuity-like property), and $\phi(p, X)$ is convex for each $(p, X) \in \mathbf{S}$. This again requires involved argument and the detailed proof can be found in the full version of this paper. Then, $\phi$ is sure to have a fixed-point which maps to CE as discussed above.

Our proof technique extends to show existence of a CE for chore division with general monotone convex disutility functions where an *agent can do only a subset of chores* and with arbitrary endowments, under a similar sufficient condition. Thus, our overall approach may be of independent interest to handle more general problems involving chores.

### 1.2.2 NP-Hardness of Determining a CE in Arbitrary Instances

We sketch the main reason why determining whether an *arbitrary* instance of chore division admits a CE is strongly NP-hard. This primarily arises due to the existence of several disconnected equilibria. We sketch a very simple scenario that could arise in chore division in the Fisher model: Consider an instance with two agents $a_1$ and $a_2$ with a fixed earning of one unit each. The disutility values are given below where $a_1$ has a disutility of 1 for $b_1$ and 3 for $b_2$, while $a_2$ has a disutility of $\infty$ for $b_1$ and 1 for $b_2$.

| | $b_1$ | $b_2$ |
|---|---|---|
| $a_1$ | 1 | 3 |
| $a_2$ | $\infty$ | 1 |

Let $p = \langle p(b_1), p(b_2)\rangle$ be an equilibrium price vector. Also, throughout this section we use the notation $MPB_a$ to denote the *minimum pain per buck* bundle for agent $a$ at the prices $p$: a chore $b \in MPB_a$ if and only if $\frac{d(a,b)}{p(b)} \le \frac{d(a,b')}{p(b')}$ for all other chores $b'$ in the instance. Observe that this small instance exhibits exactly two competitive equilibria:

- The first CE is when both $p(b_1)$ and $p(b_2)$ are set to 1. Note that only $MPB_{a_1} = \{b_1\}$ and $MPB_{a_2} = \{b_2\}$. Thus $a_1$ earns her entire one unit of money from $g_1$ and $a_2$ earns her entire one unit of money from $g_2$.
- The second CE is when $a_1$ earns from both $b_1$ and $b_2$. For this we set $p(b_1)$ to $1/2$ and $p(b_2)$ to $3/2$. Note that $MPB_{a_1} = \{b_1, b_2\}$ and $MPB_{a_2} = \{b_2\}$. Under these prices, $a_2$ earns her entire money by doing $2/3$ of $b_2$, and $a_1$ earns her money by doing all of $b_1$ and $1/3$ of $b_2$.

Also, observe that there exists no CE at any other set of prices. This is a striking difference to the scenario with only goods to divide, where all CE exists at a unique price vector. Now, let us introduce another agent $a_3$ and another chore $b_3$ in the instance. Let us say that $a_3$ has a fixed earning of one unit, and both agents $a_1$ and $a_2$ have a disutility of $\infty$ towards $b_3$. We now discuss two scenarios that may arise depending on $a_3$'s disutility towards the chores

1. $a_3$ has a disutility of 1 towards $b_3$ and $b_2$, and $\infty$ towards $b_1$.
2. $a_3$ has a disutility of 1 towards $b_3$, $\frac{1}{2}$ towards $b_1$ and $\infty$ towards $b_2$.

We will now show that, at a CE, in scenario 1, $b_2 \notin MPB_{a_1}$ and in scenario 2, $b_2 \in MPB_{a_1}$, suggesting that depending on the valuation of $a_3$, only one *local* equilibrium among the agents $a_1$, $a_2$ and chores $b_1$ and $b_2$ is admissible at a CE. Let $p(b_1)$, $p(b_2)$ and $p(b_3)$ denote the prices of chores at an equilibrium. Note that since both $a_1$ and $a_2$ have a disutility of $\infty$ for $b_3$, they only earn money from $b_1$ and $b_2$. Thus $p(b_1) + p(b_2) \ge 2$. Note that in both scenarios $b_3$ should be in $MPB_{a_3}$ as $a_3$ is the only agent with finite disutility towards it. Now,

- In scenario 1: Since $b_3 \in MPB_{a_3}$, we have $\frac{d(a_3,b_3)}{p(b_3)} \le \frac{d(a_3,b_2)}{p(b_2)}$ or equivalently $\frac{1}{p(b_3)} \le \frac{1}{p(b_2)}$, implying that $p(b_3) \ge p(b_2)$. This in turn implies that

$$\begin{aligned} p(b_2) + 2 &\le p(b_2) + (p(b_1) + p(b_2)) && (\text{as } p(b_1) + p(b_2) \ge 2) \\ &\le p(b_1) + p(b_2) + p(b_3) && (\text{as } p(b_2) \le p(b_3)) \\ &= 3. \end{aligned}$$

Thus we have $p(b_2) \le 1$, implying that $p(b_1) \ge 1$. Therefore, we can conclude that $b_2 \notin MPB_{a_1}$ as the disutility to price ratio of $b_1$ is strictly less than that of $b_2$ for agent $a_1$.

- In scenario 2: Since $b_3 \in MPB_{a_3}$, we have $\frac{d(a_3,b_3)}{p(b_3)} \leq \frac{d(a_3,b_1)}{p(b_1)}$, we have $\frac{1}{p(b_3)} \leq \frac{1}{2p(b_1)}$, implying that $p(b_3) \geq 2p(b_1)$. This in turn implies that

$$\begin{aligned} 2p(b_1)+2 &\leq 2p(b_1)+(p(b_1)+p(b_2)) && (\text{as } p(b_1)+p(b_2)\geq 2)\\ &\leq p(b_1)+p(b_2)+p(b_3) && (\text{as } 2p(b_1)\leq p(b_3))\\ &= 3. \end{aligned}$$

  Thus we have $p(b_1) \leq \frac{1}{2}$, implying that $p(b_2) \geq \frac{3}{2}$. Therefore, the disutility to price ratio of $b_2$ is at most that of $b_1$ for agent $a_1$ and thus we conclude that $b_2 \in MPB_{a_1}$.

Thus, as mentioned earlier, the valuations of the agents outside the local sub-instance, impose a specific local equilibrium (among the two disjoint local equilibria) among the agents $a_1$, $a_2$ and chores $b_1$ and $b_2$. The hardness arises from the intractability of finding the correct local equilibria when there are $n$ such local sub-instances (resulting in $2^n$ disjoint equilibria). We refer the reader to the full version of the paper for the detailed proof.

## 1.3 Further Related Work

The fair division literature is too vast to survey here, so we refer to the excellent books [6, 24, 20] and a recent survey article [21], and restrict attention to previous work that appears most relevant.

Most of the work in fair division is focused on allocating goods with a few exceptions of chores [26, 2, 6, 24]. The papers [3, 4] consider the case of mixed manna that contains both goods and bads in the Fisher model and assume all (dis)utility values to be finite. For the goods case, competitive equilibrium maximizes the Nash welfare, i.e., geometric mean of agents' utilities. In case of chores (or mixed manna), [3] shows that critical points of the geometric mean of agents' disutilities on the (Pareto) efficiency frontier are the competitive equilibrium profiles. By building on this characterization, [5] recently obtained an efficient algorithm to find an approximate competitive equilibrium (FPTAS). For the special case of constantly many agents (or chores), polynomial-time algorithms are known for computing a competitive equilibrium in the Fisher model [7, 14]. In a recent work, [9] give a simplex-like algorithm for computing a competitive equilibrium in the exchange model.

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