On Realizing a Single Degree Sequence by a Bipartite Graph

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Abstract

This paper addresses the classical problem of characterizing degree sequences that can be realized by a bipartite graph. For the simpler variant of the problem, where a partition of the sequence into the two sides of the bipartite graph is given as part of the input, a complete characterization was given by Gale and Ryser over 60 years ago. However, the general question, in which both the partition and the realizing graph need to be determined, is still open. This paper provides an overview of some of the known results on this problem in interesting special cases, including realizations by bipartite graphs and bipartite multigraphs.

2012 ACM Subject Classification Mathematics of computing → Graph theory

Keywords and phrases Degree Sequences, Graph Realization, Bipartite Graphs, Graphic Sequences, Bigraphic Sequences, Multigraph Realization

Digital Object Identifier 10.4230/LIPIcs.SWAT.2022.1

Category Invited Paper

Funding This work was supported by US-Israel BSF grant 2018043.

1 Introduction

1.1 Background and Motivation

A sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers is *graphic* if there exists an n-vertex graph G whose degree sequence $\deg(G)$ satisfies $\deg(G) = d$. The question of recognizing graphic degree sequences was studied extensively in the past six decades. Given a sequence d, the *graphic degree realization (GDR)* problem requires deciding whether d is graphic and constructing a graph G realizing it, if one exists. A complete characterization (implying also an $\mathcal{O}(n)$ time decision algorithm) for graphic degree sequences was given by Erdös and Gallai [16]. An algorithm that, given a sequence d, generates a realizing graph or proves that the sequence is not graphic, was given by Havel and Hakimi [19, 22]. This algorithm runs in time $\mathcal{O}(\sum_i d_i)$, which is optimal[1].

A natural variant of the graphic degree realization problem requires the realizing graph to be *bipartite*. A sequence admitting a bipartite realizing graph is called *bigraphic*, and the corresponding problem is called the *bigraphic degree realization (BDR)* problem. This problem has appeared as an open problem over 40 years ago [33], but did not receive a lot of attention.

[1] Note that $\sum_i d_i = 2m$ where m is the number of edges in a realizing graph if exists.

18th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2022).
Editors: Artur Czumaj and Qin Xin; Article No. 1; pp. 1:1–1:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In contrast, the simpler variant where the partition of d is given as part of the input was studied extensively. Here, the input consists of a partition of d into two sequences, $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$, and it is required to decide whether there exists a bipartite graph $G(A, B, E)$ such that $|A| = p$, $|B| = q$, and the sequences of degrees of the vertices of A and B are equal to a and b, respectively. Hereafter, we refer to such a pair (a, b) as a *bigraphic degree partition*, and to the problem as the *given partition* version of the bigraphic degree realization problem, BDR^P.

Necessary and sufficient conditions for a pair of sequences (a, b) to be a bigraphic degree partition were given in 1957 by Gale and Ryser [17, 34]. These conditions yield also a polynomial time decision algorithm for BDR^P, which can be thought of as a variant of the Havel-Hakimi algorithm for general graphs applied to one side of the partition.

An obvious question is whether the Gale-Ryser conditions can be used for attacking the (single sequence) bigraphic degree realization problem BDR. One natural strategy is to search for a bigraphic degree partition for the given sequence d, relying on the fact that $d_1 < n$ is a necessary condition for a sequence d to be graphic (or bigraphic), and for such d, a partition can be found (if one exists) in polynomial time, since the PARTITION problem is pseudo-polynomial (cf. [5, 13]). Unfortunately, it is possible that some partitions of d are bigraphic while other partitions are not (see Example 1 in Sect. 5). Moreover, the number of different partitions for a given sequence may be exponential in its length (see Examples 2 & 3 in Sect. 5). Still, one may hope that the special structure required by a bigraphic degree partition may assist us in searching for them. Unfortunately, so far we have not been able to fully characterize the class of bigraphic degree sequences, or to determine whether the problem is NP-hard. In this paper we report what we perceive to be some of the more interesting findings on the problem.

1.2 Results

We present two types of results. We first identify special instances for which one can solve the BDR problem, i.e., decide whether a given sequence is bigraphic or not and if so, generate a realizing graph. Second, we describe realizations by bipartite *multigraphs* (namely, graphs that allow parallel edges) for special instances where the BDR problem is decided in the negative or is unsolved. The multigraph realizations are generated with the objective of minimizing the *maximum* multiplicity in order to come close to resolving the bipartite realization problem, i.e., finding *approximate* realizations.

The notation of graphic and bigraphic sequences is extended to handle multigraphs. A sequence d of non-negative integers is said to be t-*graphic* (t-*bigraphic*) if it admits a (bipartite) multigraph realizations with maximum multiplicity of at most t parallel edges. If a bipartite multigraph realization is based on a partition (a, b), we say that partition (a, b) is t-bigraphic.

In the following, we classify the known results into several categories depending on the type of instances that are being considered.

Small Instances. The first category of instances concerns cases where the BDR problem can be resolved exactly due to the fact that the instance is "small" in some sense. In Section 3, we focus on two such cases. The first is when the given sequence d admits only a small number of partitions $N_{Part}(d)$. Formally, it is required that $N_{Part}(d) = \mathcal{O}(n^c)$ for some constant c. For such sequences, it is possible to exploit the fact that the PARTITION problem is pseudo-polynomial. To do that, we use an *output-sensitive* algorithm for generating all the partitions of d, namely, an algorithm requiring time $\mathcal{O}(n^{c'})$ per partition for some constant

c'. A special subcase of this case involves sequences with a constant number of distinct degrees, since a sequence with only a constant number of different degrees can have at most polynomially many different partitions.

The second case of "small" instances concerns sequences whose maximum degree is small. Specifically, we show that for every partitionable nonincreasing n-integer sequence $d = (d_1, \ldots, d_n)$, if $d_1^2 \leq t \cdot \sum_i d_i/2$, then d is a t-bigraphic degree sequence, and moreover, any partition (a, b) of d is a t-bigraphic degree partition. An alternative (weaker) condition on d_1 is that $d_1^2 \leq t \cdot n/2$.

High-Low Partitions. We then shift our attention to specific and significant types of partitions, referred to as *High-Low partitions*. A High-Low partition of a non-increasing sequence d has the form $HL(d) = (H, L)$ where $H = (d_1, \ldots, d_k)$ and $L = (d_{k+1}, \ldots, d_n)$ for some k. Clearly, this pair (H, L) is a balanced partition only if $\sum_{i=1}^{k} d_i = \sum_{i=k+1}^{n} d_i$.

For High-Low partitions, the first Gale-Ryser conditions are key to the realizability of the sequence. These conditions state that the largest degree on each side does not exceed the number of vertices on the other side (formally, $d_1 \leq n - k$ and $d_{k+1} \leq k$). A (balanced) High-Low partition (H, L) that satisfies the first Gale-Ryser conditions is referred to as a *well-behaved* High-Low partition.

The fact that a High-Low partition is well-behaved does not guarantee that it is bigraphic (see Example 4 in Sect. 5). However, as described in Sect. 4, for a non-increasing sequence d that admits a well-behaved High-Low partition (H, L), the BDR problem turns out to be solvable [4]. This follows from the fact that for a sequence d admitting a well-behaved High-Low partition (H, L), if (H, L) is not bigraphic, then *no* partition of d is bigraphic, hence d itself is not bigraphic. It follows that if d has a well-behaved High-Low partition, then it can be decided in polynomial time whether d is bigraphic or not. Moreover, if d happens to be bigraphic, a bipartite graph realizing d can be computed in polynomial time (e.g., using the adapted Havel-Hakimi algorithm described in Section 2).

In Section 4 we also discuss bipartite multigraph realizations based on High-Low partitions. It turns out that even in case a well-behaved High-Low partition fails to be bigraphic, it is still 2-*bigraphic*. More generally, defining a parameter $t(d)$ indicating the extent to which $HL(d)$ violates the first Gale-Ryser conditions, we have the following: If an r-graphic sequence d admits a High-Low partition $HL(d)$, then $HL(d)$ is t-bigraphic where $t = \max\{t(d), 2r\}$.

Equal Partitions. We explore another specific and important type of partitions, referred to as *equal partitions*. An n-integer sequence d admits an equal partition if d is *even*, namely, each integer occurring in d appears in it an even number of times. At the cost of a slight notational inconsistency, resolved by context, we adopt the compact notation $d = (d_1^{n_1}, \ldots, d_q^{n_q})$, where $\sum_{i=1}^{q} n_i = n$, for a sequence consisting of n_i copies of the integer d_i for $1 \leq i \leq q$. Then the sequence d is even if n_i is even for every $1 \leq i \leq q$. For such a sequence d, the *equal partition* is $\text{EQP}(d) = (a, b)$ where $a = b = (d_1^{n_1/2}, \ldots, d_q^{n_q/2})$.

The equal partition does not display a property similar to that of well-behaved High-Low partitions, i.e., a "well-behaved" equal partition does not allow us to resolve the BDR problem. Specifically, there are even sequences d with a well-behaved equal partition $\text{EQP}(d) = (a, b)$ such that (a, b) is not bigraphic but other partitions of d are bigraphic (see Example 5 in Sect. 5). However, as shown in [3], if the sequence d is graphic and even, then the equal partition is 2-bigraphic. More generally, for multigraph realizations with bounded maximum multiplicity, the following holds. Let d be an even and r-graphic degree sequence with equal partition $\text{EQP}(d) = (a, b)$. Then (a, b) is $2r$-bigraphic.

High-Low vs. Equal Partitions. In some sense, the High-Low partition and the equal partition are two extremes: whereas the High-Low partition tries to *differentiate* the two sides as much as possible, taking all the largest elements to one side and all the smallest elements to the other, the equal partition attempts to *equalize* the two parts as much as possible.

Interestingly, there are bigraphic even sequences for which the High-Low partition is bigraphic while the equal partition is not, or vice versa (see Examples 5 and 6 in Sect. 5). One might speculate that if d is bigraphic and has both a High-Low partition and an equal partition, then at least one of them must be bigraphic, but even that turns out to be false (see Example 1 in Sect. 5).

1.3 Related Work

The two key questions on degree sequences studied in the literature concern identifying necessary and sufficient conditions for a sequence to be graphic, and developing efficient algorithms for computing a realizing graph if exists. As mentioned above, Erdös and Gallai [16] are the first to present a characterization of graphic sequences (several alternative proofs exist, see [10, 2, 40, 14, 38, 39, 47, 26].) Havel [22] and Hakimi [19] provide a different characterization, also, implying an algorithm to construct a realizing graph.

Several related questions are considered in the literature: Given a degree sequence d, (1.) find all the (non-isomorphic) graphs that realize it. (2.) count all its (non-isomorphic) realizing graphs. (3.) sample a random realization as uniformly as possible. These questions are extensively studied, see [10, 16, 19, 22, 24, 36, 38, 44, 45, 46]. Applications to network design, randomized algorithms, social networks [6, 12, 15, 29] and chemical networks [37] exists. Miller [30] shows that only a subset of the Erdös and Gallai inequalities needs to be checked in order to decide if a degree sequence is graphic. The literature also includes surveys on degree sequences, see [41, 42, 43].

Additional intriguing directions include finding characterizations for degree sequences of specific graph families. To that end, we call a degree sequence *potentially* P-graphic if it has a realizing graph having the graph theoretic property P. Rao [33] surveys results (see references therein) on various properties like k-edge connected, k-vertex connected, hamiltonian and tournament. As an open problem characterizing potentially bipartite sequences is mentioned, i.e., the BDR problem.

Moreover, a characterization is known for trees (cf. [18]). The family of *planar* graphs was studied to some extend. The existing results provide a characterization for planar graphic k*-sequences*, where the difference between the largest and the smallest degree is bounded by k, for $k = 0, 1, 2$ [1, 35]. Full characterizations for the degree sequences of *threshold* graphs (see [20]), *split* graphs (see [21]), *matrogenic* graphs (see [28]) and *difference* graphs (see [20]) are known. Degree sequences of *chordal*, *interval*, and *perfect* graphs are considered in [9].

As mentioned above, the BDR^P was solved in [17, 34] (for an alternative proof see [27]). As the title "Combinatorial properties of matrices of zeros and ones" suggests, the problem motivating Ryser [34] has, naturally, a pair of sequences as its input. Sufficient conditions for a pair of sequences to be bigraphic were studied in [7, 48].

Owens and Trent [31] were interested in the realization problem for multigraphs. Given a degree sequence, their results provide a multigraph realizations minimizing the total number of parallel edges or loops (improved algorithms are presented in [32, 25]). The opposite objective of maximizing the total number of parallel edges is, however, proven to be NP-hard (see [23]).

2 Preliminaries

Let $H = (V, E)$ be a multigraph without loops. In this case, E is a multiset. Denote by $E_H(v, w)$ the multiset of edges connecting $v, w \in V$. The *maximum multiplicity* of H is

$$\mathsf{MaxMult}(H) = \max_{(v,w)\in E}(|E_H(v, w)|).$$

2.1 Degree Sequences of Graphs and Multigraphs

Let $d = (d_1, d_2, \dots, d_n)$ be a sequence of nonnegative integers in nonincreasing order[2]. The *volume* of d is $\sum d = \sum_{i=1}^n d_i$. Note that every graphic sequence must have even volume. We call a sequence with even volume a *degree sequence*.

The characterization of Erdös and Gallai [16] for graphic degree sequences is as follows.

▶ **Theorem 1** (Erdös-Gallai [16]). *A degree sequence $d = (d_1, d_2, \dots, d_n)$ is graphic if and only if*

$$\sum_{i=1}^{\ell} d_i \le \ell(\ell - 1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\}, \tag{1}$$

for $\ell = 1, \dots, n$.

We call Equation (1) the *ℓ-th Erdös-Gallai inequality* EG_ℓ. Theorem 1 implies an $\mathcal{O}(n)$ algorithm to verify whether a sequence is graphic.

Let r be a positive integer. Then, a degree sequence d is *r-graphic* if there exists a multigraph H such that $\deg(H) = d$ and $\mathsf{MaxMult}(H) \le r$. A characterization for r-graphic sequences was shown by Chungphaisan [11].

▶ **Theorem 2** (Chungphaisan [11]). *Let r be a positive integer. Degree sequence $d = (d_1, d_2, \dots, d_n)$ is r-graphic if and only if*

$$\sum_{i=1}^{\ell} d_i \le r\ell(\ell - 1) + \sum_{i=\ell+1}^{n} \min\{r\ell, d_i\}, \tag{2}$$

for $\ell = 1, \dots, n$.

The minimum r such that a sequence d is r-graphic can be computed in polynomial time.

2.2 Degree Sequences of Bipartite Graphs and Multigraphs

Let d be a degree sequence such that $\sum d = 2m$ for some integer m. A *block* of d is a subsequence a such that $\sum a = m$. The set of all blocks of sequence d is defined as $B(d) := \{a \subset d \mid \sum a = m\}$. For each $a \in B(d)$ there is a disjoint $b \in B(d)$ that completes it to form a partition of d (so that merging them in sorted order yields d). We call such a pair $a, b \in B(d)$ a *(balanced) partition* of d since $\sum a = \sum b$. Denote the set of all degree partitions of d by $\mathsf{BP}(d) = \{\{a, b\} \mid a, b \in B(d),\ d \setminus a = b\}$.

The Gale-Ryser theorem characterizes bigraphic degree partitions.

[2] All sequence that we consider are assumed to be in a non-increasing order.

▶ **Theorem 3** (Gale-Ryser [17, 34]). *Let d be a degree sequence and partition $(a,b) \in \mathsf{BP}(d)$ where $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots, b_q)$. The partition (a,b) is bigraphic if and only if*

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell, b_i\} \tag{3}$$

for $\ell = 1, \ldots, p$.

We refer to Equation (3) as the *ℓ-th Gale-Ryser inequality* GR_ℓ^L on the left. By symmetry, the partition (a,b) is bigraphic if and only if $\sum_{i=1}^{\ell} b_i \leq \sum_{i=1}^{p} \min\{\ell, a_i\}$, for $\ell = 1, \ldots, q$. We refer to this equation as the *ℓ-th Gale-Ryser inequality* GR_ℓ^R on the right.

Let t be a positive integer. A degree sequence d is t-bigraphic if d has a partition $(a,b) \in \mathsf{BP}(d)$ such that there is a bipartite multigraph $H = (A, B, E)$ such that $\mathsf{MaxMult}(H) \leq t$, $|A| = |a|$, $|B| = |b|$, and the sequences of degrees of the vertices of A and B are equal to a and b, respectively. We also say that partition (a,b) is t-bigraphic. Miller [30] cites the following result of Berge characterizing t-bigraphic partitions.

▶ **Theorem 4** (Berge [30]). *Consider a positive integer t, a degree sequence d and a partition $(a,b) \in \mathsf{BP}(d)$ where $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$. The partition (a,b) is t-bigraphic if and only if*

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell t, b_i\}, \tag{4}$$

for $\ell = 1, \ldots, p$.

2.3 Havel-Hakimi Algorithm for Bipartite Graphs

Kleitman and Wang [26] generalize the Havel-Hakimi theorem implying an algorithm where the 'pivot' can be chosen freely. It is folklore that the same approach can be extended to bigraphic sequences and a given partition. In the following, we formalize this result and start with introducing some notation. Let d be a degree sequence and $(a,b) \in \mathsf{BP}(d)$ where $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_q)$. We assume that partition (a,b) satisfies the first Gale-Ryser conditions, i.e., $a_1 \leq q$ and $b_1 \leq p$ hold. Let $i \leq p$ be some index. Define

$$a^{-i} = a \setminus a_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_p),$$

and

$$\mathrm{RED}(b, a_i) = (b_1 - 1, \ldots, b_{a_i} - 1, b_{a_i+1}, \ldots, b_q).$$

Moreover, let $G = (A, B, E)$ be a bipartite graph realizing (a,b). For a positive integer $\ell \leq |B|$, define the subset $\mathrm{MaxDeg}(B, \ell) \subseteq B$ to contain vertices with degrees $b_1, \ldots, b_\ell$. Ties are broken arbitrarily to ensure that $|\mathrm{MaxDeg}(B, \ell)| = \ell$.

▶ **Lemma 5.** *Let d be a degree sequence and $(a,b) \in \mathsf{BP}(d)$. Let $i \leq |a|$ be some index. If (a,b) is bigraphic, then there is a bipartite graph $G = (A, B, E)$ where vertex $v \in A$ has degree a_i and is adjacent to each vertex in MaxDeg(B, a_i).*

Proof. Let d, (a,b), and index i as in the theorem. Assume that (a,b) is bigraphic, and let $G = (A, B, E)$ be a bipartite graph realizing (a,b) where vertex $v \in A$ has degree a_i. Denote $B' = \mathrm{MaxDeg}(B, a_i)$. If v is adjacent to each vertex in B', we are done. Otherwise there are vertices $u \in B'$ and $w \in B \setminus B'$ such that $(v,u) \notin E$ and $(v,w) \in E$. By definition of B', we have that $\deg(u) \geq \deg(w)$. It follows that there is a vertex $v' \in A$ such that $(v',u) \in E$ and $(v',w) \notin E$. Now, we construct a graph $G' = (A,B,E')$ by applying an edge flip operation to G such that G' is bipartite, realizes (a,b) and v is adjacent to one more vertex in B' (comparing G and G'). For the edge flip operation, remove edges $(v,w),(v',u) \in E$ and add edges $(v,u),(v',w)$ to E'. Verify that G' satisfies the claimed properties. The operation can be applied until v is adjacent to each vertex in B', and the lemma is shown. ◀

▶ **Theorem 6.** *Let d be a degree sequence and $(a,b) \in \mathtt{BP}(d)$. Also, let $i \leq |a|$ be some index, $a' = a^{-i}$, and $b' = RED(b,a_i)$. Then, (a,b) is bigraphic if and only if (a',b') is bigraphic.*

Proof. Let d, (a,b), index i, and (a',b') be as in the theorem.

First, assume that (a,b) is bigraphic. Due to Lemma 5 there is a bipartite graph $G = (A,B,E)$ realizing (a,b) where vertex $v \in A$ has degree a_i and v is adjacent to vertices $u_1,\ldots,u_{a_i} \in B$ having degrees $b_1,\ldots,b_{a_i}$. Let G' be the result of removing vertex v and its incident edges from graph G. Verify that G' realizes (a',b'), and consequently (a',b') is bigraphic.

Now, assume that (a',b') is bigraphic. Let $G = (A,B,E)$ be a bipartite graph realizing (a',b'). Construct a graph $G' = (A',B,E')$ from G by adding a new vertex v to A, i.e., $A' = A \cup v$. Next, connect vertex v to vertices $u_1,\ldots,u_{a_i} \in B$ having degrees $b_1-1, b_2-1,\ldots,b_{a_i}-1$. Verify that G' is bipartite, vertex v has degree a_i, and that G' realizes (a,b). It follows that (a,b) is bigraphic. ◀

Similar to the Havel-Hakimi [19, 22] characterization for graphic degree sequences, Theorem 6 implies a polynomial time algorithm that, given a partition (a,b), constructs a bipartite graph realizing (a,b) or decides that (a,b) is not bigraphic.

3 Small Instances

3.1 Output-Sensitive Algorithms: Small Number of Partitions

Typically, the running time of an algorithm is measured as a function of the input size. However, in situations where the output may be very long, it is of interest to bound the time complexity also as a function of the output size. More explicitly, we say that A is a polynomial time *output-sensitive* algorithm for a problem Π if for every input I of length $\ell(I)$, whose output $\Pi(I)$ is of length $\ell(\Pi(I))$, A computes $\Pi(I)$ in time polynomial in $\max\{\ell(I),\ell(\Pi(I))\}$.

This notion may be useful in the context of the bigraphic degree realization problem. For a degree sequence d, let $N_{Part}(d) = |\mathtt{BP}(d)|$ denote the number of different block partitions of d. The BDR problem can be solved on d by enumerating all possible partitions and applying Theorem 3 for each of them. This may lead to a polynomial time output-sensitive algorithm for the problem provided that the enumeration procedure requires $\mathcal{O}(n^c)$ time, for some constant c, to generate the next partition.

We design an *output-sensitive* algorithm for *BDR* based on the natural enumeration procedure for block partitions. Recall that given an integer sequence $d = (d_1, d_2, \ldots, d_n)$, such that $d_i < n$ for every i, it is possible to find a block partition of d in polynomial time, by using the pseudo-polynomial dynamic programming algorithm for the partition problem, which in this case becomes polynomial. We claim that it is possible to find *all* block partitions of d in time $\mathcal{O}(T_{Part}(n) \cdot n \cdot N_{Part}(d))$, where $T_{Part}(n)$ is the time complexity of the best pseudo-polynomial time algorithm for deciding the PARTITION problem. To establish this, we describe a straightforward recursive algorithm. The algorithm uses a procedure $\mathsf{DP}(d, A)$ that receives as input a sequence

$$d = (d_1^{n_1}, d_2^{n_2}, \ldots, d_q^{n_q}) \quad (5)$$

of $n = n_1 + \ldots + n_q$ integers and a subsequence

$$A = (d_1^{j_1}, \ldots, d_k^{j_k}) \quad (6)$$

of d, where $0 \le k \le q$ and $0 \le j_i \le n_i$ for $i = 1, \ldots, k$. Setting

$$B = B(A, d) = (d_1^{n_1 - j_1}, \ldots, d_k^{n_k - j_k}), \quad (7)$$

the procedure decides, in time polynomial in n, if $A \parallel B$ can be completed into a partition (A', B') of d (namely, such that $A' \parallel B' = d$, $\sum A' = \sum B' = \sum d/2$, $A \subseteq A'$, $B \subseteq B'$).

Our recursive algorithm, $\mathsf{ALG}(d, A, j)$, gets as input a sequence d and a subsequence A as in Eq. (5) and (6). and $0 \le j \le n_k$. Setting $B = B(A, d)$ as in Eq. (7), the algorithm returns a set $\mathcal{P}$ containing all partitions (A', B') of d such that $A' \parallel B' = d$, $\sum A' = \sum B' = \sum d/2$, $A \circ (d_{k+1}^{j}) \subseteq A'$, $B \circ (d_{k+1}^{n_{k+1}-j}) \subseteq B'$. The set of all partitions of d is obtained by invoking $\mathsf{ALG}(d, \emptyset, 0)$ (thinking of d as augmented with an "empty prefix" $d_0^{n_0} = 0^0$).

Algorithm 1 Algorithm $\mathsf{ALG}(d, A)$.

Input: d, A as in Eq. (5) and (6).
Set $B \leftarrow B(d, A)$ as in Eq. (7).

1. Set $\mathcal{P} \leftarrow \emptyset$.
2. Invoke Procedure $\mathsf{DP}(d, A)$.
3. If the procedure returned "YES" then do:
   a. If $k = q$ then set $\mathcal{P} \leftarrow \{(A, B)\}$.
   b. Else (* $k < q$ *)
      repeat for $j_{k+1} = 0$ to n_{k+1}:
      i. Set $A' \leftarrow A \circ (d_{k+1}^{j_{k+1}})$
         Set $B' \leftarrow B \circ (d_{k+1}^{n_{k+1} - j_{k+1}})$
      ii. Recursively invoke Algorithm $\mathsf{ALG}(d, A')$, which returns $\mathcal{P}_{j_{k+1}}$.
      iii. Set $\mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{P}_{j_{k+1}}$.
4. Return $\mathcal{P}$.

▶ **Observation 7.** *The algorithm returns all partitions of d.*

▶ **Observation 8.** *The algorithm runs in time $\mathcal{O}(N_{Part}(d) \cdot n \cdot T_{Part})$.*

Note that the complexity of Algorithm ALG is dominated by the total time spent on the invocations of Procedure DP. Therefore, to prove Obs. 8, we need to show that when executing Algorithm ALG on a sequence d, the number of invocations of Procedure DP, K_{DP}, satisfies

$$K_{\mathsf{DP}} \le \mathcal{O}(N_{Part}(d) \cdot n) \quad (8)$$

To see this, let us illustrate the recursive execution of the algorithm on d by an *execution tracing tree* T_{EX} consisting of $q + 1$ levels. Each node in the tree is labeled by a pair (A, R), where $R \in \{Y, N\}$, and corresponds to one invocation of Procedure DP. The first entry in the label corresponds to the parameter A in the invocation, i.e., the root of the tree is marked $(\emptyset, R)$, and all nodes on level $1 \le k \le q$ are marked (A, R) for some subsequence A of d as in Eq. (6). The path leading to a node (A, R) in the tree captures the set A in the corresponding invocation. $R = N$ indicates that there are no partitions of d matching A and $B = B(A, d)$, whereas $R = Y$ indicates that there is at least one partition of d matching A and B. See Figure 1.

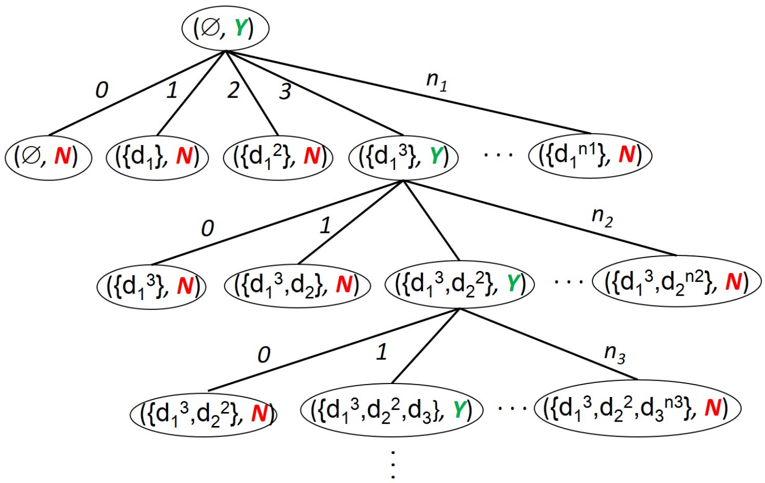

Figure 1 Illustration of the execution tracing tree T_{EX} of algorithm ALG, in the case where d has only one partition.

Note that in this tree, every node labeled (A, N) is a leaf, as the algorithm does not perform any additional recursive calls for this subsequence A. We refer to leaves labeled (A, N) (respectively, (A, Y)) as *N-leaves* (resp., *Y-leaves*), and denote their number by L_N (resp., L_Y). Also, denote the number of internal nodes on level k of the tree (which are also labeled by (A, Y)) by $I(k)$.

Observe that K_{DP}, the number of invocations of Procedure DP during the execution of Algorithm ALG, equals the number of nodes in T_{EX}, i.e.,

$$K_{\mathsf{DP}} \;=\; |V(T_{EX})| \;=\; L_Y + L_N + \sum_{k=0}^{q-1} I(k). \tag{9}$$

To bound this number, we redistribute the "charge" for the invocations of Procedure DP as follows. For $0 \le k \le q-1$, we reassign the charge for the invocation at an N-leaf v on level $k+1$ to its parent w on level k. Note that w can be charged by at most n_{k+1} N-leaves, since it has $n_{k+1}+1$ children and at least one of them is marked Y. It follows that

$$K_{\mathsf{DP}} \;\le\; L_Y + \sum_{k=0}^{q-1} I(k) \cdot (n_{k+1}+1). \tag{10}$$

Observe that $I(k) \le L_Y$ for every $0 \le k \le q-1$, since every internal node has at least one child labeled (A, Y). It follows that

$$K_{\mathsf{DP}} \;\le\; L_Y + L_Y \cdot \sum_{k=0}^{q-1} (n_{k+1}+1) \;=\; L_Y + L_Y \cdot (n+k). \tag{11}$$

Observe also that $L_Y = N_{Part}(d)$, the number of distinct partitions of d, hence

$$K_{\mathsf{DP}} \;\le\; N_{Part}(d) \cdot (n+k+1), \tag{12}$$

establishing Eq. (8) and proving Obs. 8. We get the following.

▶ **Lemma 9.** *The BDR problem admits a polynomial time output sensitive algorithm. More specifically, given an integer sequence $d = (d_1, d_2, \ldots, d_n)$, such that $d_i < n$ for every i, it is possible to find all block partitions of d in time $\mathcal{O}(T_{Part}(n) \cdot n \cdot N_{Part}(d))$, where $T_{Part}(n)$ is the time complexity of the best pseudo-polynomial time algorithm for deciding the PARTITION problem.*

Due to Theorem 4, the minimum t such that a given partition is t-bigraphic can be computed efficiently implying the following result.

▶ **Corollary 10.** *Let d be a degree sequence of length n such that $N_{Part}(d) = \mathcal{O}(n^c)$ for some constant c. Then, the minimum t such that d is t-bigraphic can be computed in polynomial time. As a special case, the BDR problem for d can be solved in polynomial time.*

We remark that a useful special subclass consists of sequences with a *constant* number of different degrees since such a sequence can have at most polynomially many different partitions.

▶ **Corollary 11.** *Let p be some constant and $d = (d_1^{n_1}, d_2^{n_2}, \ldots, d_p^{n_p})$ a degree sequence where $n = \sum_{i=1}^{p} n_i$. Then, $N_{Part}(d) = \mathcal{O}(n^c)$ for some constant c.*

3.2 Small Maximum Degree

Towards attacking the realizability problem of general bigraphic sequences, we first look at the question of bounding the total deviation of a nonincreasing sequence $d = (d_1, \ldots, d_n)$ as a function of its maximum degree, $\Delta = d_1$.

Burstein and Rubin [8] consider the realization problem for directed graphs with loops, which is equivalent to BDR^P. They give the following sufficient condition for a pair of sequences to be the in- and out-degrees of a directed graph with loops.

▶ **Theorem 12** ([8]). *Consider a degree sequence d with a partition $(a, b) \in \mathtt{BP}(d)$ assuming that a and b have the same length p. Let $\sum a = \sum b = pc$ where c is the average degree. If $a_1 b_1 \leq pc + 1$, then d is realizable by a directed graph with loops.*

In the following, their result is extended to bipartite multigraphs with bounded maximum multiplicity, i.e., to t-bigraphic sequences. We make use of the following straightforward technical claim.

▶ **Observation 13.** *Consider a nonincreasing integer sequence $x = (x_1, \ldots, x_k)$ of total sum $\sum x = X$. Then, $\sum (x[\ell]) \geq \ell X/k$, for every $1 \leq \ell \leq k$.*

Proof. Since x is nonincreasing, $\mathrm{avg}(x_1, \ldots, x_\ell) \geq \mathrm{avg}(x_{\ell+1}, \ldots, x_k)$, or more formally, $(\sum_{i=1}^{\ell} x_i)/\ell \geq (\sum_{i=\ell+1}^{k} x_i)/(k - \ell)$. Consequently,

$$X = \sum_{i=1}^{k} x_i = \sum_{i=1}^{\ell} x_i + \sum_{i=\ell+1}^{k} x_i \leq \sum_{i=1}^{\ell} x_i + \frac{k-\ell}{\ell} \sum_{i=1}^{\ell} x_i = \frac{k}{\ell} \sum_{i=1}^{\ell} x_i \ ,$$

implying the claim. ◀

▶ **Lemma 14.** *Let t be a positive integer. Consider a degree sequence d of length n with a partition $(a, b) \in \mathtt{BP}(d)$. If $a_1 \cdot b_1 \leq t \cdot \sum d/2$, then (a, b) is t-bigraphic.*

Proof. Let t, d and (a, b) as in the lemma. We first verify that the condition $a_1 \cdot b_1 \leq t \sum d/2$ implies that $b_1 \leq t|a|$. Towards a contradiction, suppose that $|a| < b_1/t$. It follows that

$$\sum a \leq a_1 \cdot |a| < a_1 \cdot b_1/t \leq \sum d/2 = \sum a,$$

a contradiction.

Let $X = \sum a = \sum b = \sum d/2$. By the assumption, $a_1 \cdot b_1 \leq tX$. Note that the conjugate sequence $\widetilde{b}$ is nonincreasing, $\sum(\widetilde{b}[b_1]) = X$, and $\widetilde{b}_i = 0$ for $i > b_1$. By Observation 13,

$$\sum(\widetilde{b}[\ell t]) \;\geq\; \ell t X/b_1 \;\geq\; \ell a_1$$

for every $1 \leq \ell \leq b_1/t$. As $a_i \leq a_1$ for every i, we also have

$$\sum(a[\ell]) \leq \ell a_1$$

for every $1 \leq \ell \leq b_1/t$. Combined, we have

$$\sum(a[\ell]) \leq \sum(\widetilde{b}[\ell t])$$

for every $1 \leq \ell \leq b_1/t$. The same inequality holds also for $|a| \geq \ell > b_1/t$, since $\sum(a[\ell]) \leq \sum a = X = \sum(\widetilde{b}[\ell t])$. Note that $\sum(\widetilde{b}[\ell t]) = \sum_{i=1}^m \min\{\ell t, b_i\}$. It follows that $\sum(a[\ell]) \leq \sum_{i=1}^m \min\{\ell t, b_i\}$ for $1 \leq \ell \leq |a|$, implying the lemma due to Theorem 4. ◀

Example 7 (see Sect. 5) establishes the following.

▶ **Lemma 15.** *The above bound is tight for some degree sequences.*

Lemma 14 is stated for a given partition (BDR^P). For BDR, we immediately have the following (with the second observation following from the first as $\sum d \geq n$).

▶ **Corollary 16.** *Let t be a positive integer. For every partitionable degree sequence $d = (d_1, \ldots, d_n)$,*

(1.) *if $d_1^2 \leq t \cdot \sum d/2$, then any partition $(a, b) \in \mathtt{BP}(d)$ is t-bigraphic.*

(2.) *if $d_1^2 \leq t \cdot n/2$, then any partition $(a, b) \in \mathtt{BP}(d)$ is t-bigraphic.*

This allows us to state the following for bounded-degree sequences whose maximum degree Δ is constant.

▶ **Corollary 17.** *Let Δ, t be positive integers. For every $n \geq 2\Delta^2/t$, and for every degree sequence $d = (d_1, \ldots, d_n)$ such that $d_1 \leq \Delta$, if d is partitionable then it is t-bigraphic.*

The extreme bound of this type is obtained when the sequence d has a balanced High-Low partition, in which case we get the following.

▶ **Corollary 18.** *Let t be a positive integer. Consider a degree sequence $d = (d_1, \ldots, d_n)$ with a balanced High-Low partition (H, L) where $H = (d_1, \ldots, d_k)$ and $L = (d_{k+1}, \ldots, d_n)$. If $d_1 \cdot d_{k+1} \leq t \cdot \sum d/2$, then (H, L) is t-bigraphic.*

4 Realizations based on the Equal or High-Low partitions

4.1 Realizations using the High-Low partition

Recall that a well-behaved High-Low partition is a balanced High-Low partition (H, L), $H = (d_1, \ldots, d_k)$ and $L = (d_{k+1}, \ldots, d_n)$, which satisfies the first Gale-Ryser conditions, i.e., $d_1 \le n - k$ and $d_{k+1} \le k$. Such a partition may or may not be bigraphic (see Example 4 in Sect. 5). However, for a non-increasing sequence d that admits a well-behaved High-Low partition (H, L), the BDR problem turns out to be solvable [4].

More explicitly, when d admits a well-behaved High-Low partition (H, L), it suffices to test the (entire collection of) Gale-Ryser conditions on (H, L). The realizability of d is then decided as follows.

- If all the Gale-Ryser conditions are met, then (H, L) is a bigraphic degree partition, hence d is a bigraphic degree sequence.
- Conversely, if one or more of the Gale-Ryser conditions is violated for (H, L), then *every* partition of d must violate one Gale-Ryser condition and d has *no* bigraphic degree partition. Consequently, d itself is not a bigraphic degree sequence.

Relying on the adapted Havel-Hakimi theorem described in Sect. 2, and on the resulting algorithm for computing a realizing bipartite graph given a bipartite degree partition, we conclude the following.

▶ **Theorem 19** ([4]). *Let d be a degree sequence with a well-behaved High-Low partition. It can be decided in polynomial time whether d is bigraphic or not. If d happens to be bigraphic, a bipartite graph realizing d can be computed in polynomial time.*

Hereafter, we examine degree sequences that have a balanced High-Low partition but are not well-behaved. Our goal is to generate bipartite multigraphs with low maximum multiplicity of parallel edges based on the High-Low partition.

In the following, let r be a positive integer, and let d be an r-graphic degree sequence with High-Low partition $HL(d) = (H, L)$ where $H = (d_1, \ldots, d_k)$ and $L = (d_{k+1}, \ldots, d_n)$, for some integer $k \in [1, n-1]$. We quantify the violation of the first Gale-Ryser conditions with the following definitions. Let

$$t_H(d) = \left\lceil \frac{d_1}{n-k} \right\rceil \qquad \text{and} \qquad t_L(d) = \left\lceil \frac{d_{k+1}}{k} \right\rceil,$$

and define $t(d) = \max\{t_H(d), t_L(d)\}$. (Note that sequence d has a well-behaved High-Low partition if $t(d) = 1$.) First, we observe that $t_H(d)$ is bounded for r-graphic sequences.

▶ **Lemma 20** ([4]). *Let d be an r-graphic degree sequence with High-Low partition $HL(d) = (H, L)$. Then, $t_H(d) \le 2r$.*

The main result is the following.

▶ **Theorem 21** ([4]). *Let d be an r-graphic degree sequence with High-Low partition $HL(d) = (H, L)$ and let $t = \max\{t(d), 2r\}$. Then, (H, L) is t-bigraphic.*

Example 8 in Sect. 5 shows that the conclusion of Theorem 21 does not hold if the degree sequence d is not r-graphic. Theorem 21 is complemented by an existential lower bound. In [4], it is shown that there are degree sequences d with High-Low partition $HL(d)$ such that $t(d) > 1$, and d is not t'-bigraphic for any $t' < t(d)$.

For graphic degree sequences, we state the following result.

▶ **Corollary 22** ([4]). *Let d be a graphic degree sequence with High-Low partition (H, L) and $t = t(d)$.*
(i) *If $t = 1$, then (H, L) is 2-bigraphic.*
(ii) *If $t > 1$, then (H, L) is t-bigraphic.*

In case there is a well-behaved High-Low partition, Theorem 19 implies the following.

▶ **Corollary 23** ([4]). *Let d be a graphic degree sequence with well-behaved High-Low partition $HL(d) = (H, L)$. Then, either*
(i) *(H, L) is bigraphic, or*
(ii) *d is not bigraphic and (H, L) is 2-bigraphic.*

To close this section, we present bounds on $t_L(d)$ and $t_H(d)$ in case degree sequence d is r-graphic or bigraphic. The next theorem establishes a bound on $t_L(d)$ for an r-graphic sequence. (A bound on $t_H(d)$ is already shown with Lemma 20.)

▶ **Theorem 24** ([4]). *Let d be an r-graphic sequence with High-Low partition (H, L). Then,*

$$t_H(d) \le 2r, \quad \text{and} \qquad t_L(d) \le \left\lceil \frac{r(k+1)}{2} \right\rceil .$$

Examples 9 & 10 in Sect. 5 show that the bound of Theorem 24 is tight and that for graphic sequences, $t_L(d) < t_H(d)$ as well as $t_H(d) < t_L(d)$ can occur.

Finally, the next theorem gives improved bounds for bigraphic degree sequences.

▶ **Theorem 25** ([4]). *Let d be a bigraphic sequence with High-Low partition $HL(d)$. Then,*

$$t_H(d) \le 1, \quad \text{and} \qquad t_L(d) \le \left\lceil \frac{k+2+1/k}{4} \right\rceil .$$

4.2 Realizations using the Equal partition

We next consider degree sequences that are even, i.e, where each degree occurs an even number of times. Let q be a positive integer, and let d be an even degree sequence consisting of n_i copies of the integer d_i for $1 \le i \le q$ where $\sum_{i=1}^{q} n_i = n$. We adopt the notation $d = (d_1^{n_1}, \ldots, d_q^{n_q})$. As n_i is even, for every $1 \le i \le q$, the equal partition EQP $(d) = (a, b)$ where $a = b = (d_1^{n_1/2}, \ldots, d_q^{n_q/2})$ is well-defined.

As mentioned earlier, a well-behaved equal partition does not seem to enable resolving the BDR problem, as there are even sequences d with an equal partition EQP $(d) = (a, b)$ that satisfies the first Gale-Ryser conditions where (a, b) is not bigraphic but other partitions of d are bigraphic (see Example 5 in Sect. 5). However, it is shown in [3] that if the sequence d is graphic and even, then the equal partition is 2-bigraphic. In general, we have the following result.

▶ **Theorem 26** ([3]). *Let d be an even and r-graphic degree sequence of length n with equal partition $EQP(d) = (a, b)$. Then, (a, b) is $2r$-bigraphic.*

5 Examples

We adopt the notation $d = (d_1^{n_1}, \ldots, d_q^{n_q})$, where $\sum_{i=1}^{q} n_i = n$, for a sequence consisting of n_i copies of the integer d_i for $1 \le i \le q$.

▶ **Example 1.** *Consider the sequence* $d = (6^2, 4^2, 2^6)$.
This sequence is even, so it has an equal partition. It also has a balanced High-Low partition, as well as a third partition:

(i.) $a = (6, 4, 2^3)$ *and* $b = (6, 4, 2^3)$ *(the equal partition),*
(ii.) $a' = (6^2, 4)$ *and* $b' = (4, 2^6)$ *(the High-Low partition),*
(iii.) $a'' = (6^2, 2^2)$ *and* $b'' = (4^2, 2^4)$.

However, only the last partition, (a'', b'')*, is bigraphic.*

▶ **Example 2.** *Consider the (non-graphic) sequence* $d = (n-1, n-2, \ldots, 3, 2, 1)$ *for* n *divisible by 4. Split* d *into length-4 subsequences*

$$B_1 = (1, 2, 3, 4), \quad B_2 = (5, 6, 7, 8), \quad \ldots$$

For each subsequence $B_j = (x, x+1, x+2, x+3)$ *for* $x = 4(j-1)+1$*, it is possible to place* $(x, x+3)$ *on one side of the partition and* $(x+1, x+2)$ *on the other side. This yields* $2^{n/4}$ *different partitions of* d.

▶ **Example 3.** *Consider the graphic sequence* $d = (n, n, n-1, n-1, \ldots, 2, 2, 1, 1)$ *of length* $2n$*, for* n *divisible by 4. Split* d *into length-8 subsequences*

$$B_1 = (1, 1, 2, 2, 3, 3, 4, 4), \quad B_2 = (5, 5, 6, 6, 7, 7, 8, 8), \quad \ldots$$

Each subsequence $B_j = (x, x, x+1, x+1, x+2, x+2, x+3, x+3)$*, for* $x = 4(j-1)+1$*, has three partitions:*

(i.) $a = (x, x+1, x+2, x+3)$ *and* $b = (x, x+1, x+2, x+3)$,
(ii.) $a' = (x, x, x+3, x+3)$ *and* $b' = (x+1, x+1, x+2, x+2)$,
(iii.) $a'' = (x+1, x+1, x+2, x+2)$ *and* $b'' = (x, x, x+3, x+3)$.

This yields $3^{n/4}$ *different partitions of* d.

▶ **Example 4.** *Consider the sequence* $d = ((6m)^m, (2m)^{5m+1}, 1^{2m})$.
This sequence has a well-behaved High-Low partition

$$H = ((6m)^m, (2m)^{m+1}), \quad L = ((2m)^{4m}, 1^{2m}),$$

but it is not bigraphic.

▶ **Example 5.** *Consider the sequence* $d = ((k^2)^k, k^{k^2}, 1^{k^2})$.
Its High-Low partition

$$H = ((k^2)^k, k^{k/2}), \quad L = (k^{k^2-k/2}, 1^{k^2})$$

is bigraphic, while its equal partition

$$a = b = ((k^2)^{k/2}, k^{k^2/2}, 1^{k^2/2})$$

is not.

▶ **Example 6.** *Consider the sequence* $d = (k^k, 1^{k^2-2k})$.
Its equal partition

$$a = b = (k^{k/2}, 1^{k^2/2-k})$$

is bigraphic, while its High-Low partition

$$H = (k^{k-1}), \quad L = (k, 1^{k^2-2k})$$

is not.

▶ **Example 7.** *Consider the sequence* $d = (t^{2k})$ *for positive integers* t, k *such that* $t > k$. *This sequence has a unique partition*

$$(a, a) \in \mathtt{BP}(d) \qquad \textit{where} \qquad a = (t^k).$$

One can verify that

$$\frac{2(a_1 \cdot a_1)}{\sum d} \;=\; \frac{t}{k} \;\leq\; \left\lceil \frac{t}{k} \right\rceil .$$

The partition (a, a) *is* $\lceil t/k \rceil$*-bigraphic but no better.*

▶ **Example 8.** *Consider the non-graphic sequence* $d = ((9m)^{m-1}, 6m+1, (3m)^{3m-1}, 1^1)$ *for some positive integer* m. *Its High-Low partition is*

$$H = ((9m)^{m-1}, 6m+1), \quad L = ((3m)^{3m-1}, 1^1).$$

We have $t_H(d) = t_L(d) = 3$, *but the conditions of Theorem 4 for* 3*-bigraphic degree sequences are violated. Specifically, the condition for index* $m-1$ *requires*

$$9m(m-1) \leq (3m-1) \cdot 3 \cdot (m-1) + 1,$$

which is false.

▶ **Example 9.** *Consider the graphic sequence* $d = (6, 3^6)$ *which has exactly one (High-Low) partition* (H, L) *with*

$$H = (6, 3^2), \quad L = (3^4).$$

One can verify that $t_L(d) = 2$ *and* $t_H(d) = 1$.

▶ **Example 10.** *Consider the degree sequence* $d' = ((\frac{k(k+1)}{2})^{k+1}, 1^{\frac{k(k+1)}{2}(k-1)})$, *for a positive integer* k. *To see that* d' *is graphic, observe that* $\sum d'$ *is even, and that the* $(k+1)$*th-EG inequality holds (For such a block sequence this is sufficient, see, e.g., [30]). The* $(k+1)$*th-EG inequality requires*

$$(k(k+1)/2) \cdot (k+1) \leq k(k+1) + (k(k+1)/2) \cdot (k-1),$$

which trivially holds. Moreover, $HL(d') = (H', L')$ *where*

$$H' = ((\frac{k(k+1)}{2})^k), \quad L' = ((\frac{k(k+1)}{2}), 1^{\frac{k(k+1)}{2}(k-1)}).$$

Hence, $|H'| = k$ *and* $d_{k+1} = \frac{k(k+1)}{2}$.

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