Multi-Dimensional Stable Roommates in 2-Dimensional Euclidean Space

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Abstract

We investigate the Euclidean d-Dimensional Stable Roommates problem, which asks whether a given set V of $\mathsf{d} \cdot n$ points from the 2-dimensional Euclidean space can be partitioned into n disjoint (unordered) subsets $\Pi = \{V_1, \ldots, V_n\}$ with $|V_i| = \mathsf{d}$ for each $V_i \in \Pi$ such that Π is *stable*. Here, *stability* means that no point subset $W \subseteq V$ is blocking Π, and W is said to be *blocking* Π if $|W| = \mathsf{d}$ such that $\sum_{w' \in W} \delta(w, w') < \sum_{v \in \Pi(w)} \delta(w, v)$ holds for each point $w \in W$, where $\Pi(w)$ denotes the subset $V_i \in \Pi$ which contains w and $\delta(a, b)$ denotes the Euclidean distance between points a and b. Complementing the existing known polynomial-time result for $\mathsf{d} = 2$, we show that such polynomial-time algorithms cannot exist for any fixed number $\mathsf{d} \geq 3$ unless P=NP. Our result for $\mathsf{d} = 3$ answers a decade-long open question in the theory of Stable Matching and Hedonic Games [18, 1, 10, 26, 21].

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1 Introduction

We study the computational complexity of a geometric and multi-dimensional variant of the classical stable matching problem, called Euclidean d-Dimensional Stable Roommates (Euclid-d-SR). This problem is to decide whether a given set V of $\mathsf{d} \cdot n$ agents, each represented by a point in the two-dimensional Euclidean space $\mathcal{R}^2$, has a d-*dimensional stable matching* (in short, d-*stable matching*). Here, each agent $x \in V$ has a preference list over all (unordered) size-d agent sets containing x which is derived from the Euclidean distances between the points. More precisely, agent x *prefers* subset S to subset T if the sum of Euclidean distances from x to S is smaller than the sum of the distances to T. We call preferences over subsets of agents which are based on the sum of Euclidean distances *Euclidean preferences*. A d-*dimensional matching* is a partition of V into n disjoint agent subsets $\Pi = \{V_1, \ldots, V_n\}$ with $|V_i| = \mathsf{d}$ for all $i \in \{1, \ldots, n\}$. In this way, each agent $v \in V$ is assigned to a subset in Π. An agent subset V' is *blocking* the d-dimensional matching Π if $|V'| = \mathsf{d}$ and each agent in V' prefers V' to its "assigned" agent subset in Π. A d-*stable matchings* is a d-matchings that is not blocked by a subset of agents of size d.

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When allowing agents to have arbitrary preferences, we arrive at the d-DIMENSIONAL STABLE ROOMMATES (d-SR) problem with 2-SR being equivalent to the classical STABLE ROOMMATES problem [14, 17]. It is well-known that *not* every instance of STABLE ROOMMATES admits a 2-stable matching, but deciding whether there exists one is polynomial-time solvable [17]. Fortunately, if we restrict the preferences to be Euclidean, then a 2-stable matching always exists and it can be found in polynomial time: Iteratively pick two remaining agents who are closest to each other and match them [1]. One may be tempted to apply this greedy approach to the case when $d = 3$. However, this would only work if it can find and match a triple of agents in each step such that this triple is the most preferred one of all three. Since such a "most-preferred" triple may not always exist, the prospects become less clear. Indeed, Arkin et al. [1] showed that not every instance of EUCLID-3-SR admits a 3-stable matching. To the best of our knowledge, nothing about the existence of EUCLID-d-SR is known for any fixed $d \geq 4$. In particular, the no instance by Arkin et al. will not work for any fixed $d \geq 4$. Arkin et al. left open the computational complexity of finding a 3-stable matching. The same question has been repeatedly asked since then [18, 21, 10, 26, 6]. Nevertheless, d-SR (i.e., for general preferences) has been known to be NP-complete for $d = 3$. Hence, it is of particular importance to search for natural restricted subcases, e.g., under Euclidean preferences, which may allow for efficient algorithms.

Our contribution. In this work, we aim at settling the computational complexity of EUCLID-d-SR for all fixed $d \geq 3$. Arkin et al. [1] showed that there is always a 3-dimensional matching which is approximately stable, which sparks hope for a polynomial-time algorithm for $d = 3$. We destroy such hope by showing that EUCLID-3-SR is NP-hard. We achieve this by reducing from an NP-complete planar variant of the EXACT COVER BY 3 SETS problem, where we make use of a novel chain gadget (see the orange and blue parts in Figure 3) and a star gadget (see Figure 1) which is adapted from the no-instance of Arkin et al. See the idea part in Section 3 for more details.

The same construction does not work for $d \geq 4$ since a no-instance for EUCLID-3-SR does not remain a no-instance for EUCLID-4-SR. However, we manage to derive two extended star structures, one for odd d and the other for even d (see the right and left figures of Figure 4, respectively), adapt the remaining component gadgets to show hardness for all fixed $d \geq 4$. Together, we show the following.

▶ **Theorem 1.** EUCLID-d-SR is NP-complete for every fixed $d \geq 3$.

Related work. Knuth [19] proposed to generalize the well-known STABLE MARRIAGE problem (a bipartite restriction of the STABLE ROOMMATES problem) to the 3-dimensional case. There are many such generalized variants in the literature, including the NP-complete 3-SR problem [18]. Huang [16] strengthen the result by showing that 3-SR remains NP-hard even for *additive* preferences. Herein, each agent $x \in V$ has cardinal preferences $\mu_x : V \setminus \{x\} \to \mathbb{R}$ over all other agents such that x *prefers* $\{x, s_1, s_2\}$ to $\{x, t_1, t_2\}$ if and only if $\mu_x(s_1) + \mu_x(s_2) > \mu_x(t_1) + \mu_x(t_2)$. Deineko and Woeginger [10] strengthen the result of Huang by showing that 3-SR remains NP-hard even for metric preferences: $\mu_x(y) = \mu_y(x) \geq 0$ and $\mu_x(y) + \mu_y(z) \leq \mu_x(z)$ such that x prefers $\{x, s_1, s_2\}$ to $\{x, t_1, t_2\}$ if and only if $\mu_x(s_1) + \mu_x(s_2) < \mu_x(t_1) + \mu_x(t_2)$. It is straightforward to see that Euclidean preferences are metric preferences and metric preferences are additive. We thus strengthen the results of Deineko and Woeginger, and Huang, by showing that the hardness remains even for Euclidean preferences. Recently, McKay and Manlove [22] strengthen the result of Huang [16] by showing that the NP-hardness remains even if the cardinal preferences are binary, i.e.,

$\mu_x(y) \in \{0, 1\}$ for all other agents y. This result is not comparable to ours since binary preferences and Euclidean preferences are not comparable. They also show that 3-SR becomes polynomial-time solvable when the preferences are binary and symmetric.

Multi-dimensional stable matchings are equivalent to the so-called *fixed-size stable cores* in hedonic games [12], where each coalition (i.e., a non-empty subset of agents) in the core must have the same size, and stability only needs to be guaranteed for any other coalition of the same size.[1] Hence, our NP-hardness result also transfers to the case of finding a fixed-size stable core in the scenario where the agents in the hedonic game have Euclidean preferences. Hedonic games have been studied under graphical preference models [11, 24], where there is an underlying social network (a directed graph) such that agents correspond to the vertices in the graph. The general idea is to assume that agents prefer to be with their own out-neighbors more than non-out-neighbors. The Euclidean preference model is related to the graphical preference model where the underlying graph is planar. However, the Euclidean model is more fine-grained and assumes that the intensity of the preferences also depends on the distance of the agents. Notably, under the graphical model, a stable core always exists and it can be found in linear time [11], but verifying whether a given partition is stable is NP-hard [7]. Hedonic games with fixed-size coalitions have been studied for other solution concepts such as strategy-proofness [27], Pareto optimality [9], and exchange stability [3].

Other generalized variants include the study of 3-stable matching with cyclic preferences [13, 4, 20], with preferences over individuals [18], and the study of the higher-dimensional case [6] and of other restricted preference domains [5]. We refer to the textbook by Manlove [21] for more references.

Paper outline. In Section 2, besides introducing necessary concepts and notations used throughout the paper, we describe a crucial star-structured instance of Euclid-3-SR (see Example 2), which serves as a tool of our NP-hardness reduction. The proof of Theorem 1 is divided into two sections: In Section 3, we consider the case of $\mathsf{d} = 3$ and show-case in detail how to combine the star-structured instance with two new gadgets, one for the local replacement and one for the enforcement, to obtain NP-hardness. In Section 4, we show how to carefully adapt the star-structured instance (which only works for $\mathsf{d} = 3$) and modify the reduction to show hardness for any fixed $\mathsf{d} \geq 4$. We conclude in Section 5. Due to space constraints, some figures, examples, and (part of) the proofs for results marked by $\star$ are deferred to the full version [8].

2 Preliminaries

Given a non-negative integer t, we use "$[t]$" (without any prefix) to denote the set $\{1, \ldots, t\}$. Throughout the paper, if not stated explicitly, we assume that ε and ε_{d} are small fractional values with $0 < \varepsilon < 0.001$ and $0 < \varepsilon_{\mathsf{d}} < \frac{1}{1000\mathsf{d}}$, where $\mathsf{d} \geq 3$. By "close to zero" we mean a value which is smaller than ε and ε_{d}.

For each fixed integer $\mathsf{d} \geq 2$, an instance of Euclidean d-Dimensional Stable Roommates (Euclid-d-SR) consists of a set $V = \{1, \ldots, \mathsf{d} \cdot n\}$ of $\mathsf{d} \cdot n$ agents and an embedding $E \colon V \to \mathbb{R}^2$ of the agents into 2-dimensional Euclidean space. We call a non-empty subset $V' \subseteq V$ of agents a *coalition*. The preference list $\succeq_x$ of each agent $x \in V$ over

[1] A stable core is a *partition* Π of the agents into disjoint coalitions such that no subset of agents would block the partition Π by forming its own new coalition.

all possible size-d coalitions containing x is derived from the sum of the Euclidean distances from x to the coalition. More precisely, for each two size-d coalitions $S = \{x, a_1, \ldots, a_{\mathsf{d}-1}\}$ and $T = \{x, b_1, \ldots, b_{\mathsf{d}-1}\}$ containing x we say that x *weakly prefers* S to T, denoted as $S \succeq_x T$, if the following holds:

$$\sum_{j\in[\mathsf{d}-1]} \delta(E(x), E(a_j)) \leq \sum_{j\in[\mathsf{d}-1]} \delta(E(x), E(b_j)),$$

where $\delta(p, q) \coloneqq \sqrt{(p[1] - q[1])^2 + (p[2] - q[2])^2}$. We use $S \succ_x T$ (i.e., x preferring S to T) and $S \sim_x T$ (i.e., x indifferent between S and T) to refer to the asymmetric and symmetric part of $\succeq_x$, respectively. To ease notation, for an agent x and a preference list $\mathcal{L}$ over a subset $\mathcal{F}$ of size-d coalitions, we use $\mathcal{L} \succ_x \cdots$ to indicate that agent x prefers every size-d coalition in $\mathcal{F}$ over every size-d coalition not in $\mathcal{F}$ and her preferences over $\mathcal{F}$ are according to $\mathcal{L}$. Further, we use the agent and her embedded points interchangeably, and the distance between two agents means the distance between their embedded points. For each agent x and each coalition $S \subseteq V$, we use $\delta(x, S)$ to refer to the sum of Euclidean distances from x to each member in S: $\delta(x, S) = \sum_{y\in S} \delta(x, y)$.

See the introduction for the definition of d-*matchings*, *blocking coalitions*, and d-*stable matchings*. Given a d-matching Π and an agent $x \in V$, let $\Pi(x)$ denote the coalition that contains x. The problem studied in this paper is defined as follows:

> Euclid-d-SR
> **Input:** An agent set $V = \{1, \ldots, \mathsf{d} \cdot n\}$ and an embedding $E \colon V \to \mathbb{R}^2$.
> **Question:** Is there a d-stable matching?

Note that since stability for each fixed d can be checked in polynomial time, Euclid-d-SR is contained in NP for every fixed d.

Not every Euclid-3-SR instance admits a 3-stable matching. Arkin et al. [1] provided a star-structured instance which does not. In Example 2, we describe an *adapted* variant of their instance, which is a decisive component of our hardness reduction.

▶ **Example 2.** Consider an instance which contains *at least* 12 agents called $W = \{0, \ldots, 11\}$ where the 12 agents are embedded as given in Figure 1. In the embedding of $W \setminus \{10, 11\}$, the five inner-most points, namely 0 to 4, form a regular pentagon with edge length a. For each $i \in \{0, \ldots, 4\}$, the three points i, $i + 1 \bmod 5$, and $i + 5$ form a triangle with side lengths a, b, c such that $a < b < c < \ell$, where ℓ denotes the diagonal of the regular pentagon. Moreover, the angle θ at points $i + 1 \bmod 5$, i, $i + 5$ is at most 90 degrees. This ensures that the distance between points $(i + 1 \bmod 5) + 5$ and i is strictly larger than ℓ (we will use this fact later). Except for point 5 (marked in red), the closest neighbor of each point $i + 5$ is i, followed by $i + 1 \bmod 5$. Point 5's two closest neighbors are points 10 and 11 with $a < \delta(5, 10) < b$ and $a < \delta(5, 11) < b$, followed by points 0 and 1. The distance between 10 and 11 is close to zero, with the intention to ensure that every 3-stable matching must match them together. The distance from 10 (resp. 11) to any agent in $W \setminus \{5, 10, 11\}$ is larger than the diagonal length ℓ while the distance from 10 (resp. 11) to any agent not in W is larger than $\delta(5, 10) - \varepsilon$. Finally, the distance between any agent from $W \setminus \{10, 11\}$ to any agent *not* from $W \setminus \{10, 11\}$ is strictly larger than ℓ.

To specify the embedding of the agents from W, we use the polar coordinate system. We first fix the embeddings of $5, 10, 11$ to ensure the distances between them are as stated above. Then, we fix points 0 and 1 and the centroid of the regular pentagon to ensure the distances satisfy $a < b < c < \ell$, and the angle θ at points $1, 0, 5$ is at most 90 degrees, and the angle at points $0, 5, j$, $j \in \{10, 11\}$, is more than 90 degrees. Once these points are fixed we can determine the other points by a simple calculation.

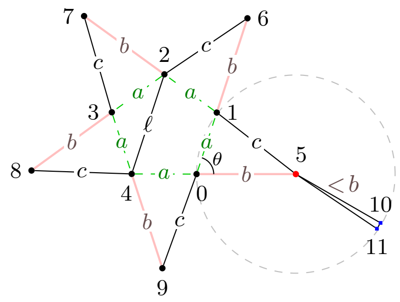


Figure 1 A star-structured instance adapted from Arkin et al. [1]; see Example 2. We use different colors to highlight the distances between the points. For instance, the smallest distance between any two points is a (highlighted in green). We also draw a dashed circle of radius b, centered at point 5 to indicate that points both 10 and 11 are with distance smaller than b to 5.

The instance of Arkin et al. [1] embeds the two extra points 10 and 11 differently than ours (see Example 2). Hence, their instance is a no-instance, while ours may be a yes-instance, provided some specific triple is matched together, formulated as follows:

▶ **Lemma 3.** *Every* 3*-stable matching of an instance satisfying the embedding described in Example 2 must contain triple* $\{5, 10, 11\}$.

Proof. Towards a contradiction, suppose that Π is a 3-stable matching with $\{5, 10, 11\} \notin \Pi$. We infer that $\{10, 11\} \subseteq \Pi(10)$ since otherwise $\{5, 10, 11\}$ is blocking Π due the following: $\delta(5, \Pi(5)) \geq \min(\delta(5, 10), \delta(5, 11)) + b > \delta(5, \{5, 10, 11\})$, and for each $x \in \{10, 11\}$ it holds that $\delta(x, \Pi(x)) \geq 2(\delta(x, 5) - \varepsilon) > \delta(x, 5) + \delta(10, 11)$ for any $\varepsilon > 0$. This implies that $\{10, 11\} \cap \Pi(5) = \emptyset$. Next, we observe that there must be a triple in Π that contains the two agents of at least one pentagon edge as otherwise $\{2, 3, 7\}$ is blocking: $\delta(2, \Pi(2)) \geq b+c > a+b$, $\delta(3, \Pi(3)) \geq b + c > a + c$, and $\delta(7, \Pi(7)) \geq b + \ell > b + c$. Thus, at least one triple in Π contains the agents of some pentagon edge, say $\{2, 3\}$; the other cases are analogous. Let $\{2, 3, x\} \in \Pi$. We distinguish between three subcases:

Case 1: $x \notin \{1, 4, 7, 8\}$**.** Then, one can verify that $\{2, 3, 7\}$ is blocking; recall that every agent not in $W \setminus \{2, 3\}$ is at distance larger than ℓ to agent 7.

Case 2: $x \in \{1, 7\}$**.** Then, $\Pi(4) = \{0, 4, 9\}$ or $\Pi(4) = \{0, 4, 8\}$ since otherwise $\{3, 4, 8\}$ blocks Π due to: $\delta(3, \Pi(3)) \geq a + \min(\delta(3, 1), \delta(3, 7)) = a + c > a + b$, $\delta(4, \Pi(4)) > a + c$, $\delta(8, \Pi(8)) > b + c$ (recall that the distance from every agent not in $W \setminus \{3, 4\}$ to agent 8 is larger than ℓ). However, both cases imply that $\{0, 1, 5\}$ is blocking since $\delta(0, \Pi(0)) \geq a + c > a + b = \delta(0, \{0, 1, 5\})$, $\delta(1, \Pi(1)) \geq a + \ell > a + c = \delta(1, \{0, 1, 5\})$, and $\delta(5, \Pi(5)) \geq c + \ell > b + c = \delta(5, \{0, 1, 5\})$; recall that $\Pi(5) \cap \{10, 11\} = \emptyset$.

Case 3: $x \in \{4, 8\}$**.** Then, $\delta(2, \Pi(2)) \geq a + \ell > a + c$. This implies that $\{0, 1, 6\} \in \Pi$ since otherwise $\{1, 2, 6\}$ is blocking Π. However, this implies that $\{0, 4, 9\}$ is blocking Π.

Since we have just shown that no agent x exists which is in the same triple as 2 and 3, no 3-stable matching exists that does not contain $\{5, 10, 11\}$. ◀

3 NP-hardness for Euclid-3-SR

In this section, we prove Theorem 1 for the case of $\mathsf{d} = 3$ by providing a polynomial reduction from the NP-complete Planar and Cubic Exact Cover by 3 Sets problem [23], which is an NP-complete restricted variant of the Exact Cover by 3 Sets problem [15].

Planar and Cubic Exact Cover by 3 Sets (PC-X3C)
Input: A $3n$-element set $X = \{1, \ldots, 3n\}$ and a collection $\mathcal{S} = (S_1, \ldots, S_m)$ of 3-element subsets of X of cardinality $3n$ such that each element occurs in exactly three sets and the associated graph is planar.
Question: Does $\mathcal{S}$ contain an *exact cover* for X, i.e., a subcollection $\mathcal{K} \subseteq \mathcal{S}$ such that each element of X occurs in exactly one member of $\mathcal{K}$?

Herein, given a PC-X3C instance $I = (X, \mathcal{S})$, the associated graph of I, denoted as $G(I)$, is a bipartite graph $G(I) = (U \uplus W, E)$ on two partite vertex sets $U = \{u_i \mid i \in X\}$ and $W = \{w_j \mid S_j \in \mathcal{S}\}$ such that there exists an edge $e = \{u_i, w_j\} \in E$ if and only if $i \in S_j$. We call the vertices in U and W the *element-vertices* and the *set-vertices*, respectively.

In our reduction, we crucially utilize the fact that the associated graph G of the input instance is planar and cubic, and hence by Valiant [25] admits a specific planar embedding in $\mathbb{Z}^2$, called *orthogonal drawing*, which maps each vertex to an integer grid point and each edge to a chain of non-overlapping horizontal and vertical segments along the grid (except at the endpoints). To simplify the description of the reduction, we use the following more restricted orthogonal drawing:

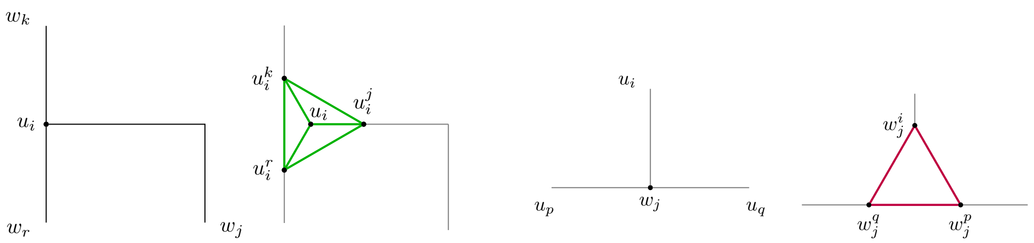
▶ **Proposition 4** ([2]). *In polynomial time, a planar graph with maximum vertex degree three can be embedded in the grid $\mathbb{Z}^2$ such that its vertices are at the integer grid points and its edges are drawn using at most one horizontal and one vertical segment in the grid.*

We call the intersection point of the horizontal and vertical segments the *bending point*.

3.1 The construction

The idea. Given an instance $I = (X, \mathcal{S})$ of PC-X3C, we first use Proposition 4 to embed the associated graph $G(I) = (U \uplus W, E)$ into a 2-dimensional grid with edges drawn using line segments of length at least $L \geq 200$, and with parallel lines at least $4L$ grid squares apart. The idea is to replace each element-vertex $u_i \in U$ with four agents which form a "star" with three close-by "leaves" (see Figure 2a). These leaves one-to-one correspond to the sets S_j with $i \in S_j$. In this way, exactly one set S_j is unmatched with the center and will be chosen to the exact cover solution. Furthermore, we replace each set-vertex $w_j \in W$ with three agents w_j^i, $i \in S_j$, which form an equilateral triangle (see Figure 2b). We replace each edge in $G(I)$ with a chain of copies of three agents, which, together with a private enforcement gadget (the star structure with a tail in Figure 3), ensure that either all three agents w_j^i are matched in the same triple (indicating that the corresponding set is in the solution) or none of them is matched in the same triple (indicating that the corresponding set is not in the solution). The agents in the star structure can be embedded "far" from other agents due to the tail.

Gadgets for the elements and the sets. For each element-vertex $u_i \in U$, assume that the three connecting edges in $G(I)$ are going horizontally to the right (rightward), vertically up (upward), and vertically down (downward); we can mirror the coordinate system if this is not the case. Let w_j, w_k, w_r denote the set-vertices on the endpoints of the rightward, upward, and downward edge, respectively. We create four element-agents, called u_i, u_i^j, u_i^k, and u_i^r. We embed them into $\mathbb{R}^2$ in such a way that u_i^j, u_i^k, u_i^r are on the segment of the rightward, upward, and downward edge, respectively, and are of equal distance 8 to each other. Agent u_i is in the center of the other three agents. See Figure 2a for an illustration.



(a) Gadget (right) for an element vertex u_i (left) s.t. element i belongs to sets S_j, S_k, S_r.

(b) Gadget (right) for a set-vertex w_j^i for which the set S_j consists of three elements i, p, q.

Figure 2 Element- and set-gadgets described in Subsection 3.1.

Similarly, for each set-vertex $w_j \in W$, assume that the three connecting edges in $G(I)$ are going rightward, leftward, and upward, connecting the element-vertices u_i, u_p, u_q, respectively. We create three set-agents, called w_j^i, w_j^p, w_j^q. We embed them into $\mathbb{R}^2$ in such a way that w_j^i, w_j^p, w_j^q are on the segment of the rightward, leftward, and upward edge, respectively, and are of equidistance 10 to each other. See Figure 2b for an illustration.

The edge- and the enforcement gadget. For each edge $e = \{u_i, w_j\}$ in $G(I)$, we create $\hat{n}$ (a constant value to be determined later) copies of the triple $A_i^j[z] = \{\alpha_i^j[z], \beta_i^j[z], \gamma_i^j[z]\}$, $1 \le z \le \hat{n}$, of agents and embed them around the line segments of edge e in the grid (refer to Figure 3). To connect to the set-gadget, we merge agent $\gamma_i^j[\hat{n}]$ and set-agent w_j^i together. For technical reasons, we also use $\gamma_i^j[0]$ to refer to u_i^j. To define the distances, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2\hat{n}}$ be a sequence of increasing positive values with $2(2\hat{n}-1)/(2\hat{n}+1) \le \varepsilon_{2\hat{n}-1} \le 2(2\hat{n}-1)/(2\hat{n}) < \varepsilon_{2\hat{n}} = 2 - \varepsilon$. Now, we embed the newly added agents so that the distances between "consecutive agents" on the line increase with $z \in [\hat{n}]$:

- The distance between agents $\alpha_i^j[z]$ and $\beta_i^j[z]$ (marked in blue) is close to zero.
- The distance between agents $\alpha_i^j[z]$ (resp. $\beta_i^j[z]$) and $\gamma_i^j[z]$ is $8 + \varepsilon_{2z}$.
- The distance between $\alpha_i^j[z]$ (resp. $\beta_i^j[z]$) and $\gamma_i^j[z-1]$ is $8 + \varepsilon_{2z-1}$.

In this manner, we will ensure that either all $A_i^j[z]$, $z \in [\hat{n}-1]$, or all $\{\gamma_i^j[z-1], \alpha_i^j[z], \beta_i^j[z]\}$, $i \in [\hat{n}]$ belong to a 3-stable matching (to be proved later).

To determine the value $\hat{n}$, let the lengths of the segments for edge $\{u_i, w_j\}$ in the orthogonal drawing of graph $G(I)$ be L_1 and L_2, respectively; L_2 is zero if there is only one straight segment. We set $\hat{n}$ to the largest value satisfying $\sum_{z=1}^{2\hat{n}}(8+0.01\cdot z) \le L_1 + L_2$, which is clearly a constant. For brevity's sake, when using $\hat{n}$, we mean the constant associated to an edge $\{u_i, w_j\}$ in the drawing which will be clear from the context. It is also fairly straightforward to check that one can choose the sequence ε_i so that the bending point of the chain is some agent $\gamma_i^j[z']$, $z' \in [n-1]$ as shown in Figure 3.

By the construction of the gadgets above, each set-agent w_j^i strictly prefers triple $A_i^j[\hat{n}]$ to triple $\{w_j^i, w_j^p, w_j^q\}$ since $\delta(w_j^i, x) < 10 = \delta(w_j^i, y)$ for all $x \in \{\alpha_i^j[\hat{n}], \beta_i^j[\hat{n}]\}$ and $y \in \{w_j^p, w_j^q\}$; recall that $w_j^i = \gamma_i^j[\hat{n}]$ and $\delta(x, \gamma_i^j[\hat{n}]) = 10 - \varepsilon$. To ensure that exactly one of the two triples is chosen, we make use of the star-gadget from Example 2. More precisely, we introduce an agent triple $H_j^i = \{f_j^i, g_j^i, h_j^i\}$ and embed them in such a way that the distances between two "consecutive" agents on the line towards the star-gadget increase:

- The distance between f_j^i and g_j^i is close to zero.
- The distance between agent h_j^i and each of $\{f_j^i, g_j^i\}$ is $10 + 2\varepsilon$.
- The distance between f_j^i (resp. g_j^i) and each of $A_i^j[\hat{n}]$ is in range $[10+\varepsilon, 10+2\varepsilon)$.

This means that the most preferred triple of agent h_j^i is H_j^i, while both f_j^i and g_j^i prefer triple S to H_j^i where $S = \{f_j^i, g_j^i, x\}$ and $x \in A_i^j[\hat{n}]$.

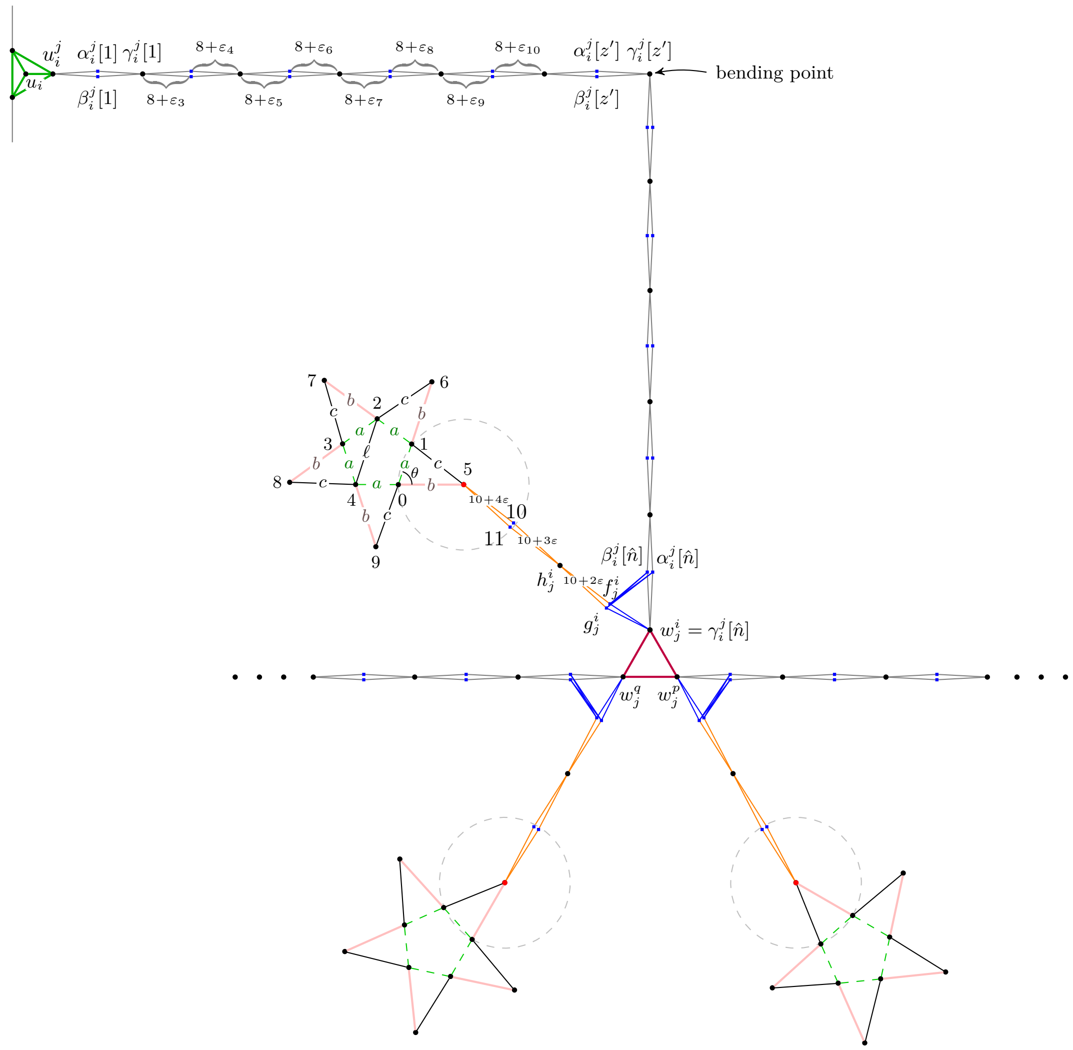


Figure 3 Gadget for edge $\{u_i, w_j\}$ in $G(I)$ with $S_j = \{i, p, q\}$. Here, the fractional values ε_z satisfy $0 < \varepsilon_1 < \cdots < \varepsilon_{\hat{n}} = 2 - \varepsilon$. The star-gadget, adapted from Arkin et al. [1], is described in Example 2. To highlight the distances between the points in the star-gadget, we use different colors. For instance, the smallest distance between any two points in the star is a (highlighted in green). We also draw a dashed circle of radius b, centered at point 5 to indicate that points both 10 and 11 are with distance smaller than b to 5.

Finally, we create 12 agents, namely, $W = \{0, \ldots, 11\}$, according to Example 2 such that agents 10 and 11's most preferred triple is $\{10, 11, h^i_j\}$, followed by $\{5, 10, 11\}$. More precisely:

- The distance between agent 10 (resp. agent 11) and h^i_j is $10 + 3\varepsilon$.
- The distance between agent 10 (resp. agent 11) and 5 is $10 + 4\varepsilon$.
- The five agents from $\{0, \ldots, 4\}$ form a regular pentagon with edge length a. Each two agents on the pentagon form with a private agent a triangle with edge lengths a (marked in green), b (marked in red), and c. We set $b = 10.1$ and $c = 10.2$. The length of the diagonal of the pentagon is ℓ.

Altogether, the lengths satisfy the relation $a < b < c < \ell$ and the specific angle θ is at most 90 degrees. Due to the chain, including f^i_j, g^i_j, and h^i_j, the distance from every agent not from $W \cup \{h^i_j\}$ to every agent from W is larger than ℓ. We call the gadget, consisting of the star-agents and the triple H^i_j, the *star-gadget* for set-agent w^i_j and element-agent u^j_i. Figure 3 provides an illustration of how the element-gadget, the set-gadget, and the star-gadget are embedded. Note that since the angle between any two line segments is 90 degrees and the line segment has length at least 200, we can make sure that such embedding is feasible.

This completes the description of the construction, which clearly can be done in polynomial time. In total, we constructed $O(4 \cdot 3n + 3 \cdot 3n + 3 \cdot 2\hat{n} \cdot 3n + 15 \cdot 3n) = O(n)$ agents. Note that we only need to have a good approximation of the embedding of the agents in the star-gadget and the equilateral triangle.

3.2 The correctness proof for $\mathbf{d = 3}$

Before we proceed with the correctness proof, we summarize the preferences derived from the embedding via the the following observation.

▶ **Observation 5** (⋆)**.** *For each element $i \in X$ and each set $S_j \in \mathcal{S}$ with $S_j = \{i, p, q\}$, let $0, \dots, 11$ denote the 12 agents in the associated star-gadget. Then, the following holds.*

(i) *The preference list of each agent $x \in \{10, 11\}$ satisfies $\{h^i_j, 10, 11\} \succ_x \cdots$.*

(ii) *For each triple $B \neq \{h^i_j, f^i_j, g^i_j\}$ with $B \succeq_{h^i_j} \{h^i_j, 10, 11\}$ it holds that $B \cap \{10, 11\} \neq \emptyset$.*

(iii) *For each agent $x \in \{f^i_j, g^i_j\}$ and each triple $B \neq \{f^i_j, g^i_j, h^i_j\}$:*
- *If $B = \{f^i_j, g^i_j, y\}$ (where $y \in \{\alpha^j_i[\hat{n}], \beta^i_j[\hat{n}], \gamma^i_j[\hat{n}]\}$), then $B \succ_x \{f^i_j, g^i_j, h^i_j\}$.*
- *If $B \succeq_x \{f^i_j, g^i_j, h^i_j\}$, then $B = \{f^i_j, g^i_j, y\}$ for some $y \in \{\alpha^j_i[\hat{n}], \beta^i_j[\hat{n}], \gamma^i_j[\hat{n}]\}$.*

(iv) *For each $z \in [\hat{n}]$ the preference list of agent $\gamma^j_i[z]$ satisfies $\{\alpha^j_i[z], \beta^j_i[z], \gamma^j_i[z]\} \succ_{\gamma^j_i[z]} \cdots$.*

(v) *For each $z \in [\hat{n}]$ the preference list of each agent $x \in \{\alpha^j_i[z], \beta^j_i[z]\}$ satisfies $\{\alpha^j_i[z], \beta^j_i[z], \gamma^j_i[z-1]\} \succ_x \{\alpha^j_i[z], \beta^j_i[z], \gamma^j_i[z]\} \succ_x \cdots$.*

(vi) *For each $z \in [\hat{n} - 1]$ and each triple $B \neq \{\alpha^j_i[z+1], \beta^j_i[z+1], \gamma^j_i[z]\}$ with $B \succeq_{\gamma^j_i[z]} \{\alpha^j_i[z+1], \beta^j_i[z+1], \gamma^j_i[z]\}$ it holds that $B \cap \{\alpha^j_i[z], \beta^j_i[z]\} \neq \emptyset$.*

(vii) *For each $B \neq \{w^i_j, w^p_j, w^q_j\}$ with $B \succeq_{w^i_j} \{w^i_j, w^p_j, w^q_j\}$ we have $B \cap \{\alpha^i_j[\hat{n}], \beta^i_j[\hat{n}]\} \neq \emptyset$.*

Finally, we show the correctness, i.e., "$I = (X, \mathcal{S})$ admits an exact cover if and only if the constructed instance admits a 3-stable matching" via the following lemmas. Lemma 6 shows the "only if" direction and Lemma 8 the other.

▶ **Lemma 6** (⋆)**.** *If $\mathcal{K} \subset \mathcal{S}$ is an exact cover of I, then the following 3-matching Π is stable.*
- *For each $S_j \in \mathcal{K}$ with $S_j = \{i, p, q\}$ add $\{w^i_j, w^p_j, w^q_j\}$ to Π.*
- *For each element $i \in X$ and each set $S_j \in \mathcal{S}$ with $i \in S_j$, call the agents in the associated star-gadget along with the tail agents $0, \dots, 11, h^i_j, f^i_j$, and g^i_j.*
  - *Add H^i_j, $\{5, 10, 11\}$, $\{1, 6, 8\}$, $\{2, 3, 7\}$, and $\{0, 4, 9\}$ to Π.*
  - *If $S_j \in \mathcal{K}$, then add all triples $\{\alpha^j_i[z], \beta^j_i[z], \gamma^j_i[z-1]\}$, $z \in [\hat{n}]$, to Π. Otherwise, add all triples $A^j_i[z]$, $z \in [\hat{n}]$, to Π.*
- *For each element $i \in X$ let S_k, S_r be the two sets which contain i, but are not chosen in the exact cover $\mathcal{K}$. Add $\{u_i, u^k_i, u^r_i\}$ to Π.*

The proof of the other direction is based on the following properties.

▶ **Lemma 7** (⋆)**.** *Let Π be a 3-stable matching of the constructed instance. For each element $i \in X$ and each set S_j with $S_j = \{i, p, q\}$, the following holds:*

(i) $H^i_j \in \Pi$.

(ii) Π *contains either all triples $\{\alpha^j_i[z], \beta^j_i[z], \gamma^j_i[z]\}$ or all triples $\{\alpha^j_i[z], \beta^j_i[z], \gamma^j_i[z-1]\}$, $z \in [\hat{n}]$.*

Now, we consider the "if" direction.

▶ **Lemma 8.** *If* Π *is a* 3*-stable matching, then the subcollection* $\mathcal{K}$ *with* $\mathcal{K} = \{S_j \in \mathcal{S} \mid \{\alpha_i^j[1], \beta_i^j[1], \gamma_i^j[0]\} \in \Pi$ *for some* $i \in S_j\}$ *is an exact cover.*

Proof. First of all, for each two chosen $S_j, S_k \in \mathcal{K}$ we observe that it cannot happen that $S_j \cap S_k \neq \emptyset$ as otherwise $\{u_i, u_i^j, u_i^k\}$ is a blocking triple; recall that $\gamma_i^j[0] = u_i^j$ and $\gamma_i^k[0] = u_i^k$. It remains to show that $\mathcal{K}$ covers each element at least once.

Now, for each element $i \in X$, let S_j, S_k, S_r denote the three sets that contain i. We claim that at least one of S_j, S_k, S_r belongs to $\mathcal{K}$ because of the following. If $S_j \notin \mathcal{K}$, then by construction, it follows that $T = \{\alpha_i^j[1], \beta_i^j[1], \gamma_i^j[0]\} \notin \Pi$. By Lemma 7(ii), it follows that $A_i^j[1] \in \Pi$. Since T is the most-preferred triple of both $\alpha_i^j[1]$ and $\beta_i^j[1]$ (see Observation 5(v)), by stability, u_i^j must be matched in a triple which she weakly prefers to T. Since $A_i^j[1] \in \Pi$, it follows that either $\{u_i^j, u_i^k, u_i^r\} \in \Pi$ or $\{u_i^j, u_i, v\} \in \Pi$ for some $v \in \{u_i^k, u_i^r\}$. It cannot happen that $\{u_i^j, u_i^k, u_i^r\} \in \Pi$ as otherwise there will be at least three blocking triples, including $\{u_i, u_i^j, u_i^r\}$. Hence, $\{u_i^j, u_i, v\} \in \Pi$ for some $v \in \{u_i^k, u_i^r\}$. Without loss of generality, assume that $v = u_i^k$. Then, it is straightforward to check that $\{u_i^r, \alpha_i^r[1], \beta_i^r[1]\} \in \Pi$. This implies that $S_r \in \mathcal{K}$.

To complete the correctness proof, we show that for each element $p \in S_r \setminus \{i\}$ it holds that $\{\alpha_p^r[1], \beta_p^r[1], \gamma_p^r[0]\} \in \Pi$. Let $S_r = \{i, p, q\}$. Since $S_r \in \mathcal{K}$, by definition and by Lemma 7(ii), we infer that $\{\alpha_i^r[\hat{n}], \beta_i^r[\hat{n}], \gamma_i^r[\hat{n}-1]\} \in \Pi$ (for some constant $\hat{n}$ defined in the construction). We infer that $\{w_r^i, w_r^p, w_r^q\} \in \Pi$ due to the following: By Lemma 7(i), we know that $H_r^i \in \Pi$; recall that $H_r^i = \{f_r^i, g_r^i, h_r^i\}$. Since both f_r^i and g_r^i prefer $\{f_r^i, g_r^i, w_r^i\}$ to H_r^i (see the first part of Observation 5(iii)), it follows by stability that $\Pi(w_r^i) \succeq_{w_r^i} \{f_r^i, g_r^i, w_r^i\}$. By Observation 5(vii), we infer that $\Pi(w_r^i) = \{w_r^i, w_r^p, w_r^q\}$ since $\alpha_i^r[\hat{n}]$ and $\beta_i^r[\hat{n}]$ are not available anymore. This means that $A_p^r[\hat{n}'], A_q^r[\hat{n}''] \notin \Pi$ since $w_r^p = \gamma_p^r[\hat{n}']$ and $w_r^q = \gamma_p^r[\hat{n}'']$ (for some constants $\hat{n}'$ and $\hat{n}''$). Consequently, we infer by Lemma 7(ii) that $\{\alpha_p^r[1], \beta_p^r[1], \gamma_p^r[0]\}, \{\alpha_q^r[1], \beta_q^r[1], \gamma_q^r[0]\} \in \Pi$, as desired. ◀

This concludes the proof of Theorem 1 for $\mathsf{d} = 3$.

4 Euclid-d-SR with $\mathsf{d} \geq 4$

In this section we look at the cases where $\mathsf{d} \geq 4$, and let $\kappa := \lfloor (\mathsf{d} - 1)/2 \rfloor$. The general idea of the reduction is similar to the case where $\mathsf{d} = 3$, and we still reduce from PC-X3C. Briefly put, we adapt the star-gadget from Example 2. However, depending on whether d is even or not, we need to carefully revise the star-gadget from Example 2 to make sure the enforcement gadget works. We will replace each pentagon-agent with a subset of agents of size κ, and each further agent from the triangle with two agents if d is even. We also need to update both the replacement and the enforcement gadget. In Subsection 4.1, we describe in detail what the new star-gadgets and the the remaining gadgets look like, and how they are connected to each other. In Subsection 4.2 we show the correctness.

4.1 The construction

We first describe the adapted star-gadgets through the following example (also see Figure 4).

▶ **Example 9.** We first consider the construction for even d, i.e., $\mathsf{d} = 2\kappa + 2$. Consider an instance with $7\kappa + 11$ agents called W where 5κ agents are embedded as the five vertices of a pentagon with κ agents at each vertex of the pentagon. We denote the five sets of points at the five vertices of the pentagon as $X_0, \ldots, X_4$. All points in each cluster X_i, $0 \leq i \leq 4$,

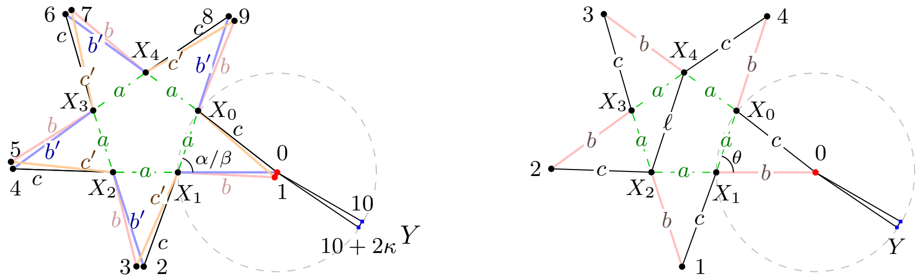


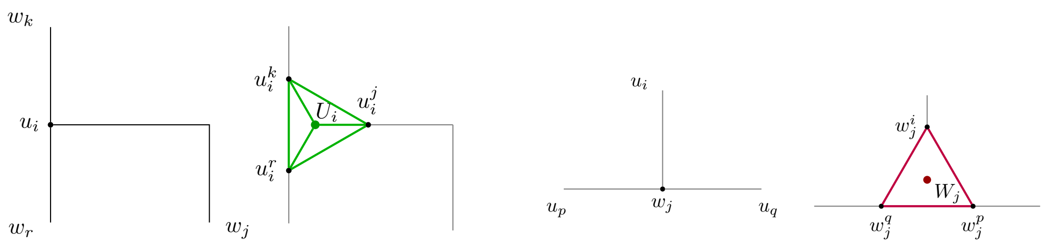
Figure 4 A star-structured instance adapted from Arkin et al. [1], similar to Example 2. The left one is for even d, while the right one is for odd d, both described in Example 9. See the caption of Example 2 for further explanation regarding the colors of the edges.

are embedded within an enclosing circle of radius close to zero, with the intention that a d-matching is stable only if all agents in X_i are matched together. For each $i \in \{0, \dots, 4\}$, the distance between each point X_i and each point in $X_{i+1 \bmod 5}$ is in the range of $[a, a + \varepsilon_{\mathsf{d}}]$, while the distance between each point in X_i and each point in $X_{i+2 \bmod 5}$ is in the range of $[\ell, \ell + \varepsilon_{\mathsf{d}}]$. There are 10 points $\{0, \dots, 9\}$ that form a star with the pentagon, as shown in Figure 4 (left). For each $i \in \{0, \dots, 4\}$, embed the points $2i$ and $2i + 1$ as follows: point $2i$ is at a distance between c and $c + \varepsilon_{\mathsf{d}}$ to every point in X_i, and at a distance between b' and $b' + \varepsilon_{\mathsf{d}}$ to every point in $X_{i+1 \bmod 5}$. Point $2i + 1$ is at a distance between c' and $c' + \varepsilon_{\mathsf{d}}$ to every point in X_i, and at a distance between b and $b + \varepsilon_{\mathsf{d}}$ to every point in $X_{i+1 \bmod 5}$. Finally, the distance $\delta(2i, 2i+1)$ is close to 0. Here the mentioned values satisfy the following relations $a < b < c < \ell$, $b < b' < \ell$, $c < c' < \ell$, $b + b' < 3a$, $c + c' < 3a$, $b + b' < a + \ell$, and $c + c' < b + \ell$.

The remaining $2\kappa + 1$ points, denoted by $10, \dots, 10 + 2\kappa$ in the figure, are called Y; note that $|Y| = 2\kappa + 1 = \mathsf{d} - 1$. Together, $W := \bigcup_{i \in \{0,\dots,4\}} X_i \cup \{0, \dots, 9\} \cup Y$. All points in Y are embedded within an enclosing ball with radius close to zero. For each point y in Y, it holds that $b - \varepsilon_{\mathsf{d}} \le \delta(0, y) < b$ and $b - \varepsilon_{\mathsf{d}} \le \delta(1, y) < b$, and for each each point w in $W \setminus (\{0, 1\} \cup Y)$ it holds that $\delta(w, y) > \ell$. Points 0 and 1 are the two points from $W \setminus Y$ which are closest to the points in Y.

To specify the embedding, We first fix points 0, 1, and Y such that the distances between them are as stated above and they are embedded roughly around a straight line. Then, we fix the positions of X_0, X_1, and the centroid of the pentagon to ensure the values a, b, b', c, c', and ℓ satisfy the above relations. For each $i \in \{0, 1, 2, 3, 4\}$ and each two points $x \in X_i$ and $x' \in X_{i+1 \bmod 5}$, the angle α (resp. β) at the points $2i$, x, and x' (resp. $2i + 1$, x', and x) is less than 90 degrees. The angle at points y, j, and x ($y \in Y$, $\{i, j\} = \{0, 1\}$, $x \in X_i$) is more than 90 degrees. After fixing X_0, X_1, 0, and 1, we can determine the other points by simple calculations.

Now, we turn to odd d, i.e., $\mathsf{d} = 2\kappa + 1$. Instead of having ten points $\{0, \dots, 9\}$, we create five points that form a star with the pentagon. Consider an instance with $7\kappa + 5$ agents called W where 5κ agents are embedded to replace the five vertices of a pentagon with κ agents at each vertex of the pentagon. That is, each vertex of the pentagon is a cluster of points. note the five clusters of points by X_0, X_1, X_2, X_3, and X_4. There are five points $\{0, 1, 2, 3, 4\}$ that form a star with the pentagon, as in Example 2 (see Figure 4 (right)). Point i is at a distance b from X_i and c from $X_{i+1 \bmod 5}$, for each $i \in \{0, \dots, 4\}$ where $a < b < c < \ell$ and $b < 2a$.



(a) Gadget (right) for an element vertex u_i (left) s.t. element i belongs to sets S_j, S_k, S_r.

(b) Gadget (right) for a set-vertex w_j^i for which the set S_j consists of three elements i, p, q.

Figure 5 Element and set gadgets described in Subsection 4.1.

The remaining 2κ points are called Y. Together, $W := \bigcup_{i\in\{0,\ldots,4\}} X_i \cup \{0,1,2,3,4\} \cup Y$. All points in Y are embedded within an enclosing circle with radius close to zero. For each point y in Y, it holds that $b-\varepsilon \le \delta(0,y) < b$, and for each each point w in $W \setminus (\{0\}\cup Y)$ it holds that $\delta(y,w) > \ell$. Point 0 is the only point from $W \setminus Y$ which is closest to the points in Y. The remaining unmentioned points are at distance at least $b/2$ to the points Y. We specify the embeddings of the agents similarly to the one for even d.

Using a similar reasoning as to Example 2, we claim that the above embeddings are feasible.

Since the distance between each two points in X_i is close to zero, we assume it to be 0 for ease of reasoning. The following lemma summarizes the crucial effect of the star-gadget.

▶ **Lemma 10** ($\star$). *Every* d*-stable matching* Π *of the instance in Example 9 satisfy the following.*

- *If* d *is even, then* $\Pi(0) \cap Y \neq \emptyset$ *or* $\Pi(1) \cap Y \neq \emptyset$.
- *If* d *is odd, then* $\Pi(0) = Y \cup \{0\}$.

The remaining gadgets. Let $I = (X, \mathcal{S})$ be an instance of PC-X3C. Similarly to the case with $\mathsf{d} = 3$, we first embed the associated graph $G(I) = (U \uplus W, E)$ into a 2-dimensional grid with edges drawn using line segments of length at least $L \ge 200$, and with parallel lines at least $4L$ grid squares apart. The element- and the edge-gadget are almost the same as the ones describe in Subsection 3.1. The only difference is that we replace each element-agent u_i (for $u_i \in U$) with a size-($\mathsf{d}-2$) coalition U_i that are embedded so close to each other that any stable matching must match them together. Similarly, for each $z \in [\hat{n}]$ (recall that $\hat{n}$ is a constant as defined in the Subsection 3.1) and $w_j \in W$, we replace the two agents $\alpha_i^j[z]$ and $\beta_i^j[z]$ with a size-($\mathsf{d}-1$) coalition $\hat{A}_i^j[z]$ such that the distance between each pair of points in $\hat{A}_i^j[z]$ is close to zero, and define $A_i^j[z] := \hat{A}_i^j[z] \cup \{\gamma_i^j[z]\}$. For each set-vertex $w_j \in W$, assume that the three connecting edges in $G(I)$ are going rightward, leftward, and upward, connecting the element-vertices u_i, u_p, u_q, respectively. We create three set-agents, called w_j^i, w_j^p, w_j^q, and an additional coalition W_j of size $\mathsf{d}-3$ and as before, define $w_j^i = \gamma_j^i[\hat{n}]$. We embed them into $\mathbb{R}^2$ in such a way that w_j^i, w_j^p, w_j^q are on the segment of the rightward, leftward, and upward edge, respectively, and are of equidistance 17.5 to each other, and the coalition W_j is embedded in the center so that the distance between any two of them is close to zero. Moreover, the largest distance from any agent of W_j to any agent of $\{w_j^i, w_j^p, w_j^q\}$ is 10. See Figure 5 for an illustration.

We remark that by the construction of the set-gadget and the edge-gadget, each set-agent w_j^i prefers coalition $A_i^j[\hat{n}]$ (recall that $\gamma_i^j[z] = w_j^i$) to coalition $\{w_j^i, w_j^p, w_j^q\} \cup W_j$ since the sum of distances from w_j^i to the latter coalition is $17.5 + 17.5 + 10(\mathsf{d}-3) > (\mathsf{d}-1)\cdot(10-\varepsilon)$.

To ensure that one of the two coalitions is chosen, we make use of the star-gadgets from Example 9. Define $b := 22.6$ and $c := 22.7$. We create an agent-subset F^i_j of size $\mathsf{d}-1$ and agent h^i_j and a star-gadget W as described in Example 9, with Y being the extra $\mathsf{d}-1$ agents such that the most preferred coalition of each agent in Y is $Y \cup \{h^i_j\}$. Note that F^i_j has the same role as $\{f^i_j, g^i_j\}$ in the case for $\mathsf{d}=3$.

- The distance between each two agents in F^i_j is close to zero.
- The distance from each agent in F^i_j to each agent in $\hat{A}^j_i[\hat{n}]$ is in the range of $[10+\varepsilon, 10+2\varepsilon)$.
- The distance from each agent in F^i_j to agent w^i_j is $10 + \frac{15}{\mathsf{d}-1}$.
- The distance from each agent in F^i_j to agent h^i_j is $15 + 2\varepsilon$.
- The distance from agent h^i_j and each agent Y is $15 + 3\varepsilon$.
- The distance from each agent Y to 0 (and also to 1 if d is even) is $15 + 4\varepsilon$.

Finally, we create two types of garbage collector agents to match with some left over agents. For each added star gadget corresponding to S_j and $i \in S_j$, we create $O(\kappa)$ garbage collector agents R^i_j as follows: If d is odd, set $|R^i_j| := \mathsf{d} - \kappa - 2$. Otherwise if $\mathsf{d} \le 6$, set $|R^i_j| := 2\mathsf{d} - \kappa - 5$, and otherwise set $|R^i_j| := \mathsf{d} - \kappa - 5$. These agents have distance close to zero to each other. For each $y \in R^i_j$ it holds that $\ell < \delta(y,x) < 2\ell < \delta(y,x')$, where x is an agent from the same star and x' is an agent from neither R^i_j or the same star. It is straightforward to see that the distance between any two agents from different star-gadgets is larger than ℓ, and the distance from an agent in W to an agent to a set-gadget is at larger ℓ, where a, b, b', c', and ℓ are as defined in Example 9. Lastly, we add $m-n$ triples of additional garbage collector agents. The agents in each triple have distance close to zero to each other but is far away from the other agents. Note that each triple will be matched to some W_j whenever S_j is not chosen to the exact cover. See Figure 6 (for even d, without the garbage collector agents) for an illustration. This completes the description of the construction, which clearly can be done in polynomial time.

4.2 The correctness proof for $\mathsf{d} \ge 4$

The reasoning for the correctness is similar to the one for $\mathsf{d}=3$. For the forward direction, assume that $(X, \mathcal{S})$ admits an exact cover $\mathcal{K}$. Then, using a reasoning similar to the one for $\mathsf{d}=3$, one can verify that the following d-matching Π is stable; recall that $\kappa = \lfloor(\mathsf{d}-1)/2\rfloor$.

- For each $S_j \in \mathcal{K}$ with $S_j = \{i,p,q\}$ add $\{w^i_j, w^p_j, w^q_j\} \cup W_j$ to Π.
- For each element $i \in X$ let S_k, S_r be the two sets which contain i, but are not chosen in the exact cover $\mathcal{K}$. Add $U_i \cup \{u^k_i, u^r_i\}$ to Π. For each $S_j \notin \mathcal{K}$, take a triple of garbage collector agents (of the second type) and match them with W_j.
- For each element $i \in X$ and each set $S_j \in \mathcal{S}$ with $i \in S_j$, call the agents in the associated star-gadget along with the tail $X_0 \cup \dots \cup X_4 \cup \{0,1,2,3,4,h^i_j\} \cup Y \cup \{F^i_j\} \cup \{5,6,7,8,9 \mid \text{if } \mathsf{d} \text{ odd}\}$. If $S_j \in \mathcal{K}$, then add all $\hat{A}^i_j[z] \cup \{\gamma^i_j[z-1]\}$, $z \in [\hat{n}]$, to Π. Otherwise, add all $A^j_i[z]$, $z \in [\hat{n}]$, to Π. Add $F^i_j \cup \{h^i_j\}$ and $Y \cup \{0\}$ to Π. If d is odd, add $X_1 \cup X_2 \cup \{1\}$ and $X_3 \cup X_4 \cup \{3\}$ to Π. Otherwise, add $X_1 \cup X_2 \cup \{2,3\}$ and $X_3 \cup X_4 \cup \{6,7\}$ to Π. Next, if $\mathsf{d} \le 6$, then match X_0 with $\mathsf{d}-\kappa$ agents from $(1,8,9,4)$ (in this sequence) to Π. In any case, match the remaining star-agents with R^i_j.

The proof for the backward direction works analogously to $\mathsf{d}=3$ and is deferred to the full version [8].

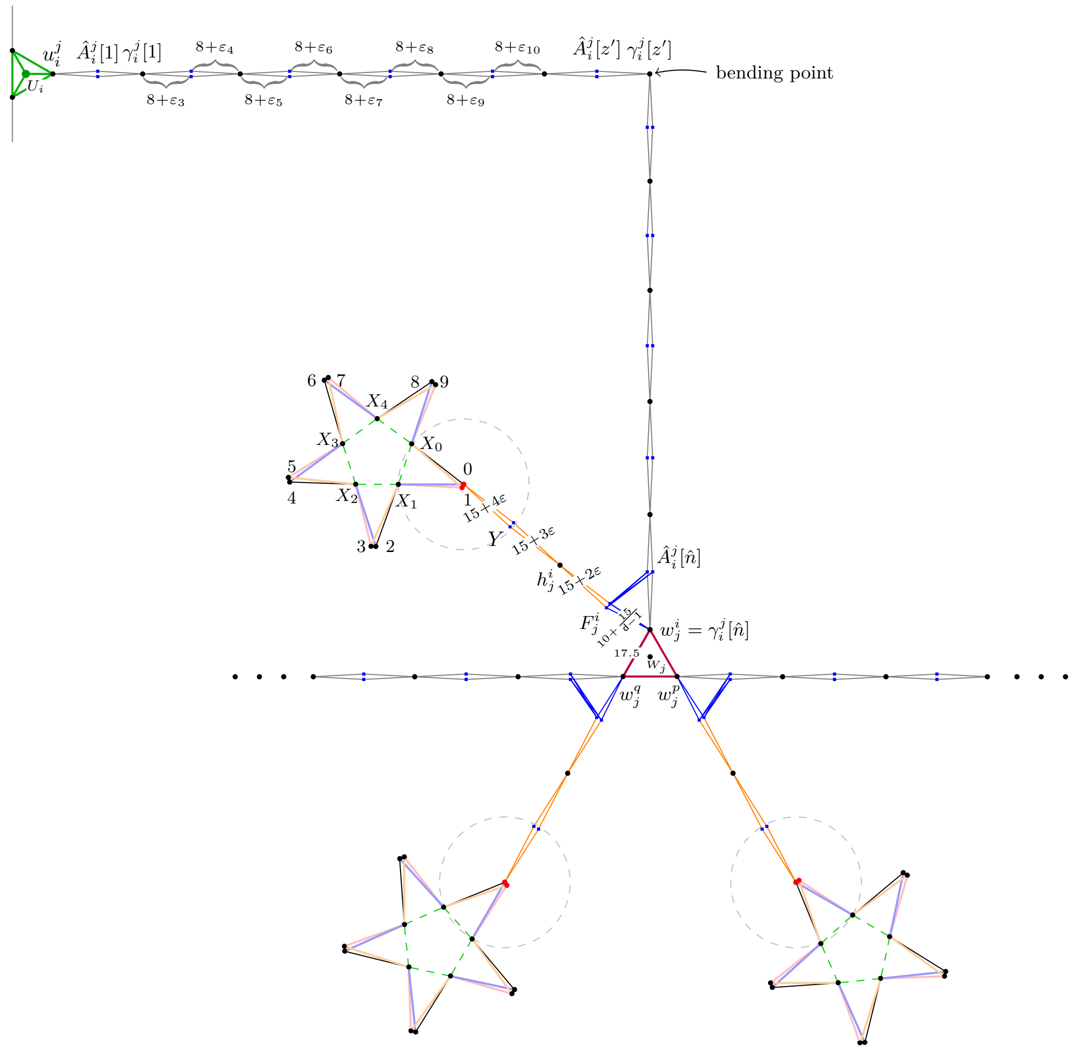


Figure 6 Gadget for edge $\{u_i, w_j\}$ in $G(I)$ with $S_j = \{i, p, q\}$ for the case when d is even, omitting the garbage collector agents for the sake of brevity.

5 Conclusion and Outlook

Establishing the first complexity results in the study of multi-dimensional stable matchings for Euclidean preferences, we show that d-SR remains NP-hard for Euclidean preferences and for all fixed $d \geq 3$. The gadgets in the reductions may be useful for other matching and hedonic games problems with Euclidean preferences.

Our Euclidean preference model assumes that the preferences over coalitions are based on the sum of distances to all individual agents in the coalition. It would be interesting to see whether taking the maximum or the minimum distance to the coalition members instead of the sum would change the complexity. Furthermore, it would be interesting to see whether restricting the agents' embedding to 1-dimensional Euclidean space could lower the complexity. We were not able to identify the complexity for this restricted variant, but conjecture that it can be solved in polynomial time. Note that in 1-dimensional Euclidean space, a 3-stable matching for the maximum distance setting always exists, which can be found by greedily finding three consecutive agents which are closest to each other and matching them.

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