Parameterized Complexity of Incomplete Connected Fair Division

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Abstract

Fair division of resources among competing agents is a fundamental problem in computational social choice and economic game theory. It has been intensively studied on various kinds of items (*divisible* and *indivisible*) and under various notions of *fairness*. We focus on CONNECTED FAIR DIVISION (CFD), the variant of fair division on graphs, where the *resources* are modeled as an *item graph*. Here, each agent has to be assigned a connected subgraph of the item graph, and each item has to be assigned to some agent.

We introduce a generalization of CFD, termed INCOMPLETE CFD (ICFD), where exactly p vertices of the item graph should be assigned to the agents. This might be useful, in particular when the allocations are intended to be "economical" as well as fair. We consider four well-known notions of fairness: PROP, EF, EF1, EFX. First, we prove that EF-ICFD, EF1-ICFD, and EFX-ICFD are W[1]-hard parameterized by p plus the number of agents, even for graphs having constant *vertex cover number* (vcn). In contrast, we present a randomized FPT algorithm for PROP-ICFD parameterized only by p. Additionally, we prove both positive and negative results concerning the kernelization complexity of ICFD under all four fairness notions, parameterized by p, vcn, and the total number of different valuations in the item graph (val).

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1 Introduction

Allocating indivisible goods among competing agents in a "fair" manner is a fundamental research problem in computational social choice [10]. Classically, this resource allocation problem is referred to as FAIR DIVISION, and it has been intensively studied in the literature under various notions of *fairness* and *efficiency* [3, 4, 7, 11, 14, 16, 17, 29].

It might be desirable, sometimes, that the allocations respect some connectivity measure. For example, while allocating offices to various research teams in a university campus, it might be desirable to provide adjoining offices to the members of the same team. Similarly, while dividing farmlands among heirs, it might be desirable to provide a connected piece to each individual. Another example can be when the items are people and the graph is a social network, and we want each team to consist of members that know each other (not necessarily directly). Thus, Bouveret et al. [9] introduced CONNECTED FAIR DIVISION (CFD), an adaptation of FAIR DIVISION to graphs. In this problem, we are given (1) a set of items represented as vertices of an input graph G, (2) a set of agents A, and (3) a set $\mathcal{U}$ of utility

Table 1 Summary of our results.

Fairness	$p + \mathsf{vcn} + \lvert A\rvert$	$p + \mathsf{vcn} + \lvert A\rvert + \mathsf{val}$	
	Parameterization	**Exponential Kernel**	**No-Poly Kernel**
EF	**W[1]-hard [Theorem 1]**	**[Theorem 3]**	**[Theorem 4]**
EF1	**W[1]-hard [Theorem 2]**	**[Theorem 3]**	**[Theorem 4]**
EFX	**W[1]-hard [Theorem 2]**	**[Theorem 3]**	**[Theorem 4]**
PROP	**FPT by p [Theorem 5]**	**[Theorem 3]**	**[Theorem 4]**

functions specifying the utility of each item (vertex) for each agent. The goal is to decide whether there exists an allocation of vertices to the agents which is (i) *fair* under some notion of fairness, and (ii) *connected* (i.e., for each agent, the subgraph induced by the set of vertices assigned to the agent is connected). Fixing a fairness notion φ, CFD is referred to as φ-CFD. Note that Fair Division is the special case of CFD restricted on complete graphs. Since its introduction, CFD has received a significant amount of interest [3, 6, 20, 28].

We introduce an "economic" generalization of CFD, termed Incomplete CFD (ICFD), where we have to allocate exactly p vertices of G to the agents (i.e., $|V(G)| - p$ vertices remain unassigned), and where each agent must be assigned at least one vertex. Note that CFD is the special case of ICFD when $p = |V(G)|$. Leaving some items unassigned has recently gained significant attention in context of classical Fair Division [5, 8, 15, 18]. Here, the motivation is to attain *fairness* by donating some items to *charity*. We note that the variation of ICFD that asks to allocate at most (rather than exactly) k vertices can be modeled using ICFD (by repeating for $p = 1$ to k).

Our Contribution. In this paper, we study the parameterized complexity of ICFD considering four well-known notions of fairness: PROP, EF, EF1, and EFX (defined in Section 2). An overview of our results is given in Table 1. We remark that parameterized analysis of fair allocations is an extensively studied research subject [7, 8, 20, 24, 21]. We first prove that EF-ICFD remains W[1]-hard when parameterized by $\mathsf{vcn} + p + |A|$, where vcn is the *vertex cover number* of the item graph and $|A|$ is the total number of agents. To this end, we provide a simple parameterized reduction from the (k, M)-SUM problem (parameterized by k) to EF-ICFD on a star graph parameterized by $p + |A|$. Moreover, we have $|A| = 2$, and both agents have *identical valuation* in the reduced instance.

▶ **Theorem 1.** EF-ICFD *is $W[1]$-hard parameterized by $p + |A|$ even for star graphs.*

Next, we extend this result to the fairness notions EF1 and EFX. Specifically, we show that EF1-ICFD and EFX-ICFD are W[1]-hard when parameterized by $\mathsf{vcn} + p + |A|$, even on graphs having $\mathsf{vcn} = 2$. To this end, we provide a non-trivial parameterized reduction from (k, M)-SUM problem (parameterized by k) to EF1-ICFD (resp., EFX-ICFD) parameterized by $p + |A|$ on a graph having $\mathsf{vcn} = 2$ (and $|A| = 3$).

▶ **Theorem 2.** EF1-ICFD *and* EFX-ICFD *are $W[1]$-hard parameterized by $p + |A| + \mathsf{vcn}$ even for graphs having $\mathsf{vcn} = 2$.*

Observe that φ-ICFD, for $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, is trivially in the class XP when parameterized by p alone. Since φ-ICFD, for $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, is W[1]-hard when parameterized by $p + \mathsf{vcn} + |A|$, we include another parameter in quest of tractability. Let val be the total number of distinct valuations that agents assign to items (val is the range of the function

$\mathcal{U}$). We note that val is used to achieve tractability of CFD by Deligkas et al. [20]. We establish that φ-ICFD, for $\varphi \in \{\mathsf{PROP}, \mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, is FPT parameterized by $\mathsf{vcn} + \mathsf{val} + p$ by designing exponential kernels.

▶ **Theorem 3.** *For $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-ICFD admits a kernel with at most $p2^{\mathsf{vcn}}\mathsf{val}^{\mathsf{val}} + \mathsf{vcn}$ vertices. Moreover, PROP-ICFD admits a kernel with at most $p2^{\mathsf{vcn}}\mathsf{val}^{\mathsf{val}} + \mathsf{vcn} + p$ vertices.*

Next, we complement our exponential kernels by showing that it is unlikely for φ-ICFD, for $\varphi \in \{\mathsf{PROP}, \mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, parameterized by $\mathsf{vcn} + \mathsf{val} + |A| + p$ to admit a polynomial compression. To this end, we provide *polynomial parameter transformations* from RED-BLUE DOMINATING SET.

▶ **Theorem 4.** *For $\varphi \in \{\mathsf{PROP}, \mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-ICFD parameterized by $\mathsf{vcn} + \mathsf{val} + |A| + p$ does not admit polynomial compression, unless NP $\subseteq$ coNP/poly.*

Finally, we establish that PROP-ICFD is FPT when parameterized by p alone. To this end, we provide a color-coding based randomized algorithm with constant success probability[1].

▶ **Theorem 5.** *There exists a randomized algorithm that solves PROP-ICFD, in time $e^p p^{\mathcal{O}(p \log p)} m^{\mathcal{O}(1)}$ with success probability at least $1 - \frac{1}{e}$.*

The choice of vcn as a structural parameter is quite natural for our results. Arguably, vcn is the best structural parameter for providing negative results – indeed, our negative results on parameterized complexity and kernelization complexity of ICFD parameterized by vcn (plus other relevant parameters) imply the same for other smaller parameters such as treewidth, cliquewidth, treedepth, and feedback vertex set (plus other relevant parameters) [19]. Furthermore, vcn is one of the most efficiently computable parameters from both approximation [34] and parameterized [19] points of view, making it fit from an applicative perspective even when a vertex cover is not given along with the input.

Brief Survey. The need for fairly dividing resources among competing agents is one of the oldest problems in human civilization. The famous cut-and-choose method to remove envy between two agents dates back to the Book of Genesis. Fairly dividing a divisible item among agents is a very old and classical problem, also referred to as *cake-cutting*, and is studied extensively in the literature [13, 33]. When the items are indivisible, the problem is well studied under the name FAIR DIVISION [3, 4, 7, 11, 12, 14, 16, 17, 29]. The study of CFD was initiated by Bouveret et al. [9]. Later, their work was extended to include the notion of *chores* as well by Aziz et al. [3]. Various notions of fairness like maximin share allocation (Greco and Scarcello [27]) and Pareto-optimal allocations (Igarashi and Peters [28]) are also studied. Bilò et al. [6] studied the conditions that guarantee various notions of fair allocations. Recently, Deligkas et al. [20] provided a comprehensive picture of the parameterized complexity of CFD.

Motivation. A wide range of considerations motivates our definition of ICFD. First, it is often the case that the allocator wants to save some resources for later. For instance, consider the example of a university campus where the professors and research groups are first allocated offices while saving some rooms for various administrative purposes and classrooms, or for future professors in case the university is expanding. In these settings, the requirement of connectivity might be desirable.

[1] Clearly, repetition allows to boost the success probability to any constant.

Second, partial allocations are specifically useful when the allocator wants to save some resources for the future. For instance, it is practical for a university department to not allocate all of its resources (like computer equipment and travel funding) in one round of allocations and save some for future use. The notion of connectivity can be introduced based on the graph of devices that work well with each other. A similar setting arises while allocating ground to companies/people to build buildings while saving some ground for the future. Here, the connectivity requirement comes naturally.

Third, it saves resources from the viewpoint of the allocators: using this framework, they can assign the least amount of resources that makes the agents "happy". This is best exemplified when the allocator actually needs to buy the items. For example, when a company actually needs to rent offices in an office space or has to buy some equipment for its employees.

For another example, consider a park where different groups want to organize picnics on a specific day, and the park committee has to allocate these picnic spots. Allocating a connected spot to each group makes good sense here. Further, each day, the committee receives multiple applications with the preferences of each group. Now, it may be impractical to delay the allocation process till the very last day. So, it may be a good idea to allocate these spots in phases such that in each phase, the applicants are provided the spots "fairly" while saving a sufficient amount of spots for further phases.

Due to the connectivity requirement, it might so happen that there are some "problematic" vertices, which, when assigned to one of the agents, may deem the allocation "unfair" in CFD. For example, consider a vertex v such that ℓ degree-one vertices are attached to it. Moreover, let n be the number of agents, and ℓ is much larger than n. In this case, since each vertex is to be assigned to some agent in CFD, at least $\ell - n$ of these degree-one vertices must be assigned to the same agent that is assigned the vertex v, possibly making other agents "envious" of this agent. In these scenarios, it is a practical (and desirable) question to seek a fair allocation by leaving some vertices unassigned.

2 Preliminaries

For $\ell \in \mathbb{N}$, let $[\ell] = \{1, \ldots, \ell\}$ and $[0, \ell] = \{0\} \cup [\ell]$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Due to space constraints, the proofs of the statements marked with $(*)$, or for which we only give a proof sketch, can be found in the full version [26].

Graph Theory. We consider finite, simple, and connected graphs. For a graph G, let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For a vertex $v \in V(G)$, let $N(v)$ denote the *open neighborhood* of v, that is, $N(v) = \{u \mid uv \in E(G)\}$. For a subset $X \subseteq V(G)$, let $N_X(v) = N(v) \cap X$. Moreover, let $G[X]$ denote the subgraph of G induced by vertices in X, and let $G - X = G[V(G) \setminus X]$ of G. A set $U \subseteq V(G)$ is a *vertex cover* of G if for every edge in G, at least one of its endpoints is in U. The minimum cardinality of a vertex cover of G is its *vertex cover number* (vcn). Given a graph G, let $d_G(v) = |N(v)|$. For standard graph theoretic terminology not defined explicitly in this paper, we refer to the book by Diestel [22].

Incomplete Connected Fair Division. An instance of Incomplete Connected Fair Division (ICFD) consists of $(G, A, \mathcal{U}, p)$ where G is the *utility graph* on m vertices, $A = [n]$ is the set of n *agents*, $\mathcal{U}$ is the set of *utility functions* $\{u_i : V(G) \to \mathbb{N}_0\}_{i \in [n]}$, and $p \in [m]$. When clear from context, we denote $|V(G)|$ by m and $|A|$ by n. Every vertex $v \in V(G)$

corresponds to an *item*. It is a standard assumption in the literature, and we assume it too, that the agents have *additive valuations*, i.e., for every $X \subseteq V(G)$ and for every $i \in A$, we have $u_i(X) = \sum_{v \in X} u_i(v)$. An *allocation* of items is a tuple $\Pi = (\pi_1, \dots, \pi_n)$ such that:

1. For $i \in A$: $\pi_i \subseteq V(G)$, $|\pi_i| \geq 1$, and $G[\pi_i]$ is connected.
2. For $i, j \in A$ such that $i \neq j$: $\pi_i \cap \pi_j = \emptyset$.
3. $|\bigcup_{i \in A} \pi_i| = p$.

We say that the *bundle* π_i is *assigned* to agent i in Π. For a bundle π_i, let $\tau_i = \{v \in \pi_i \mid G[\pi_i \setminus \{v\}]$ is connected$\}$. Next, we have the following notions of *fairness* (that we consider in this paper). An allocation $\Pi = (\pi_1, \dots, \pi_n)$ is:

- *proportional* (PROP) if for every $i \in A$, $u_i(\pi_i) \geq \frac{u_i(V(G))}{n}$;
- *envy-free* (EF) if for every $i, j \in A$, $u_i(\pi_i) \geq u_i(\pi_j)$;
- *envy-free up to one item* (EF1) if for every $i, j \in A$, $u_i(\pi_i) \geq u_i(\pi_j) - \max_{v \in \tau_j} u_i(v)$;
- *envy-free up to any item* (EFX) if for every $i, j \in A$, $u_i(\pi_i) \geq u_i(\pi_j) - \min_{v \in \tau_j} u_i(v)$.

For a fairness criterion $\varphi \in \{\mathsf{PROP}, \mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, $\varphi-$ICFD asks whether there exists an allocation Π for $(G, A, \mathcal{U}, p)$ satisfying φ. An allocation Π that satisfies φ is termed as *φ-allocation.* Agents i and j have *same type* if for every $v \in V(G)$, $u_i(v) = u_j(v)$. Let $\mathcal{A}$ denote the set of all agent types, i.e., $\mathcal{A}$ is the partition of A such that agents i and j are in the same part $\mathsf{a} \in \mathcal{A}$ if and only if i and j have the same type. The type of an agent i is the type $\mathsf{a} \in \mathcal{A}$ such that $i \in \mathsf{a}$; let a_i denote the type of agent i. Since the agents of each type have the same valuation function, we let u_{a} denote the valuation function u_i where $i \in \mathsf{a}$. When $|\mathcal{A}| = 1$, we say that the agents have *identical valuations* and for each vertex v and agent i, we also denote $u_i(v)$ by $u(v)$ for brevity. We say that an instance $(G, A, \mathcal{U}, p)$ admits *unary valuation* (resp., *binary valuation*) if the valuations (in the valuation function $\mathcal{U}$) are encoded in unary (resp., binary), i.e., there is some polynomial (resp., exponential) function f such that $\max_{i \in A, v \in V(G)} u_i(v) \leq f(|V(G)|)$.

Parameterized Complexity. In parameterized complexity, each instance of a problem Π is associated with a non-negative integer *parameter* k. A parametrized problem Π_p is *fixed-parameter tractable* (FPT) if there is an algorithm that, given an instance (I, k) of Π_p, solves it in time $f(k) \cdot |I|^{\mathcal{O}(1)}$, for some computable function f. Central to parameterized complexity is the following hierarchy of complexity classes: $\mathsf{FPT} \subseteq \mathsf{W[1]} \subseteq \mathsf{W[2]} \subseteq \ldots \subseteq \mathsf{XP}$. We note that $\mathsf{FPT} \neq \mathrm{W[1]}$ under ETH.

Two instances I and I' (possibly of different problems) are *equivalent* when I is a Yes-instance if and only if I' is a Yes-instance. A *compression* of a parameterized problem Π_1 into a (possibly non-parameterized) problem Π_2 is a polynomial-time algorithm that maps each instance (I, k) of Π_1 to an equivalent instance I' of Π_2 such that size of I' is bounded by $g(k)$, for a computable function $g(\cdot)$. If $g(\cdot)$ is polynomial, then the problem is said to admit a *polynomial compression.* When $\Pi_1 = \Pi_2$, compression is also termed as *kernelization.* A *polynomial parameter transformation* from Π_1 to Π_2 is a polynomial time algorithm that given an instance (I, k) of Π_1 generates an equivalent instance (I', k') of Π_2 such that $k' \leq p(k)$, for some polynomial $p(\cdot)$. Here, if Π_1 does not admit a polynomial compression, then Π_2 does not admit a polynomial compression [19]. We refer to the books by Cygan et al. [19] and Fomin et al. [25] for more details on parameterized complexity.

Preliminary Results and Observations. Consider an instance $(G, A, \mathcal{U}, p)$ of ICFD. For each $i \in A$, we assume that $u_i(V(G)) > 0$. Moreover, since each agent must be assigned at least one vertex, we assume that $|A| > p$. We have the following basic observations and lemmas.

▶ **Observation 6** (∗). *Let Π be an allocation for an instance $(G, A, \mathcal{U}, p)$ of* EF-ICFD *with identical valuations (i.e., $|\mathcal{A}| = 1$). Then, Π admits* EF *if and only if for every $i, j \in A$, $u(\pi_i) = u(\pi_j)$.*

▶ **Lemma 7** (∗). *Consider an instance $(G, A, \mathcal{U}, p)$ of φ-*ICFD *where $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$. Let H be an induced subgraph of G. If $(H, A, \mathcal{U}, p)$ is a Yes-instance of φ-*ICFD*, then $(G, A, \mathcal{U}, p)$ is a Yes-instance of φ-*ICFD.

We remark that for an induced subgraph H of G, it might happen that $(H, A, \mathcal{U}, p)$ is a Yes-instance of PROP-ICFD but $(G, A, \mathcal{U}, p)$ is a No-instance of PROP-ICFD. To see this, consider a Yes-instance $(H, A, \mathcal{U}, p)$ of PROP-ICFD such that $H = G - \{v\}$ and $|A| > 1$. Now, for each agent $i \in A$, let $u_i(v) = (n + 1) \sum_{x \in V(H)} u_i(x)$. Now, consider the instance $(G, A, \mathcal{U}, p)$. Since $|A| > 1$, there is at least one agent, say, j, such that the vertex v is not assigned to the agent j. Observe that even if $\pi_j = V(G) \setminus \{v\}$, $u_j(\pi_j) < \frac{u_j(V(G))}{n}$ (assuming $u_j(V(H)) > 0$). Hence, $(G, A, \mathcal{U}, p)$ is a No-instance. Next, we have the following observation concerning PROP-ICFD.

▶ **Observation 8** (∗). *Let $p < p' \leq |V(G)|$. Moreover, let $(G, A, \mathcal{U}, p)$ be a Yes-instance of* PROP-ICFD*. Then, $(G, A, \mathcal{U}, p')$ is a Yes-instance of* PROP-ICFD *as well.*

The following observation follows directly from the definition of EF1-ICFD and EFX-ICFD.

▶ **Observation 9.** *Consider an allocation $\Pi = (\pi_1, \ldots, \pi_n)$ for an instance $(G, A, \mathcal{U}, p)$. If there are two agents i and j such that for each vertex $v \in \pi_j$, $u_i(\pi_i) < u_i(\pi_j) - u_i(v)$, then Π is neither an* EF1 *allocation nor an* EFX *allocation.*

3 W[1]-hardness Results

In this section, we show that φ-ICFD where $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$ is W[1]-hard when parameterized by $p + \mathsf{vcn} + |A|$. For this purpose, we first define the following problem. The (k, M)-SUM problem is to determine, given N integers $a_1, \ldots, a_N \in [0, M]$ and a target integer $t \in [0, M]$, whether there exists $S \subseteq N$ such that $|S| = k$ and $\sum_{i \in S} a_i = t$. We also assume that $2 \leq k \leq N - 2$ (as otherwise, we can solve the problem in polynomial time). Abboud et al. [1] proved the following:

▶ **Proposition 10** ([1]). *(k, N^{2k})-SUM parameterized by k is $W[1]$-complete.*

3.1 EF-ICFD

First, we provide a parameterized reduction from (k, M)-SUM to EF-ICFD on a star graph such that $p = k + 2$ and $|A| = 2$. We begin by explaining our construction.

Construction. Let $(a_1, \ldots, a_N, t)$ be an instance of (k, M)-SUM. We create an instance $(G, A, \mathcal{U}, p)$ of EF-ICFD in the following manner. Let G be a star on $N + 2$ vertices such that $V(G) = \{v_1, \ldots, v_N\} \cup \{d_1, d_2\}$ and $E(G) = \{d_1 d_2\} \cup \{d_1 v_i \mid i \in [N]\}$ (i.e., G is a star with $N + 1$ leaves and d_1 as the center vertex). Let $A = \{1, 2\}$. Moreover, for $j \in A$ and $i \in [N]$, let $u_j(v_i) = a_i$, $u_j(d_1) = 1 + \sum a_i$, and $u_j(d_2) = u_j(d_1) + t$ (here, $|\mathcal{A}| = 1$). Finally, we set $p = k + 2$.

Now, we have the following lemma.

▶ **Lemma 11.** *$(a_1, \ldots, a_N, t)$ is a Yes-instance of (k, M)-SUM if and only if $(G, A, \mathcal{U}, p)$ is a Yes-instance of* EF-ICFD.

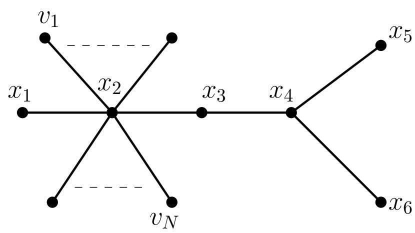

Figure 1 Illustration for the construction of graph G used in the proof of Theorem 2.

Proof Sketch. In one direction, let $S \subseteq [N]$ is a set such that $|S| = k$ and $\sum_{i \in S} a_i = t$. Then, observe that the allocation $\Pi = (\pi_1, \pi_2)$ as $\pi_1 = \{d_1\} \cup \{v_i \mid i \in S\}$ and $\pi_2 = d_2$ satisfies EF.

In the other direction, suppose $(G, A, \mathcal{U}, p)$ is a Yes-instance of EF-ICFD and $\Pi = (\pi_1, \pi_2)$ is an allocation for $(G, A, \mathcal{U}, p)$ satisfying EF. Since G is a star graph and $p \geq 4$ (as $p = k+2$ and $k \geq 2$), at least one of the agents, wlog let π_1, is assigned at least two vertices, one of which is vertex d_1. Now, $u(\pi_1) \geq 1 + \sum_{i \in [N]} u(v_i)$. Observe that the vertex $d_2 \in \pi_2$, as otherwise $u(\pi_2) \leq \sum_{i \in [N]} u(v_i) < u(\pi_1)$, contradicting that Π satisfies EF (due to Observation 6). Since $d_1 \in \pi_1$, no other vertex can be assigned to π_2 while keeping $G[\pi_2]$ connected. Hence, $\pi_2 = \{d_2\}$ and $u(\pi_2) = u(d_1) + t$. Since $|\pi_2| = 1$, note that $|\pi_1| = p - 1 = k + 1$. Hence, exactly k vertices are assigned to π_i other than d_1. Moreover, since Π satisfies EF, due to Observation 6, we have $u(\pi_1) = u(\pi_2)$. Therefore, there is a set $S \subseteq [N]$ such that $|S| = k$ and $\sum_{i \in S} u(v_i) = u(\pi_2) - u(d_1) = t$. This implies that $\sum_{i \in S} a_i = t$ (since $a_i = u(v_i)$). Hence $(a_1, \ldots, a_N, t)$ is a Yes-instance. ◀

We have the following theorem due to the above construction, Proposition 10 and Lemma 11.

▶ **Theorem 1.** EF-ICFD *is $W[1]$-hard parameterized by $p + |A|$ even for star graphs.*

3.2 EF1-ICFD and EFX-ICFD

Next, we prove that EF1-ICFD and EFX-ICFD are W[1]-hard when parameterized by $p +$ vcn $+ |A|$. To this end, we provide a parameterized reduction from (k, M)-SUM to EF1-ICFD (resp.,EFX-ICFD) on a graph with vcn $= 2$, while $|A| = 3$, and $p = k + 6$. First, we explain our construction.

Construction. Let $(a_1, \ldots, a_N, t)$ be an instance of (k, M)-SUM. First, we will define how to construct the graph G as it is the same for both EF1-ICFD and EFX-ICFD. Consider a path on five vertices $x_1, \ldots, x_5$ and attach a vertex x_6 to x_4. Now, we add N vertices $v_1, \ldots, v_N$ and attach them to x_2 by an edge. More formally, let $V(G) = \{x_1, \ldots, x_6, v_1, \ldots, v_N\}$ and $E(G) = \{x_2 v_i \mid i \in [N]\} \cup \{x_4 x_6\} \cup \{x_i x_{i+1} \mid 1 \leq i \leq 4\}$. See Figure 1 for a reference. Next, we have $A = [3]$ and $p = k + 6$. Finally, we define the valuation functions $\mathcal{U}$ and $\mathcal{U}'$ for EF1-ICFD and EFX-ICFD, respectively. These functions will be identical except for the valuation $u_3(x_1)$ and $u'_3(x_1)$. Moreover, let $C = \sum_{i \in [N]} a_i$. The valuation functions are defined as follows:

- $u_i(v_j) = C + a_i$ for $i \in [3]$ and $j \in [N]$,
- $u_i(x_2) = 2NC$ for $i \in [3]$,
- $u_1(x_3) = u_2(x_3) = 0$ and $u_3(x_3) = 2NC + kC + t$,
- $u_1(x_4) = u_2(x_4) = 3NC + kC + t$ and $u_3(x_4) = 0$,

- $u_i(x_5) = 0$ for $i \in [3]$,
- $u_i(x_6) = 0$ for $i \in [3]$,
- $u_i(x_1) = NC$ for $i \in [3]$. Finally, $u'_3(x_1) = 0$.

First, we have the following straightforward lemma that proves one side of our reduction.

▶ **Lemma 12.** *If $(a_1, \ldots, a_N, t)$ is a Yes-instance of (k, M)-SUM, then $(G, A, \mathcal{U}, p)$ and $(G, A, \mathcal{U}', p)$ are Yes-instances of* EF1-ICFD *and* EFX-ICFD*, respectively.*

Proof Sketch. Let $S \subseteq \{a_1, \ldots a_N\}$ be a set such that $\sum_{a \in S} a = t$ and $|S| = k$. Then, consider the allocation $\Pi = (\pi_1, \pi_2, \pi_3)$ such that $\pi_1 = S \cup \{x_1, x_2\}$, $\pi_2 = \{x_4, x_5, x_6\}$, and $\pi_3 = \{x_3\}$. It is easy to see that $|\pi_1| + |\pi_2| + |\pi_3| = |S| + 6 = k + 6 = p$, and to verify that Π is an EF1-ICFD and EFX-ICFD for $(G, A, \mathcal{U}, p)$ and $(G, A, \mathcal{U}', p)$, respectively. ◀

Some Useful Observations. Next, we prove some observations that will be useful for proving the other direction of our reduction. For the observations where we do not use any of $u_3(x_1)$ and $u'_3(x_1)$, we will use only the valuation functions $\mathcal{U}$ to ease the presentation. Moreover, we remark that these observations are valid for both EF1-ICFD (assuming valuations $\mathcal{U}$) and EFX-ICFD (assuming valuations $\mathcal{U}'$). First, observe that agents 1 and 2 have identical valuations, $\{x_2, x_4\}$ is a vertex cover of G and hence $G[V(G) \setminus \{x_2, x_4\}]$ is an independent set.. Moreover, since $k \geq 2$, note that $p \geq 8$.

For the rest of this section, let $\Pi = (\pi_1, \pi_2, \pi_3)$ be an allocation that is an EF1-ICFD for $(G, A, \mathcal{U}, p)$ (resp., EFX-ICFD for $(G, A, \mathcal{U}', p)$). First, we have the following easy observation.

▶ **Observation 13 (∗).** *If $x_2 \in \pi_i$, for $i \in [3]$, then $x_2 \notin \tau_i$.*

Next, we have the following observation.

▶ **Observation 14 (∗).** *$x_2 \in \pi_1 \cup \pi_2$.*

Next, we have the following crucial observation.

▶ **Observation 15 (∗).** *$\pi_3 = \{x_3\}$ and $\pi_2 = \{x_4, x_5, x_6\}$.*

Finally, we have our main lemma, which implies the other side of the reduction.

▶ **Lemma 16 (∗).** *If $(G, A, \mathcal{U}, p)$ (resp., $(G, A, \mathcal{U}', p)$) is a Yes-instance of* EF1-ICFD *(resp.,* EFX-ICFD*), then $(a_1, \ldots, a_N, t)$ is a Yes-instance of (k, M)-SUM.*

The following theorem is a consequence of our construction, Proposition 10, and Lemmas 12 and 16.

▶ **Theorem 2.** EF1-ICFD *and* EFX-ICFD *are $W[1]$-hard parameterized by $p + |A| +$ vcn even for graphs having* vcn $= 2$.

Theorems 1 and 2 establish that, for $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-ICFD is W[1]-hard parameterized by $p + |A| + t$ when the valuations are encoded in binary. We remark that these results cannot be directly generalized to unary valuations as (k, M)-SUM is in P when M is encoded in unary.

4 Kernelization by $\mathsf{vcn} + p + \mathsf{val}$

Let $(G, A, \mathcal{U}, p)$ be an instance of ICFD. We consider the parameterization of ICFD parameterized by $\mathsf{vcn} + p + \mathsf{val}$. Recall that vcn is the vertex cover of the input graph G, $\mathcal{A}$ is the set of all agent types, and $\mathsf{val} = |\{u_i(v) \mid i \in [n], v \in V(G)\}|$. Moreover, let $\mathsf{VAL} = \{u_i(v) \mid i \in [n], v \in V(G)\}$.

Let U be a vertex cover of size t. If no such vertex cover is given, then we can compute a vertex cover U of size $t \leq 2\mathsf{vcn}$ using a polynomial-time approximation algorithm [34]. Let I be the independent set $V(G) \setminus U$. Our kernelization algorithm uses techniques similar to the ones used by Deligkas et al. [20] to provide an FPT algorithm for CFD parameterized by $\mathsf{vcn} + |\mathcal{A}| + \mathsf{val}$. We partition the vertices of I into *equivalence classes* such that for any two vertices u and v of the same equivalence class, $N_U(u) = N_U(v)$ and $u_i(v) = u_i(u)$ for each $i \in A$. Since vertices of each equivalence class are "indistinguishable" for the agents, we keep only at most p vertices from each equivalence class. We then establish that we can have at most $2^t \mathsf{val}^{|\mathcal{A}|}$ many equivalence classes, thus giving us a kernel with at most $2^t \mathsf{val}^{|\mathcal{A}|} p + t$ vertices. Below, we discuss these ideas formally.

Kernel for EF-ICFD, EF1-ICFD, and EFX-ICFD. We have the following reduction rule.

▶ **Reduction Rule 1** (RR1). *Let $(G, A, \mathcal{U}, p)$ be an instance of φ-ICFD where $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$. Let $S \subseteq I$ be a set of vertices such that $|S| > p$ and for any two vertices $u, v \in S$, $N_U(u) = N_U(v)$ and for each agent $i \in A$, $u_i(u) = u_i(v)$. Moreover, let $v_1, \dots v_{|S|}$ be an ordering of vertices in S. Then, let $H \Leftarrow G - \{v_{p+1}, \dots, v_{|S|}\}$.*

We have the following lemmas to prove RR1 is safe and to bound the size of our kernel.

▶ **Lemma 17** (∗). *RR1 is safe.*

▶ **Lemma 18** (∗). *Let $(G, A, \mathcal{U}, p)$ be an instance of φ-ICFD such that RR1 cannot be applied to $(G, A, \mathcal{U}, p)$. Then, $|V(G)| \leq 2^t \cdot val^{|\mathcal{A}|} \cdot p + t$.*

Since $|\mathcal{A}| \leq \mathsf{val}$, RR1, along with Lemma 17 and Lemma 18, imply that for $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-ICFD admits a kernel with at most $p2^{\mathsf{vcn}}\mathsf{val}^{\mathsf{val}} + \mathsf{vcn}$ vertices. This gives us the desired kernel for φ-ICFD, for $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, from Theorem 3.

Kernel for PROP-ICFD. Next, we design a kernel for PROP-ICFD. First, we give an overview of the ideas leading to this kernel. As remarked in Section 2, since it might so happen that PROP-ICFD exists for a subgraph H of G but not for G, we cannot simply delete vertices as before (which might decrease the overall valuation of the graph for some agents) to get our kernel. So, we have to keep track of the "valuation that is lost" for each agent while deleting the vertices. In this quest, we augment the graph with some dummy vertices and assign them the valuation for each deleted vertex. Hence, even if we start with a unary (resp., binary) valuation instance, we might end up with an instance that does not admit unary (resp., binary) valuations. We tackle this problem by establishing that if the valuation for a dummy vertex gets "too large", then we are dealing with a No-instance. This ensures that we have a kernel that respects the unary (resp., binary) valuation if the original instance respects unary (resp., binary) valuation. Now, we discuss the details of our kernelization algorithm.

Let $(G, A, \mathcal{U}, p)$ be an instance of PROP-ICFD. First, we use the following preprocessing step to generate an augmented instance $(G', A', \mathcal{U}', p')$ in the following manner.

Preprocessing Step: We get $(G', A', \mathcal{U}', p')$ by adding n dummy agents, n dummy vertices, and setting $p' = p + n$. More formally:

1. Let $V(G') = V(G) \cup \{d_1, \ldots, d_n\}$. Fix a vertex $w \in U$. Now, let $E(G) = E(G) \cup \{d_i d_j \mid i, j \in A\} \cup \{d_1 w\}$. Moreover, let $A' = A \cup \{n+1, \ldots, 2n\}$ and $p' = p + n$.
2. For $i \in [n]$ and $v \in V(G)$, let $u'_i(v) = 2u_i(v)$. Moreover, for $i \in [n]$ and $v \in \{d_1, \ldots, d_n\}$, let $u'_i(v) = 0$ and $u'_{n+i}(d_i) = 1$. Furthermore, for $n < i \leq 2n$ and $v \notin \{d_1, \ldots, d_n\}$, let $u'_i(v) = 0$.

Now, we have the following lemma.

▶ **Lemma 19** $(*)$. *$(G, A, \mathcal{U}, p)$ is a Yes-instance of* PROP-ICFD *if and only if $(G', A', \mathcal{U}', p')$ is a Yes-instance of* PROP-ICFD.

Let $D = \{d_1, \ldots, d_n\}$ be the set of dummy vertices added in the Preprocessing Step. Now, we have the following reduction rule.

▶ **Reduction Rule 2** (RR2). *Let $(G', A', \mathcal{U}', p')$ be an instance of* PROP-ICFD *(obtained from an initial instance $(G, A, \mathcal{U}, p)$ after applying Preprocessing Step and possibly RR2). Let $S \subseteq I$ be a set of vertices such that $|S| > p$ and for any two vertices $u, v \in S$, $N_U(u) = N_U(v)$ and for each agent $i \in A$, $u'_i(u) = u'_i(v)$. Moreover, let $v_1, \ldots v_{|S|}$ be an ordering of vertices in S. Then, let $G'' \Leftarrow G' - \{v_{p+1}, \ldots, v_{|S|}\}$. Moreover, we modify $\mathcal{U}'$ to get $\mathcal{U}''$ in the following manner. For $i \in A'$ and $v \in V(G'')$, $u''_i(v) \Leftarrow u'_i(v)$ and, finally, for $i \in A$ ($i \in [n]$), $u''_i(d_i) \Leftarrow u''_i(d_i) + \sum_{p+1 \leq j \leq |S|} u'_i(v_j)$.*

We have the following lemma to prove that RR2 is safe.

▶ **Lemma 20** $(*)$. *RR2 is safe.*

Next, we have the following rule that we apply after we have applied RR2 exhaustively. This rule ensures that the valuation on any vertex does not get "much larger" than the size of the kernel.

▶ **Reduction Rule 3** (RR 3). *Let $(G', A', \mathcal{U}', p')$ be an instance we get after applying Preprocessing Step and then an exhaustive application of RR2. If for some $i \in [n]$, $u_i(d_i) > u_i(V(G') \setminus D)$, then report a No-instance.*

▶ **Lemma 21** $(*)$. *RR3 is safe.*

Next, we establish a bound on the size of the reduced instance. We have the following lemma.

▶ **Lemma 22** $(*)$. *Let $(G, A, \mathcal{U}, p)$ be an instance of* PROP-ICFD *such that G has a vertex cover U of size t. Let $(G', A', \mathcal{U}', p')$ be the instance we get after applying the Preprocessing Step and an exhaustive application of RR2. Then, $|V(G')| \leq 2^t val^{|\mathcal{A}|} p + t + n$ and $p' \leq p + n$.*

Since $|A| = n \leq p$ and $|\mathcal{A}| \leq val$, RR2, along with Lemmas 19, 20, and 22, imply that PROP-ICFD admits a $2^t val^{val} p + t + p$ vertex kernel. Moreover, due to RR3 and Lemma 21, if the initial instance respects unary (resp., binary) valuation, then the kernel also respects unary (resp., binary) valuation. Finally, the kernelization algorithms presented in this section imply the following theorem.

▶ **Theorem 3.** *For $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-ICFD admits a kernel with at most $p2^{\mathsf{vcn}}\mathsf{val}^{\mathsf{val}} + \mathsf{vcn}$ vertices. Moreover,* PROP-ICFD *admits a kernel with at most $p2^{\mathsf{vcn}}\mathsf{val}^{\mathsf{val}} + \mathsf{vcn} + p$ vertices.*

5 Incompressibility

In this section, we complement our exponential kernels by showing that it is unlikely that φ-ICFD admits a polynomial kernel parameterized by $\mathsf{vcn} + val + p + |A|$, where $\varphi \in \{\mathsf{PROP}, \mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$. We first define the following problem. In Red-Blue Dominating Set, we are given a bipartite graph G with a vertex bipartition $V(G) = T \cup N$ and a non-negative integer k. A set of vertices $N' \subseteq N$ is said to be an *RBDS* if each vertex in T has a neighbour in N'. The aim of Red-Blue Dominating Set is to decide whether there exists an *RBDS* of size at most k in G. We assume that $k < |N|$ as otherwise, the problem becomes trivial. Dom et al. [23] proved the following:

▶ **Proposition 23** ([23]). *Red-Blue Dominating Set parameterized by $|T| + k$ does not admit a polynomial compression, unless NP $\subseteq$ coNP/poly.*

5.1 Incompressibility for PROP and EF

In this section, we provide a polynomial parameter transformation from Red-Blue Dominating Set parameterized by $|T| + k$ to PROP-ICFD and EF-ICFD parameterized by $\mathsf{vcn} + val + p + |A|$.

Polynomial Parameter Transformation. Suppose $((G, k), |T| + k)$ is an instance of Red-Blue Dominating Set parameterized by $|T| + k$ such that $V(G) = T \cup N$. For brevity, let P denote the value of the parameter $\mathsf{vcn} + \mathsf{val} + p + |A|$. First we construct a graph G' with $V(G') = T' \cup N'$ from G by adding two dummy vertices d_1 and d_2 such that $T' = T \cup \{d_1, d_2\}$, $N' = N$, and $E(G') = E(G) \cup \{d_1x, d_2x \mid x \in N'\}$. Now, we construct the instance $((G', A, \mathcal{U}, p), P)$ of ICFD such that $((G, k), |T| + k)$ is a Yes-instance if and only if $((G', A, \mathcal{U}, p), P)$ is a Yes-instance and $|P| \leq h(|T| + k)$ where h is a polynomial function. Let $p = |T| + k + 2$ and $A = \{1, 2\}$. We will define the valuation function $\mathcal{U}$ based on the fairness criterion.

Incompressiblity of PROP-ICFD. For each vertex $v \in N'$, let $u_1(v) = u_2(v) = 0$. For each vertex $v \in T' \setminus \{d_1\}$, let $u_1(v) = u_2(v) = 1$ and $u_1(d_1) = u_2(d_1) = |T| + 1$. Note that $\mathsf{val} = 3$ and $|\mathcal{A}| = 1$ (i.e., identical valuation). Observe that our reduction is a polynomial parameter transformation since $P = \mathsf{vcn} + \mathsf{val} + p + |A| \leq |T'| + 3 + |T| + k + 2 + 2 = 2|T| + k + 9$ (as T' is a vertex cover of G' and $|T'| = |T| + 2$). Hence, the following lemma implies the incompressibility of PROP-ICFD.

▶ **Lemma 24.** *$((G', A, \mathcal{U}, p), P)$ is a Yes-instance of* PROP-ICFD *if and only if $((G, k), t + k)$ is a Yes-instance of Red-Blue Dominating Set.*

Proof Sketch. In one direction, let G has an *RBDS* S of size k. Then, observe that $\Pi = (\pi_1, \pi_2)$ such that $\pi_1 = \{d_1\}$ and $\pi_2 = (T' \setminus \{d_1\}) \cup S$ is a PROP allocation for $((G', A, \mathcal{U}, p), P)$.

In the reverse direction, let $((G', A, \mathcal{U}, p), P)$ be a Yes-instance of PROP-ICFD. Then, since $u_1(V(G')) = u_2(V(G')) = 2|T| + 2$ and both agents have identical valuation, for PROP-ICFD, $u_i(\pi_1) = u_i(\pi_2) = |T| + 1$. The only way to achieve this is to assign the vertex d_1 (possibly along with some vertices from N') to one of the agents and the vertices in $T' \setminus \{d_1\}$ (along with some vertices from N') to the other agent. WLOG, assume that $d_1 \in \pi_1$ and $T' \setminus \{d_1\} \subseteq \pi_2$. Observe that $|\pi_1| \geq 1$. Therefore, $|\pi_2| \leq p - 1 = |T| + k + 1$. Since $T' \setminus \{d_1\} \subseteq \pi_2$, $|\pi_2 \cap N'| \leq k$. Moreover, since $G'[\pi_2]$ is connected, observe that each vertex in $T' \setminus \{d_1\}$ shares an edge with at least one vertex in $\pi_2 \cap N'$. Hence, $\pi_2 \cap N'$ is an *RBDS* of G containing at most k vertices. ◀

Incompressibility of EF-ICFD. Here, we distinguish two cases depending on whether $|T| + k + 2$ is odd or even. The two cases have similar proofs and hence we consider only the case when $|T| + k + 2$ is odd, and the case when $|T| + k + 2$ is even is considered in the full version [26].

Case 1: $|T| + k + 2$ is odd. In this case, for each vertex $v \in N'$, let $u_1(v) = u_2(v) = 5|T|$. For each vertex $v \in T' \setminus \{d_1\}$, let $u_1(v) = u_2(v) = 1$ and $u_1(d_1) = u_2(d_1) = 5k|T| + |T| + 1$. Note that $\mathsf{val} = 3$ and $|\mathcal{A}| = 1$ (i.e., identical valuation). Moreover, similarly to the above case for PROP-ICFD, here also $P \leq 2|T| + k + 9$. Now, we have the following lemma.

▶ **Lemma 25.** *$((G', A, \mathcal{U}, p), P)$ is a Yes-instance of* EF-ICFD *if and only if $((G, k), |T| + k)$ is a Yes-instance of* Red-Blue Dominating Set.

Proof. In one direction, let S be an *RBDS* in G with k vertices. Then,observe that $\Pi = (\pi_1, \pi_2)$ such that $\pi_1 = \{d_1\}$ and $\pi_2 = (T' \setminus \{d_1\}) \cup S$ is an EF allocation for $((G', A, \mathcal{U}, p), P)$.

In the reverse direction, let $\Pi = (\pi_1, \pi_2)$ be an EF allocation for $((G', A, \mathcal{U}, p), P)$. First, we show that $d_1 \in \pi_1 \cup \pi_2$. Targeting contradiction, assume that $d_1 \notin \pi_1 \cup \pi_2$. Since $|\pi_1| + |\pi_2| = |T| + k + 2$ is an odd number, $|\pi_1| \neq |\pi_2|$. As the valuations are identical, we have $u(\pi_1) = u(\pi_2)$ (Observation 6). Let $|\pi_1 \cap N'| = \alpha$ and $|\pi_2 \cap N'| = \beta$. Then, $u(\pi_1) = u(\pi_1 \cap N') + u(\pi_1 \cap T') = 5\alpha|T| + (|\pi_1| - \alpha) \leq 5\alpha|T| + |T| + 1$. Similarly, $u(\pi_2) \leq 5\beta|T| + |T| + 1$. We have the following two cases:

1. $\alpha \neq \beta$: Without loss of generality assume that $\alpha > \beta$. Then, $u(\pi_1) - u(\pi_2) \geq 5\alpha|T| - 5\beta|T| - |T| - 1 \geq 5|T| - |T| - 1 \geq 4|T| - 1 > 0$ (for $|T| > 0$). Hence $u(\pi_1) \neq u(\pi_2)$, which contradicts the fact that Π is an EF allocation (due to Observation 6).
2. $\alpha = \beta$: Since $|\pi_1| \neq |\pi_2|$ and $\alpha = \beta$, observe that $|\pi_1| - \alpha \neq |\pi_2| - \beta$. Hence, $u(\pi_1) - u(\pi_2) = 5\alpha|T| + (|\pi_1| - \alpha) - 5\beta|T| - (|\pi_2| - \beta) = (|\pi_1| - \alpha) - (|\pi_2| - \beta) \neq 0$. Hence $u(\pi_1) \neq u(\pi_2)$, which contradicts the fact that Π is an EF allocation (due to Observation 6).

Hence, $d_1 \in \pi_1 \cup \pi_2$. WLOG, assume $d_1 \in \pi_1$. Thus, $u(\pi_1) \geq 5k|T| + |T| + 1$. Moreover, let π_1 be assigned α vertices from N' and β vertices from $T' \setminus \{d_1\}$. (We will argue that $\alpha = \beta = 0$.) Then, $u(\pi_1) = 5k|T| + |T| + 1 + 5\alpha|T| + \beta = 5(k + \alpha)|T| + |T| + \beta + 1$. Now, let π_2 is assigned α' vertices from N' and β' vertices from $T' \setminus \{d_1\}$. Therefore, $u(\pi_2) = 5\alpha'|T| + \beta'$ where $\beta' \leq |T| + 1$. We again have the following two cases depending on whether α' equals $k + \alpha$ or not.

1. $\alpha' = k + \alpha$. In this case, note that $u(\pi_1) = u(\pi_2)$ if and only if $|T| + \beta + 1 = \beta'$. It is only possible if $\beta = 0$ and $\beta' = |T| + 1$ (since $\beta' \leq |T| + 1$).
2. $\alpha' \neq k + \alpha$. We will show that, in this case, $u(\pi_1) \neq u(\pi_2)$. Note that $u(\pi_1) - u(\pi_2) = 5(k+\alpha)|T| + |T| + \beta + 1 - 5\alpha'|T| - \beta' = 5(k+\alpha-\alpha')|T| + |T| + 1 + \beta - \beta'$. Since $\alpha' \neq k+\alpha$, $k+\alpha-\alpha' \neq 0$, and hence, $-1 \geq k+\alpha-\alpha' \geq 1$. Therefore, $-5|T| + |T| + 1 + \beta - \beta' \geq u(\pi_1) - u(\pi_2) \geq 5|T| + |T| + 1 + \beta - \beta'$, i.e., $-4|T| + 1 + \beta - \beta' \geq u(\pi_1) - u(\pi_2) \geq 6|T| + 1 + \beta - \beta'$. Since $0 \leq \beta \leq |T| + 1$ and $0 \leq \beta' \leq |T| + 1$, we have that $-|T| \leq 1 + \beta - \beta' \leq |T| + 2$. Therefore, $u(\pi_1) \neq u(\pi_2)$.

Hence, if $u(\pi_1) = u(\pi_2)$, then $\alpha' = \alpha + k$, $\beta = 0$, and $\beta' = |T| + 1$. Moreover, $1 + \alpha + \beta + \alpha' + \beta' = p = |T| + k + 1$. Therefore, $1 + \alpha + 0 + \alpha' + |T| + 1 = |T| + k + 1$, i.e, $\alpha + \alpha' = k$. Solving $\alpha' = \alpha + k$ and $\alpha + \alpha' = k$ gives us $\alpha' = k$ and $\alpha = 0$. Therefore, $\pi_1 = \{d_1\}$ and $\pi_2 = (T' \setminus \{d_1\}) \cup S$ such that $S \subseteq N'$ and $|S| = k$. Since $G'[\pi_2]$ is connected, each element in $T' \setminus \{d_1\}$ has a neighbour in S. Since $T \subseteq T' \setminus \{d_1\}$, S is an *RBDS* of G and $|S| = k$. ◀

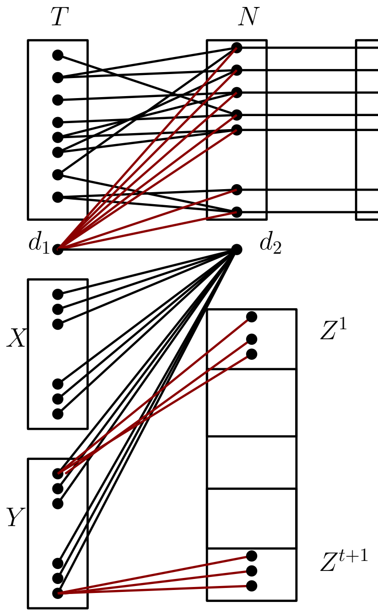

Figure 2 Illustration for the construction of G'.

5.2 Incompressibility for EFX and EF1

In this section, we provide a polynomial parameter transformation from RED-BLUE DOMINATING SET parameterized by $|T|+k$ to EF1 (resp., EFX) parameterized by $P = \mathsf{vcn}+\mathsf{val}+|A|+p$. First, we provide our construction. Let $t = |T|$. Moreover, we assume that $k > 1$ and $t > 1$.

Construction. Suppose $((G, k), t + k)$ is an instance of RED-BLUE DOMINATING SET parameterized by $t + k$ such that $V(G) = T \cup N$. See Figure 2 for a reference. First, we construct a graph G' with $V(G') = T' \cup N'$ from G by adding some dummy vertices. First, we add two dummy vertices d_1 and d_2 such that $T' = T \cup \{d_1\}$, $N' = N \cup \{d_2\}$, and $E(G') = E(G) \cup \{d_1x \mid x \in N'\}$ (d_1d_2 is also an edge). Next, we add two sets of vertices $X = \{x_1, \dots, x_{t+3}\}$ and $Y = \{y_1, \dots, y_{t+1}\}$ of size $t + 3$ and $t + 1$, respectively, to T' and attach each x_i and y_j to d_2, i.e., $T' = T' \cup \{x_1, \dots, x_{t+3}\} \cup \{y_1, \dots, y_{t+1}\}$ and $E(G') = E(G') \cup \{x_id_2, y_jd_2 \mid i \in [t + 3], j \in [t + 1]\}$. Next, for each y_j, $j \in [t + 1]$, we add $32kt$ (here, we want a large enough number) vertices $Z^j = \{z^j_1, \dots, z^j_{32kt}\}$ to N' and for each z^j_ℓ, $\ell \in [32kt]$, we add the edge $y_jz^j_\ell$ to $E(G')$. Finally, let $\{v_1, \dots, v_{|N|}\}$ be an ordering of vertices of N. For each v_i, $i \in [|N|]$, we add a vertex w_i and an edge v_iw_i to G'. Let W denote the set of vertices $\{w_1, \dots, w_{|N}\}$. This completes our construction. Note that $|V(G')| = |V(G)| + |N| + 2 + t + 3 + t + 1 + (t + 1)32kt$. Next, we set $A = [2t + 6]$ and $p = 4k(2t + 6) = 8kt + 24k$. We define the valuation function $\mathcal{U}$ below. Moreover, for ease of presentation, let us denote the set $T \cup N \cup W \cup \{d_1\}$ by B' and the set $V(G') \setminus (B' \cup \{d_2\})$ by B.

Valuation Function. We have two types of agents: Type-I and Type-II, each with cardinality $t + 3$, such that agents of the same type have identical valuations. Recall that $|A| = 2t + 6$. Hence, without loss of generality, assume that $\{1, \dots, t + 3\}$ are Type-I agents and $\{t + 4, \dots, 2t + 6\}$ are Type-II agents. For ease of presentation, we will use u_{I} and u_{II} to represent the valuation function for all Type-I and Type-II agents, respectively. The valuation function $\mathcal{U}$ is defined as follows:

- for $v \in T$, $u_{\mathrm{I}}(v) = u_{\mathrm{II}}(v) = 1$,
- $u_{\mathrm{I}}(d_1) = u_{\mathrm{II}}(d_1) = 1$,
- for $v \in N$, $u_{\mathrm{I}}(v) = 0$ and $u_{\mathrm{II}}(v) = 5kt$,
- for $v \in W$, $u_{\mathrm{I}}(v) = 5kt$ and $u_{\mathrm{II}}(v) = 0$,
- $u_{\mathrm{I}}(d_2) = u_{\mathrm{II}}(d_2) = 10kt$,
- for $i \in [t+3]$, $u_{\mathrm{I}}(x_i) = 0$ and $u_{\mathrm{II}}(x_i) = 5kt + t + 1$,
- for $i \in [t+1]$, $u_{\mathrm{I}}(y_i) = 5kt + t + 1$ and $u_{\mathrm{II}}(y_i) = 0$, and
- for $i \in [t+1]$, $j \in [32kt]$, $u_{\mathrm{I}}(z^i_j) = u_{\mathrm{II}}(z^i_j) = 0$.

First, we have the following straightforward observation concerning our construction and $\mathcal{U}$.

▶ **Observation 26.** *For each connected component C in $G[B]$, $u_I(C) \leq 5kt + t + 1$ and $u_{II}(C) \leq 5kt + t + 1$.*

Now, we will show that $((G', A, \mathcal{U}, p), P)$ is Yes-instance of EF1-ICFD (resp., EFX-ICFD) if and only if $((G, k), t + k)$ is a Yes-instance of Red-Blue Dominating Set. First, we prove one side of this argument in the following lemma.

▶ **Lemma 27** ($*$). *If $((G, k), t + k)$ is a Yes-instance of Red-Blue Dominating Set, then $((G', A, \mathcal{U}, p), P)$ is Yes-instance of* EF1-ICFD *as well as of* EFX-ICFD*.*

Next, we will prove the other side of our argument. For the rest of this section, let $((G', A, \mathcal{U}, p), P)$ be Yes-instance of EF1-ICFD (resp., EFX-ICFD) and $\Pi = (\pi_1, \ldots, \pi_{2t+6})$ be an allocation for $((G', A, \mathcal{U}, p), P)$ that satisfies EF1 (resp., EFX). We will need the following observations about the properties of Π. These observations will be valid for both EF1 as well as EFX.

Important Observations. First, we have the following easy observation.

▶ **Observation 28** ($*$). *Let there be an i such that for any vertex $v \in \tau_i$, $u_I(\pi_i) - u_I(v) \geq 5t+1$ (resp., $u_{II}(\pi_i) - u_{II}(v) \geq 5t + 1$), then at most $t + 1$ agents of Type-I (resp., Type-II) can be assigned vertices only from $B' = T \cup N \cup W \cup \{d_1\}$.*

Next, we have the following crucial observation concerning Π.

▶ **Observation 29** ($*$). *If $d_2 \in \pi_i$ for some $i \in A$, then $d_2 \in \tau_i$.*

The following observation establishes a necessary condition for Π to satisfy EF1 (resp., EFX).

▶ **Observation 30** ($*$). *If for every vertex $v \in \tau_i$, $\pi_i \setminus \{v\}$ contains at least $k + 1$ vertices from N, then Π is not an* EF1 *(resp.* EFX*) allocation.*

The following observation establishes that "too many" vertices from B' cannot be allocated to agents in Π.

▶ **Observation 31** ($*$). *For $i \in A$, π_i can be assigned at most k vertices from W and at most $k + 1$ vertices from N.*

Our next observation establishes that at least one agent of Type-I is assigned a vertex from $\{y_1, \ldots, y_{t+1}\}$, say y_i, along with some vertices (at least two) of the form z^i_j.

▶ **Observation 32** (∗). *There is at least one Type-I agent, say i, such that π_i contains at least one vertex from Y, say y_j, such that $y_j \notin \tau_i$. Equivalently, $u_I(\pi_i) - \min_{v\in\tau_i} u_I(v) \geq u_I(\pi_i) - \max_{v\in\tau_i} u_I(v) \geq 5kt + t + 1$.*

Finally, we have the following lemma that proves the other side of the reduction.

▶ **Lemma 33** (∗). *If $((G', A, \mathcal{U}, p), P)$ is Yes-instance of* EF1-ICFD *(resp.,* EFX-ICFD*), then $((G, k), t + k)$ is a Yes-instance of* Red-Blue Dominating Set.

Finally, we have the following theorem as a consequence of Proposition 23, and Lemma 24, Lemma 25, Lemma 27, Lemma 33, and the fact in all our constructions in Section 5, the value of $\mathsf{vcn} + \mathsf{val} + |A| + p$ is bounded by a function polynomial in $k + |T|$.

▶ **Theorem 4.** *For $\varphi \in \{\mathsf{PROP}, \mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-*ICFD *parameterized by $\mathsf{vcn} + \mathsf{val} + |A| + p$ does not admit polynomial compression, unless* NP $\subseteq$ coNP/poly.

6 Color Coding For PROP-ICFD

Here, we present a color-coding [2] based FPT algorithm for PROP-ICFD parameterized by p. An (n, k)-*perfect hash family* $\mathcal{F}$ is a family of functions from $[n]$ to $[k]$ such that for every set $S \subseteq [n]$ of size k, there exists a function $f \in \mathcal{F}$ that is injective on S. We shall use the following result.

▶ **Proposition 34** ([31]). *For any $n, k \geq 1$, we can construct an (n, k)-perfect hash family of size $e^k k^{\mathcal{O}(\log k)} \log n$ and in $e^k k^{\mathcal{O}(\log k)}\ n \log n$ time.*

▶ **Definition 35** (Coloring). *For a graph G, a* coloring *of G* using *k* colors *is a function $col : V(G) \to [k]$. A connected subgraph H of G is* colorful *if col is injective on $V(H)$.*

In the following lemma, we solve a sub-problem that we will use for our algorithm for PROP-ICFD using dynamic programming and color-coding.

▶ **Lemma 36** (∗). *Let G be a vertex-weighted graph with weight function $w : V(G) \to \mathbb{N}_0$. Then, given numbers $t, k \in \mathbb{N}$, we can decide in* FPT *time whether there exists a connected subgraph H of G such that $|V(H)| = k$ and $\sum_{v\in V(H)} w(v) \geq t$.*

Let $((G, A, \mathcal{U}, p), p)$ be an instance of PROP-ICFD. Let the vertices of G be colored using $|A|$ colors ($n = |A|$). A coloring is *suitable* if there exists a PROP allocation $\Pi = (\pi_1, \ldots, \pi_n)$ such that each vertex in π_i is colored with the color i. Now, we have the following lemma.

▶ **Lemma 37** (∗). *Given a coloring $col : V(G) \to A$ for an instance $(G, A, \mathcal{U}, p)$ of* PROP-ICFD*, we can decide whether this is a suitable coloring for $(G, A, \mathcal{U}, p)$ in $e^p p^{\mathcal{O}(p \log p)} m^{\mathcal{O}(1)}$ time.*

Next, we establish the probability of getting a suitable coloring of G, using $|A|$ colors uniformly at random, given that $(G, A, \mathcal{U}, p)$ is a Yes-instance of PROP-ICFD, in the following lemma.

▶ **Lemma 38** (∗). *If $(G, A, \mathcal{U}, p)$ is a Yes-instance of* PROP-ICFD*, then coloring the vertices of G uniformly at random using $|A|$ colors returns a suitable coloring with probability at least $(\frac{1}{|A|})^p$.*

Finally, we have the following theorem.

▶ **Theorem 5.** *There exists a randomized algorithm that solves* PROP-ICFD*, in time* $e^p p^{\mathcal{O}(p \log p)} m^{\mathcal{O}(1)}$ *with success probability at least* $1 - \frac{1}{e}$.

Proof Sketch. Using Lemma 37 and 38, we have an algorithm that given a Yes-instance, returns a solution with probability at least $\frac{1}{n^p}$. Clearly, by repeating this algorithm n^p times, the success probability of our algorithm is least $1 - \frac{1}{e}$. Since one execution of Lemma 37 requires $e^p p^{\mathcal{O}(p \log p)} m^{\mathcal{O}(1)}$ time and we repeat it $n^p \leq p^p$ times (since $n \leq p$), we get the desired running time. ◀

7 Conclusion

In this paper, we defined ICFD, a generalization of CFD, and conducted a comprehensive parameterized analysis of the problem. On the one hand, we proved that for $\varphi \in \{\mathsf{EF}, \mathsf{EF1}, \mathsf{EFX}\}$, φ-ICFD remains W[1]-hard parameterized by $p + \mathsf{vcn} + |A|$, even when $\mathsf{vcn} + |A|$ is a small constant. On the other hand, we proved that PROP-ICFD admits a randomized FPT algorithm parameterized by p alone. We also presented positive as well as negative results on the kernelization complexity of ICFD (for all fairness notions considered) parameterized by $\mathsf{vcn} + \mathsf{val} + p$.

There are several other generalizations of ICFD (or CFD) that can be practically well-motivated as well as theoretically interesting. For instance, the requirement of connectivity can be "relaxed" (for example, by requiring that each agent receives at most c connected components where $c \geq 1$) as well as "restricted" (for example, by requiring that each agent should receive a connected component that is 2-connected, or, more generally, c-connected where $c \geq 1$). Moreover, it might be desirable sometimes that the diameter of each connected component (or the sum of diameters of all connected components) assigned to an agent is bounded by a parameter d. For the examples we have already described in the Introduction, it is easy to see that such variants make sense.

Furthermore, several other notions of *fairness*, *welfare*, and *chores* were studied with respect to classical Fair Division. Hence, it might be interesting to consider ICFD in light of these notions. Moreover, it might be interesting to study ICFD by considering the parameter $|V(G) - p|$ along with other parameters. This might be specifically applicable when the allocator wants to allocate as much resources as possible.

In the classical fair division (ICFD on a clique with $p = |V(G)|$), it is known that EF1 always exists and can be computed in polynomial time using the famous envy-cycles procedure [30]. On the other hand, the guarantee of the existence of EFX exists for very restricted settings (for example, 4 agents with additive valuations [5] and 2 agents with arbitrary valuations [32]). It will be interesting to study if this difference in the "difficulty" of the two notions extends to the setting of ICFD and/or to the setting of CFD.

Finally, we discuss a natural variant of PROP-ICFD, namely PRP-ICFD, where every agent measures its happiness not in the context of its valuation of the whole graph G, but the subgraph G' of G that is assigned to the agents. More formally, an allocation $\Pi = (\pi_1, \ldots, \pi_n)$ respects PRP-ICFD if for every $i \in A$, $u_i(\pi_i) \geq \frac{u_i(V(G'))}{n}$, where $G' = G[\pi_1 \cup \cdots \cup \pi_n]$. Observe that, for PRP-ICFD, RR 1 is safe, and is sufficient to get an exponential kernel parameterized $\mathsf{vcn} + \mathsf{val} + p$, unlike PROP-ICFD, which requires more careful reduction rules. Moreover, the color-coding based FPT algorithm for PROP-ICFD is unlikely to work for PRP-ICFD since the key ingredient of this algorithm is that we know beforehand the exact valuation that will make an agent "happy", which is not the case for PRP-ICFD. In this regard, PRP-ICFD behaves more like EF, EF1, and EFX ICFD. Hence, it will be interesting to figure out whether PRP-ICFD is FPT or W[1]-hard when parameterized by $p + \mathsf{vcn}$.

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