Geometric Covering via Extraction Theorem

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Abstract

In this work, we address the following question. Suppose we are given a set D of positive-weighted disks and a set T of n points in the plane, such that each point of T is contained in at least two disks of D. Then is there always a subset S of D such that the union of the disks in S contains all the points of T and the total weight of the disks of D that are not in S is at least a constant fraction of the total weight of the disks in D?

In our work, we prove the Extraction Theorem that answers this question in the affirmative. Our constructive proof heavily exploits the geometry of disks, and in the process, we make interesting connections between our work and the literature on local search for geometric optimization problems.

The Extraction Theorem helps to design the first polynomial-time $O(1)$-approximations for two important geometric covering problems involving disks.

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1 Introduction

Geometric covering is a popular research topic in computer science which is widely used in areas such as wireless and sensor networks, robotics, data mining, computational biology, image processing, and VLSI design [7, 38, 14, 39, 28, 20, 27, 25]. Given a geometric object O and a point p, O is said to *cover* p if p is contained in O. A set $\mathcal{O}$ of objects *covers* a set P of points, if each point $p \in P$ is covered by an object in $\mathcal{O}$. One frequently studied and fundamental covering problem is the following.

▶ **Problem 1** (*Minimum Weight Unit Disk Cover* (WUDC)). *Given a set $\mathcal{D}$ of unit disks along with a weight function $w : \mathcal{D} \to \mathbb{R}^+$ and a set T of points in the plane, the goal is to find a subset $\mathcal{D}' \subseteq \mathcal{D}$, such that $\mathcal{D}'$ covers T and the sum of the weights of the disks in $\mathcal{D}'$ is minimized.*

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When all the weights are 1, the problem is popularly known as *Discrete Unit Disk Cover* (DUDC), which is already NP-complete [26]. In a seminal work, Mustafa and Ray [34] obtained a PTAS for DUDC. However, their result is based on a local search scheme that is not sufficient to handle weights. Li and Jin [30] obtained the first PTAS for WUDC based on the approach of partitioning the plane into squares and solving a restricted problem in those squares.

Another fundamental covering problem is *Maximum Coverage with Unit Disks*.

▶ **Problem 2** (*Maximum Coverage with Unit Disks* (MCUD))**.** *Given a set $\mathcal{D}$ of unit disks and a set T of points in the plane, and an integer $k > 0$, the goal is to find a subset $\mathcal{D}' \subseteq \mathcal{D}$ of size at most k, such that the number of points of T covered by $\mathcal{D}'$ is maximized.*

This problem is also known to be NP-complete [26]. In a recent work, Chaplick et al. [13] designed a local search based PTAS for this problem.

In this work, we study the *Discrete Covering with Two Types of Radii* problem, a combination of DUDC and MCUD.

For any point x in the plane and any real number $\rho > 0$, let $D(x, \rho)$ denote the closed disk with center x and radius ρ.

▶ **Problem 3** (*Discrete Covering with Two Types of Radii* (DC-2))**.** *We are given two point sets P (of n "users") and $A = \{a_1, a_2, \ldots, a_m\}$ (of m "access points") in the plane, and two real numbers ρ_1 and ρ_2, such that $0 < \rho_1 < \rho_2$ and $P \subseteq \bigcup_{i=1}^{m} D(a_i, \rho_2)$. The goal is to select for each $i = 1, 2, \ldots, m$, a value $r_i \in \{\rho_1, \rho_2\}$ such that $P \subseteq \bigcup_{i=1}^{m} D(a_i, r_i)$ and the size of the set $\{p \in P :$ there is an index $1 \leq i \leq m$ such that $r_i = \rho_1$ and $p \in D(a_i, r_i)\}$ is maximized.*

We refer to the disk $D(a_i, \rho_1)$ as the *small* disk centered at a_i. Similarly, we refer to $D(a_i, \rho_2)$ as the *large* disk centered at a_i. Using this terminology, the goal is to select for each $1 \leq i \leq m$, either the small disk centered at a_i or the large disk centered at a_i, such that the set P is covered by the chosen disks and the number of points in P each of which is contained in at least one chosen small disk is maximized.

The DC-2 problem appears naturally in wireless networks [5]. In this setting, a user receives data from an *access point* a_i if it is within a certain distance from a_i. The data is received at high-speed if the user is close to a_i. Each access point can work at a single frequency assigned to it from a range, which we assume, for simplicity, has only two values: high and low. High frequency corresponds to high-speed, but smaller coverage area. Similarly, low frequency corresponds to low-speed, but larger coverage area. The goal is to assign frequencies to the access points such that each user is within the coverage area of some access point and the number of users within the coverage area of high-frequency access points is maximized.

Despite the significance of the DC-2 problem in wireless networks, the problem was introduced to the theory community only recently by Maheshwari *et al.* [31]. They presented polynomial-time algorithms (based on dynamic programming) for the one-dimensional case. On the other hand, they proved that the DC-2 problem is NP-complete. The natural question that arises from their work is whether DC-2 admits a constant-approximation.

Informally speaking, the DC-2 problem encapsulates the two fundamental covering problems DUDC and MCUD: we need to select disks satisfying the mixed goal of covering all points and maximizing the number of points covered by the chosen small disks. As mentioned before, both DUDC and MCUD admit PTASes. However, due to the mixed flavor of DC-2, none of these approximation schemes seem to be extendable to DC-2. Handling the two different, but dependent tasks together turns out to be the biggest hurdle in approaching this problem. Hence, the problem has posed new challenges in geometric covering and warrants the development of novel tools and techniques.

Even though DC-2 already seems difficult, one might be more interested in a natural extension of this problem where we have a set of an arbitrary $t \geq 2$ number of frequencies available to select from for each access point. Moreover, there is a quality function associated with each frequency or radius, which defines the gain of an access point if that particular frequency is selected.

▶ **Problem 4** (*Discrete Covering with t Types of Radii* (DC)). *We are given two point sets $P = \{p_1, p_2, \ldots, p_n\}$ (of n "users") and $A = \{a_1, a_2, \ldots, a_m\}$ (of m "access points") in the plane, t real numbers $\rho_1, \ldots, \rho_t$, such that $0 < \rho_1 < \ldots < \rho_t$ and $P \subseteq \bigcup_{i=1}^{m} D(a_i, \rho_t)$, and a function $\mathsf{quality} : \{1, \ldots, t\} \to \mathbb{R}^+ \cup \{0\}$. A feasible solution constitutes a value $r_i \in \{\rho_1, \ldots, \rho_t\}$ for each $i = 1, 2, \ldots, m$ such that $P \subseteq \bigcup_{i=1}^{m} D(a_i, r_i)$. The gain of a feasible solution is defined by the expression $\sum_{j=1}^{n} \max_{\{l : \exists i \text{ s.t. } p_j \in D(a_i, r_i) \text{ and } r_i = \rho_l\}} \mathsf{quality}(l)$, i.e., each point p_j contributes a value to the gain that is equal to the quality of the maximum quality radius ρ_l such that there is a disk $D(a_i, \rho_l)$ in the solution containing p_j. The goal is to compute a feasible solution that maximizes the gain.*

It is not hard to see that DC-2 is a restricted version of DC with $t = 2$ where $\mathsf{quality}(1) = 1$ and $\mathsf{quality}(2) = 0$. An interesting question about DC is whether it admits an approximation in polynomial time for any arbitrary number t where the approximation factor is a true constant that does not depend on t.

1.1 Our contributions

In our work, we design a polynomial-time true constant-approximation algorithm for DC with the approximation factor of 6.328. For DC-2, we obtain an improved 4-approximation. Both the approximations are based on a fundamental theorem regarding the weights of geometric set covers, which we refer to as the Extraction Theorem. For any set of weighted objects S, let $W(S)$ denote the sum of the weights of the objects.

▶ **Theorem 5** (Extraction Theorem). (Informal) *Suppose we are given a set $\mathcal{D}$ of disks along with a weight function $w : \mathcal{D} \to \mathbb{R}^+$ and a set T of points in the plane, such that each point of T is contained in at least two disks of $\mathcal{D}$. Then there exists a subset $\mathsf{sol} \subset \mathcal{D}$ such that sol covers T and $W(\mathcal{D} \setminus \mathsf{sol}) \geq W(\mathcal{D})/4$. Moreover, such a subset sol can be computed in polynomial time.*

DC-2. Given the Extraction Theorem, our approximation for DC-2 is based on a straightforward combinatorial algorithm. We assign each point that is in at least one small disk to an arbitrary small disk. The number of points assigned to each small disk is its gain. Assign a weight to each large disk whose value is equal to the gain of the corresponding small disk. If we select the large disk for an access point, we lose its weight. Otherwise, the gain of the small disk is at least the weight. Moreover, the total weight of the large disks is at least the optimal gain of the instance. Applying the Extraction Theorem gives us a set of large disks that covers all the points and its complement set of disks has a gain of at least one-fourth of the optimal gain.

DC. The main challenge that we now face for DC is this: Suppose that we somehow decide to use some access point a_i to maximize the gain and not to guarantee coverage. It is still not clear which disk at a_i we need to open, in contrast to DC-2 where we can simply open the small disk. Therefore, to obtain the approximation for DC, we use a more involved combinatorial algorithm, which has two major steps. We define a point to be a *private* point if it is covered by the disks of exactly one access point. Otherwise, it is a *non-private* point. In the first step, we consider the problem of selecting one disk for each access point such that each private point is covered and the overall gain is maximized. In particular, we model this problem as a submodular maximization problem subject to a matroid. Subsequently, we use an algorithm for the latter problem due to Calinescu et al. [10], which yields an $O(1)$-approximation for our problem in the first step. Naturally, this step does not ensure that every non-private point is covered by the chosen disks. In the second step, we apply the Extraction Theorem in a careful manner such that all points are covered. This step is similar to the algorithm for DC-2. In particular, the algorithm for DC-2 can be seen as a simpler version of the algorithm for DC without the first step.

We note that our result for DC also holds for a somewhat more general setting, where the set of allowed radii and corresponding qualities need not be the same for each access point.

Proof of the Extraction Theorem. The proof is indeed simple if all the disks have the same radius. Compute a Delaunay triangulation of the unit disks, which is known to be a planar graph. Hence, its vertices (or the disks) can be colored by 4 colors in polynomial time. It follows that there is an independent set in this graph of weight at least one-fourth of the total weight of the disks. The complement set is a vertex cover of the graph. One can prove that the disks corresponding to any vertex cover in this graph cover all the points. This follows due to the properties of Delaunay triangulations. Notably the factor 4 is tight (see Figure 1). The proof for arbitrary disks goes along the same lines. Here also we construct a planar graph, extract a vertex cover, and argue that the disks that correspond to this vertex cover also cover all the points. However, the construction of the planar graph in this case is much more involved. During the process of the planar graph construction, we make interesting connections between the Extraction Theorem and analyses of local search algorithms for geometric optimization problems [34, 33, 12, 3, 9]. In particular, the analysis of such a local search algorithm warrants the existence of a bipartite support graph such that the vertices are the objects from a local search solution and an optimal solution, and the graph has a small-sized vertex separator. Interestingly, most of the analyses show the existence of such a support graph that is planar. The connection helps us borrow ideas from these analyses for the construction of our planar graph.

Extraction theorem for more general objects. Note that the Extraction Theorem is not-necessarily true if we have a combinatorial set system instead of geometric objects and points. Consider the complete graph K_m on m vertices and set the weight of each vertex to be one. Then every vertex cover must contain at least $m-1$ vertices, and thus the total weight of any such cover is at least $m-1$ or $(1-1/m)$-fraction of the total weight. Consequently, one might wonder about the most general class of objects for which such an Extraction Theorem would be true. Motivated by this, we proved an Extraction Theorem for any class of regions (equivalently objects) that have linear union complexity, albeit with a constant larger than 4. This leads to extraction theorems for regions such as fat triangles of similar size and unit axis-parallel cubes in $\mathbb{R}^3$. The proof of the general theorem is different from the proof for disks and is based on a result due to Ene et al. [21]. In the process of proving this theorem, we establish an interesting connection to their work.

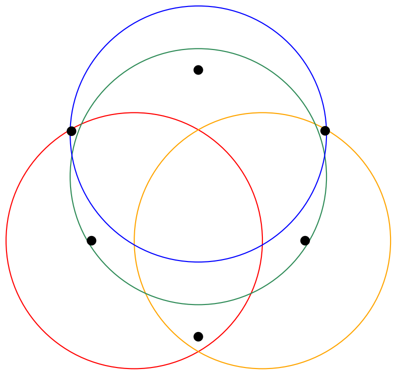

Figure 1 Set a weight of 1 to each disk. Then the weight of every subset of disks that covers all the points is at least 3 or 3/4 of the total weight.

Other applications of Extraction theorems. Our Extraction theorems naturally lead to a max-gain version of geometric set cover. Given a set $\mathcal{D}$ of unit disks along with a weight function $w : \mathcal{D} \to \mathbb{R}^+$ and a set T of n points in the plane, the goal is to find a subset $\mathcal{D}' \subseteq \mathcal{D}$, such that $\mathcal{D}'$ covers T and the sum of the weights of the disks in $\mathcal{D} \setminus \mathcal{D}'$ is maximized. It is not hard to see that *Minimum Weight Unit Disk Cover* (WUDC) is the dual of this max-gain version in the sense that for an instance I of WUDC, if the minimum solution has the value Γ, then the maximum solution for the max-gain version on I has the value $W(\mathcal{D}) - \Gamma$. Similarly, for an instance I' of the max-gain version, if the maximum solution has the value Γ', then the minimum solution for WUDC on I' has the value $W(\mathcal{D}) - \Gamma'$. Indeed, one can define this max-gain covering problem for any class of geometric objects. As far as we know, such a geometric version has not been studied before. However, for the more general combinatorial version with sets and elements, a Δ-approximation is known [4, 17], where Δ is the maximum cardinality of the sets. Our general Extraction Theorem directly yields $O(1)$-approximations for such a natural problem with a wide range of geometric objects mentioned above. Further consequences for the max-gain covering problem are discussed in Section 6.

1.2 Related work

Designing $O(1)$-approximation for geometric set cover is a popular research direction in computational geometry. Earlier works focused on the unweighted case. In a classic work, Brönnimann and Goodrich [8] (see also [24]) linked the integrality gap of the canonical LP relaxation of set cover to the existence of small-sized ϵ-nets, and obtained $O(1)$-approximations for disks in $\mathbb{R}^2$ and halfspaces in $\mathbb{R}^3$. Clarkson and Varadarajan [15] showed a novel connection between the size of ϵ-nets and the union complexity of objects, leading to $O(1)$-approximations for more general objects, such as similar-sized fat triangles and pseudodisks. Basu Roy et al. [37] obtained a local search based PTAS for non-piercing regions (e.g., pseudodisks); their work builds on the approach pioneered by Mustafa and Ray [34] in their PTAS for disks in the plane. Handling the weighted case turned out to be much more challenging. $O(1)$-approximations were known only for unit disks [2, 16, 18, 19, 23, 29, 41] and unit squares [22]. In a breakthrough, Varadarajan [40] proposed a quasi-uniform sampling based approach that achieved improved approximations in the weighted case for several classes of objects. His approach is again based on a randomized ϵ-net construction. Chan et al. [11] further refined and generalized this approach to obtain an $O(1)$-approximation for weighted disks. Mustafa et al. [33] obtained a quasi-PTAS for pseudodisks based on the separator theorem of Adamaszek and Wiese [1].

Almost all results on geometric covering are based on either LP rounding, local search, or separator theorems. In particular, the $O(1)$-approximation for weighted disk cover should be contrasted with our $O(1)$-approximation for the max-gain version (dual) which is obtained via a straightforward algorithm (based on the Extraction Theorem for disks) and with a reasonably small constant factor of 4.

Organization. In Section 2 and 3, we describe the algorithms for DC-2 and DC, respectively. In Section 4, we prove the Extraction Theorem for unit disks and arbitrary disks. The proof of the general Extraction Theorem appears in Section 5. In Section 6, we conclude with some open questions.

2 A 4-approximation for DC-2

In this section, we describe a combinatorial algorithm achieving the 4-approximation for DC-2. For any access point $a_i \in A$, we denote the disk having the larger radius by $\mathrm{lrg}(i)$ and the disk having the smaller radius by $\mathrm{sml}(i)$. The gain of a solution S, denoted by $\mathrm{gain}(S)$, is the number of points covered by the small disks in S. We say that a point p in P is *vulnerable*, if p is not contained in any of the small disks, i.e., $p \notin \bigcup_{i=1}^{m} \mathrm{sml}(i)$. Let $P_v \subseteq P$ be the set of vulnerable points in P, and let $P_n = P \setminus P_v$ be the complement set consisting of the non-vulnerable points in P.

▶ **Observation 6.** *No vulnerable point contributes to the gain of any feasible solution. Each non-vulnerable point is covered in every solution irrespective of which disks are chosen. Also, each non-vulnerable point p contributes* 1 *to the gain of any feasible solution if p is covered by a chosen small disk.*

By the above observation, we need to consider covering only vulnerable points. We can assume that each point of P_v is in at least two large disks. Indeed, assume that $p_j \in P_v$ is contained in $\mathrm{lrg}(i)$, but not $\mathrm{lrg}(i')$ for any $i' \neq i$. Any feasible solution must select large disk $\mathrm{lrg}(i)$ in order to cover p_j. We can pick $\mathrm{lrg}(i)$ corresponding to access point a_i, and remove a_i and every point in P that (a) is covered by $\mathrm{lrg}(i)$ but (b) is not covered by $\mathrm{sml}(i')$ for any $i' \neq i$. This removal gives us a smaller instance of the problem to solve. Note that each removed point contributes 0 to the gain of any feasible solution to the original instance.

Henceforth, we assume that each point of P_v is in at least two large disks.

2.1 Solving DC-2 using Extraction Theorem

We assign each non-vulnerable point p_j to some access point $a_i \in A$ such that p_j is covered by $\mathrm{sml}(i)$; if multiple small disks contain p_j we break the tie arbitrarily. Let $w(i)$ denote the number of points assigned to a_i; we think of $w(i)$ as a weight associated with $\mathrm{lrg}(i)$. Let OPT denote the optimal gain of the DC-2 instance, and let $\mathcal{S}$ be the set of all large disks. We have the following observation.

▶ **Observation 7.** $W(\mathcal{S}) = \sum_{a_i \in A} w(i) = |P_n| \geq \mathrm{OPT}$.

We apply our Extraction Theorem 19 on the set of disks $\mathcal{S}$, the set of points P_v, and the weight function $w(.)$. The Extraction Theorem returns a subset $\mathcal{S}' \subset \mathcal{S}$ such that (a) $\mathcal{S}'$ covers P_v, and (b) $W(\mathcal{S}') \leq (3/4) \cdot W(\mathcal{S})$. By Observation 7, we have the following observation.

▶ **Observation 8.** $W(\mathcal{S} \setminus \mathcal{S}') \geq W(\mathcal{S})/4 \geq \mathrm{OPT}/4$.

We construct a solution to DC-2 as follows. For each $a_i \in A$, if $\text{lrg}(i)$ is in $\mathcal{S}'$, we set $r_i = \rho_2$. Otherwise, we set $r_i = \rho_1$. That is, the large disks in our constructed solution $\{D(a_i, r_i) \mid a_i \in A\}$ are precisely those that are in S'. The Extraction Theorem guarantees that the set S' covers P_v, the set of vulnerable points. All non-vulnerable points are also covered by our solution.

We now analyze the gain of our solution to DC-2. This is the number of points in P_n that are covered by the small disks in our solution, which, by the way we define weights, is at least

$$\sum_{a_i \in A: lrg(i) \notin \mathcal{S}'} w(i) = W(\mathcal{S} \setminus \mathcal{S}') \geq \text{OPT}/4.$$

▶ **Theorem 9.** *There is a polynomial-time 4-approximation algorithm for DC-2.*

3 A 6.328-approximation for DC

We have t classes of disks $1, 2, \ldots, t$, ranging from the smallest to the largest. The goal is to determine, for each access point $a_i \in A$, the class that we need to "open", such that all points in P are covered and the coverage quality is maximized. Recall that $\mathsf{quality}(k)$ denotes the quality provided by a disk of class $k \in \{1, 2, \ldots, t\}$. Let $\mathsf{disk}(i, k)$ denote the disk of class k centered at access point a_i.

We say that a point $p \in P$ is *private* to access point a_i if p can only be covered by a disk at access point a_i. That is, such a point p is not in the class t disk centered at access point $b \in A$ for each $b \neq a_i$. The points in P that are private to a_i impose a restriction on the class of disk that can be opened at a_i. Let $\mathsf{allowed}(i)$ denote the classes of disks thus restricted. Formally, $\mathsf{allowed}(i) = \{k, k+1, \ldots, t\}$, where $1 \leq k \leq t$ is the smallest integer such that every point $p \in P$ private to a_i is in $\mathsf{disk}(i, k)$. If there is no point private to a_i, then $\mathsf{allowed}(i) = \{1, \ldots, t\}$. To cover private points of a_i, we must pick a disk from $\mathsf{allowed}(i)$. Henceforth, we assume that we can only open a disk from $\mathsf{allowed}(i)$ at a_i. Note that this guarantees coverage of private points. We denote the set of private points by P'. A point in P is called *non-private* if it is not private to any access point.

Our approximation algorithm has two steps. In the first step, we consider the problem of selecting one disk from $\mathsf{allowed}(i)$ for each access point a_i such that the overall gain is maximized. Naturally, this step does not ensure that every non-private point is covered by the chosen disks. In the second step, we apply Extraction Theorem in a careful manner such that all points are covered. In the following, we describe these two steps in detail.

3.1 Step 1: Maximizing Gain while Covering Private Points

Let us consider the following slightly modified problem.

▶ **Problem 10** (DC-private-coverage). *We are given two point sets $P = \{p_1, p_2, \ldots, p_n\}$ (of n "users") and $A = \{a_1, a_2, \ldots, a_m\}$ (of m "access points") in the plane, t real numbers $\rho_1, \ldots, \rho_t$, such that $0 < \rho_1 < \ldots < \rho_t$ and $P \subseteq \bigcup_{i=1}^{m} D(a_i, \rho_t)$, and a function $\mathsf{quality} : \{1, \ldots, t\} \to \mathbb{R}^+ \cup \{0\}$. A feasible solution constitutes of an index $k(i) \in \mathsf{allowed}(i)$ for each $i = 1, 2, \ldots, m$. The gain of a feasible solution is $\sum_{j=1}^{n} \max_{\{1 \le i \le m | p_j \in \mathit{disk}(i,k(i))\}} \mathsf{quality}(k(i))$, i.e., each point p_j contributes a value to the gain that is equal to the quality of the maximum quality radius $\rho_{k(i)}$ correspond to an index i such that $\mathsf{disk}(i, k(i))$ contains p_j. The goal is to compute a feasible solution that maximizes the gain.*

Notably the only difference between DC-private-coverage and DC is that DC-private-coverage guarantees the coverage of all private points, but not of all non-private points. We construct an instance of DC-private-coverage given the instance of DC in the natural way. Let OPT and OPT′ be the optimal gain of the instance of DC and DC-private-coverage, respectively.

▶ **Observation 11.** $\mathrm{OPT}' \geq \mathrm{OPT}$.

The above observation follows from the fact that any feasible solution to DC is also a feasible solution to DC-private-coverage. To obtain a solution for DC with sufficient gain, we first consider the DC-private-coverage problem. In particular, we represent the latter problem as an instance of submodular maximization subject to a matroid.

Let X be the union of the disks $\mathsf{disk}(i, k)$ such that $a_i \in A$ and $k \in \mathsf{allowed}(i)$. We define a function $f : 2^X \to \mathcal{R}^+$ in the following. For each $S \in 2^X$,

$$f(S) = \sum_{j=1}^{n} \max_{\{l : \exists i \text{ s.t. } \mathsf{disk}(i,l) \in S \text{ and } p_j \in \mathsf{disk}(i,l)\}} \mathsf{quality}(l).$$

We refer to the term for p_j in the above summation (i.e., $\mathsf{quality}(l)$) as the quality gain of p_j. Now, define the partition matroid $M = (X, \mathcal{I})$ such that each set in $\mathcal{I}$ contains exactly one $\mathsf{disk}(i, k)$ correspond to each $a_i \in A$, where $k \in \mathsf{allowed}(i)$. Then, the DC-private-coverage problem is equivalent to maximizing the function f subject to the matroid M. To solve this latter maximization problem, we need good characterization of the function f. Calinescu et al. [10] showed that one can obtain a $(1 - 1/e)$-approximation if f is monotone and submodular. In particular, they prove the following theorem.

▶ **Proposition 12** ([10]). *There is a polynomial-time, randomized algorithm giving a $(1-1/e)$-approximation (in expectation) to the problem $\max\{f(S) : S \in \mathcal{I}\}$, where $f : 2^X \to \mathbb{R}^+$ is a monotone submodular function given by a value oracle, and $M = (X, \mathcal{I})$ is a matroid given by a membership oracle.*

In our case, given $S \subseteq X$, $f(S)$ can be easily computed. So, we can assume that f is given by a value oracle. Also, whether $S \in \mathcal{I}$ or not can be determined easily, and thus M can be assumed to be given by a membership oracle. In our case, f is monotonically non-decreasing, as adding more disks to a set of disks cannot decrease the quality gain of any point in P. Next, we prove that f is also submodular.

▶ **Lemma 13.** *f is submodular.*

Proof. To prove that f is submodular, we show that for any $A \subseteq B \subseteq X$ and $\mathsf{disk}(i, k) \in X \backslash B$, $f(A \cup \{\mathsf{disk}(i, k)\}) - f(A) \geq f(B \cup \{\mathsf{disk}(i, k)\}) - f(B)$.

Let $Q \subseteq P$ be the subset of points that are in both $\mathsf{disk}(i,k)$ and $\{p_j \mid p_j \in \mathsf{disk}(i',l') \in B\}$. Also let $R \subseteq P$ be the subset of points that are in $\mathsf{disk}(i,k)$, but not in $\{p_j \mid p_j \in \mathsf{disk}(i',l') \in B\}$. Notably, $\{Q,R\}$ is a partition of the points in $\mathsf{disk}(i,k)$. First, note that the quality gain of a point in $f(B \cup \{\mathsf{disk}(i,k)\})$ (resp. $f(A \cup \{\mathsf{disk}(i,k)\})$) cannot be smaller than its quality gain in $f(B)$ (resp. $f(A)$). Moreover, the quality gain of a point p_j can increase going from $f(B)$ to $f(B \cup \{\mathsf{disk}(i,k)\})$ only if it is in $\mathsf{disk}(i,k)$. We consider two disjoint cases: $p_j \in Q$ and $p_j \in R$.

First, suppose $p_j \in Q$. If the quality gain of p_j increases going from $f(B)$ to $f(B \cup \{\mathsf{disk}(i,k)\})$, then it becomes exactly $\mathsf{quality}(k)$, which is larger than the $\mathsf{quality}$-value of any disk in B that contains p_j. As $A \subseteq B$, the quality gain of p_j also increases going from $f(A)$ to $f(A \cup \{\mathsf{disk}(i,k)\})$ and becomes $\mathsf{quality}(k)$. Moreover, the maximum $\mathsf{quality}$-value of any disk in A that contains p_j is at most the maximum $\mathsf{quality}$-value of any disk in B that contains p_j. Hence, the increase in the quality gain of p_j going from $f(A)$ to $f(A \cup \{\mathsf{disk}(i,k)\})$ is at least that going from $f(B)$ to $f(B \cup \{\mathsf{disk}(i,k)\})$.

Now, consider any $p_j \in R$. The quality gain of p_j in $f(B)$ is 0, as p_j is not in any of the disks in B. As $A \subseteq B$, p_j is also not in any of the disks in A and its quality gain in $f(A)$ is also 0. Moreover, the quality gain of p_j in both $f(A \cup \{\mathsf{disk}(i,k)\})$ and $f(B \cup \{\mathsf{disk}(i,k)\})$ is $\mathsf{quality}(k)$. Hence, the increase in the quality gain of p_j going from $f(A)$ to $f(A \cup \{\mathsf{disk}(i,k)\})$ is equal to that going from $f(B)$ to $f(B \cup \{\mathsf{disk}(i,k)\})$.

Summing over all points $p_j \in P$, we obtain that $f(A \cup \{\mathsf{disk}(i,k)\}) - f(A) \geq f(B \cup \{\mathsf{disk}(i,k)\}) - f(B)$. This completes the proof of the lemma. ◀

By the above lemma, Proposition 12, and Observation 11, we directly obtain the following lemma.

▶ **Lemma 14.** *There is a randomized algorithm that given an instance of* DC-private-coverage *returns a feasible solution S of expected gain at least* $(1 - 1/e)\cdot\mathrm{OPT}$*, where* OPT *is the optimal gain of the corresponding* DC *instance.*

3.2 Step 2: Covering all points

Henceforth, we denote by $\hat{\mathcal{S}}$ the set of disks computed by Step 1, that is, the algorithm of Lemma 14. For each $1 \leq i \leq m$, let $k(i)$ be the index of the disk in $\hat{\mathcal{S}}$ chosen for a_i. The disks in $\hat{\mathcal{S}}$ cover all the private points. The expected value of the gain $f(\hat{\mathcal{S}})$ is at least $(1 - 1/e)\cdot\mathrm{OPT}$.

In Step 2, we apply Extraction Theorem to additionally cover the non-private points. The application is similar to the 2-radii case. Let $\mathcal{S} = \{\mathsf{disk}(i,t) \mid 1 \leq i \leq m\}$. We assign each point p_j that is covered by $\hat{\mathcal{S}}$ to a disk in $\mathcal{S}$ as follows: Suppose that $\mathsf{disk}(i,k(i)) \in \hat{\mathcal{S}}$ is the highest quality disk in $\hat{\mathcal{S}}$ that covers p_j (we break ties arbitrarily). We assign p_j to the corresponding largest disk $\mathsf{disk}(i,t)$ centered at access point a_i.

For $\mathsf{disk}(i,t) \in \mathcal{S}$, we define its weight $w(i)$ to be the product of the number of points assigned to $\mathsf{disk}(i,t)$ and $\mathsf{quality}(k(i))$. The following observation is straightforward.

▶ **Observation 15.** $W(\mathcal{S}) = f(\hat{\mathcal{S}})$.

We apply our Extraction Theorem 19 on the set of disks $\mathcal{S}$, the set of non-private points in P, and the weight function $w(.)$. The Extraction Theorem returns a subset $\mathcal{S}' \subset \mathcal{S}$ such that (a) $\mathcal{S}'$ covers the non-private points in P, and (b) $W(\mathcal{S}') \leq (3/4) \cdot W(\mathcal{S})$. By Observation 15, we have the following observation.

▶ **Observation 16.** $W(\mathcal{S} \setminus \mathcal{S}') \geq W(\mathcal{S})/4 \geq f(\hat{\mathcal{S}})/4$.

We construct a solution to DC as follows. For each $a_i \in A$, if $\mathsf{disk}(i,t)$ is in $\mathcal{S}'$, we set $r_i = \rho_t$, that is, we pick $\mathsf{disk}(i,t)$ among the disks centered at access point a_i. Otherwise, we set $r_i = k(i)$, that is, we pick $\mathsf{disk}(i,k(i))$ among the disks centered at a_i. Extraction Theorem guarantees that the set $\mathcal{S}'$ covers the set of non-private points, as by definition each such point is contained in at least two disks of $\mathcal{S}$. Our solution to DC also covers these points, as it contains $\mathcal{S}'$. All private points are also covered by our solution, as we pick a $\mathsf{disk}(i,k)$ for each $a_i \in A$ such that $k \in \mathsf{allowed}(i)$.

We can lower bound the gain of our solution by considering the contribution of the set

$$\mathcal{D} = \{\mathsf{disk}(i,k(i)) \mid \mathsf{disk}(i,t) \in \mathcal{S} \setminus \mathcal{S}'\}.$$

The gain of our solution is at least

$$f(\mathcal{D}) \geq W(\mathcal{S} \setminus \mathcal{S}') \geq f(\hat{\mathcal{S}})/4.$$

Here, the first inequality follows from the way we define weights, and the second from Observation 16 . Thus, the expected gain of our solution is at least

$$f(\hat{\mathcal{S}})/4 \geq (1 - 1/e) \cdot \mathrm{OPT}/4 \geq \mathrm{OPT}/6.328.$$

Hence, we obtain the desired theorem.

▶ **Theorem 17.** *There is a polynomial-time, randomized algorithm giving a 6.328-approximation (in expectation) to the* DC *problem.*

▶ Remark 18. One might note that our algorithm works for a much more general model where the radii of the disks at any pair of access points are not-necessarily the same as in *DC*. In that case, Step 1 works unchangeably. In Step 2, the largest disks for which we apply *Extraction Theorem*, can now have arbitrary radii. But, then we can use the more general Theorem 23.

4 Proof of Extraction Theorems

In this section, we first prove the Extraction Theorem for unit disks. We subsequently establish the Extraction Theorem for disks of arbitrary radii.

4.1 The Extraction Theorem for Weighted Unit-Disks

▶ **Theorem 19.** *Let $\mathcal{T}$ be a set of n points in the plane, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ be a set of m weighted unit-disks (i.e., disks of radius one) in the plane. For each i with $1 \leq i \leq m$, let $w_i \in \mathbb{R}^+$ denote the weight of the disk D_i, and let $W = \sum_{i=1}^{m} w_i$ denote the total weight of all disks in $\mathcal{D}$. Assume that each point of $\mathcal{T}$ is contained in at least two disks of $\mathcal{D}$. Then there exists a subset $\mathcal{D}'$ of $\mathcal{D}$ such that*
1. *the disks in $\mathcal{D}'$ cover all points of $\mathcal{T}$ and*
2. *the total weight of all disks in $\mathcal{D}'$ is at most $(3/4) \cdot W$.*

Moreover, such a subset $\mathcal{D}'$ can be computed in polynomial time.

We introduce the following notation:
- For each i with $1 \leq i \leq m$, we denote the center of the disk D_i by c_i.
- Let $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ be the set of all center points.
- Let $\mathsf{DT}(\mathcal{C})$ be the Delaunay triangulation of the set $\mathcal{C}$.

▶ **Lemma 20.** *Let $\mathcal{V}$ be a vertex cover of* $\mathsf{DT}(\mathcal{C})$. *Then the disks in $\mathcal{D}$ whose centers are in $\mathcal{V}$ cover all points of $\mathcal{T}$.*

Proof. Let p be a point in $\mathcal{T}$. By assumption, there are two distinct indices i and j, such that p is in both the disks D_i and D_j. Let D be the unit-disk centered at the point p. Then both c_i and c_j are contained in D.

It is well-known that $\mathsf{DT}(\mathcal{C})$ contains a path Π from c_i to c_j that is completely inside D. A proof can be found in [6].

Let c_k be the second point on the path Π. Then (c_i, c_k) is an edge in $\mathsf{DT}(\mathcal{C})$ and both c_i and c_k are contained in D. Thus, p is contained in both D_i and D_k. Since $\mathcal{V}$ is a vertex cover, at least one of c_i and c_k is in $\mathcal{V}$. ◀

▶ Remark 21. We can choose c_i to be the point in $\mathcal{C}$ that is closest to p. Since p is in at least two disks of $\mathcal{D}$, there is an index j with $j \neq i$, such that p is in the disk D_j. Since $|pc_i| \leq |pc_j| \leq 1$, p is in the disk D_i.

Proof of Theorem 19.
- Since $\mathsf{DT}(\mathcal{C})$ is a planar graph, it is 4-colorable in polynomial time [36].
- Let $\mathcal{I}$ be the set of vertices in $\mathsf{DT}(\mathcal{C})$ in the color class of largest weight. Observe that $\mathcal{I}$ is an independent set in $\mathsf{DT}(\mathcal{C})$ and $\mathcal{V} = \mathcal{C} \setminus \mathcal{I}$ is a vertex cover of $\mathsf{DT}(\mathcal{C})$.
- Consider the subset of all disks in $\mathcal{D}$ whose centers are in $\mathcal{I}$. The total weight of all disks in this subset is at least $W/4$.
- Since $\mathcal{V}$ is a vertex cover of $\mathsf{DT}(\mathcal{C})$, it follows from Lemma 20 that the disks in

  $$\mathcal{D}' = \{D_i \mid c_i \in \mathcal{V}\}$$

  cover all points of $\mathcal{T}$.
- The total weight of all disks in $\mathcal{D}'$ is at most $(3/4) \cdot W$.

As all the steps in our proof is constructive in polynomial time, the moreover part also follows. ◀

▶ Remark 22. Theorem 19 also holds for axes-parallel unit-squares. To prove this, we use the L_∞-Delaunay triangulation of the centers of the squares.

4.2 The Extraction Theorem for Arbitrary Disks

In this section, we will prove the following result.

▶ **Theorem 23.** *Let $\mathcal{T} = \{t_1, t_2, \ldots, t_n\}$ be a set of n points in the plane, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ be a set of m weighted disks in the plane. For each i with $1 \leq i \leq m$, let $w_i \in \mathbb{R}^+$ denote the weight of the disk D_i, and let $W = \sum_{i=1}^{m} w_i$ denote the total weight of all disks in $\mathcal{D}$. Assume that each point of $\mathcal{T}$ is contained in at least two disks of $\mathcal{D}$. Then there exists a subset $\mathcal{D}'$ of $\mathcal{D}$ such that*
1. *the disks in $\mathcal{D}'$ cover all points of $\mathcal{T}$ and*
2. *the total weight of all disks in $\mathcal{D}'$ is at most $(3/4) \cdot W$.*

Moreover, such a subset $\mathcal{D}'$ can be computed in polynomial time.

Consider the following transformation f which maps points in $\mathbb{R}^2$ to half-spaces in $\mathbb{R}^3$, and maps disks in $\mathbb{R}^2$ to points in $\mathbb{R}^3$:
- Any point $t = (a, b)$ in $\mathbb{R}^2$ is mapped to the half-space $f(t)$ in $\mathbb{R}^3$ defined by

  $$2ax + 2by + z \geq a^2 + b^2.$$

- Any disk D in $\mathbb{R}^2$ with center (c, d) and radius R is mapped to the point

$$f(D) = \left(c, d, R^2 - c^2 - d^2\right) \text{ in } \mathbb{R}^3.$$

▶ **Lemma 24.** *Let t be a point in $\mathbb{R}^2$ and let D be a disk in $\mathbb{R}^2$. Then t is contained in D if and only if the point $f(D)$ is contained in the half-space $f(t)$.*

Proof. Let $t = (a, b)$ and let D have center (c, d) and radius R. The point t is in the disk D if and only if

$$(a - c)^2 + (b - d)^2 \leq R^2. \tag{1}$$

The point $f(D)$ is contained in the half-space $f(t)$ if and only if

$$2ac + 2bd + \left(R^2 - c^2 - d^2\right) \geq a^2 + b^2. \tag{2}$$

It is obvious that (1) and (2) are equivalent. ◀

Consider the point set $\mathcal{T} = \{t_1, t_2, \ldots, t_n\}$ and the disk set $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ in Theorem 23. We map each point t_i to the half-space $H_i = f(t_i)$ and we map each disk D_i to the point $p_i = f(D_i)$. We give each point p_i the weight w_i, i.e., the same weight as the corresponding disk D_i. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ and $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$.

Recall that each point of $\mathcal{T}$ is contained in at least two disks of $\mathcal{D}$. Lemma 24 implies that each half-space in $\mathcal{H}$ contains at least two points of $\mathcal{P}$. Since each half-space in $\mathcal{H}$ is unbounded in the positive z-direction, we have

$$\bigcup_{i=1}^{n} H_i \neq \mathbb{R}^3.$$

Therefore, Theorem 23 will follow from the following theorem.

▶ **Theorem 25.** *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a set of n half-spaces in $\mathbb{R}^3$, and let $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ be a set of m weighted points in $\mathbb{R}^3$. For each i with $1 \leq i \leq m$, let $w_i \in \mathbb{R}^+$ denote the weight of the point p_i, and let $W = \sum_{i=1}^{m} w_i$ denote the total weight of all points in $\mathcal{P}$. Assume that each half-space in $\mathcal{H}$ contains at least two points of $\mathcal{P}$ and*

$$\bigcup_{i=1}^{n} H_i \neq \mathbb{R}^3.$$

Then there exists a subset $\mathcal{P}'$ of $\mathcal{P}$ such that
1. *every half-space in $\mathcal{H}$ contains at least one point of $\mathcal{P}'$ and*
2. *the total weight of all points in $\mathcal{P}'$ is at most $(3/4) \cdot W$.*

Moreover, such a subset $\mathcal{P}'$ can be computed in polynomial time.

4.3 Proof of Theorem 25

For proving Theorem 25, we take inspiration from analyses of local search algorithms [34, 33, 12, 3, 9]. In particular, to analyze such a local search algorithm, one needs to show the existence of a bipartite support graph whose vertices are the objects from a local search solution and an optimal solution, and the graph has a small-sized vertex separator. Mustafa and Ray [34] gave such a planar bi-partite graph construction for the problem of hitting halfplanes in $\mathbb{R}^3$ by points. Our construction of the planar graph is inspired by their construction. However, we will present a more detailed proof.

▶ **Lemma 26.** *Assume we can define a planar graph G with vertex set $\mathcal{P}$ such that each half-space in $\mathcal{H}$ contains at least two points of $\mathcal{P}$ such that there is an edge in G between their corresponding vertices. Then the claim in Theorem 25 holds.*

Proof. Since G is a planar graph, it is 4-colorable. Let $\mathcal{I}$ be the set of vertices in G in the color class of largest weight. Observe that $\mathcal{I}$ is an independent set in G and $\mathcal{P}' = \mathcal{P} \setminus \mathcal{I}$ is a vertex cover of G. Also, since the total weight of all points in $\mathcal{I}$ is at least $W/4$, the total weight of all points in $\mathcal{P}'$ is at most $(3/4) \cdot W$.

Let H_i be a half-space in $\mathcal{H}$. By assumption, H_i contains an edge, say $p_j p_k$, of the graph G. Since $\mathcal{P}'$ is a vertex cover of G, at least one of p_j and p_k is in $\mathcal{P}'$. ◀

▶ Remark 27. A graph as in Lemma 26 is known as a *planar support* for the half-spaces in $\mathcal{H}$. See Raman and Ray [35]. Indeed, one approach to prove our extraction theorem is to use their planar support theorem for non-piercing regions, instead of the planar support for half-spaces that we pursue here.

It remains to define the planar graph G that satisfies the property in Lemma 26. Let $\mathsf{CH}(\mathcal{P})$ denote the convex hull of the point set $\mathcal{P}$. Suppose each point of $\mathcal{P}$ that is a vertex of $\mathsf{CH}(\mathcal{P})$ is colored *red*, whereas each other point of $\mathcal{P}$ is colored *blue*. The graph G contains the skeleton of $\mathsf{CH}(\mathcal{P})$. Note that this is a planar graph on the red points of $\mathcal{P}$. Below, we will show how to add edges from each blue point to some red points, such that the resulting graph is still planar.

Since, by assumption, $\bigcup_{i=1}^{n} H_i \neq \mathbb{R}^3$, we can choose a point o in $\mathbb{R}^3$ that is not in any half-space of $\mathcal{H}$. For each blue point b in $\mathcal{P}$, consider the ray with direction $\overrightarrow{ob}$ that emanates from b. Let $\pi(b)$ be the point on this ray that is on the boundary of $\mathsf{CH}(\mathcal{P})$.

For each face $\Delta = (r_1, r_2, r_3)$ of $\mathsf{CH}(\mathcal{P})$, we do the following. Let

$$\mathcal{Q} = \{b \in \mathcal{P} : \ b \text{ is a blue point and } \pi(b) \in \Delta\}.$$

Assume that $\mathcal{Q} \neq \emptyset$. We will add edges to G between the points in $\mathcal{Q}$ and the corners of Δ such that the following two properties hold:

P.1: At most one point in $\mathcal{Q}$ has edges to all three corners of Δ.

P.2: Each other point of $\mathcal{Q}$ has edges to exactly two corners of Δ.

▶ **Lemma 28.** *Assume that properties P.1 and P.2 hold for each face Δ of $\mathsf{CH}(\mathcal{P})$. Then the graph G is planar.*

Proof. As mentioned above, the graph G contains the skeleton of $\mathsf{CH}(\mathcal{P})$, which is planar. Consider a face Δ and the corresponding subset $\mathcal{Q}$ of blue points in $\mathcal{P}$. Each point b in $\mathcal{Q}$ is embedded as a point b' in the face Δ, as indicated in Figure 2. Each edge br in G between a point b in $\mathcal{Q}$ and a point r in $\{r_1, r_2, r_3\}$ is drawn as the line segment $b'r$ in Δ. In this way, we obtain a crossing-free embedding of G on the surface of $\mathsf{CH}(\mathcal{P})$. ◀

Consider again the face $\Delta = (r_1, r_2, r_3)$ of $\mathsf{CH}(\mathcal{P})$ and the corresponding subset $\mathcal{Q}$ of blue points in $\mathcal{P}$. To establish properties P.1 and P.2, we categorize each point of $\mathcal{Q}$ in the following way.

- A point b in $\mathcal{Q}$ is called *bad*, if for every corner c of Δ, there exists a half-space H in $\mathbb{R}^3$, such that

  $$H \cap (\mathcal{Q} \cup \{r_1, r_2, r_3, o\}) = \{b, c\}.$$

  Note that H is not necessarily a half-space in $\mathcal{H}$.

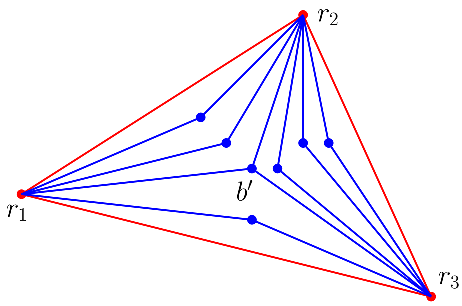

Figure 2 The blue points in the set $\mathcal{Q}$ are embedded in the face $\Delta = (r_1, r_2, r_3)$. The blue point b' is the embedding of the blue point b in $\mathcal{Q}$ that satisfies property P.1. Each other blue point is the embedding of a blue point in $\mathcal{Q}$ that satisfies property P.2.

- A point b in $\mathcal{Q}$ is called *good*, if there exists a corner c of Δ, such that for every half-space H in $\mathbb{R}^3$,

  $$H \cap (\mathcal{Q} \cup \{r_1, r_2, r_3, o\}) \neq \{b, c\}.$$

  For each point b in $\mathcal{Q}$, we add the following edges to the graph G.

- If b is a bad point, then we add edges between b and all three corners of Δ.
- Assume that b is a good point. Let c be a corner of Δ in the definition of b being good. Then we add edges between b and the other two corners of Δ.

The following lemma states that there cannot be more than one bad point in $\mathcal{Q}$. This will imply that properties P.1 and P.2 are satisfied. The proof of this lemma will be given in Section 4.4.

▶ **Lemma 29.** *For any face Δ of $\mathsf{CH}(\mathcal{P})$, the set $\mathcal{Q}$ contains at most one bad point.*

We have defined the planar graph G with vertex set $\mathcal{P}$. It remains to prove that this graph satisfies the property in Lemma 26.

Let H_i be an arbitrary half-space in $\mathcal{H}$. We will show that H_i contains at least one edge of the graph G.

Recall that, by assumption, H_i contains at least two points of $\mathcal{P}$. Thus, H_i contains at least one vertex of $\mathsf{CH}(\mathcal{P})$. First assume that H_i contains at least two vertices of $\mathsf{CH}(\mathcal{P})$. Then H_i contains at least one edge of $\mathsf{CH}(\mathcal{P})$, which is an edge in G.

From now on, we assume that H_i contains exactly one vertex, say r, of $\mathsf{CH}(\mathcal{P})$. Thus, H_i contains at least one blue point of $\mathcal{P}$, i.e., a point in the interior of $\mathsf{CH}(\mathcal{P})$. Let H be a translate of H_i such that (i) r is the only red point in H and (ii) H contains exactly one blue point, say b.

Recall that the point o is not in H_i and, therefore, not in H. Let Δ be the face that contains the point $\pi(b)$. Then r is a corner of Δ.

If b is a bad point, then, by construction, br is an edge in G. This edge is in H and, thus, in H_i.

Now assume that b is a good point. Let r_1, r_2, and r_3 be the three corners of Δ. We already saw that $r \in \{r_1, r_2, r_3\}$. Recall that

$$\mathcal{Q} = \{b \in \mathcal{P} : \ b \text{ is a blue point and } \pi(b) \in \Delta\}.$$

The point b is in this set $\mathcal{Q}$. For the translated half-space H of H_i introduced above, we have

$$H \cap (\mathcal{Q} \cup \{r_1, r_2, r_3, o\}) = \{b, r\}.$$

Thus, the corner c in the definition of b being good is not equal to r. By construction, br is an edge in G. This edge is in H and, thus, in H_i.

This concludes the proof that the planar graph G satisfies the property in Lemma 26. As a result, we have proved Theorem 25. As the construction of G can be done in polynomial time, the moreover part also follows.

4.4 Proof of Lemma 29

Proof (Lemma 29). Consider a face $\Delta = (r_1, r_2, r_3)$ of $\mathsf{CH}(\mathcal{P})$ and the corresponding subset

$$\mathcal{Q} = \{b \in \mathcal{P} : \ b \text{ is a blue point and } \pi(b) \in \Delta\}$$

of blue points in $\mathcal{P}$. We will show that $\mathcal{Q}$ contains at most one bad point. The proof is by contradiction. Thus, we assume that $\mathcal{Q}$ contains two bad points, say b_1 and b_2. Let $F = \{b_1, b_2, r_1, r_2, r_3\}$. We distinguish two cases.

Case 1. The point set F is in convex position.

By Radon's Theorem (see, e.g., Matoušek [32, Section 1.3]), F can be partitioned into two subsets F_1 and F_2, such that $\mathsf{CH}(F_1) \cap \mathsf{CH}(F_2) \neq \emptyset$. We may assume without loss of generality that $|F_1| < |F_2|$.

Since F is in convex position, and b_1 and b_2 are on the same side of the plane through r_1, r_2, and r_3, each of the following three cases implies that $\mathsf{CH}(F_1) \cap \mathsf{CH}(F_2) = \emptyset$: (i) $|F_1| = 1$ and $|F_2| = 4$, (ii) $F_1 = \{b_1, b_2\}$ and $F_2 = \{r_1, r_2, r_3\}$, (iii) F_1 contains two elements of $\{r_1, r_2, r_3\}$, say r_1 and r_2, and $F_2 = \{b_1, b_2, r_3\}$.

Thus, we may assume without loss of generality that $F_1 = \{b_1, r_1\}$ and $F_2 = \{b_2, r_2, r_3\}$. Since b_1 is a bad point, by taking the corner $c = r_1$, there exists a half-space H in $\mathbb{R}^3$, such that

$$H \cap (Q \cup \{r_1, r_2, r_3, o\}) = \{b_1, r_1\}.$$

The bounding plane of H separates the points b_1 and r_1 from the points b_2, r_2, and r_3. Therefore, $\mathsf{CH}(F_1) \cap \mathsf{CH}(F_2) = \emptyset$, which is a contradiction.

Case 2. The point set F is not in convex position.

Since b_1 and b_2 are on the same side of the plane through r_1, r_2, and r_3, we may assume without loss of generality that

$$b_1 \in \mathsf{CH}(o, r_1, r_2, r_3)$$

and

$$b_2 \in \mathsf{CH}(b_1, r_1, r_2, r_3);$$

see the left part of Figure 3.

Consider the ray with direction $\overrightarrow{ob_1}$ that emanates from o. Let p be the first point on this ray that is on the boundary of $\mathsf{CH}(b_2, r_1, r_2, r_3)$. Note that p is contained in one of the three triangles (b_2, r_1, r_2), (b_2, r_1, r_3), and (b_2, r_2, r_3). We may assume without loss of generality that p is in (b_2, r_1, r_2); see the left part of Figure 3.

Since b_1 is on the line segment op, we have

$$b_1 \in \mathsf{CH}(o, b_2, r_1, r_2);$$

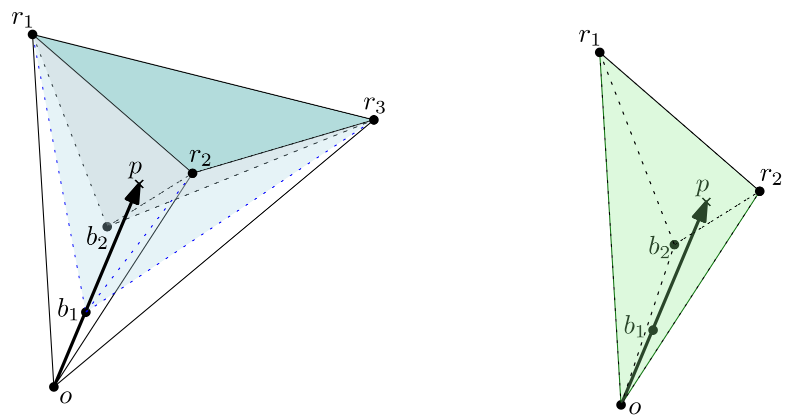

Figure 3 Illustrating Case 2 in the proof of Lemma 29.

see the right part of Figure 3. It follows that every half-space in $\mathbb{R}^3$ that contains b_1 also contains at least one of the points o, b_2, r_1, and r_2.

Since b_1 is a bad point, by taking the corner $c = r_3$, there exists a half-space H in $\mathbb{R}^3$, such that

$$H \cap (Q \cup \{r_1, r_2, r_3, o\}) = \{b_1, r_3\}.$$

This is a contradiction, because H contain at least one of the points o, b_2, r_1, and r_2. ◀

5 Extraction Theorem for Objects with Linear Union Complexity

In this section, we prove the following theorem.

▶ **Theorem 30.** *Let $\mathcal{T}$ be a set of n points, and let $\mathcal{D} = \{D_1, D_2, \dots, D_m\}$ be a set of m weighted regions such that any subset $\mathcal{D}_1 \subseteq \mathcal{D}$ of regions has $O(|\mathcal{D}_1|)$ union complexity. For each i with $1 \le i \le m$, let $w_i \in \mathbb{R}^+$ denote the weight of the region D_i, and let $W = \sum_{i=1}^m w_i$ denote the total weight of all regions in $\mathcal{D}$. Assume that each point of $\mathcal{T}$ is contained in at least two regions of $\mathcal{D}$. Then there exists a subset $\mathcal{D}'$ of $\mathcal{D}$ such that*

1. *the regions in $\mathcal{D}'$ cover all points of $\mathcal{T}$ and*
2. *the total weight of all regions in $\mathcal{D}'$ is $(1 - \Omega(1)) \cdot W$.*

Moreover, a subset $\mathcal{D}'$ of $\mathcal{D}$ can be computed in polynomial time that satisfies Property 1, and the expected total weight of all regions in $\mathcal{D}'$ is $(1 - \Omega(1)) \cdot W$.

We prove this theorem using a result due to Ene et al. [21]. In this process, we establish a connection between their work and Extraction Theorem for geometric regions. Ene et al. studied the following packing problem.

▶ **Problem 31** (PackRegions)**.** *Given a set $\mathcal{R}$ of regions and a set P of points such that each region r has a weight $w(r)$ and each point p has a capacity $\#(p)$, find a maximum weight subset $X \subseteq \mathcal{R}$ of the regions such that, for each point p, the number of regions in X that contain p is at most its capacity $\#(p)$.*

Considering this problem they design LP rounding based approximation algorithms for a wide range of regions. In particular, they obtain $O(1)$-approximations for regions having linear union complexity. We will make use of this result to prove the Extraction Theorem for regions having linear union complexity, albeit with a constant larger than 4.

To prove Theorem 30, first we construct an instance $\mathcal{I}$ of the PackRegions problem. The set of regions $\mathcal{R} := \mathcal{D}$ and the set of points $P := \mathcal{T}$. The weight of each region $r \in \mathcal{R}$ is w_i where $r = D_i \in \mathcal{D}$. For each point p, its capacity $\#(p)$ is set to be the number of regions it is contained in minus 1. Then we have the following observation.

▶ **Observation 32.** *Consider the instance $\mathcal{I}$ and suppose $X \subseteq \mathcal{R}$ be such that, for each point $p \in P$, the number of regions in X that contain p is at most its capacity $\#(p)$. Then $\mathcal{R} \setminus X$ covers the points of P.*

Proof. The observation precisely follows from the way we set the capacity for each point. In particular, for each point p, there is at least one region not in X that contains p. Hence, the complement set of regions $\mathcal{R} \setminus X$ covers p. ◀

In the following, we show a polynomial-time construction of a set X as in the above observation that has an expected total weight of $\Omega(W)$. This will complete the proof of Theorem 30 by setting $\mathcal{D}' = \mathcal{R} \setminus X$. In the rest of this section, we describe this construction.

The $O(1)$-approximation result in [21] is based on the following natural LP relaxation for PackRegions.

$$\begin{array}{lll} \text{maximize} & \sum_{r\in\mathcal{R}} w(r)\cdot x_r & \text{(PackRegions-LP)} \\ \text{subject to} & \sum_{p\in r} x_r \le \#(p) & \forall p \in P, \\ & 0 \le x_r \le 1 & \forall r \in \mathcal{R}. \end{array}$$

The following result is due to Ene et al. [21].

▶ **Proposition 33.** *Let $\mathcal{R}$ be a set of regions having linear union complexity. Then there is a polynomial-time randomized scheme that rounds any optimal solution to* PackRegions-LP *having value* opt *to an integral solution with expected value* $\Omega(\text{opt})$*.*

To obtain the desired subset $X \subseteq \mathcal{R}$, we first show that the value of any optimal fractional solution to PackRegions-LP for our instance $\mathcal{I}$ is at least $W/2$. Then by Proposition 33 it follows that there is a subset $X \subseteq \mathcal{R}$ as in Observation 32 whose total weight is $\Omega(W)$. By the same proposition, a subset $X \subseteq \mathcal{R}$ as in Observation 32 can be computed in polynomial time whose expected value is $\Omega(W)$. Hence, the moreover part of Theorem 30 also follows.

▶ **Lemma 34.** *The value of any optimal fractional solution to* PackRegions-LP *for the instance $\mathcal{I}$ is at least $W/2$.*

Proof. To prove the lemma, we show the existence of a fractional solution having value exactly $W/2$. In fact, we construct such a solution in the following simple way. Set the value of each x_r to $1/2$. The value of this solution is $W/2$. To prove that it is a feasible solution, let us consider any point p. Let $\mathcal{R}_p$ be the set of regions that contain p. Note that $\#(p) = |\mathcal{R}_p| - 1$. Now,

$$\sum_{p\in r} x_r = |\mathcal{R}_p|/2 \le |\mathcal{R}_p| - 1 = \#(p).$$

The above inequality follows, as the size of $\mathcal{R}_p$ is at least 2. This completes the proof of the lemma. ◀

6 Conclusions

In this work, we designed polynomial-time constant-approximations for two important geometric covering problems, namely DC-2 and DC. In the process of designing our algorithms, we proved a fundamental theorem called Extraction Theorem. Such a theorem helps us obtain $O(1)$-approximations for several natural geometric covering problems involving fairly general regions. Many questions have been left open by our work. In the following, we list some of those.

A natural question is to find out whether DC-2 admits a PTAS. If we have t arbitrary radii as in DC, does the problem become APX-hard?

We also proved an Extraction theorem for more general geometric objects, albeit with a large constant. Proving an Extraction theorem for general objects with a reasonably better constant remains an interesting open question.

Lastly, our Extraction theorems lead to a max-gain version of geometric covering and directly yield an $O(1)$-approximation for this version with a wide range of objects. The extraction theorems have other implications for the approximability of max-gain geometric covering, which we explain using the example of disks in the plane. Given a set $\mathcal{D}$ of disks along with a weight function $w : \mathcal{D} \to \mathbb{R}^+$ and a set T of n points in the plane, the goal in max-gain covering is to find a subset $\mathcal{D}' \subseteq \mathcal{D}$, such that $\mathcal{D}'$ covers T and the sum of the weights of the disks in $\mathcal{D} \setminus \mathcal{D}'$ is maximized. The dual problem of minimum weight disk cover seeks to minimize the sum of the weights of the disks in $\mathcal{D}'$. Let us say that a disk D_i is *necessary* if there is some point $p \in T$ such that D_i is the only disk in $\mathcal{D}$ that covers p. Any cover of T must include all the necessary disks. Therefore, we remove the necessary disks and all points covered by them to obtain a reduced instance $(\mathcal{D}', T')$. In this reduced instance, every point in T' is covered by at least two disks in $\mathcal{D}'$. Thus, our extraction theorems tell us that there is a cover for the reduced instance such that the total weight of the disks not in the cover is at least $W(\mathcal{D}')/4$.

While this gives a 4-approximation for the max-gain problem, it also follows that to obtain a $(1 + \epsilon)$ approximation it suffices to have an approximation algorithm for the reduced problem with *additive error* $\epsilon \cdot W(\mathcal{D}')$. For the case of unit disks, such an additive approximation readily follows because the dual problem of minimum weight disk cover has a PTAS [30]. For arbitrary disks also, such an additive approximation seems within reach using known techniques. It would therefore be interesting to investigate the approximability of the max-gain covering problem for other families of objects, beyond disks and pseudo-disks.

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