A Natural Intuitionistic Modal Logic: Axiomatization and Bi-Nested Calculus

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Abstract

We introduce **FIK**, a natural intuitionistic modal logic specified by Kripke models satisfying the condition of forward confluence. We give a complete Hilbert-style axiomatization of this logic and propose a bi-nested calculus for it. The calculus provides a decision procedure as well as a countermodel extraction: from any failed derivation of a given formula, we obtain by the calculus a finite countermodel of it directly.

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1 Introduction

Intuitionistic modal logic (**IML**) has a long history, starting from the pioneering work by Fitch [5] in the late 40's and Prawitz [12] in the 60's. Along the time, two traditions emerged that led to the study of two different families of systems. The first tradition, called *intuitionistic modal logics*, has been introduced by Fischer Servi [13, 14, 15], Plotkin and Stirling [11] and then systematized by Simpson [16]. Its main goal is to define an analogous of classical modalities justified from an intuitionistic meta-theory. The basic modal logic in this tradition, **IK**, is intended to be the intuitionistic counterpart of the minimal normal modal logic **K**. The second tradition leads to so-called *constructive modal logics* that are mainly motivated by their applications in computer science such as type-theoretic interpretations, verification and knowledge representation (contextual reasoning). This second tradition has been developed independently, first by Wijesekera [17] who proposed the system **CCDL** (Constructive Concurrent Dynamic logic), and then by Bellin, De Paiva, and Ritter [2], among others who proposed the logic **CK** (Constructive **K**) as the basic system for a constructive account of modality.

But putting aside the historical perspective, we can consider naively the following question: how can we build "from scratch" an **IML**? Since both modal logic and intuitionistic logic enjoy Kripke semantics, we can think of combining them together in order to define an intuitionistic modal logic. The simplest proposal is to consider Kripke models equipped with two relations, $\leq$ for intuitionistic implication and R for modalities. Propositional intuitionistic connectives (in particular implication) have their usual interpretations. We request that every valid formula or rule scheme of propositional intuitionistic logic **IPL** is also valid in **IML**. To reach this goal, we must ensure the *hereditary property*, which means for any formula A, if A is forced by a world, it will also be forced also by all its uppers worlds, namely:

if $x \Vdash A$ and $x \leq y$ then also $y \Vdash A$.

Thus the question becomes how to define modalities in order to ensure this property. The simplest solution is to build the hereditary property in the forcing conditions for $\Box$ and $\Diamond$:

(1) $x \Vdash \Box A$ iff for all x' with $x' \geq x$, for all y with $Rx'y$ it holds $y \Vdash A$ and
(1') $x \Vdash \Diamond A$ iff for all x' with $x' \geq x$, there exists y with $Rx'y$ s.t. $y \Vdash A$.

Observe that the definition of $\Box A$ is reminiscent of the definition of $\forall$ in intuitionistic first-order logic. This logic is nothing else than the propositional part of Wijeskera's **CCDL** mentioned above and is *non-normal* as it does not contain all formulas of the form

(DP) $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$.

Moreover, the logic does not satisfy the maximality criteria, one of the criteria stated by Simpson [16, Chapter 3] for a "good" **IML** since by adding any classical principle to it, we cannot get the classical normal modal logic **K**. In addition, **CCDL** has also been criticized for being *too strong*, as it still satisfies the *nullary* $\Diamond$ distribution: $\Diamond\bot \supset \bot$. By removing this last axiom, the constructive modal logic **CK** is obtained.

However, the opposite direction is also possible: we can make local the definition of $\Diamond$ (pursuing the analogy with $\exists$ in intuitionistic first-order logic **FOIL**) exactly as in classical **K**, that is:

(2) $x \Vdash \Diamond A$ iff there exists y with Rxy s.t. $y \Vdash A$.

In this way we recover $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$, making the logic *normal*. But there is a price to pay: nothing ensures that the hereditary property holds for $\Diamond$-formulas. In order to solve this problem, we need to postulate some frame conditions. The most natural (and maybe the weakest) condition is simply that if $x' \geq x$ and x has an R-accessible y then also x' must have an R-accessible y' which refines y, which means $y' \geq y$. This condition is called *Forward Confluence* in [1]. It is not new as it is also called (F1) by Simpson [16, Chapter 3] and together with another frame conditions (F2) characterizes the very well-known system **IK** by Fischer-Servi and Simpson. Although from a meta-theoretical point of view **IK** can be justified by its standard translation in first-order intuitionistic logic, it does not seem to be the minimal system allowing the definition of modalities as in (1) and (2) above.

This paper attempts to fill the gap by studying a weaker logic for which the forcing conditions for modalities are just (1) and (2) above and we assume *only* Forward Confluence for the frames. We call this logic **FIK** for *forward confluenced* **IK**. As far as we know, this logic has never been studied before. And we think it is well worth being studied since it seems to be the minimal logic defined by bi-relational models with forcing conditions (1) and (2) which preserves intuitionistic validity.

In the following sections, we first give a sound and complete Hilbert axiomatization of **FIK**. We show that **FIK** finds its place in the **IML**/constructive family: it is strictly stronger than **CCDL** (whence than **CK**) and strictly weaker than **IK**. At the same time **FIK** seems acceptable to be regarded as an **IML** since it satisfies *all* criteria proposed by Simpson, including the one about maximality, which means by adding any classical principle to **FIK**, we can get the classical normal modal logic **K**. All in all **FIK** seems to be a respectable intuitionistic modal logic and is a kind of "third way" between intuitionistic **IK** and constructive **CCDL**/**CK**.

We then investigate **FIK** from a proof-theoretic viewpoint. We propose a nested sequent calculus $\mathbf{C_{FIK}}$ which makes use of two kinds of nestings, one for representing $\geq$-upper worlds and the other for R-related worlds. A nested sequent calculus for (first-order) intuitionistic logic that exploits the first type of nesting has been proposed in [6], so our calculus can be seen as an extension of the propositional part of it. More recently in [4], the authors present a sequent calculus with the same kind of nesting to capture the **IML** logic given by $\mathbf{CCDL} + (DP)$ [1].

As mentioned, our calculus contains a double type of nesting. The use of this double nesting is somewhat analogous to the labelled calculus proposed in [10] which introduces two kinds of relations on labels in the syntax. However, the essential ingredient of our calculus $\mathbf{C_{FIK}}$ is the *interactive rule* between the two kinds of nested sequents that captures the specific Forward Confluence condition.

We also prove that the calculus $\mathbf{C_{FIK}}$ provides a decision procedure for the logic **FIK**. In addition, since the rules of $\mathbf{C_{FIK}}$ are invertible, we show that from a failed derivation under a suitable strategy, it is possible to extract a finite countermodel of the formula or sequent at the root of the derivation. This result allows us to obtain a constructive proof of the finite model property, which means if a formula is not valid then it has a finite countermodel.

2 A natural intuitionistic modal logic

Firstly, we present the syntax and semantics of forward confluenced intuitionistic modal logic **FIK**. Secondly, we present an axiom system and we prove its soundness and completeness. Thirdly, we discuss whether **FIK** satisfies the properties that are expected from intuitionistic modal logics.

▶ **Definition 1** (Formulas). *The set $\mathcal{L}$ of all formulas (denoted as A, B, etc.) is generated by the following grammar: $A ::= \ p \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$ where p ranges over a countable set of atomic propositions At. We omit parentheses for readability. For all formulas A, we write $\neg A$ instead of $A \supset \bot$. For all formulas A, B, we write $A \equiv B$ instead of $(A \supset B) \wedge (B \supset A)$. The size of a formula A is denoted $|A|$.*

▶ **Definition 2** (Bi-relational model). *A bi-relational model is a quadruple $\mathcal{M} = (W, \leq, R, V)$ where W is a nonempty set of worlds, $\leq$ is a pre-order on W, R is a binary relation on W and $V : \ W \longrightarrow \wp(\mathsf{At})$ is a valuation on W satisfying the following hereditary condition:*

$$\forall x, y \in W, \ (x \leq y \ \Rightarrow \ V(x) \subseteq V(y)).$$

The triple $(W, \leq, R)$ is called a frame. For all $x, y \in W$, we write $x \geq y$ instead of $y \leq x$. Moreover, we say "y is a successor of x" when Rxy.

[1] A calculus for **IK** with the same kind of nesting was also preliminarily considered in [9]

It is worth mentioning that an upper world of a successor of a world is not necessarily a successor of an upper world of that world. However, from now on in this paper, we only consider models $\mathcal{M} = (W, \leq, R, V)$ that satisfy the following condition called *Forward Confluence* as in [1]:

(FC) $\forall x, y \in W,\ (\exists z \in W,\ (x \geq z \ \&\ Rzy) \ \Rightarrow\ \exists t \in W,\ (Rxt \ \&\ t \geq y))$.

▶ **Definition 3** (Forcing relation). *Let $\mathcal{M} = (W, \leq, R, V)$ be a bi-relational model and $w \in W$. The forcing conditions are the usual ones for atomic propositions and for formulas constructed by means of the connectives $\bot, \top$, $\wedge$ and $\vee$. For formulas constructed by means of the connectives $\supset$, $\Box$ and $\Diamond$, the forcing conditions are as follows:*
- $\mathcal{M}, w \Vdash B \supset C$ *iff for all $w' \in W$ with $w \leq w'$ and $\mathcal{M}, w' \Vdash B$, $\mathcal{M}, w' \Vdash C$;*
- $\mathcal{M}, w \Vdash \Box B$ *iff for all $w', v' \in W$ with $w \leq w'$ and $Rw'v'$, $v' \Vdash B$;*
- $\mathcal{M}, w \Vdash \Diamond B$ *iff there exists $v \in W$ with Rwv and $\mathcal{M}, v \Vdash B$.*

We also abbreviate $\mathcal{M}, w \Vdash A$ as $w \Vdash A$ if the model is clear from the context.

▶ **Proposition 4.** *Let $(W, \leq, R, V)$ be a bi-relational model. For all formulas A in $\mathcal{L}$ and for all $x, y \in W$ with $x \leq y$, $x \Vdash A$ implies $y \Vdash A$.*

Proposition 4 is proved by induction on the size of A using (FC) for the case of $A = \Diamond B$.

▶ **Definition 5** (Validity). *A formula A in $\mathcal{L}$ is valid, denoted $\Vdash A$, if for any bi-relational model $\mathcal{M}$ and any world w in it, $\mathcal{M}, w \Vdash A$. Let **FIK** be the set of all valid formulas.*

Obviously, **FIK** contains all standard axioms of **IPL**. Moreover, **FIK** is closed with respect to the following inference rules:

$$\frac{p \supset q, p}{q}\ (\mathbf{MP}) \qquad \frac{p}{\Box p}\ (\mathbf{NEC})$$

Finally, **FIK** contains the following formulas:

$(\mathbf{K}_\Box)$ $\Box(p \supset q) \supset (\Box p \supset \Box q)$,
$(\mathbf{K}_\Diamond)$ $\Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$,
$(\mathbf{N})$ $\neg\Diamond\bot$,
$(\mathbf{DP})$ $\Diamond(p \vee q) \supset \Diamond p \vee \Diamond q$,
$(\mathbf{wCD})$ $\Box(p \vee q) \supset ((\Diamond p \supset \Box q) \supset \Box q)$.

We only show the validity of $(\mathbf{wCD})$. Suppose $\nVdash \Box(p \vee q) \supset ((\Diamond p \supset \Box q) \supset \Box q)$. Hence, there exists a model $(W, \leq, R, V)$ and $w \in W$ such that $w \Vdash \Box(p \vee q)$, $w \Vdash \Diamond p \supset \Box q$ and $w \nVdash \Box q$. Thus, let $u, v \in W$ be such that $w \leq u$, Ruv and $v \nVdash q$. Since $w \Vdash \Box(p \vee q)$, $v \Vdash p \vee q$. Since $v \nVdash q$, $v \Vdash p$. Since Ruv, $u \Vdash \Diamond p$. Since $w \Vdash \Diamond p \supset \Box q$ and $w \leq u$, $u \Vdash \Diamond p \supset \Box q$. Since $u \Vdash \Diamond p$, $u \Vdash \Box q$. Since Ruv, $v \Vdash q$: a contradiction.

▶ **Definition 6** (Axiom system). *Let $\mathbf{D}_{\boldsymbol{FIK}}$ be the Hilbert-style axiom system consisting of all standard axioms of **IPL**, the inference rules* $(\mathbf{MP})$ *and* $(\mathbf{NEC})$ *and the formulas* $(\mathbf{K}_\Box)$, $(\mathbf{K}_\Diamond)$, $(\mathbf{N})$, $(\mathbf{DP})$ *and* $(\mathbf{wCD})$ *considered as axioms. Derivations are defined as usual. For all formulas A, we write $\vdash A$ when A is $\mathbf{D}_{\boldsymbol{FIK}}$-derivable. The set of all $\mathbf{D}_{\boldsymbol{FIK}}$-derivable formulas will also be denoted $\mathbf{D}_{\boldsymbol{FIK}}$.*

The formulas $(\mathbf{K}_\Box)$, $(\mathbf{K}_\Diamond)$, $(\mathbf{DP})$ and $(\mathbf{N})$ are not new, seeing that they have already been used by many authors as axioms in multifarious variants of **IML**. As for the formula $(\mathbf{wCD})$, as far as we are aware, it is used here for the first time as an axiom of an **IML** variant. Indeed, $(\mathbf{wCD})$ is derivable in **IK**. Moreover, it is a weak form of the *Constant Domain* axiom $(\mathbf{CD}): \Box(p \vee q) \supset \Diamond p \vee \Box q$ used in [1]. In other respect, $(\mathbf{wCD})$ is derivable in $\mathbf{IK}$, whereas it is not derivable in $\mathbf{CCDL}/\mathbf{CK}$. As for the $\mathbf{IK}$ axiom $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$, it is not in $\mathbf{FIK}$ as it will be also constructively shown by using the calculus presented in next section. Therefore, we get $\mathbf{CK} \subset \mathbf{CCDL} \subset \mathbf{FIK} \subset \mathbf{IK}$. We can consider also the logic $\mathbf{CCDL} + (\mathbf{DP})$ $(= \mathbf{CK} + (\mathbf{N}) + (\mathbf{DP}))$ recently studied in [4], according to the results in that paper, we get that $\mathbf{CCDL} + (\mathbf{DP}) \subset \mathbf{FIK}$.

▶ **Theorem 7** (Soundness). $\mathbf{D}_{\boldsymbol{FIK}} \subseteq \boldsymbol{FIK}$, *i.e. for all formulas* A, *if* $\vdash A$ *then* $\Vdash A$.

Theorem 7 can be proved by induction on the length of the derivation of A. Later, we will prove the converse inclusion (Completeness) saying that $\mathbf{FIK} \subseteq \mathbf{D}_{\mathbf{FIK}}$. At the heart of our proof of completeness, will be the concept of theory.

▶ **Definition 8** (Theories). *A theory is a set of formulas containing* $\mathbf{D}_{\boldsymbol{FIK}}$ *and closed with respect to* $\mathbf{MP}$. *A theory* Γ *is proper if* $\bot \notin \Gamma$. *A proper theory* Γ *is prime if for all formulas* A, B, *if* $A \vee B \in \Gamma$ *then either* $A \in \Gamma$, *or* $B \in \Gamma$. *For all theories* Γ *and for all formulas* A, *let* $\Gamma + A = \{B \in \mathcal{L} : A \supset B \in \Gamma\}$ *and* $\Box\Gamma = \{A \in \mathcal{L} : \Box A \in \Gamma\}$.

Obviously, $\mathbf{D}_{\mathbf{FIK}}$ is the least theory and $\mathcal{L}$ is the greatest theory. Moreover, for all theories Γ, Γ is proper if and only if $\Gamma \neq \mathcal{L}$ if and only if $\Diamond\bot \notin \Gamma$.

▶ **Lemma 9.** *For all theories* Γ *and for all formulas* A, *(i)* $\Gamma + A$ *is the least theory containing* Γ *and* A; *(ii)* $\Gamma + A$ *is proper if and only if* $\neg A \notin \Gamma$; *(iii)* $\Box\Gamma$ *is a theory.*

Lemma 9 can be proved by using standard axioms of $\mathbf{IPL}$, inference rules $(\mathbf{MP})$ and $(\mathbf{NEC})$ and axiom $\mathbf{K}_{\Box}$.

▶ **Lemma 10** (Lindenbaum's Lemma). *Let* A *be a formula. If* $A \notin \mathbf{D}_{\boldsymbol{FIK}}$ *then there exists a prime theory* Γ *such that* $A \notin \Gamma$.

▶ **Definition 11** (Canonical model). *Let* $\bowtie$ *be the binary relation between sets of formulas such that for all sets* Δ, Λ *of formulas,* $\Delta \bowtie \Lambda$ *iff for all formulas* B, *the following conditions hold: (i) if* $\Box B \in \Delta$ *then* $B \in \Lambda$ *and (ii) if* $B \in \Lambda$ *then* $\Diamond B \in \Delta$.

Let $(W_c, \leq_c, R_c)$ *be the frame such that* W_c *is the set of all prime theories,* $\leq_c$ *is the inclusion relation on* W_c *and* R_c *is the restriction of* $\bowtie$ *to* W_c. *For all* $\Gamma, \Delta \in W_c$, *we write "*$\Gamma \geq_c \Delta$*" instead of "*$\Delta \leq_c \Gamma$*". Let* $V_c : W_c \longrightarrow \wp(\mathit{At})$ *be the valuation on* W_c *such that for all* Γ *in* W_c, $V_c(\Gamma) = \Gamma \cap \mathit{At}$.

By Theorem 7, $\bot \notin \mathbf{D}_{\mathbf{FIK}}$. Hence, by Lemma 10, W_c is nonempty.

▶ **Lemma 12.** $(W_c, \leq_c, R_c, V_c)$ *satisfies the frame condition* $(\mathbf{FC})$.

The proof of the completeness will be based on the following lemmas.

▶ **Lemma 13** (Existence Lemma). *Let* Γ *be a prime theory and* B, C *be formulas.*

1. *If* $B \supset C \notin \Gamma$ *then there exists a prime theory* Δ *such that* $\Gamma \subseteq \Delta$, $B \in \Delta$ *and* $C \notin \Delta$,
2. *if* $\Box B \notin \Gamma$ *then there exists prime theories* Δ, Λ *such that* $\Gamma \subseteq \Delta$, $\Delta \bowtie \Lambda$ *and* $B \notin \Lambda$,
3. *if* $\Diamond B \in \Gamma$ *then there exists a prime theory* Δ *such that* $\Gamma \bowtie \Delta$ *and* $B \in \Delta$.

▶ **Lemma 14** (Truth Lemma). *For all formulas* A *and for all* $\Gamma \in W_c$, $A \in \Gamma$ *if and only if* $\Gamma \Vdash A$.

The proof of Lemma 14 can be done by induction on the size of A. The case when A is an atomic proposition is by definition of V_c. The cases when A is of the form $\bot, \top$, $B \wedge C$ and $B \vee C$ are as usual. The cases when A is of the form $B \supset C$, $\Box B$ and $\Diamond B$ use the Existence Lemma.

As for the proof of Theorem 15, it can be done by contraposition. Indeed, if $\nvdash A$ then by Lemma 10, there exists a prime theory Γ such that $A \notin \Gamma$. Thus, by Lemma 14, $\Gamma \nVdash A$. Consequently, $\nVdash A$.

▶ **Theorem 15** (Completeness). ***FIK*** $\subseteq$ $\mathbf{D}_{\boldsymbol{FIK}}$*, i.e. for all formulas A, if $\Vdash A$ then $\vdash A$.*

As mentioned above, there exists many variants of **IML**. Therefore, one may ask how *natural* is the variant we consider here. Simpson [16, Chapter 3] discusses the formal features that might be expected of an **IML L**:

(C_1) **L** is conservative over **IPL**,
(C_2) **L** contains all substitution instances of **IPL** and is closed under (**MP**),
(C_3) for all formulas A, B, if $A \vee B$ is in **L** then either A is in **L**, or B is in **L**,
(C_4) the addition of the law of excluded middle to **L** yields modal logic **K**,
(C_5) $\Box$ and $\Diamond$ are independent in **L**.

The fact that $\mathbf{D}_{\mathbf{FIK}}$ satisfies features (C_1) and (C_2) is an immediate consequence of Theorems 7 and 15. The fact that $\mathbf{D}_{\mathbf{FIK}}$ satisfies feature (C_3) will be proved in Section 3. Concerning feature (C_4), let $\mathbf{D}_{\mathbf{FIK}}{}^{+}$ be the Hilbert-style axiom system consisting of $\mathbf{D}_{\mathbf{FIK}}$ plus the law $p \vee \neg p$ of excluded middle. The set of all $\mathbf{D}_{\mathbf{FIK}}{}^{+}$-derivable formulas will also be denoted $\mathbf{D}_{\mathbf{FIK}}{}^{+}$. Obviously, $\mathbf{D}_{\mathbf{FIK}}{}^{+}$ contains all substitution instances of **CPL** and is closed under (**MP**). Moreover, it contains all substitution instances of $(\mathbf{K}_{\Box})$ and is closed under (**NEC**). Therefore, in order to prove that $\mathbf{D}_{\mathbf{FIK}}$ satisfies feature (C_4), it suffices to prove

▶ **Lemma 16.** *$\Diamond p \equiv \neg\Box\neg p$ is in $\mathbf{D}_{\boldsymbol{FIK}}{}^{+}$.*

The fact that $\mathbf{D}_{\mathbf{FIK}}$ satisfies feature (C_5) is a consequence of

▶ **Lemma 17.** *Let p be an atomic proposition. There exists no $\Box$-free A such that $\Box p \equiv A$ is in $\mathbf{D}_{\boldsymbol{FIK}}$ and there exists no $\Diamond$-free A such that $\Diamond p \equiv A$ is in $\mathbf{D}_{\boldsymbol{FIK}}$.*

Consequently, $\mathbf{D}_{\mathbf{FIK}}$ can be considered as a natural intuitionistic modal logic. [2]

3 A bi-nested sequent calculus

In this section, we present a bi-nested calculus for **FIK**. The calculus is two-sided and it makes use of two kinds of nestings, also called blocks $\langle\cdot\rangle$ and $[\cdot]$. The former is called an *implication* block and the latter a *modal* block. The intuition is that implication blocks correspond to upper worlds while modal blocks correspond to R-successors in a bi-relational model. The calculus we present is a conservative extension (with some notational difference) of the nested sequent calculus for **IPL** presented in [6].

[2] Simpson considers a further requirement (C6), in our opinion more controversial: "there is an intuitionistically comprehensible explanation of the meaning of the modalities, relative to which IML is sound and complete". He interprets this as the requirement of soundness and completeness with respect to the obvious (the same as in the classical case) translation of the modalities into first-order intuitionistic logic. The logic **IK** is sound and complete with respect to such a translation, whereas evidently no weaker logic, whence neither **CK**, **CCDL**, nor **FIK** is. However, this does not mean that any other translation is impossible. A wider discussion will be deferred to further work.

▶ **Definition 18** (Bi-nested sequent). *A bi-nested sequent S is defined as follows:*

- $\Rightarrow$ *is a bi-nested sequent (the empty sequent);*
- $\Gamma \Rightarrow B_1, \ldots, B_k, [S_1], \ldots, [S_m], \langle T_1 \rangle, \ldots, \langle T_n \rangle$ *is a bi-nested sequent if $S_1, \ldots, S_m$, $T_1, \ldots$, T_n are bi-nested sequents where $m, n \geq 0$, and Γ is a finite (possibly empty) multi-set of formulas and $B_1, \ldots, B_k$ are formulas.*

We use S, T to denote bi-nested sequents and to simplify wording we will call bi-nested sequents simply by sequents in the rest of this paper. We denote by $|S|$ *the size* of a sequent S intended as the length of S as a string of symbols.

As usual with nested calculi, we need the notion of context in order to specify the rules, as they can be applied to sequents occurring inside other sequents. A *context* is of the form $G\{\}$, in which G is a part of a sequent, $\{\cdot\}$ is regarded as a placeholder that needs to be filled by another sequent in order to complete G. $G\{S\}$ is the sequent obtained by replacing the occurrence of the symbol $\{\}$ in $G\{\}$ by the sequent S.

▶ **Definition 19** (Context). *A context $G\{\}$ is inductively defined as follows:*

- $\{\}$ *is a context (the empty context).*
- *if $\Gamma \Rightarrow \Delta$ is a sequent and $G'\{\}$ is a context then $\Gamma \Rightarrow \Delta, \langle G'\{\}\rangle$ is a context.*
- *if $\Gamma \Rightarrow \Delta$ is a sequent and $G'\{\}$ is a context then $\Gamma \Rightarrow \Delta, [G'\{\}]$ is a context.*

For example, given a context $G\{\} = A \wedge B, \Box C \Rightarrow \langle \Box A \Rightarrow [\Rightarrow B]\rangle, [\{\}]$ and a sequent $S = A \Rightarrow \Delta, [C \Rightarrow B]$, we have $G\{S\} = A \wedge B, \Box C \Rightarrow \langle \Box A \Rightarrow [\Rightarrow B]\rangle, [A \Rightarrow \Delta, [C \Rightarrow B]]$.

The two types of blocks interact by the (inter) rule. In order to define this rule, we need the following:

▶ **Definition 20** ($*$-operator). *Let $\Lambda \Rightarrow \Theta$ be a sequent, we define Θ^* as follows:*

- $\Theta^* = \emptyset$ *if Θ is $[\cdot]$-free;*
- $\Theta^* = [\Phi_1 \Rightarrow \Psi_1^*], \ldots, [\Phi_k \Rightarrow \Psi_k^*]$ *if $\Theta = \Theta_0, [\Phi_1 \Rightarrow \Psi_1], \ldots, [\Phi_k \Rightarrow \Psi_k]$ and Θ_0 is $[\cdot]$-free.*

By definition, given a sequent $\Lambda \Rightarrow \Theta$, Θ^* is a multi-set of modal blocks. Denote the sequent $G\{S\}$ in the previous example for context by $\Lambda \Rightarrow \Theta$, then by definition, we can see $\Lambda \Rightarrow \Theta^* = A \wedge B, \Box C \Rightarrow [A \Rightarrow [C \Rightarrow]]$.

Now we can give a bi-nested sequent calculus for **FIK** as follows.

▶ **Definition 21.** *The calculus $\mathbf{C}_{\boldsymbol{FIK}}$ is given in Figure 1.*

Here is a brief explanation of these rules. As usual, the (id) axiom can be generalized from atoms to formulas. The logical rules, except $(\supset_R)$, are just the standard rules of intuitionistic logic in their nested version. From a backward direction and a semantic point of view, the rule $(\supset_R)$ introduces an implication block, which corresponds to an upper world (in the pre-order). The modal rules create new modal blocks or propagate modal formulas into existing ones, which correspond to R-accessible worlds. The (trans) rule transfers formulas (forced by) lower worlds to upper worlds following the pre-order. This rule is called (Lift) in [6]. Finally, the (inter) rule encodes the (FC) frame condition. It partially transfers "accessible" modal blocks from lower worlds to upper ones and creates new accessible worlds from upper worlds fulfilling the (FC) condition.

We define the modal degree of a sequent, which will be useful when discussing termination.

▶ **Definition 22** (Modal degree). *Modal degree for a formula F, denoted as $md(F)$, is defined as usual:*

- $md(p) = md(\bot) = md(\top) = 0$;
- $md(A \circ B) = max(md(A), md(B))$, *for $\circ \in \{\wedge, \vee, \supset\}$;*
- $md(\Box A) = md(\Diamond A) = md(A) + 1$.

Axioms:

$$\dfrac{}{G\{\Gamma, \bot \Rightarrow \Delta\}}\,(\bot_L) \qquad \dfrac{}{G\{\Gamma \Rightarrow \top, \Delta\}}\,(\top_R) \qquad \dfrac{}{G\{\Gamma, p \Rightarrow \Delta, p\}}\,(\mathrm{id})$$

Logical rules:

$$\dfrac{G\{A, B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}}\,(\wedge_L) \qquad \dfrac{G\{\Gamma \Rightarrow \Delta, A\} \qquad G\{\Gamma \Rightarrow \Delta, B\}}{G\{\Gamma \Rightarrow \Delta, A \wedge B\}}\,(\wedge_R)$$

$$\dfrac{G\{\Gamma, A \Rightarrow \Delta\} \qquad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \vee B \Rightarrow \Delta\}}\,(\vee_L) \qquad \dfrac{G\{\Gamma \Rightarrow \Delta, A, B\}}{G\{\Gamma \Rightarrow \Delta, A \vee B\}}\,(\vee_R)$$

$$\dfrac{G\{\Gamma, A \supset B \Rightarrow A, \Delta\} \qquad G\{\Gamma, B \Rightarrow \Delta\}}{G\{\Gamma, A \supset B \Rightarrow \Delta\}}\,(\supset_L) \qquad \dfrac{G\{\Gamma \Rightarrow \Delta, \langle A \Rightarrow B\rangle\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}}\,(\supset_R)$$

$$\dfrac{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]\}}{G\{\Gamma, \Box A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\,(\Box_L) \qquad \dfrac{G\{\Gamma \Rightarrow \Delta, \langle \Rightarrow [\Rightarrow A]\rangle\}}{G\{\Gamma \Rightarrow \Delta, \Box A\}}\,(\Box_R)$$

$$\dfrac{G\{\Gamma \Rightarrow \Delta, [A \Rightarrow]\}}{G\{\Gamma, \Diamond A \Rightarrow \Delta\}}\,(\Diamond_L) \qquad \dfrac{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi, A]\}}{G\{\Gamma \Rightarrow \Delta, \Diamond A, [\Sigma \Rightarrow \Pi]\}}\,(\Diamond_R)$$

Transferring and interactive rules:

$$\dfrac{G\{\Gamma, \Gamma' \Rightarrow \Delta, \langle \Gamma', \Sigma \Rightarrow \Pi\rangle\}}{G\{\Gamma, \Gamma' \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle\}}\,(\mathrm{trans}) \qquad \dfrac{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi, [\Lambda \Rightarrow \Theta^*]\rangle, [\Lambda \Rightarrow \Theta]\}}{G\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Lambda \Rightarrow \Theta]\}}\,(\mathrm{inter})$$

Figure 1 $\mathbf{C_{FIK}}$.

Further, let Γ *be a finite set of formulas, define* $md(\Gamma) = md(\bigwedge \Gamma)$*. As for a nested sequent* S *of the following form*

$$S = \Gamma \Rightarrow \Delta, [S_1], \dots, [S_m], \langle T_1\rangle, \dots, \langle T_n\rangle,$$

we set $md(S) = \max\{md(\Gamma), md(\Delta), md(S_1) + 1, \dots, md(S_m) + 1, md(T_1), \dots, md(T_n)\}$.

▶ **Example 23.** Axiom (**wCD**) in $\mathbf{D_{FIK}}$ is provable in $\mathbf{C_{FIK}}$.

Proof. To prove this, it suffices to prove $S = \Diamond p \supset \Box q, \Box(p \vee q) \Rightarrow \Box q$. Let $\Gamma = \Diamond p \supset \Box q, \Box(p \vee q)$ and then a derivation for S, i.e. $\Gamma \Rightarrow \Box q$ is given as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow \langle \Gamma \Rightarrow \Diamond p, [p \Rightarrow q, p]\rangle}\,(\mathrm{id})}{\Gamma \Rightarrow \langle \Gamma \Rightarrow \Diamond p, [p \Rightarrow q]\rangle}\,(\Diamond_R) \qquad \dfrac{\dfrac{}{\Gamma \Rightarrow \langle \Box q, \Box(p \vee q) \Rightarrow [q, p \Rightarrow q]\rangle}\,(\mathrm{id})}{\Gamma \Rightarrow \langle \Box q, \Box(p \vee q) \Rightarrow [p \Rightarrow q]\rangle}\,(\Box_L)}{\Gamma \Rightarrow \langle \Gamma \Rightarrow [p \Rightarrow q]\rangle}\,(\supset_L) \qquad \dfrac{}{\Gamma \Rightarrow \langle \Gamma \Rightarrow [q \Rightarrow q]\rangle}\,(\mathrm{id})}{\Gamma \Rightarrow \langle \Gamma \Rightarrow [p \vee q \Rightarrow q]\rangle}\,(\vee_L)}{\Gamma \Rightarrow \langle \Gamma \Rightarrow [\Rightarrow q]\rangle}\,(\Box_L)}{\Gamma \Rightarrow \langle \Rightarrow [\Rightarrow q]\rangle}\,(\mathrm{trans})}{\Gamma \Rightarrow \Box q}\,(\Box_R)$$

Then we are done. ◀

▶ **Example 24.** The formula $(\neg\Box\bot \supset \Box\bot) \supset \Box\bot$ is provable in $\mathbf{C_{FIK}}$.[3].

[3] Note that this $\Diamond$-free formula is unprovable in **CK** (whence the $\Diamond$-free fragments of these two logics are different, see [4]).

Proof. To prove this, it suffices to prove $S = \neg\Box\bot \supset \Box\bot \Rightarrow \Box\bot$. Let $\Gamma = \neg\Box\bot \supset \Box\bot$ and then a derivation for S, i.e. $\Gamma \Rightarrow \Box\bot$ is given as below.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$(\bot_L)$}
\UnaryInfC{$\Gamma \Rightarrow \langle \Gamma \Rightarrow \langle \Box\bot \Rightarrow \bot, [\bot \Rightarrow] \rangle, [\Rightarrow \bot] \rangle$}
\RightLabel{$(\Box_L)$}
\UnaryInfC{$\Gamma \Rightarrow \langle \Gamma \Rightarrow \langle \Box\bot \Rightarrow \bot, [\Rightarrow] \rangle, [\Rightarrow \bot] \rangle$}
\RightLabel{(inter)}
\UnaryInfC{$\Gamma \Rightarrow \langle \Gamma \Rightarrow \langle \Box\bot \Rightarrow \bot \rangle, [\Rightarrow \bot] \rangle$}
\RightLabel{$(\supset_R)$}
\UnaryInfC{$\Gamma \Rightarrow \langle \Gamma \Rightarrow \neg\Box\bot, [\Rightarrow \bot] \rangle$}
\AxiomC{}
\RightLabel{$(\bot_L)$}
\UnaryInfC{$\Gamma \Rightarrow \langle \Box\bot \Rightarrow [\bot \Rightarrow \bot] \rangle$}
\RightLabel{$(\Box_L)$}
\UnaryInfC{$\Gamma \Rightarrow \langle \Box\bot \Rightarrow [\Rightarrow \bot] \rangle$}
\RightLabel{$(\supset_L)$}
\BinaryInfC{$\Gamma \Rightarrow \langle \Gamma \Rightarrow [\Rightarrow \bot] \rangle$}
\RightLabel{(trans)}
\UnaryInfC{$\Gamma \Rightarrow \langle \Rightarrow [\Rightarrow \bot] \rangle$}
\RightLabel{$(\Box_R)$}
\UnaryInfC{$\Gamma \Rightarrow \Box\bot$}
\end{prooftree}
$$

Then we are done. ◀

We now show that the calculus $\mathbf{C_{FIK}}$ enjoys the disjunctive property, which means if $A \vee B$ is provable, then either A or B is provable. This fact is an immediate consequence of the following lemma. Its general form is due to the fact that backwards expansion of a sequent with empty antecedent will (only) treat/introduce formulas and implication blocks in the consequent.

▶ **Lemma 25.** *Suppose that a sequent $S = \Rightarrow A_1, \ldots, A_m, \langle G_1 \rangle, \ldots, \langle G_n \rangle$ is provable in $\mathbf{C_{FIK}}$, where the A_i's are formulas. Then either for some A_i, sequent $\Rightarrow A_i$ is provable or for some G_j, sequent $\Rightarrow \langle G_j \rangle$ is provable.*

Since $\Rightarrow A \vee B$ is provable if and only if $\Rightarrow A, B$ from the lemma we immediately obtain:

▶ **Proposition 26.** *For any formulas A, B, if $\Rightarrow A \vee B$ is provable in $\mathbf{C_{FIK}}$, then either $\Rightarrow A$ or $\Rightarrow B$ is provable.*

By the soundness and completeness of $\mathbf{C_{FIK}}$ with respect to **FIK** proved in the following, we will conclude that the logic **FIK** enjoys the disjunctive property.

Next, we prove the soundness of the calculus $\mathbf{C_{FIK}}$. To achieve this aim, we need to define the semantic interpretation of sequents, whence their validity. We first extend the forcing relation $\Vdash$ to sequents and blocks therein.

▶ **Definition 27.** *Let $\mathcal{M} = (W, \leq, R, V)$ be a bi-relational model and $x \in W$. The relation $\Vdash$ is extended to sequents as follows:*

$\mathcal{M}, x \nVdash \emptyset$
$\mathcal{M}, x \Vdash [T]$ *if for every y with Rxy, $\mathcal{M}, y \Vdash T$*
$\mathcal{M}, x \Vdash \langle T \rangle$ *if for every x' with $x \leq x'$, $\mathcal{M}, x' \Vdash T$*
$\mathcal{M}, x \Vdash \Gamma \Rightarrow \Delta$ *if either $\mathcal{M}, x \nVdash A$ for some $A \in \Gamma$ or $\mathcal{M}, x \Vdash \mathcal{O}$ for some $\mathcal{O} \in \Delta$*

We say S is valid *in $\mathcal{M}$ iff $\forall w \in W$, we have $\mathcal{M}, w \Vdash S$. S is* valid *iff it is valid in every bi-relational model.*

Whenever the model $\mathcal{M}$ is clear, we omit it and write simply $x \Vdash \mathcal{O}$, where $\mathcal{O}$ is either formula, or a sequent, or a block. Moreover, given a sequent $S = \Gamma \Rightarrow \Delta$, we write $x \Vdash \Delta$ if there is an $\mathcal{O} \in \Delta$ s.t. $x \Vdash \mathcal{O}$ and write $x \nVdash \Delta$ if the previous condition does not hold.

The following lemma gives a semantic meaning to the $*$-operation used in (inter).

▶ **Lemma 28.** *Let $\mathcal{M} = (W, \leq, R, V)$ be a bi-relational model and $x, x' \in W$ with $x \leq x'$. Let $S = \Gamma \Rightarrow \Delta$ be any sequent, if $x \nVdash \Delta$ then $x' \nVdash \Delta^*$.*

In order to prove soundness we first show that the all rules are *forcing-preserving*.

▶ **Lemma 29.** *Given a model $\mathcal{M} = (W, \leq, R, V)$ and $x \in W$, for any rule (r) of the form* $\frac{G\{S_1\} \quad G\{S_2\}}{G\{S\}}$ *or* $\frac{G\{S_1\}}{G\{S\}}$*, if $x \Vdash G\{S_i\}$, then $x \Vdash G\{S\}$.*

Proof of this lemma proceeds by induction on the structure of the context $G\{\}$. The the base of the induction (that is $G = \emptyset$) is the important one, we check rule by rule and in the case of (inter) we make use of Lemma 28.

By Lemma 29, the soundness of $\mathbf{C_{FIK}}$ is proved as usual by a straightforward induction on the length of derivations.

▶ **Theorem 30** (Soundness). *If a sequent S is provable in $\mathbf{C_{FIK}}$, then it is valid.*

4 Termination and completeness for $\mathrm{C_{FIK}}$

In this section, we provide a terminating proof-search procedure based on $\mathbf{C_{FIK}}$, whence a decision procedure for **FIK**; it will then be used to prove that $\mathbf{C_{FIK}}$ is complete with respect to **FIK** bi-relational semantics. Here is a roadmap. First we introduce a set-based variant of the calculus where all rules are cumulative (or kleen'ed), in the sense that principal formulas are kept in the premises. With this variant, we formulate saturation conditions on a sequent associated to each rule. Saturation conditions are needed for both termination and completeness in order to prevent "redundant" application of the rules as the source of non-termination. In the meantime saturation conditions also ensure that a saturated sequent satisfies the truth conditions specified by the semantics (which will be presented in the truth lemma), so it can be seen as a countermodel.

The reformulation of the calculus by means of set-based sequents is motivated as usual by the following consideration: while multisets are the natural data-structure for any proof-system (at least with commutative $\wedge$, $\vee$), set-based sequents are needed to bound the size of sequents occurring in a derivation in terms of *subsets of subformulas* of the formula or sequent at the root of the derivation (see for instance [3]).

With this in mind, we first present the following $\mathbf{CC_{FIK}}$, a variant of $\mathbf{C_{FIK}}$ where sequents are set-based rather than multi-set based and the rules are cumulative.

▶ **Definition 31.** $\mathbf{CC_{FIK}}$ *acts on set-based sequents, where a set-based sequent $S = \Gamma \Rightarrow \Delta$ is defined as in definition 18, but Γ is a* set *of formulas and Δ is a* set *of formulas and/or blocks (containing set-based sequents). The rules are as follows:*

- *It keeps the rules $(\bot_L)$, $(\top_R)$, (id), $(\Box_L)$, $(\Diamond_R)$, (trans) and (inter) of $\mathbf{C_{FIK}}$.*
- *$(\supset_R)$ is replaced by the two rules dealing with cases of $A \in \Gamma$ and $A \notin \Gamma$ respectively,*

$$\frac{G\{\Gamma \Rightarrow \Delta, A \supset B, B\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}}\ (\supset_{R_1}) \qquad \frac{G\{\Gamma \Rightarrow \Delta, A \supset B, \langle A \Rightarrow B\rangle\}}{G\{\Gamma \Rightarrow \Delta, A \supset B\}}\ (\supset_{R_2})$$

- *Other rules $(\wedge_L)$, $(\wedge_R)$, $(\vee_L)$, $(\vee_R)$, $(\supset_L)$, $(\Box_R)$ and $(\Diamond_L)$ in $\mathbf{C_{FIK}}$ are modified by keeping the principal formula in the premise(s). For example, the cumulative version of $(\supset_L)$ is*

$$\frac{G\{\Gamma, A \supset B \Rightarrow A, \Delta\} \qquad G\{\Gamma, A \supset B, B \Rightarrow \Delta\}}{G\{\Gamma, A \supset B \Rightarrow \Delta\}}\ (\supset_L)$$

and the cumulative versions of $(\wedge_L)$ and $(\Box_R)$ are

$$\frac{G\{A, B, A \wedge B, \Gamma \Rightarrow \Delta\}}{G\{A \wedge B, \Gamma \Rightarrow \Delta\}}\ (\wedge_L) \qquad \frac{G\{\Gamma \Rightarrow \Delta, \Box A, \langle \Rightarrow [\Rightarrow A]\rangle\}}{G\{\Gamma \Rightarrow \Delta, \Box A\}}\ (\Box_R)$$

The following proposition is a consequence of the admissibility of weakening and contraction of $\mathbf{C_{FIK}}$ which can be done by a standard proof.

▶ **Proposition 32.** *A sequent S is provable in $\mathbf{C_{FIK}}$ if and only if S is provable in $\mathbf{CC_{FIK}}$.*

From now on we only consider $\mathbf{CC_{FIK}}$. We introduce the notion of *structural inclusion* between sequents. It is used in the definition of saturation conditions as well as the model construction presented at the end of the section.

▶ **Definition 33** (Structural inclusion $\subseteq^{\mathbf{S}}$)**.** *Let $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2$ be two sequents. $\Gamma_1 \Rightarrow \Delta_1$ is said to be structurally included in $\Gamma_2 \Rightarrow \Delta_2$, denoted as $\Gamma_1 \Rightarrow \Delta_1 \subseteq^{\mathbf{S}} \Gamma_2 \Rightarrow \Delta_2$, if:*

- *$\Gamma_1 \subseteq \Gamma_2$ and*
- *for each $[\Lambda_1 \Rightarrow \Theta_1] \in \Delta_1$, there exists $[\Lambda_2 \Rightarrow \Theta_2] \in \Delta_2$ such that $\Lambda_1 \Rightarrow \Theta_1 \subseteq^{\mathbf{S}} \Lambda_2 \Rightarrow \Theta_2$.*

It is easy to see that $\subseteq^{\mathbf{S}}$ is reflexive and transitive; moreover if $\Gamma_1 \Rightarrow \Delta_1 \subseteq^{\mathbf{S}} \Gamma_2 \Rightarrow \Delta_2$, then $\Gamma_1 \subseteq \Gamma_2$.

We define now the saturation conditions associated to each rule of $\mathbf{CC_{FIK}}$.

▶ **Definition 34** (Saturation conditions)**.** *Let $\Gamma \Rightarrow \Delta$ be a sequent where Γ is a set of formulas and Δ is a set of formulas and blocks. Saturation conditions associated to a rule in the calculus are given as below.*

($\bot_L$) $\bot \notin \Gamma$.
($\top_R$) $\top \notin \Delta$.
(id) $\mathsf{At} \cap (\Gamma \cap \Delta)$ *is empty.*
($\wedge_R$) *If $A \wedge B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$.*
($\wedge_L$) *If $A \wedge B \in \Gamma$, then $A \in \Gamma$ and $B \in \Gamma$.*
($\vee_R$) *If $A \vee B \in \Delta$, then $A \in \Delta$ and $B \in \Delta$.*
($\vee_L$) *If $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$.*
($\supset_R$) *If $A \supset B \in \Delta$, then either $A \in \Gamma$ and $B \in \Delta$, or there is $\langle \Sigma \Rightarrow \Pi \rangle \in \Delta$ with $A \in \Sigma$ and $B \in \Pi$.*
($\supset_L$) *If $A \supset B \in \Gamma$, then $A \in \Delta$ or $B \in \Gamma$.*
($\Box_R$) *If $\Box A \in \Delta$, then either there is $[\Lambda \Rightarrow \Theta] \in \Delta$ with $A \in \Theta$, or there is $\langle \Sigma \Rightarrow [\Lambda \Rightarrow \Theta], \Pi \rangle \in \Delta$ with $A \in \Theta$.*
($\Box_L$) *If $\Box A \in \Gamma$ and $[\Sigma \Rightarrow \Pi] \in \Delta$, then $A \in \Sigma$.*
($\Diamond_R$) *If $\Diamond A \in \Delta$ and $[\Sigma \Rightarrow \Pi] \in \Delta$, then $A \in \Pi$.*
($\Diamond_L$) *If $\Diamond A \in \Gamma$, then there is $[\Sigma \Rightarrow \Pi] \in \Delta$ with $A \in \Sigma$.*
(trans) *If Δ is of form $\Delta', \langle \Sigma \Rightarrow \Pi \rangle$, then $\Gamma \subseteq \Sigma$.*
(inter) *If Δ is of form $\Delta', \langle \Sigma \Rightarrow \Pi \rangle, [\Lambda \Rightarrow \Theta]$, then there is $[\Phi \Rightarrow \Psi] \in \Pi$ with $\Lambda \Rightarrow \Theta \subseteq^{\mathbf{S}} \Phi \Rightarrow \Psi$.*

Concerning the (inter)-saturation, observe that in the (inter) rule we have $\Lambda \Rightarrow \Theta \subseteq^{\mathbf{S}} \Lambda \Rightarrow \Theta^*$, thus the saturation condition actually generalizes the situation.

▶ **Proposition 35.** *Let $\Gamma \Rightarrow \Delta$ be a sequent saturated with respect to both (trans) and (inter). If $\langle \Sigma \Rightarrow \Pi \rangle \in \Delta$, then $\Gamma \Rightarrow \Delta \subseteq^{\mathbf{S}} \Sigma \Rightarrow \Pi$.*

In order to define a terminating proof-search procedure based on $\mathbf{CC_{FIK}}$ (like for any calculus with cumulative rules), as usual we say that the backward application of a rule (R) to a sequent S is *redundant* if S satisfies the corresponding saturation condition for that application of (R) and we impose the following constraints:

(i) *No rule is applied to an axiom* and
(ii) *No rule is applied redundantly.*

However, the restrictions above are not sufficient to ensure the termination of the procedure.

▶ **Example 36.** Let us consider the sequent $S = \Box a \supset \bot, \Box b \supset \bot \Rightarrow p$, where we abbreviate by Γ the antecedent of S. Consider the following derivation, we only show the leftmost branch (the others succeed), we collapse some steps:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{(2)\ \Gamma \Rightarrow p, \Box a, \Box b, \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a], \langle \Rightarrow [\Rightarrow b]\rangle\rangle, \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b]\rangle}}{(1)\ \Gamma \Rightarrow p, \Box a, \Box b, \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a]\rangle, \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b]\rangle}\,(\Box_R)}{\Gamma \Rightarrow p, \Box a, \Box b, \langle \Gamma \Rightarrow [\Rightarrow a]\rangle, \langle \Gamma \Rightarrow [\Rightarrow b]\rangle}\,(\supset_L) \times 4}{\Gamma \Rightarrow p, \Box a, \Box b, \langle \Rightarrow [\Rightarrow a]\rangle, \langle \Rightarrow [\Rightarrow b]\rangle}\,(trans) \times 2}{\Gamma \Rightarrow p, \Box a, \Box b}\,(\Box_R) \times 2}{\Gamma \Rightarrow p}\,(\supset_L) \times 2}{}
$$

$$
\dfrac{\vdots}{(3)\ \Gamma \Rightarrow p, \Box a, \Box b, \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b]\rangle\rangle, \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b]\rangle}
$$

Observe that in the first implication block of sequent (1) $(\Box_R)$ can only be applied to $\Box b$, creating the nested block $\langle \Rightarrow [\Rightarrow b]\rangle$ in (2), as it satisfies the saturation condition for $\Box a$. This block will be further expanded to $\langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b]\rangle$ in (3) that satisfies the saturation condition for $\Box b$, but not for $\Box a$, whence it will be further expanded, and so on. Thus the branch does not terminate.

In order to deal with this case of non-termination, intuitively we need to block the expansion of a sequent that occurs nested in another sequent whenever the former has already been expanded and the latter is "equivalent" in some sense to the former. To realize this purpose we first introduce a few notions.

▶ **Definition 37** ($\in^{\langle\cdot\rangle}, \in^{[\cdot]}, \in^{+}$-relation)**.** *Let $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2$ be two sequents. We denote $\Gamma_1 \Rightarrow \Delta_1 \in_0^{\langle\cdot\rangle} \Gamma_2 \Rightarrow \Delta_2$ if $\langle \Gamma_1 \Rightarrow \Delta_1\rangle \in \Delta_2$ and let $\in^{\langle\cdot\rangle}$ be the transitive closure of $\in_0^{\langle\cdot\rangle}$. Relations $\in_0^{[\cdot]}$ and $\in^{[\cdot]}$ for modal blocks are defined similarly. Besides, let $\in_0^{+} = \in_0^{\langle\cdot\rangle} \cup \in_0^{[\cdot]}$ and finally let $\in^{+}$ be the reflexive-transitive closure of $\in_0^{+}$.*

Observe that when we say $S' \in^{+} S$, it is equivalent to say that for some context G, $S = G\{S'\}$.

We introduce the operator $\sharp$ for the succedent of a sequent, it is used to remove implication blocks but retain all the other formulas and modal blocks.

▶ **Definition 38** ($\sharp$-operator)**.** *Let $\Lambda \Rightarrow \Theta$ be a sequent. We define $\Theta^{\sharp}$ as follows:*
- **(i)** $\Theta^{\sharp} = \Theta$ *if* Θ *is block-free;*
- **(ii)** $\Theta^{\sharp} = \Theta_0^{\sharp}, [\Phi \Rightarrow \Psi^{\sharp}]$ *if* $\Theta = \Theta_0, [\Phi \Rightarrow \Psi]$*;*
- **(iii)** $\Theta^{\sharp} = \Theta_0^{\sharp}$ *if* $\Theta = \Theta_0, \langle \Phi \Rightarrow \Psi\rangle$*.*

We can compare this $\sharp$-operator with $*$ in Definition 20. For example, let $\Delta = b, [c \Rightarrow d, [e \Rightarrow f], \langle g \Rightarrow h\rangle], \langle t \Rightarrow [p \Rightarrow q]\rangle, [m \Rightarrow n]$, then $\Delta^{\sharp} = b, [c \Rightarrow d, [e \Rightarrow f]], [m \Rightarrow n]$, while $\Delta^{*} = [c \Rightarrow [e \Rightarrow]], [m \Rightarrow]$.

Intuitively speaking, if a sequent $S = \Lambda \Rightarrow \Theta$ describes a model rooted in S and specifies formulas forced and not forced in S, then $\Lambda \Rightarrow \Theta^{\sharp}$, describes the chains of R-related worlds to S by specifying all formulas forced and not forced in each one of them, but ignores upper worlds in the pre-order, the latter being represented by implication blocks.

We use the $\sharp$-operator to define an equivalence relation between sequents. The equivalence relation will be used to detect loops in a derivation as in the example above.

▶ **Definition 39** ($\sharp$-equivalence)**.** *Let S_1, S_2 be two sequents where $S_1 = \Gamma_1 \Rightarrow \Delta_1, S_2 = \Gamma_2 \Rightarrow \Delta_2$. We say S_1 is $\sharp$-equivalent to S_2, denoted as $S_1 \simeq S_2$, if $\Gamma_1 = \Gamma_2$ and $\Delta_1^{\sharp} = \Delta_2^{\sharp}$.*

In order to define a proof-search procedure, we divide rules of $\mathbf{CC_{FIK}}$ into three groups and define correspondingly three levels of saturation.

(R1) basic rules: all propositional and modal rules except $(\supset_R)$ and $(\Box_R)$;
(R2) rules that transfer formulas and blocks into implication blocks: (trans) and (inter);
(R3) rules that create implication blocks: $(\Box_R)$ and $(\supset_R)$.

▶ **Definition 40** (Saturation). *Let $S = \Gamma \Rightarrow \Delta$ be a sequent and not an axiom. S is called:*

- R1-saturated *if $\Gamma \Rightarrow \Delta^\sharp$ satisfies all the saturation conditions of R1 rules;*
- R2-saturated *if S is R1-saturated and S satisfies saturation conditions of R2 rules for blocks $S_1 \in_0^{\langle\cdot\rangle} S$ and $S_2 \in_0^{[\cdot]} S$.*
- R3-saturated *if S is R2-saturated and S satisfies saturation conditions of R3 rules for formulas $\Box A, B \supset C \in \Delta$.*

We can finally define when a sequent is blocked, the intention is that it will not be expanded anymore by the proof-search procedure.

▶ **Definition 41** (Blocked sequent). *Given a sequent S and $S_1, S_2 \in^+ S$, with $S_1 = \Gamma_1 \Rightarrow \Delta_1, S_2 = \Gamma_2 \Rightarrow \Delta_2$. We say S_2 is blocked by S_1 in S, if S_1 is R3-saturated, $S_2 \in^{\langle\cdot\rangle} S_1$ and $S_1 \simeq S_2$. We say that a sequent S' is* blocked *in S if there exists $S_1 \in^+ S$ such that S' is blocked by S_1 in S.*

Observe that if S is finite, then for any $S' \in^+ S$ checking whether S' is blocked in S can be effectively decided. We will say just that S' is blocked when S is clear.

▶ **Example 42.** We reconsider the example 36. The sequent (3) will be further expanded to

$$
\begin{aligned}
(4)\ \Gamma \Rightarrow\ & p, \Box a, \Box b, \\
& \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a] \rangle^{(ii)} \rangle \rangle^{(i)}, \\
& \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b] \rangle
\end{aligned}
$$

We have marked by (i) and (ii) the relevant blocks. Observe that the sequent $S_2 = \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a]$ in the block marked (ii) is blocked by the sequent $S_1 = \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a] \rangle \rangle$ marked (i), since S_1 is R3-saturated, $S_2 \in^{\langle\cdot\rangle} S_1$ and $S_1 \simeq S_2$, as in particular $(\Box a, \Box b, [\Rightarrow a], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow b], \langle \Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a] \rangle \rangle)^\sharp = (\Gamma \Rightarrow \Box a, \Box b, [\Rightarrow a])^\sharp$.

We finally define three global saturation conditions.

▶ **Definition 43** (Global saturation). *Let S be a sequent and not an axiom. S is called :*

- global-R1-saturated *if for each $T \in^+ S$, T is either R1-saturated or blocked;*
- global-R2-saturated *if for each $T \in^+ S$, T is either R2-saturated or blocked;*
- global-saturated *if for each $T \in^+ S$, T is either R3-saturated or blocked.*

In order to specify the proof-search procedure, we make use of three sub-procedures that extend a given derivation $\mathcal{D}$ by expanding a leaf S, each procedure applies rules *non-redundantly* to some $T := \Gamma \Rightarrow \Delta \in^+ S$, that we recall it means that $S = G\{T\}$, for some context G . We define :

1. $\mathbf{EXP1}(\mathcal{D}, S, T) = \mathcal{D}'$ where $\mathcal{D}'$ is the extension of $\mathcal{D}$ obtained by applying R1 rules to every formula in $\Gamma \Rightarrow \Delta^\sharp$.
2. $\mathbf{EXP2}(\mathcal{D}, S, T) = \mathcal{D}'$ where $\mathcal{D}'$ is the extension of $\mathcal{D}$ obtained by applying R2-rules to blocks $\langle T_i \rangle, [T_j] \in \Delta$.
3. $\mathbf{EXP3}(\mathcal{D}, S, T) = \mathcal{D}'$ where $\mathcal{D}'$ is the extension of $\mathcal{D}$ obtained by applying R3-rules to formulas $\Box A, A \supset B \in \Delta$.

The three procedures are used as macro-steps in the proof search procedure defined next. We are going to prove that the three sub-procedures terminate, this is stated in Proposition 46 below. The claim is obvious for $\mathbf{EXP2}(\mathcal{D}, S, T)$, $\mathbf{EXP3}(\mathcal{D}, S, T)$ as only finitely many blocks or formulas in T are processed. For $\mathbf{EXP1}(\mathcal{D}, S, T)$, the claim is not so trivial, since the rules are applied also deeply within $\Gamma \Rightarrow \Delta^\sharp$. But notice that **EXP1** only applies the rules (both left and right) for $\wedge, \vee, \Diamond$ and $\supset_L, \Box_L$ while ignores implication blocks, we can see $\mathbf{EXP1}(\mathcal{D}, S, T)$ produces exactly the same expansion of $\mathcal{D}$ that we would obtain by applying the same rules of a nested sequent calculus for classical modal logic $\mathbf{K}$ and we know that that procedure terminates (see [3], Lemma 7).

In order to give a proof of the claim for $\mathbf{EXP1}(\mathcal{D}, S, T)$ precisely we introduce the following definition.

▶ **Definition 44.** *Given a sequent S, the tree $\mathcal{T}_S$ is defined as follows: (i) the root of $\mathcal{T}_S$ is S; (ii) if $S_1 \in_0^{[\cdot]} S_2$, then S_1 is a child of S_2.*

We denote the height of $\mathcal{T}_S$ as $h(\mathcal{T}_S)$. It is easy to verify that $h(\mathcal{T}_S) \leq md(S)$. Moreover we denote by $Sub(A)$ the set of subformulas of a formula A and for a sequent $S = \Gamma \Rightarrow \Delta$ we use the corresponding notations $Sub(\Gamma)$, $Sub(\Delta)$, $Sub(S)$. Finally, we recall that $Card(Sub(S)) = O(|S|)$.

By estimating the size of the tree associated to a sequent, we can get the following *rough* bound of the size of any sequent occurring in a derivation by R1-rules.

▶ **Proposition 45.** *Let $\mathcal{D}o$ be a derivation with root a non-axiomatic sequent $T = \Gamma \Rightarrow \Delta$ obtained by applying R1-rules to $\Gamma \Rightarrow \Delta^\sharp$, then any T' occurring in $\mathcal{D}o$ has size $O(|T|^{|T|+1})$.*

We can now prove the following proposition.

▶ **Proposition 46.** *Given a finite derivation $\mathcal{D}$, a finite leaf S of $\mathcal{D}$ and $T \in^+ S$, then each $\mathbf{EXP1}(\mathcal{D}, S, T)$, $\mathbf{EXP2}(\mathcal{D}, S, T)$,$\mathbf{EXP3}(\mathcal{D}, S, T)$ terminates by producing a finite expansion of $\mathcal{D}$ where all sequents in it are finite.*

We present below the proof-search procedure PROCEDURE(A), that given an input formula A it returns either a proof of A or a finite derivation tree in which all non-axiomatic leaves are global-saturated.

Note that the proof-search algorithm we give is breadth-first, as we can see in line 8, we expand all such non-axiomatic leaves in parallel. As a result, in line 5, the output is a fully-saturated derivation, which means each non-axiomatic leaf in it is global-saturated. Actually it is also possible to redesign the algorithm in a depth-first way by working with one leaf exhaustively at each time and then the procedure for a unprovable formulas terminates once the first global-saturated leaf is constructed.

An important property of the proof-search procedure is that saturation and blocking are preserved through sequent expansion, in other words, they are *invariant* of the repeated loop of the procedure.

▶ **Lemma 47** (Invariant). *Let S be a leaf of a derivation $\mathcal{D}$ with root $\Rightarrow A$:*

1. *Let $T \in^+ S$, where $T = \Gamma \Rightarrow \Delta$, for every rule (R) if T satisfies the R-saturation condition on some formulas A_i and/or blocks $\langle T_j \rangle, [T_k]$* before *the execution of (the body of) the repeat loop (lines 3-14), then T satisfies the R-condition on the involved $A_i, \langle T_j \rangle, [T_k]$* after *the execution of it.*
2. *Let $T \in^+ S$, if T is blocked in S* before *the execution of (the body of) the repeat loop, then it is still so* after *it.*

Algorithm 1 PROCEDURE(A).

Input: $\mathcal{D}_0 = \Rightarrow A$
initialization $\mathcal{D} = \mathcal{D}_0$;
repeat
  if *all the leaves of* $\mathcal{D}$ *are axiomatic* **then**
    return "PROVABLE" and $\mathcal{D}$
  else if *all the non-axiomatic leaves of* $\mathcal{D}$ *are global-saturated* **then**
    return "UNPROVABLE" and $\mathcal{D}$
  else
    for all non-axiomatic leaves S of $\mathcal{D}$ that are not global-saturated
    if *S is global-R2-saturated* **then**
      for all $T \in^+ S$ such that T is a $\in^{\langle\cdot\rangle}$-minimal and not R3-saturated, check whether T is blocked in S, if not, let $\mathcal{D} = \mathbf{EXP3}(\mathcal{D}, S, T)$
    else if *S is global-R1-saturated* **then**
      for all $T \in^+ S$ such that T is not R2-saturated, let $\mathcal{D} = \mathbf{EXP2}(\mathcal{D}, S, T)$
    else
      for all $T \in^+ S$ such that T is not R1-saturated, let $\mathcal{D} = \mathbf{EXP1}(\mathcal{D}, S, T)$
until *FALSE*;

The last ingredient in order to prove termination is that in a derivation of a formula, there can only be finitely many non-blocked sequents.

▶ **Lemma 48.** *Given a formula A, let $\boldsymbol{Seq}(A)$ be the set of sequents that may occur in any possible derivation with root $\Rightarrow A$. Let $\boldsymbol{Seq}(A)/_{\simeq}$ be the quotient of $\boldsymbol{Seq}(A)$ with respect to $\sharp$-equivalence $\simeq$ as defined in Definition 39. Then $\boldsymbol{Seq}(A)/_{\simeq}$ is finite.*

Intuitively, the termination of the procedure is based on the fact that the procedure cannot run forever by building an infinite derivation. The reason is that the built derivation cannot contain any infinite branch, because (i) once that a sequent satisfies a saturation condition for a rule (R), further expansions of it will still satisfy that condition (whence will not be reconsidered for the application of (R)); (ii) if a sequent is blocked, further application or rules cannot "unblock" it; (iii) the number of non-equivalent, whence unblocked sequents is finite.

▶ **Theorem 49** (Termination)**.** *Let A be a formula. Proof-search for the sequent $\Rightarrow A$ terminates with a finite derivation in which any leaf is either an axiom or global-saturated.*

Next, we prove the completeness of $\mathbf{CC_{FIK}}$. Given a finite global-saturated leaf S of the derivation $\mathcal{D}$ produced by PROCEDURE(A), we can define a model $\mathcal{M}_S$ as follows, which will be shown as a countermodel for A.

▶ **Definition 50.** *The model $\mathcal{M}_S = (W_S, \leq_S, R_S, V_S)$ determined by S is defined as follows:*

- $W_S = \{x_{\Phi\Rightarrow\Psi} \mid \Phi \Rightarrow \Psi \in^+ S\}$.
- $x_{S_1} \leq_S x_{S_2}$ *if* $S_1 \subseteq^{\mathbf{S}} S_2$.
- $R_S x_{S_1} x_{S_2}$ *if* $S_2 \in_0^{[\cdot]} S_1$.
- *for each* $x_{\Phi\Rightarrow\Psi} \in W_S$*, let* $V_S(x_{\Phi\Rightarrow\Psi}) = \{p \mid p \in \Phi\}$.

We give some brief remarks on the model construction. Obviously $\mathcal{M}_S$ is finite, and each world in W_S corresponds to either a R3-saturated or a blocked sequent, that is nonetheless saturated with respect to (inter) and (trans). Moreover, by Proposition 35, we can see if $x_{\Gamma\Rightarrow\Delta',\langle\Sigma\Rightarrow\Pi\rangle} \in W_S$ then $x_{\Sigma\Rightarrow\Pi} \in W_S$, and $x_{\Gamma\Rightarrow\Delta',\langle\Sigma\Rightarrow\Pi\rangle} \leq_S x_{\Sigma\Rightarrow\Pi}$. Lastly, by the property of structural inclusion $\subseteq^{\mathbf{S}}$, we have that $\leq_S$ is a pre-order.

▶ **Proposition 51.** *$\mathcal{M}_S$ satisfies the hereditary property (HP) and forward confluence (FC).*

▶ **Lemma 52** (Truth Lemma). *Let S be a global-saturated sequent and $\mathcal{M}_S$ be defined as above. (a). If $A \in \Phi$, then $\mathcal{M}_S, x_{\Phi\Rightarrow\Psi} \Vdash A$; (b). If $A \in \Psi$, then $M_S, x_{\Phi\Rightarrow\Psi} \nVdash A$.*

From the truth lemma we immediately obtain the completeness of $\mathbf{CC_{FIK}}$.

▶ **Theorem 53.** *For any formula $A \in \mathcal{L}$, if $\Vdash A$, then $\Rightarrow A$ is provable in $\mathbf{CC_{FIK}}$.*

▶ **Example 54.** We show how to build a countermodel of the formula $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$ by $\mathbf{CC_{FIK}}$ (due to space limit, we do not present the full derivation here). By backward application of the rules, one branch of the derivation ends up with the the saturated sequent S_0:

$$S_0 = \ \Diamond p \supset \Box q \Rightarrow \Diamond p, \Box(p \supset q), \langle \Diamond p \supset \Box q \Rightarrow \Diamond p, [\Rightarrow p \supset q, \langle p \Rightarrow q\rangle, p]\rangle \text{ and let}$$

$$S_1 = \Diamond p \supset \Box q \Rightarrow \Diamond p, [\Rightarrow p \supset q, \langle p \Rightarrow q\rangle, p], \ \ S_2 = \Rightarrow p \supset q, \langle p \Rightarrow q\rangle, p, \ \ S_3 = p \Rightarrow q$$

We then get the model $M_{S_0} = (W, \leq, R, V)$ where

- $W = \{x_{S_0}, x_{S_1}, x_{S_2}, x_{S_3}\}$,
- $x_{S_0} \leq x_{S_1}$, $x_{S_2} \leq x_{S_0}$, $x_{S_2} \leq x_{S_3}$,
- $Rx_{S_1}x_{S_2}$,
- $V(x_{S_0}) = V(x_{S_1}) = V(x_{S_2}) = \emptyset$ and $V(x_{S_3}) = \{p\}$.

It is easy to see that $x_{S_0} \nVdash (\Diamond p \supset \Box q) \supset \Box(p \supset q)$.

▶ **Example 55.** Consider another example $\neg\neg\Box\neg p \supset \Box\neg p$ which shows that the $\Diamond$-free fragment of **FIK** is weaker than the same fragment of **IK**. The formula is presented in [4] and is provable in **IK**. By building a derivation with the root $\Rightarrow ((\Box(p \supset \bot) \supset \bot) \supset \bot) \supset \Box(p \supset \bot)$, we generate a saturated sequent

$$S_0 = F_1 \Rightarrow \Box(p \supset \bot), F_2, \langle S_1\rangle, \langle S_6\rangle,$$

where $F_1 = (\Box(p \supset \bot) \supset \bot) \supset \bot, F_2 = \Box(p \supset \bot) \supset \bot$, and

$$\begin{aligned} &S_1 = F_1 \Rightarrow F_2, [\Rightarrow \langle p \Rightarrow \bot\rangle], \langle S_4\rangle, \quad S_4 = F_1, \Box(p \supset \bot) \Rightarrow \bot, F_2, [p \supset \bot \Rightarrow p], \\ &S_6 = F_1, \Box(p \supset \bot) \Rightarrow \bot, F_2, \\ &S_2 = \Rightarrow \langle p \Rightarrow \bot\rangle, \quad S_3 = p \Rightarrow \bot, \quad S_5 = p \supset \bot \Rightarrow p. \end{aligned}$$

We then get the model $M_{S_0} = (W, \leq, R, V)$ where

- $W = \{x_{S_0}, \ldots, x_{S_6}\}$,
- $x_{S_0} \leq x_{S_1}, x_{S_0} \leq x_{S_6}, x_{S_1} \leq x_{S_4}, x_{S_6} \leq x_{S_4}$, $x_{S_2} \leq x_{S_3}$, $x_{S_2} \leq x_{S_5}$, $x_{S_2} \leq x_{S_0}$,
- $Rx_{S_1}x_{S_2}$, $Rx_{S_4}x_{S_5}$,
- $V(x_{S_i}) = \emptyset$ if $i \neq 3$ and $V(x_{S_3}) = \{p\}$.

It is easy to see that $x_{S_0} \nVdash ((\Box(p \supset \bot) \supset \bot) \supset \bot) \supset \Box(p \supset \bot)$.

5 Conclusion and future work

We have proposed **FIK**, a natural variant of Intuitionistic modal logic characterized by forward confluent bi-relational models. **FIK** is intermediate between constructive modal logic **CK** and intuitionistic modal logic **IK** and it satisfies all the expected criteria for **IML**. We have presented a sound and complete axiomatization of it and a bi-nested calculus $\mathbf{C_{FIK}}$ which provides a decision procedure together with a finite countermodel extraction.

There are many topics for further research. First we may study extensions of **FIK** with the standard axioms from the modal cube. To obtain decidability and terminating proof systems for transitive logics (e.g. the **4**-extension) might be difficult and it may be worthwhile to study an embedding of our nested sequent calculus into a labelled calculus and then adapt the techniques and results in [7]. More generally, we can also explore extensions of **FIK** whose accessibility relation is defined by Horn properties and the nested sequent calculi might be obtained by means of the refinement technique proposed in [8]. Lastly we can consider other bi-relational frame conditions relating to the pre-order and the accessible (including the one for **IK**) and see how they can be captured uniformly in bi-nested calculi with suitable "interactive rules".

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A Appendix

The appendix includes the proofs of some of our results.

Proof of Lemma 12. Let $\Gamma, \Delta, \Lambda \in W_c$ be such that $\Gamma \geq_c \Delta$ and $\Delta R_c \Lambda$. Hence, $\Gamma \supseteq \Delta$ and $\Delta \bowtie \Lambda$. Let $A_1, A_2, \ldots$ be an enumeration of $\Box\Gamma$ and $B_1, B_2, \ldots$ be an enumeration of Λ. Obviously, for all $n \in \mathbb{N}$, $\Box(A_1 \wedge \ldots \wedge A_n) \in \Gamma$ and $B_1 \wedge \ldots \wedge B_n \in \Lambda$. Since $\Delta \bowtie \Lambda$, for all $n \in \mathbb{N}$, $\Diamond(B_1 \wedge \ldots \wedge B_n) \in \Delta$. For all $n \in \mathbb{N}$, let $\Theta_n = \mathbf{D_{FIK}} + A_1 \wedge \ldots \wedge A_n \wedge B_1 \wedge \ldots \wedge B_n$. Obviously, $(\Theta_n)_{n\in\mathbb{N}}$ is a chain of theories such that $\bigcup\{\Theta_n : n \in \mathbb{N}\} \supseteq \Lambda$.

We claim that for all formulas C, if $\Box C \in \Gamma$ then $C \in \bigcup\{\Theta_n : n \in \mathbb{N}\}$. If not, there exists a formula C such that $\Box C \in \Gamma$ and $C \notin \bigcup\{\Theta_n : n \in \mathbb{N}\}$. Thus, $C \in \Box\Gamma$. Consequently, let $n \in \mathbb{N}$ be such that $A_n = C$. Hence, $A_1 \wedge \ldots \wedge A_n \wedge B_1 \wedge \ldots \wedge B_n \to C$ is in $\mathbf{D_{FIK}}$. Thus, $C \in \Theta_n$. Consequently, $C \in \bigcup\{\Theta_n : n \in \mathbb{N}\}$: a contradiction. Hence, for all formulas C, if $\Box C \in \Gamma$ then $C \in \bigcup\{\Theta_n : n \in \mathbb{N}\}$.

We claim that for all formulas C, if $C \in \bigcup\{\Theta_n : n \in \mathbb{N}\}$ then $\Diamond C \in \Gamma$. If not, there exists $n \in \mathbb{N}$ and there exists a formula C such that $C \in \Theta_n$ and $\Diamond C \notin \Gamma$. Thus, $A_1 \wedge \ldots \wedge A_n \wedge B_1 \wedge \ldots \wedge B_n \to C$ is in $\mathbf{D_{FIK}}$. Consequently, $B_1 \wedge \ldots \wedge B_n \to (A_1 \wedge \ldots \wedge A_n \to C)$ is in $\mathbf{D_{FIK}}$. Hence, $\Diamond(B_1 \wedge \ldots \wedge B_n) \supset \Diamond(A_1 \wedge \ldots \wedge A_n \supset C)$ is in $\mathbf{D_{FIK}}$. Since $\Diamond(B_1 \wedge \ldots \wedge B_n) \in \Delta$, $\Diamond(A_1 \wedge \ldots \wedge A_n \supset C) \in \Delta$. Since $\Gamma \supseteq \Delta$, $\Diamond(A_1 \wedge \ldots \wedge A_n \supset C) \in \Gamma$. Thus, $\Box(A_1 \wedge \ldots \wedge A_n) \supset \Diamond C \in \Gamma$. Since $\Box(A_1 \wedge \ldots \wedge A_n) \in \Gamma$, $\Diamond C \in \Gamma$: a contradiction. Consequently, for all formulas C, if $C \in \bigcup\{\Theta_n : n \in \mathbb{N}\}$ then $\Diamond C \in \Gamma$.

Let $\mathcal{S} = \{\Theta :\ \Theta$ is a theory such that **(1)** $\Gamma \bowtie \Theta$ and **(2)** $\Theta \supseteq \Lambda\}$. Obviously, $\bigcup\{\Theta_n : n \in \mathbb{N}\} \in \mathcal{S}$. Hence, $\mathcal{S}$ is nonempty. Moreover, for all nonempty chains $(\Pi_i)_{i\in I}$ of elements of $\mathcal{S}$, $\bigcup\{\Pi_i : i \in I\}$ is an element of $\mathcal{S}$. Thus, by Zorn's Lemma, $\mathcal{S}$ possesses a maximal element Θ. Consequently, Θ is a theory such that $\Gamma \bowtie \Theta$ and $\Theta \supseteq \Lambda$. Hence, it only remains to be proved that Θ is proper and prime.

We claim that Θ is proper. If not, $\bot \in \Theta$. Since $\Gamma \bowtie \Theta$, $\Diamond\bot \in \Gamma$: a contradiction. Thus, Θ is proper.

We claim that Θ is prime. If not, there exists formulas C, D such that $C \vee D \in \Theta$, $C \notin \Theta$ and $D \notin \Theta$. Consequently, by the maximality of Θ in $\mathcal{S}$, $\Theta + C \notin \mathcal{S}$ and $\Theta + D \notin \mathcal{S}$. Hence, there exists a formula E such that $E \in \Theta + C$ and $\Diamond E \notin \Gamma$ and there exists a formula F such that $F \in \Theta + D$ and $\Diamond F \notin \Gamma$. Thus, $C \supset E \in \Theta$ and $D \supset F \in \Theta$. Consequently, $C \vee D \supset E \vee F \in \Theta$. Since $C \vee D \in \Theta$, $E \vee F \in \Theta$. Since $\Gamma \bowtie \Theta$, $\Diamond(E \vee F) \in \Gamma$. Hence, either $\Diamond E \in \Gamma$, or $\Diamond F \in \Gamma$: a contradiction. Thus, Θ is prime. ◀

Proof of Lemma 13. We only show the case of $\square$ here. Suppose $\square B \notin \Gamma$. Let $\mathcal{S} = \{\Delta : \ \Delta$ is a theory such that **(1)** $\Gamma \subseteq \Delta$ and **(2)** $\square B \notin \Delta\}$.

Since $\square B \notin \Gamma$, $\Gamma \in \mathcal{S}$. Hence, $\mathcal{S}$ is nonempty. Moreover, for all nonempty chains $(\Delta_i)_{i\in I}$ of elements of $\mathcal{S}$, $\bigcup\{\Delta_i : \ i \in I\}$ is an element of $\mathcal{S}$. Thus, by Zorn's Lemma, $\mathcal{S}$ possesses a maximal element Δ. Consequently, Δ is a theory such that $\Gamma \subseteq \Delta$ and $\square B \notin \Delta$.

We claim that Δ is proper. If not, then $\Delta = \mathcal{L}$. Hence, $\square B \in \Delta$: a contradiction. Thus, Δ is proper.

We claim that Δ is prime. If not, there exists formulas C, D such that $C \vee D \in \Delta$, $C \notin \Delta$ and $D \notin \Delta$. Consequently, by the maximality of Δ in $\mathcal{S}$, $\Delta + C \notin \mathcal{S}$ and $\Delta + D \notin \mathcal{S}$. Hence, $\square B \in \Delta + C$ and $\square B \in \Delta + D$. Thus, $C \supset \square B \in \Delta$ and $D \supset \square B \in \Delta$. Consequently, $C \vee D \supset \square B \in \Delta$. Since $C \vee D \in \Delta$, $\square B \in \Delta$: a contradiction. Hence, Δ is prime.

We claim that for all formulas C, if $C \vee B \in \square\Delta$ then $\lozenge C \in \Delta$. If not, there exists a formula C such that $C \vee B \in \square\Delta$ and $\lozenge C \notin \Delta$. Thus, by the maximality of Δ in $\mathcal{S}$, $\Delta + \lozenge C \notin \mathcal{S}$. Consequently, $\square B \in \Delta + \lozenge C$. Hence, $\lozenge C \supset \square B \in \Delta$. Since $C \vee B \in \square\Delta$, $\square(C \vee B) \in \Delta$. Since $\lozenge C \supset \square B \in \Delta$, $\square B \in \Delta$: a contradiction. Thus, for all formulas C, if $C \vee B \in \square\Delta$ then $\lozenge C \in \Delta$.

Let $\mathcal{T} = \{\Lambda : \ \Lambda$ is a theory such that **(1)** $\square\Delta \subseteq \Lambda$, **(2)** for all formulas C, if $C \vee B \in \Lambda$ then $\lozenge C \in \Delta$ and **(3)** $B \notin \Lambda\}$. Since $\square B \notin \Delta$, $B \notin \square\Delta$. Consequently, $\square\Delta \in \mathcal{T}$. Hence, $\mathcal{T}$ is nonempty. Moreover, for all nonempty chains $(\Lambda_i)_{i\in I}$ of elements of $\mathcal{T}$, $\bigcup\{\Lambda_i : \ i \in I\}$ is an element of $\mathcal{T}$. Thus, by Zorn's Lemma, $\mathcal{T}$ possesses a maximal element Λ. Consequently, Λ is a theory such that $\square\Delta \subseteq \Lambda$, for all formulas C, if $C \vee B \in \Lambda$ then $\lozenge C \in \Delta$ and $B \notin \Lambda$. Hence, it only remains to be proved that Λ is proper and prime and $\Delta \bowtie \Lambda$.

We claim that Λ is proper. If not, $\Lambda = \mathcal{L}$. Thus, $B \in \Lambda$: a contradiction. Consequently, Λ is proper.

We claim that Λ is prime. If not, there exists formulas C, D such that $C \vee D \in \Lambda$, $C \notin \Lambda$ and $D \notin \Lambda$. Hence, by the maximality of Λ in $\mathcal{T}$, $\Lambda + C \notin \mathcal{T}$ and $\Lambda + D \notin \mathcal{T}$. Thus, either there exists a formula E such that $E \vee B \in \Lambda + C$ and $\lozenge E \notin \Delta$, or $B \in \Lambda + C$ and either there exists a formula F such that $F \vee E \in \Lambda + D$ and $\lozenge F \notin \Delta$, or $B \in \Lambda + D$. Consequently, we have to consider the following four cases.

(1) Case "there exists a formula E such that $E \vee B \in \Lambda + C$ and $\lozenge E \notin \Delta$ and there exists a formula F such that $F \vee B \in \Lambda + D$ and $\lozenge F \notin \Delta$": Hence, $C \supset E \vee B \in \Lambda$ and $D \supset F \vee B \in \Lambda$. Thus, $C \vee D \supset E \vee F \vee B \in \Lambda$. Since $C \vee D \in \Lambda$, $E \vee F \vee B \in \Lambda$. Consequently, $\lozenge(E \vee F) \in \Delta$. Hence, either $\lozenge E \in \Delta$, or $\lozenge F \in \Delta$: a contradiction.

(2) Case "there exists a formula E such that $E \vee F \in \Lambda + C$ and $\lozenge E \notin \Delta$ and $B \in \Lambda + D$": Thus, $C \supset E \vee B \in \Lambda$ and $D \supset B \in \Lambda$. Consequently, $C \vee D \supset E \vee B \in \Lambda$. Since $C \vee D \in \Lambda$, $E \vee B \in \Lambda$. Hence, $\lozenge E \in \Delta$: a contradiction.

(3) Case "$B \in \Lambda + C$ and there exists a formula F such that $F \vee B \in \Lambda + D$ and $\lozenge F \notin \Delta$": Thus, $C \supset B \in \Lambda$ and $D \supset F \vee B \in \Lambda$. Consequently, $C \vee D \supset F \vee B \in \Lambda$. Since $C \vee D \in \Lambda$, $F \vee B \in \Lambda$. Hence, $\lozenge F \in \Delta$: a contradiction.

(4) Case "$B \in \Lambda + C$ and $B \in \Lambda + D$": Thus, $C \supset B \in \Lambda$ and $D \supset B \in \Lambda$. Consequently, $C \vee D \supset B \in \Lambda$. Since $C \vee D \in \Lambda$, $B \in \Lambda$: a contradiction.

Hence, Λ is prime.

Lastly, we claim that $\Delta \bowtie \Lambda$. If not, there exists a formula C such that $C \in \Lambda$ and $\lozenge C \notin \Delta$. Thus, $C \vee B \in \Lambda$. Consequently $\lozenge C \in \Delta$: a contradiction. Hence, $\Delta \bowtie \Lambda$. ◀

Proof of Lemma 28. By induction on the structure of Δ^*. If $\Delta^* = \emptyset$ it follows by definition. Otherwise $\Delta^* = [\Phi_1 \Rightarrow \Psi_1^*], \ldots, [\Phi_k \Rightarrow \Psi_k^*]$ where $\Delta = \Delta_0, [\Phi_1 \Rightarrow \Psi_1], \ldots, [\Phi_k \Rightarrow \Psi_k]$ and Δ_0 is $[\cdot]$-free. By hypothesis $x \not\Vdash \Delta$, thus $x \not\Vdash [\Phi_i \Rightarrow \Psi_i]$ for $i = 1, \ldots, k$. Therefore there are $y_1, \dots, y_k$ with Rxy_i for $i = 1, \dots, k$ such that $y_i \nVdash \Phi_i \Rightarrow \Psi_i$. This means that (a) $y_i \Vdash C$ for every $C \in \Phi_i$ and (b) $y_i \nVdash \Psi_i$. By (FC) property there are $y'_1, \dots, y'_k$ such that $Rx'y'_i$ and $y'_i \geq y_i$ for $i = 1, \dots, k$. By (a) it follows that (c) $y'_i \Vdash C$ for every $C \in \Phi_i$; moreover by induction hypothesis it follows that (d) $y'_i \nVdash \Psi^*_i$. Thus from (c) and (d) we have $y'_i \nVdash \Phi_i \Rightarrow \Psi^*_i$, whence $x' \nVdash [\Phi_i \Rightarrow \Psi^*_i]$ for for $i = 1, \dots, k$, which means that $x' \nVdash \Delta^*$. ◀

Proof of Theorem 49. (Sketch) We prove that PROCEDURE(A) terminates producing a finite derivation, in this case all leaves are axioms or global-saturated. A non-axiomatic leaf S is necessarily global-saturated, otherwise S would be further expanded in Step 8 of PROCEDURE(A) and it would not be a leaf. Thus it suffices to prove that the procedure produces a finite derivation. Let $\mathcal{D}$ built by PROCEDURE(A). First we claim that all branches of $\mathcal{D}$ are finite. Suppose for the sake of a contradiction that $\mathcal{D}$ contains an infinite branch $\mathcal{B} = S_0, \dots, S_i, \dots$, with $S_0 \Longrightarrow A$. The branch is generated by applying repeatedly $\mathbf{EXP1}(\cdot), \mathbf{EXP2}(\cdot)$ and $\mathbf{EXP3}(\cdot)$ to each S_i (or more precisely to some $T_i \in^+ S_i$) . Since each one of these sub-procedures terminates, the three of them must infinitely alternate on the branch. By (invariant) Lemma, if $T_i \in^+ S_i$ satisfies a saturation condition for a rule (R) or is blocked in (S_i) it will remain so in all S_j with $j > i$. That is to say, further steps in the branch cannot "undo" a fulfilled saturation condition or "unblock" a blocked sequent. We can conclude that the branch must contain infinitely many phases of $\mathbf{EXP3}(\cdot)$ each time applied to an unblocked sequent in some S_i. This entails that $\mathcal{B}$ contains infinitely many sequents that are not $\simeq$-equivalent, but this contradicts previous lemma 48. Thus each branch of the derivation $\mathcal{D}$ built by PROCEDURE(A) is finite. To conclude the proof, just observe that $\mathcal{D}$ is a tree whose branches have a finite length and is finitely branching (namely each node/sequent has at most 2 successors, as the rules of $\mathbf{CC_{FIK}}$ are at most binary), therefore $\mathcal{D}$ is finite. ◀

Proof of Proposition 46. We only prove the claim for $\mathbf{EXP1}(\mathcal{D}, S, T)$, the other cases being obvious. To this purpose we show that any derivation $\mathcal{D}o$, with root $\Gamma \Rightarrow \Delta^\sharp$ and generated by R1-rules, is finite. Then the claim follows since $\mathbf{EXP1}(\mathcal{D}, S, T)$ is obtained simply by "appending" $\mathcal{D}o$ to $\mathcal{D}$, where we replace every sequent T' in $\mathcal{D}o$ by $G\{T'\}$, as $S = G\{T\}$. In order to prove that $\mathcal{D}o$ is finite, notice that (i) all R1-rules are at most binary, (ii) the length of a branch of $\mathcal{D}o$ is bounded by the size of the maximal sequent that can occur in it because of non-redundancy restriction. But by proposition 45, every sequent T' in $\mathcal{D}o$ has a bounded size (namely $O(|T|^{|T|+1})$), whence we get a bound on the length of any branch of $\mathcal{D}o$. In conclusion $\mathcal{D}o$ is a finitely-branching tree, whose branches have a finite length, whence it is finite. ◀

In the following proofs, we abbreviate $R_S, \leq_S$ as R and $\leq$ respectively for readability.

Proof of Proposition 51. For (HP), take arbitrary $x_{S_1}, x_{S_2} \in W_S$ with $x_{S_1} \leq x_{S_2}$. Suppose S_1, S_2 are of form $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ respectively, then $\Gamma_1 \Rightarrow \Delta_1 \subseteq^{\mathbf{S}} \Gamma_2 \Rightarrow \Delta_2$. By definition, it follows $\Gamma_1 \subseteq \Gamma_2$. As $V_S(x_{S_1}) = \{p \mid p \in \Gamma_1\}$ and $V_S(x_{S_2}) = \{p \mid p \in \Gamma_2\}$, we have $V_S(x_{S_1}) \subseteq V_S(x_{S_2})$.

For (FC), take arbitrary $x_{\Gamma \Rightarrow \Delta}, x_{\Sigma \Rightarrow \Pi}, x_{\Lambda \Rightarrow \Theta} \in W_S$ where $x_{\Gamma \Rightarrow \Delta} \leq x_{\Sigma \Rightarrow \Pi}$ as well as $Rx_{\Gamma \Rightarrow \Delta}x_{\Lambda \Rightarrow \Theta}$, our goal is to find some $x_0 \in W_S$ s.t. both $x_{\Lambda \Rightarrow \Theta} \leq x_0$ and $Rx_{\Sigma \Rightarrow \Pi}x_0$ hold. Since $Rx_{\Gamma \Rightarrow \Delta}x_{\Lambda \Rightarrow \Theta}$, by the definition of R, we see that $[\Lambda \Rightarrow \Theta] \in \Delta$ and hence $\Gamma \Rightarrow \Delta$ can be written explicitly as $\Gamma \Rightarrow \Delta', [\Lambda \Rightarrow \Theta]$. Meanwhile, since $x_{\Gamma \Rightarrow \Delta} \leq x_{\Sigma \Rightarrow \Pi}$, by the definition of $\leq$, we have $\Gamma \Rightarrow \Delta', [\Lambda \Rightarrow \Theta] \subseteq^{\mathbf{S}} \Sigma \Rightarrow \Pi$. By the definition of structural inclusion, there is a block $[\Phi \Rightarrow \Psi] \in \Pi$ s.t. $\Lambda \Rightarrow \Theta \subseteq^{\mathbf{S}} \Phi \Rightarrow \Psi$. Since $\Phi \Rightarrow \Psi \in^+ \Sigma \Rightarrow \Pi \in^+ S$ and $\in^+$ is transitive, we see that $x_{\Phi \Rightarrow \Psi} \in W_S$ as well. Take $x_{\Phi \Rightarrow \Psi}$ to be x_0, by the construction of $\mathcal{M}_S$, it follows directly $x_{\Lambda \Rightarrow \Theta} \leq x_0$ and $Rx_{\Sigma \Rightarrow \Pi}x_0$. ◀

Proof of Lemma 52. We prove the lemma by induction on the complexity of A. For convenience, we abbreviate $x_{\Phi\Rightarrow\Psi}$ as x.

We only show the case when A is of the form $\Box B$. For (a), let $\Box B \in \Phi$. $\Phi \Rightarrow \Psi$ satisfies the saturation condition associated with $(\Box_R)$ for $\Box B$ regardless of whether the sequent itself is blocked or not. Assume for the sake of a contradiction that $x \nVdash \Box B$. Then there exists $x_{\Sigma\Rightarrow\Pi}, x_{\Lambda\Rightarrow\Theta}$ denoted as x_1, x_2 s.t. $x \leq x_1, Rx_1x_2$ and $x_2 \nVdash B$. By IH, we see that $B \notin \Lambda$. Meanwhile, according to the model construction, we see that $\Phi \Rightarrow \Psi \subseteq^{\mathbf{S}} \Sigma \Rightarrow \Pi$ and $[\Lambda \Rightarrow \Theta] \in \Pi$. Moreover we have $\Phi \subseteq \Sigma$, thus $\Box B \in \Sigma$ as well. Also, since $\Sigma \Rightarrow \Pi$ is of form $\Sigma \Rightarrow \Pi', [\Lambda \Rightarrow \Theta]$, by the saturation condition associated with $(\Box_L)$, we have $B \in \Lambda$, which leads to a contradiction.

For (b), let $\Box B \in \Psi$. We distinguish whether $\Phi \Rightarrow \Psi$ is blocked or not. Assume that $\Phi \Rightarrow \Psi$ is not blocked, then it satisfies the one of the two saturation conditions associated with $(\Box_R)$ for $\Box B$:

(1) there is a block $[\Lambda \Rightarrow \Theta] \in \Psi$ with $B \in \Theta$. By IH, we have $x_{\Lambda\Rightarrow\Theta} \nVdash B$. By reflexivity $x \leq x$ and model construction $Rxx_{\Lambda\Rightarrow\Theta}$, so that $x \nVdash \Box B$.

(2) there is a block $\langle \Omega \Rightarrow [\Lambda \Rightarrow \Theta], \Xi \rangle \in \Psi$ with $B \in \Theta$. Denote the sequent $\Omega \Rightarrow [\Lambda \Rightarrow \Theta], \Xi$ by S_0. Since $\Phi \Rightarrow \Psi$ is saturated with (trans) and (inter), by Proposition 35, we have $\Phi \Rightarrow \Psi \subseteq^{\mathbf{S}} S_0$. According to the model construction, we see that $x \leq x_{S_0}$ and $Rx_{S_0}x_{\Lambda\Rightarrow\Theta}$. Since $B \in \Theta$, by IH we have $x_{\Lambda\Rightarrow\Theta} \nVdash B$ and we can conclude $x \nVdash \Box B$.

Assume that $\Phi \Rightarrow \Psi$ is blocked and does not satisfy condition (1) for $\Box B$, otherwise the proof proceeds as in case (1) above. Then there is an unblocked sequent $\Sigma \Rightarrow \Pi \in^{+} S$ such that $\Phi \Rightarrow \Psi$ is blocked by it. Then $\Sigma \Rightarrow \Pi \simeq \Phi \Rightarrow \Psi$, which implies $\Pi^\sharp = \Psi^\sharp$, so $\Box B \in \Pi$ as well. Moreover, by definition, we have $\Phi \Rightarrow \Psi \subseteq^{\mathbf{S}} \Sigma \Rightarrow \Pi$, whence by model construction (**) $x \leq x_{\Sigma\Rightarrow\Pi}$. Given that $\Sigma \Rightarrow \Pi$ is R3-saturated, it satisfies the saturation condition associated with $(\Box_R)$ for $\Box B$, but since $\Sigma \Rightarrow \Pi \simeq \Phi \Rightarrow \Psi$, we have that $\Sigma \Rightarrow \Pi$ does not satisfy condition (1), thus it must satisfy condition (2). Therefore there is a block $\langle \Omega \Rightarrow [\Lambda \Rightarrow \Theta], \Xi \rangle \in \Pi$, such that $B \in \Theta$. Letting $S_0 = \Omega \Rightarrow [\Lambda \Rightarrow \Theta], \Xi$, we have $x_{\Sigma\Rightarrow\Pi} \leq x_{S_0}$ and $Rx_{S_0}x_{\Lambda\Rightarrow\Theta}$. By (**) we have also $x \leq x_{S_0}$ and we conclude as in case (2) above. ◀