Energy Games over Totally Ordered Groups

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Abstract

Kopczyński (ICALP 2006) conjectured that prefix-independent half-positional winning conditions are closed under finite unions. We refute this conjecture over finite arenas. For that, we introduce a new class of prefix-independent bi-positional winning conditions called energy conditions over totally ordered groups. We give an example of two such conditions whose union is not half-positional. We also conjecture that every prefix-independent bi-positional winning condition coincides with some energy condition over a totally ordered group on periodic sequences.

2012 ACM Subject Classification Theory of computation → Logic

Keywords and phrases Games on graphs, half-positionality, ordered groups

Digital Object Identifier 10.4230/LIPIcs.CSL.2024.34

Related Version *Previous Version*: https://doi.org/10.48550/arXiv.2205.04508

Funding *Alexander Kozachinskiy*: the author is funded by ANID – Millennium Science Initiative Program – Code ICN17002, and the National Center for Artificial Intelligence CENIA FB210017, Basal ANID.

1 Introduction

This paper is devoted to positional determinacy in turn-based infinite-duration games. An arena is a (possibly, infinite) directed graph whose edges are colored into elements of some set of colors C and whose nodes are partitioned between two players called Eve and Adam. They play by traveling over the nodes of the arena. In each turn, one of the players chooses an edge from the current node, and the players move toward the endpoint of this edge. Whether it is an Eve's or an Adam's turn to choose depends on whether the current node is an Eve's node or an Adam's node. This continues for infinitely many turns. As a result, the players obtain an infinite word over C (by concatenating the colors of edges that appear in the play). A *winning condition* W, which is a set of infinite words over C, defines the aims of the players. Eve wants to obtain an infinite word that belongs to W, while Adam wants it to be outside W.

A vast amount of literature in this area is devoted to *positional strategies*. A strategy of Eve or Adam is positional if, for every node controlled by the player in question, there exists an out-going edge which is always played by this strategy at this node. Implementing such strategies is easy because we only have to specify one edge for each node of the corresponding player. This makes these strategies relevant for such areas as controller synthesis [2], where an implementation of a controller can be seen as its strategy against an environment.

Correspondingly, of great interest are winning conditions for which positional strategies are always sufficient to play optimally (for one of the players or even for both of them). This area has the following terminology. A winning condition W is *half-positional* if in every arena Eve has a positional strategy σ such that for every node of the arena the following holds: if σ is not winning w.r.t. W if the game starts at this node, then Adam has a winning strategy w.r.t. W (not necessarily positional) from this node. To put it simply, σ must be winning everywhere where Adam does not have a winning strategy. If this condition holds in all finite arenas (but possibly does not hold in some infinite arenas), then W is called half-positional over finite arenas. A winning condition W is *bi-positional* (over all or over finite arenas) if additionally the same requirement as for Eve holds for Adam (or, in other words, if both W and its complement are half-positional). The family of parity condition, which is of great interest due to its applications in logic [20, 14], is known to be bi-positional over finite and infinite arenas [9]. There exist winning conditions that are bi-positional over finite arenas, but not even half-positional over infinite arenas, for instance, certain variants of the mean-payoff condition [18].

Winning conditions, bi-positional over infinite arenas, are understood quite well. For instance, it is known that all of them are ω-regular [4]. In turn, parity conditions are exactly those winning conditions that are bi-positional over infinite arenas and are prefix-independent, that is, closed under adding or removing finite prefixes [7].

Bi-positionality over finite arenas was studied by Gimbert and Zielonka [11, 12]. In [11], they gave a simple sufficient condition for bi-positionality over finite arenas, suitable for the majority of the applications, and in [12], they gave a condition which is sufficient and necessary (but far more complex). Their sufficient and necessary condition has a corollary called *1-to-2-player lifting*, which is of great interest in practice. It states that as long as a winning condition is half-positional for Eve in finite arenas without Adam and half-positional for Adam in finite arenas without Eve, it is bi-positional in all finite arenas.

Recently, Ohlmann [18] obtained a sufficient and necessary condition for half-positionality over infinite arenas. As for finite arenas, several sufficient conditions for half-positionality were obtained in the literature [15, 1], but none of them is necessary.

Are there set operations under which bi-positional and half-positional winning conditions are closed? Bi-positional winning conditions are closed under complement by definition. At the same time, bi-positional winning conditions are not closed under intersection (and hence under union) [17].

Closure properties of half-positional winning conditions were first addressed by Kopczyński [15]. He conjectured that prefix-independent half-positional winning conditions are closed under union. This conjecture has many variants, depending on whether we mean half-positionality over finite or infinite arenas, and whether we consider arbitrary unions or only finite ones. Kopczyński himself refuted a variant for infinite arenas and uncountable unions. He also noticed that dropping the prefix-independence assumption or changing union to intersection immediately makes the conjecture false.

No counter-example, refuting it for finite or even countable unions, had been found. On the positive side, several classes of prefix-independent half-positional winning conditions that are closed under union were identified in the literature, including concave conditions (over finite arenas and arbitrary unions) of Kopczyński [15] and conditions "excluding healing" of Ohlmann [18] (for infinite arenas but at most countable unions).

In this paper, we refute the Kopczyński's conjecture for *finite* arenas and *finite* unions. Moreover, we present two winning conditions that are *bi-positional* over finite arenas and whose union is not half-positional over finite arenas.

Kopczyński's conjecture over infinite arenas for finite/countable unions remains open. Additionally, there has been an interest in whether some variant of the Kopczyński's conjecture holds in a restriction to ω-regular condition. Bouyer et al. [3] obtained that prefix-independent ω-regular conditions, recognizable by deterministic Büchi automata (DBA), are closed under finite union. In fact, they simply show that every prefix-independent DBA-recognizable ω-regular condition can be given as a set of sequences, having infinitely many occurrences of some fixed subset of colors. Such conditions are trivially closed under finite unions.

Our technique. We introduce a new class of bi-positional winning conditions called *energy conditions over totally ordered groups*, or ETOG conditions for short. They are defined as follows (see more details in Section 3). We consider elements of some totally ordered group (we stress that it should be bi-ordered) as colors of edges. Given an infinite sequence of these elements, we arrange them into a formal series. Eve wants the sequence of its partial sums to have an infinite decreasing subsequence. Canonical energy conditions [5] can be defined in this way over $\mathbb{Z}$ with the standard ordering.

In Section 4, we establish the bi-positionality (over finite arenas) of the ETOG conditions using a sufficient condition of Gimbert and Zielonka. In Section 5, we refute the Kopczyński's conjecture over finite arenas and for finite unions. A key factor allowing us to do this is that free groups can be totally ordered. We construct two energy conditions over a free group with 2 generators whose union is not half-positional. We also observe in Section 5 that energy conditions over free groups are non-permuting, and that they can be used to refute 1-to-2-player lifting for half-positionality.

We believe that the class of energy conditions over totally ordered groups is interesting on its own. Namely, we find this class suitable for the following conjecture.

▶ **Conjecture 1.** *Every prefix-independent winning condition, bi-positional over finite arenas, coincides on periodic sequences with some energy condition over a totally ordered group.*

We cannot expect it to hold for all sequences, but periodic ones are sufficient, say, for algorithmic applications. If our conjecture is true, it gives an explicit description of the class of bi-positional prefix-independent winning condition. This would be in line with an explicit description of the class of *continuous* bi-positional payoffs from [16]. We discuss our conjecture in more detail in Section 6, where we reduce it to a problem about free groups.

2 Preliminaries

From now on, we restrict ourselves to finite arenas and to bi(half)-positionality over finite arenas.

If C is a set, we denote by C^* (resp., by C^ω) the set of all finite (resp., infinite) words over C. For $x \in C^*$, by $|x|$ we denote the length of x. Additionally, by C^+ we denote the set of all finite non-empty words over C. If $x \in C^+$, then by x^ω we denote an infinite word obtained by repeating x infinitely many times. The free group over C is denoted by F_C.

An arena $\mathcal{A}$ over a non-empty finite set (of colors) C is a tuple $\langle V_A, V_B, E\rangle$, where V_A and V_B are disjoint finite sets and $E \subseteq (V_A \cup V_E) \times C \times (V_A \cup V_B)$ is such that for every $s \in V_A \cup V_B$ there exist $c \in C$ and $t \in V_A \cup V_B$ for which $(s, c, t) \in E$. Elements of V_A are called Eve's nodes, and elements of V_B are called Adam's nodes. Elements of E are called edges of $\mathcal{A}$. An edge $e = (s, c, t) \in E$ is represented as a c-colored arrow from s to t. We use the notation $\mathsf{source}((s, c, t)) = s, \mathsf{col}((s, c, t)) = c$ and $\mathsf{target}((s, c, t)) = t$. Our definition guarantees that every node $v \in V_A \cup V_B$ has an out-going edge, that is, an edge e such that $\mathsf{source}(e) = v$.

An infinite-duration game over $\mathcal{A}$ from a node $s \in V_A \cup V_B$ is played as follows. At the beginning, one of the players chooses an edge $e_1 \in E$ with $\mathsf{source}(e_1) = s$. Namely, if $s \in V_A$, then Eve chooses e_1, and if $s \in V_B$, then Adam chooses e_1. More generally, in the first n turns players choose n edges $e_1, e_2, \ldots, e_n \in E$, one edge per turn. These edges always form a *path* in $\mathcal{A}$, that is, we have $\mathsf{target}(e_1) = \mathsf{source}(e_2), \ldots, \mathsf{target}(e_{n-1}) = \mathsf{source}(e_n)$. Then the $(n + 1)$st turn is played as follows. Players consider the endpoint node of the current path, which is $\mathsf{target}(e_n)$. One of the players chooses an edge e_{n+1} with $\mathsf{source}(e_{n+1}) = \mathsf{target}(e_n)$.

Namely, if $\mathsf{target}(e_n) \in V_A$, then Eve chooses e_{n+1}, and if $\mathsf{target}(e_n) \in V_B$, then Adam chooses e_{n+1}. After infinitely many turns, players get an infinite sequence of edges $p = (e_1, e_2, e_3, \ldots)$ called a play (it forms an infinite path in $\mathcal{A}$).

A winning condition over a set of colors C is a subset $W \subseteq C^\omega$. A strategy of Eve is winning from $s \in V_A \cup V_B$ w.r.t. W if any play $p = (e_1, e_2, e_3, \ldots)$ with this strategy in the infinite-duration game over $\mathcal{A}$ from s is such that its sequence of colors $\mathsf{col}(e_1)\mathsf{col}(e_2)\mathsf{col}(e_3)\ldots$ belongs to W. Similarly, a strategy of Adam is winning from $s \in V_A \cup V_B$ w.r.t. W if any play $p = (e_1, e_2, e_3, \ldots)$ with this strategy in the infinite-duration game over $\mathcal{A}$ from s is such that $\mathsf{col}(e_1)\mathsf{col}(e_2)\mathsf{col}(e_3)\ldots \notin W$.

A positional strategy of Eve is a function $\sigma\colon V_A \to E$ such that $\mathsf{source}(\sigma(u)) = u$ for any $u \in V_A$. It is interpreted as follows: for any $u \in V_A$, whenever Eve has to choose an edge from u, she chooses $\sigma(u)$. Similarly, a positional strategy of Adam is a function $\tau\colon V_B \to E$ such that $\mathsf{source}(\tau(u)) = u$ for any $u \in V_B$. It is interpreted analogously.

A winning condition $W \subseteq C^\omega$ is *half-positional* if for every finite arena $\mathcal{A}$ over C there exists a positional strategy σ of Eve such that for every node s of $\mathcal{A}$ the following holds: if σ is not winning w.r.t. W from s, then Adam has a winning strategy w.r.t. W from s. A winning condition W is *bi-positional* if both W and its complement $C^\omega \setminus W$ are half-positional.

A winning condition $W \subseteq C^\omega$ is *prefix-independent* if for all $x \in C^*$ and $\alpha \in C^\omega$ we have $\alpha \in W \iff x\alpha \in W$.

We state the following sufficient condition for bi-positionality due to Gimbert and Zielonka.

▶ **Definition 2.** *Let $W \subseteq C^\omega$ be a winning condition over a finite set of colors C. We call W* ***fairly mixing*** *if the following 3 conditions hold:*

- *A) For every $x \in C^*$ and $\alpha, \beta \in C^\omega$ we have that*

$$(x\alpha \notin W \wedge x\beta \in W) \implies (\alpha \notin W \wedge \beta \in W).$$

- *B) For every $S \in \{W, C^\omega \setminus W\}$, for every $x \in C^+$ and for every $\alpha \in C^\omega$ we have that*

$$(x^\omega \in S, \alpha \in S) \implies (x\alpha \in S).$$

- *C) For every $S \in \{W, C^\omega \setminus W\}$ and for every infinite sequence $x_1, x_2, x_3, \ldots \in C^+$ it holds that:*

$$\Big[(x_1x_3x_5\ldots \in S) \wedge (x_2x_4x_6\ldots \in S) \wedge (\forall n \geq 1\ x_n^\omega \in S)\Big] \implies x_1x_2x_3\ldots \in S.$$

▶ **Theorem 3** ([11])**.** *Any fairly mixing winning condition is bi-positional over finite arenas.*

3 Definition of Energy Games over Totally Ordered Groups

A totally ordered group [8] is a triple $(G, +, \leq)$, where $(G, +)$ is a group and $\leq$ is a total order on G such that

$$a \leq b \implies x + a + y \leq x + b + y \qquad \text{for all } a, b, x, y \in G.$$

The neutral element of G is denoted by 0. We do not assume that $+$ is commutative[1].

[1] Possibly, more common is to use the multiplicative notation for non-Abelian groups. However, we find additive notation more suitable due to the intuition that comes with the standard energy games.

Consider any finite set C of colors and any totally ordered group $(G, +, \leq)$. By a *valuation of colors* over $(G, +, \leq)$ we mean any function $\mathbf{val}\colon C \to G$. It can be extended to a homomorphism $\mathbf{val}\colon C^* \to G$ by setting

$$\mathbf{val}(\text{empty word}) = 0, \qquad \mathbf{val}(c_1 c_2 \dots c_n) = \mathbf{val}(c_1) + \mathbf{val}(c_2) + \ldots + \mathbf{val}(c_n).$$

Additionally, for every infinite sequence of colors $c_1 c_2 c_3 \ldots \in C^\omega$, we denote by $\overline{\mathbf{val}}(c_1 c_2 c_3 \ldots)$ the sequences of valuations of its finite prefixes:

$$\overline{\mathbf{val}}(c_1 c_2 c_3 \ldots) = \{\mathbf{val}(c_1 \ldots c_n)\}_{n=1}^{\infty}.$$

In other words, $\overline{\mathbf{val}}(c_1 c_2 c_3 \ldots)$ is the sequence of partial sums of the formal series $\sum_{n=1}^{\infty} \mathbf{val}(c_n)$.

An energy condition over $(G, +, \leq)$, defined by a valuation of colors $\mathbf{val}\colon C \to G$, is the set $W \subseteq C^\omega$ of all $\alpha \in C^\omega$ such that $\overline{\mathbf{val}}(\alpha)$ has an infinite decreasing subsequence. It is immediate that any energy condition over a totally ordered group is prefix-independent.

As an illustration, we show that parity conditions fall into this definition. The parity condition over d priorities is a winning condition $W^d_{par} \subseteq \{1, 2, \ldots, d\}^\omega$,

$$W^d_{par} = \{c_1 c_2 c_3 \ldots \in \{1, 2, \ldots, d\}^\omega \mid \limsup_{n\to\infty} c_i \text{ is odd}\}.$$

Observe that W^d_{par} is an energy condition over $\mathbb{Z}^d$ with the lexicographic ordering, defined by the following valuation:

$$\begin{aligned}
\mathbf{val}(d) &= ((-1)^d, 0, \ldots 0)\\
\mathbf{val}(d-1) &= (0, (-1)^{d-1}, \ldots 0)\\
&\vdots\\
\mathbf{val}(1) &= (0, 0, \ldots, -1).
\end{aligned}$$

As far as we know, the most general class of bi-positional prefix-independent winning conditions that were previously considered are *priority mean payoff conditions* [13]. They can also be defined as energy conditions over $\mathbb{Z}^d$. Moreover, to define them, it is sufficient to consider only valuations that map each color to a vector with at most 1 non-zero coordinate, as in the case of parity conditions.

4 Bi-positionality of Energy Conditions over Totally Ordered Groups

In this section, we establish

▶ **Theorem 4.** *Every ETOG condition is bi-positional over finite arenas.*

We derive it from the following technical result (which will also be useful in Section 6). If C is a non-empty finite set and $W \subseteq C^\omega$, define $\mathsf{per}(W) = \{x \in C^+ \mid x^\omega \in W\}$ to be the set of periods of periodic words from W.

▶ **Proposition 5.** *Let C be a non-empty finite set. Consider any set $P \subseteq C^+$ such that both P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts. Define a winning condition $W_P \subseteq C^\omega$ as follows:*

$$W_P = \{x y_1 y_2 y_3 \ldots \mid x \in C^*, y_1, y_2, y_3, \ldots \in P\}.$$

Then W_P is a prefix-independent fairly mixing winning condition with $P = \mathsf{per}(W_P)$.

Let us start with a derivation of Theorem 4.

Proof of Theorem 4 (modulo Proposition 5). Assume that $W \subseteq C^\omega$ is an energy condition over a totally ordered group $(G, +, \leq)$, defined by a valuation of colors $\mathbf{val}\colon C \to G$. Set $P = \{y \in C^+ \mid \mathbf{val}(y) < 0\}$. We claim that $W = W_P$. Indeed, W consists of all $\alpha = c_1c_2c_3 \ldots \in C^\omega$ such that

$$\overline{\mathbf{val}}(\alpha) = (\mathbf{val}(c_1), \mathbf{val}(c_1c_2), \mathbf{val}(c_1c_2c_3), \ldots)$$

has an infinite decreasing subsequence. Consider any $i < j$. Observe that the jth element of $\overline{\mathbf{val}}(\alpha)$ is smaller than the ith element of $\overline{\mathbf{val}}(\alpha)$ if and only if

$$-\mathbf{val}(c_1 \ldots c_i) + \mathbf{val}(c_1 \ldots c_j) = \mathbf{val}(c_{i+1} \ldots c_j) < 0.$$

In other words, $\overline{\mathbf{val}}(\alpha)$ has an infinite decreasing subsequence if and only if $\alpha = c_1c_2c_3 \ldots$ can be represented, except for some finite prefix, as a as a sequence of words with negative valuations. This means that $W = W_P$.

We now show that both P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts. By Proposition 5, this would imply that $W = W_P$ is fairly mixing. In turn, by Theorem 3, this implies that W is bi-positional.

Consider any two words $x, y \in C^+$. Obviously:

$$\mathbf{val}(x) < 0, \mathbf{val}(y) < 0 \implies \mathbf{val}(xy) = \mathbf{val}(x) + \mathbf{val}(y) < 0,$$
$$\mathbf{val}(x) \geq 0, \mathbf{val}(y) \geq 0 \implies \mathbf{val}(xy) = \mathbf{val}(x) + \mathbf{val}(y) \geq 0.$$

This demonstrates that both P and $C^+ \setminus P$ are closed under concatenations. Now, we claim that $\mathbf{val}(c_1c_2 \ldots c_n) < 0 \iff \mathbf{val}(c_2 \ldots c_nc_1) < 0$ for any word $c_1c_2 \ldots c_n \in C^+$ (this implies that both P and $C^+ \setminus P$ are closed under cyclic shifts). Indeed,

$$\begin{aligned}
&\mathbf{val}(c_1) + \mathbf{val}(c_2) + \ldots + \mathbf{val}(c_n) < 0 \\
&\iff -\mathbf{val}(c_1) + (\mathbf{val}(c_1) + \mathbf{val}(c_2) + \ldots + \mathbf{val}(c_n)) + \mathbf{val}(c_1) < -\mathbf{val}(c_1) + 0 + \mathbf{val}(c_1) \\
&\iff \mathbf{val}(c_2) + \ldots + \mathbf{val}(c_n) + \mathbf{val}(c_1) < 0.
\end{aligned}$$

Proof of Proposition 5. Prefix-independence of W_P is immediate. We now show that $P = \mathsf{per}(W_P)$. We have $z^\omega \in W_P$ for any $z \in P$ by definition. Hence, $P \subseteq \mathsf{per}(W_P)$. Now, take any $z \in \mathsf{per}(W_P)$. We show that $z \in P$. By definition of $\mathsf{per}(W_P)$, we have $z^\omega = xy_1y_2y_3 \ldots$ for some $x \in C^*$ and $y_1, y_2, y_3 \ldots \in P$. There exist $i < j$ such that $|xy_1 \ldots y_i|$ and $|xy_1 \ldots y_j|$ are equal modulo $|z|$. This means that $y_{j+1} \ldots y_j$ must be a multiple of some cyclic shift of z. We have that $y_{j+1} \ldots y_j \in P$ because P is closed under concatenations. This means that this cyclic shift of z also belongs to P. Indeed, otherwise, we could write $y_{j+1} \ldots y_j$ as a multiple of some word from $C^+ \setminus P$, and this is impossible because $C^+ \setminus P$ is closed under concatenations. Since P is closed under cyclic shifts, we obtain $z \in P$.

Finally, we show that W_P is fairly mixing. Since W_P is prefix-independent, we should care only about the third item of Definition 2. That is, we only have to show the following two claims:

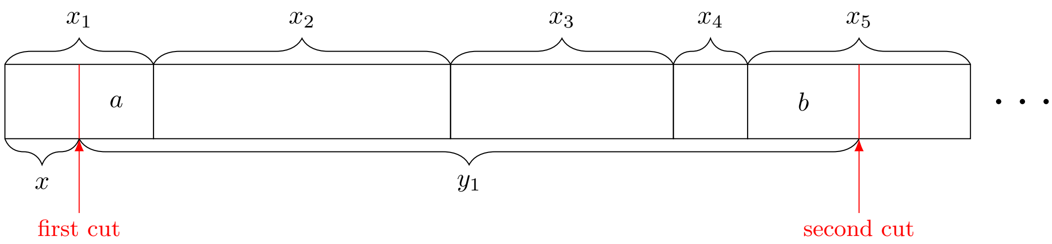
$$\Big[(x_1x_3x_5 \ldots \in W_P) \wedge (x_2x_4x_6 \ldots \in W_P) \wedge (\forall n \geq 1 \; x_n^\omega \in W_P)\Big] \implies x_1x_2x_3 \ldots \in W_P, \tag{1}$$

$$\Big[(x_1x_3x_5 \ldots \in \overline{W_P}) \wedge (x_2x_4x_6 \ldots \in \overline{W_P}) \wedge (\forall n \geq 1 \; x_n^\omega \in \overline{W_P})\Big] \implies x_1x_2x_3 \ldots \in \overline{W_P}, \tag{2}$$

for every infinite sequence of words $x_1, x_2, x_3, \ldots \in C^+$. Here, for brevity, by $\overline{W_P}$ we denote $C^\omega \setminus W_P$.

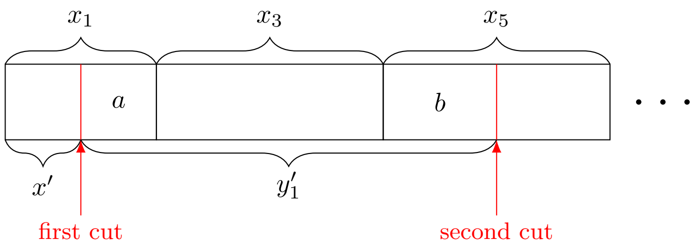
We first show (1). If $x_n^\omega \in W_P$ for every n, then $x_n \in \mathsf{per}(W_P) = P$ for every n, and hence $x_1x_2x_3\ldots \in W_P$ by definition.

A proof of (2) is more elaborate. Assume for contradiction that $x_1x_2x_3\ldots \in W_P$. Then we can write $x_1x_2x_3\ldots = xy_1y_2y_3\ldots$ for some $x \in C^*$ and $y_1, y_2, y_3, \ldots \in P$. One can represent the equality as a sequence of "cuts" inside $x_1x_2x_3\ldots$, as on the following picture:



Either there are infinitely many cuts inside x_n with odd indices, or there are infinitely many cuts inside x_n with even indices. Without loss of generality, we may assume that we only have cuts inside x_n with odd indices, and at most one for each n. Indeed, if necessary, we can join several successive y_i's into one word (this is legal because P is closed under concatenations).

We can now write each y_i as $y_i = ax_{2k}x_{2k+1}\ldots x_{2m}b$ for some $a, b \in C^*$ and $1 \le k \le m$. Now, let $y_i' = ax_{2k+1}x_{2k+3}\ldots x_{2m-1}b$ be a word which can be obtained from y_i by removing x_n with even indices. Additionally, we let $x' \in C^*$ be a word which can be obtained from x in the same way. Since each x_n with an even index lies entirely in some y_i or in x, we have that $x_1x_3x_5\ldots = x'y_1'y_2'y_3'\ldots$, as the following picture illustrates:



We will show that $y_i' \in P$ for every P. This would contradict the fact that $x_1x_3x_5\ldots \in \overline{W_P}$.

First, observe that $x_n \notin P$ for every n. Indeed, we are given that $x_n^\omega \in \overline{W_P}$ for every n. Hence, $x_n \notin \mathsf{per}(W_P) = P$, as required.

Assume for contradiction that $y_i' = ax_{2k+1}x_{2k+3}\ldots x_{2m-1}b \notin P$. Using the fact that $C^+ \setminus P$ is closed under concatenations and cyclic shifts, we obtain:

$$
\begin{aligned}
& y_i' = ax_{2k+1}x_{2k+3}\ldots x_{2m-1}b \notin P & \\
\Longrightarrow\ & x_{2k+1}x_{2k+3}\ldots x_{2m-1}ba \notin P & \\
\Longrightarrow\ & x_{2k}x_{2k+1}x_{2k+3}\ldots x_{2m-1}ba \notin P & \text{because } x_{2k} \notin P \\
\Longrightarrow\ & x_{2k+3}\ldots x_{2m-1}bax_{2k}x_{2k+1} \notin P & \\
\Longrightarrow\ & x_{2k+2}x_{2k+3}\ldots x_{2m-1}bax_{2k}x_{2k+1} \notin P & \text{because } x_{2k+2} \notin P \\
& \vdots & \\
\Longrightarrow\ & x_{2m}bax_{2k}x_{2k+1}\ldots x_{2m-1} \notin P & \text{because } x_{2m} \notin P \\
\Longrightarrow\ & y_i = ax_{2k}x_{2k+1}\ldots x_{2m}b \notin P, &
\end{aligned}
$$

contradiction. ◀

5 Refuting Kopczyński's conjecture

▶ **Theorem 6.** *There exist two ETOG conditions whose union is not half-positional over finite arenas.*

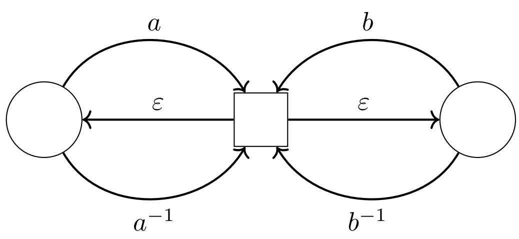
This theorem, together with the result that ETOG conditions are bi-positional over finite arenas (Theorem 4), refutes the Kopczyński's conjecture over finite arenas for finite unions.

Proof of Theorem 6. Consider the free group $F_{\{a,b\}}$ with 2 generators a, b. As was proved by Shimbireva [19], see also [8, Page 18], free groups can be totally ordered. We take an arbitrary total ordering $\leq$ of $F_{\{a,b\}}$. We also consider its inverse $\leq^{-1}$, which is also a total ordering of $F_{\{a,b\}}$. Define a set of colors $C = \{a, a^{-1}, b, b^{-1}, \varepsilon\}$. Here a^{-1}, b^{-1} are inverses of a, b in $F_{\{a,b\}}$, and ε is the identity element of $F_{\{a,b\}}$.

Let $W_1 \subseteq C^\omega$ be an energy condition over $(F_{\{a,b\}}, \leq)$, defined by a (suggestive) valuation of colors which interprets elements of C as corresponding elements of $F_{\{a,b\}}$. Similarly, we let $W_2 \subseteq C^\omega$ be an energy condition over $(F_{\{a,b\}}, \leq^{-1})$, defined by the same valuation. The only difference between W_1 and W_2 is that they are defined w.r.t. different total orderings of $F_{\{a,b\}}$ (one ordering is the inverse of the other one).

We show that the union $W_1 \cup W_2$ is not half-positional. It consists of all $\alpha \in C^\omega$ such that $\overline{\mathbf{val}}(\alpha)$ contains either an infinite decreasing subsequence w.r.t. $\leq$ or an infinite decreasing subsequence w.r.t. $\leq^{-1}$. In other words, it consists of all $\alpha \in C^\omega$ such that $\overline{\mathbf{val}}(\alpha)$ contains either an infinite decreasing subsequence or an infinite *increasing* subsequence w.r.t. $\leq$.

We show that $W_1 \cup W_2$ is not half-positional in the following finite arena.



Here, Eve controls the square and Adam controls the two circles. Assume that the game starts in the square. We show that Eve has a winning strategy w.r.t. $W_1 \cup W_2$, but not a positional one.

Eve has two positional strategies in this arena: always go to the left and always go to the right. Consider, for example, the first one. Adam has the following counter-strategy which wins against it: alternate the a-edge with the a^{-1}-edge. We get the following sequence of colors in the play of these two strategies:

$$\varepsilon a \varepsilon a^{-1} \varepsilon a \varepsilon a^{-1} \ldots$$

This sequence does not belong to $W_1 \cup W_2$ because

$$\overline{\mathbf{val}}(\varepsilon a \varepsilon a^{-1} \varepsilon a \varepsilon a^{-1} \ldots) = \varepsilon, a, a, \varepsilon, \varepsilon, a, a, \varepsilon, \ldots$$

There are only two distinct elements of $F_{\{a,b\}}$ occurring in $\overline{\mathbf{val}}(\varepsilon a \varepsilon a^{-1} \varepsilon a \varepsilon a^{-1} \ldots)$. Hence, it neither has an infinite decreasing subsequence nor an infinite increasing subsequence. By the same argument, the second positional strategy of Eve (always go to the right) is not winning w.r.t. $W_1 \cup W_2$ either.

On the other hand, Eve has the following winning strategy: alternate the edge to the left circle with the edge to the right circle. Consider any play with this strategy. Its sequence of colors looks as follows:

$$\varepsilon a^{\pm 1}\varepsilon b^{\pm 1}\varepsilon a^{\pm 1}\varepsilon b^{\pm 1}\ldots$$

We show that this sequence belongs to $W_1 \cup W_2$. A restriction of $\overline{\mathbf{val}}(\varepsilon a^{\pm 1}\varepsilon b^{\pm 1}\varepsilon a^{\pm 1}\varepsilon b^{\pm 1})$ to elements with even indices looks like this:

$$a^{\pm 1},\ a^{\pm 1}b^{\pm 1},\ a^{\pm 1}b^{\pm 1}a^{\pm 1},\ a^{\pm 1}b^{\pm 1}a^{\pm 1}b^{\pm 1}\ldots \tag{3}$$

All elements of (3) are distinct. Hence, by the Infinite Ramsey Theorem, it either has an infinite decreasing subsequence or an infinite increasing subsequence w.r.t. $\preceq$. Indeed, consider an infinite complete graph over $\{1, 2, 3, \ldots\}$, whose edges are colored green and red as follows. Pick any $i, j \in \{1, 2, 3, \ldots\}$, $i < j$. If the ith element of (3) is bigger than the jth element of (3), then color the edge between i and j into green. Otherwise, color this edge red (in this case, the ith element of (3) is smaller than the jth element of (3)). Our graph has an infinite induced subgraph in which all edges are of the same color. If they are all green (resp., red), then this subgraph defines an infinite decreasing (resp., increasing) subsequence of (3). ◀

Additional remarks. Energy conditions over free groups are interesting because they are *non-permuting* (if there is more than one generator). A prefix-independent winning condition is permuting if it is closed under permuting periods of periodic sequences. All previously known prefix-independent bi-positional winning conditions were permuting. This is because they can be seen as energy conditions over Abelian groups (on periodic sequences). In a talk of Colcombet and Niwiński [6] it was asked whether there exists a non-permuting bi-positional prefix-independent winning condition. The answer is "yes". For example, take W_1 as above in this section. Without loss of generality, we may assume that $aba^{-1}b^{-1}$ is negative w.r.t. $\preceq$ (otherwise we can consider its inverse). Then $(aba^{-1}b^{-1})^\omega \in W_1$, but $(aa^{-1}bb^{-1})^\omega \notin W_1$.

Additionally, the winning condition $W_1 \cup W_2$ is interesting because it refutes 1-to-2-player lifting for half-positionality. Namely, it is easy to see that $W_1 \cup W_2$ is positional for Eve in all arenas, where there are no nodes of Adam. This is because she can win in such arenas if and only if there is a reachable non-zero simple cycle. But as we have shown, $W_1 \cup W_2$ is not positional for Eve in the presence of Adam. Previously, there were examples that refute 1-to-2-player lifting for half-positionality in stochastic games [10].

6 Discussing Conjecture 1

First, it is useful to understand how prefix-independent bi-positional winning conditions are arranged on periodic sequences. Luckily, Proposition 5 gives an answer.

▶ **Proposition 7.** *Let C be a finite non-empty set. Then for any $P \subseteq C^+$ the following two conditions are equivalent:*

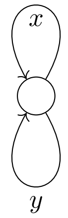
- *A) $P = \mathsf{per}(W)$ for some prefix-independent bi-positional (over finite arenas) winning condition $W \subseteq C^\omega$;*
- *B) P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts;*

Proof. The fact that the second item implies the first item follows from Proposition 5. Indeed, if P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts, then $P = \mathsf{per}(W_P)$ for a prefix-independent fairly mixing winning condition W_P, which is bi-positional by Theorem 3.

We now show that the first item implies the second item. The fact that P and $C^+ \setminus P$ are closed under cyclic shifts is a consequence of the prefix-independence of W:

$$\begin{aligned} c_1c_2 \dots c_n \in P &\iff (c_1c_2 \dots c_n)^\omega \in W \iff c_n(c_1c_2 \dots c_n)^\omega = (c_nc_1 \dots c_{n-1})^\omega \in W \\ &\iff c_nc_1 \dots c_{n-1} \in P. \end{aligned}$$

We now show that P is closed under concatenations (there is a similar argument for $C^+ \setminus P$). Take any $x, y \in P$. Consider the following arena.



It has a central circle node that lies on two simple cycles, one of which is colored by x and the other one by y. All nodes are controlled by Adam. Since, $x, y \in P$, we have that $x^\omega, y^\omega \in W$. Hence, Adam does not have a positional winning strategy w.r.t. W from the central circle. Since W is bi-positional, Adam has no winning strategy from the central circle w.r.t. W. Now, assume that Adam alternates the x-cycle with the y-cycle. He obtains $(xy)^\omega$ as a sequence of colors. Since this strategy is not winning, we have $xy \in P$. ◀

In turn, periods of periodic sequences of ETOG conditions are arranged as follows.

▶ **Proposition 8.** *Let C be a non-empty finite set and $W \subseteq C^\omega$ be an energy condition over a totally ordered group $(G, +, \leq)$, defined by a valuation of colors $\mathbf{val}\colon C \to G$. Then* $\mathsf{per}(W) = \{x \in C^+ \mid \mathbf{val}(x) < 0\}$.

Proof. Define $P = \{x \in C^+ \mid \mathbf{val}(x) < 0\}$. By the argument from the derivation of Theorem 4, we have $W = W_P$. Moreover, it was shown there that P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts. Finally, by Proposition 5, we have that $P = \mathsf{per}(W_P) = \mathsf{per}(W)$. ◀

Thus, Conjecture 1 is equivalent to the following conjecture.

▶ **Conjecture 9.** *Let C be any non-empty finite set. Then for any $P \subseteq C^+$ such that P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts there exists a totally ordered group $(G, +, \leq)$ and a valuation of colors $\mathbf{val}\colon C \to G$ such that $P = \{x \in C^+ \mid \mathbf{val}(x) < 0\}$.*

It might be concerning that P and $C^+ \setminus P$ are interchangeable in Conjecture 9, while **val** treats them asymmetrically. Namely, we require it to be negative on P and *non-negative* on $C^+ \setminus P$. However, **val** can always be made strictly positive on $C^+ \setminus P$. Namely, instead of G, consider the direct product $G \times \mathbb{Z}$ with the lexicographic order, and define a new valuation of colors $\mathbf{val}'\colon C \to G \times \mathbb{Z}$, $\mathbf{val}'(c) = (\mathbf{val}(c), 1)$.

Finally, we notice that our conjecture can be reduced to a reasoning about free groups.

▶ **Definition 10.** *A subset S of a group G is called an* ***invariant sub-semigroup*** *of G if the following two conditions hold:*

- *A) $xy \in S$ for all $x, y \in S$ (closure under multiplications);*
- *B) $gxg^{-1} \in S$ for all $g \in G, x \in S$ (closure under conjugations with elements of G).*

▶ **Conjecture 11.** *Consider an arbitrary non-empty finite set C and any $P \subseteq C^+$ such that P and $C^+ \setminus P$ are closed under concatenations and cyclic shifts. Then there exists an invariant sub-semigroup S of the free group F_C such that, first, $C^+ \setminus P$ is a subset of S, second, P is disjoint with S, and third, for every $g \in F_C$ either $g \in S$ or $g^{-1} \in S$ (in particular, S must have the neutral element).*

▶ **Proposition 12.** *Conjecture 9 is equivalent to Conjecture 11.*

Proof. Consider an arbitrary non-empty finite set C. It is sufficient to show that for any $P \subseteq C^+$ the following two conditions are equivalent:

- A) there exist a totally ordered group $(G, +, \leq)$ and a valuation of colors $\mathbf{val}\colon C \to G$ such that $P = \{x \in C^+ \mid \mathbf{val}(x) < 0\}$.
- B) there exists an invariant sub-semigroup S of the free group F_C such that, first, $C^+ \setminus P$ is a subset of S, second, P is disjoint with S, and third, for every $g \in F_C$ either $g \in S$ or $g^{-1} \in S$.

We first establish A) $\implies$ B). Extend $\mathbf{val}$ to a homomorphism from F_C to G by setting $\mathbf{val}(c^{-1}) = -\mathbf{val}(c)$ for $c \in C$. Set $S = \{g \in F_C \mid \mathbf{val}(g) \geq 0\}$. It is easy to check that all conditions on S are satisfied.

Now we establish B) $\implies$ A). Let S be as in B). Consider a binary relation $\sim$ on F_C, defined by $f \sim g \iff fg^{-1}, gf^{-1} \in S$ for $f, g \in F_C$. The fact that S is an invariant sub-semigroup with the neutral element implies that $\sim$ is a congruence on the group F_C. Let $G = F_C/\sim$ be the corresponding quotient group. Now, consider a binary relation $\preceq$ on F_C, defined by $f \preceq g \iff gf^{-1} \in S$ for $f, g \in F_C$ (observe that $f \sim g \iff f \preceq g, g \preceq f$). It is easy to see that $\preceq$ is correctly defined over $F_C/\sim$, whose elements are equivalence classes of $\sim$. More formally, it holds that if $a \sim b, x \sim y$, then $a \preceq x \iff b \preceq y$ (it can again be derived from the fact that S is an invariant sub-semigroup). It is also routine to check that $\preceq$ defines a total ordering on G. We need a condition that either $g \in S$ or $g^{-1} \in S$ for every $g \in F_C$ only to show the totality of our order. Namely, to show that there are no $f, g \in F_C$ with $f \not\preceq g$ and $g \not\preceq f$, we notice that otherwise neither gf^{-1} nor $fg^{-1} = (gf^{-1})^{-1}$ are in S. Observe that the equivalence class of $g \in F_C$ w.r.t. $\sim$ is non-negative in $(G, \preceq)$ if and only if $g \in S$. Now, recall that $C^+ \setminus P$ is a subset of S and P is disjoint with P. Hence, if we consider a valuation of colors $\mathbf{val}\colon C \to G$, which maps $c \in C$ to its equivalence class w.r.t. $\sim$, then P would be the set of words from C^+ whose valuation is negative w.r.t. $\preceq$. ◀

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