Limitations of Game Comonads for Invertible-Map Equivalence via Homomorphism Indistinguishability

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Abstract

Abramsky, Dawar, and Wang (2017) introduced the pebbling comonad for k-variable counting logic and thereby initiated a line of work that imports category theoretic machinery to finite model theory. Such game comonads have been developed for various logics, yielding characterisations of logical equivalences in terms of isomorphisms in the associated co-Kleisli category. We show a first limitation of this approach by studying linear-algebraic logic, which is strictly more expressive than first-order counting logic and whose k-variable logical equivalence relations are known as invertible-map equivalences (IM). We show that there exists no finite-rank comonad on the category of graphs whose co-Kleisli isomorphisms characterise IM-equivalence, answering a question of Ó Conghaile and Dawar (CSL 2021). We obtain this result by ruling out a characterisation of IM-equivalence in terms of homomorphism indistinguishability and employing the Lovász-type theorem for game comonads established by Reggio (2022). Two graphs are *homomorphism indistinguishable* over a graph class if they admit the same number of homomorphisms from every graph in the class. The IM-equivalences cannot be characterised in this way, neither when counting homomorphisms in the natural numbers, nor in any finite prime field.

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1 Introduction

Logic fragments such as k-variable first-order logic with or without counting quantifiers induce equivalence relations on graphs, or more generally, on structures: Two structures are equivalent in this sense if they satisfy exactly the same sentences of the respective logic fragment. Such equivalence relations are approximations of the isomorphism relation. The more expressive the logic fragment, the more non-isomorphic structures are distinguished by it. Classical model-comparison games and counterexamples like the Cai–Fürer–Immerman (CFI) construction show that k-variable first-order logic (even with counting) does not distinguish all pairs of non-isomorphic structures. Hence, the induced equivalence is indeed strictly coarser than isomorphism. Such approximations of isomorphism can be studied from many different angles. For example, it is well-known that counting logic equivalence is the same as indistinguishability by the Weisfeiler–Leman graph isomorphism test [6].

Another perspective to approximations of isomorphism is offered by homomorphism indistinguishability: Two graphs G and H are *homomorphism indistinguishable* over a class of graphs $\mathcal{F}$ if for all graphs $F \in \mathcal{F}$ the number of homomorphisms from F to G is equal to the number of homomorphisms from F to H. Equivalence relations with respect to many logic fragments can be characterised as homomorphism indistinguishability relations over some graph class. For example, two graphs are counting logic equivalent if and only if they are homomorphism indistinguishable over all graphs of bounded treewidth [13, 12]. Besides counting logic equivalence, many other natural equivalence relations between graphs, including isomorphism [24], quantum isomorphism [26], cospectrality [12], and feasibility of integer programming relaxations for graph isomorphism [12, 18, 33] have been characterised as homomorphism indistinguishability relations over various graph classes. Characterising (logical) equivalences as homomorphism indistinguishability relations is desirable because such characterisations allow to compare the expressive power of logics solely by comparing the graph classes from which homomorphisms are counted [33, 32]. In this way, deep results from structural graph theory are made available for studying the expressive power of logics [34].

It is natural to ask whether this approach can be extended to interesting logics that are more expressive than counting logic, as they are for example studied in the quest for a logic for Ptime. Such examples are *rank logic* [9, 16] and the more general *linear-algebraic logic* (LA) [8]. We answer this question in the negative. The *invertible-map equivalence* $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$, as the equivalence of the k-variable fragment of LA is called, cannot be characterised as a homomorphism indistinguishability relation.

▶ **Theorem 1.** *For every $k \geq 6$, $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ is not a homomorphism indistinguishability relation.*

The proof relies on a CFI-like construction similar to the one used by the first author to separate rank logic from polynomial time [23]. We combine this with results by Roberson [32] in order to obtain graphs that are invertible-map equivalent but not quantum isomorphic. As shown by the third author [34], this suffices to conclude that invertible-map equivalence is not a homomorphism indistinguishability relation – if it were, then it would have to be a refinement of quantum isomorphism.

Theorem 1 implies a negative answer to a question posed by Ó Conghaile and Dawar [30]. Their work is part of a recent line of research that explores connections between methods from finite model theory, descriptive complexity, and category theory [3]. One goal of these efforts is to characterise logical equivalences using *game comonads*. Concretely, Ó Conghaile and Dawar asked whether this is possible for linear-algebraic logic. Employing a categorical Lovász-type theorem [11, 31] that allows to infer a homomorphism indistinguishability characterisation from the existence of appropriate game comonads, we obtain the following result. To our knowledge, this is the first provable limitation of such comonadic characterisations.

▶ **Theorem 2.** *For every $k \geq 6$, there is no finite-rank comonad $\mathbb{C}$ on the category of graphs such that $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ coincides with the isomorphism relation in the co-Kleisli category of $\mathbb{C}$.*

In this context, the concept of a comonad is best explained by recalling the *pebbling comonad* $\mathbb{T}_k$ introduced by Abramsky, Dawar, and Wang [1]. Designed to provide a categorical formulation of the k-pebble game from finite model theory, it can be thought of as map sending structures to structures encoding Spoiler's plays in this game. Being a comonad, it gives rise to a category, its co-Kleisli category, whose objects are graphs and whose morphisms can be interpreted as winning strategies for Duplicator in the k-pebble game. Various notions from finite model theory can now be recovered from this construction: For example, a graph has treewidth less than k if and only if it admits a $\mathbb{T}_k$-coalgebra. Crucially, two graphs satisfy the same k-variable counting logic sentences if and only if they are isomorphic in the co-Kleisli category of $\mathbb{T}_k$. Subsequently, comonads for many fragments [1, 3, 27] and extensions [30] of first-order logic have been constructed. They have in common that their co-Kleisli morphisms and isomorphisms encode winning strategies for Duplicator in one-sided, symmetric, and bijective games. Our Theorem 2 rules out a characterisation of invertible-map equivalence via co-Kleisli isomorphisms. We note that our Theorem 1 does not exclude characterisations involving other comonadic constructions.

Comonads on the category of graphs and homomorphism indistinguishability are intimately connected. Every homomorphism indistinguishability relation over a graph class with mild closure properties can be characterised as co-Kleisli isomorphism over a comonad [2]. Conversely, the existence of co-Kleisli isomorphisms over a finite-rank comonad can be characterised as a homomorphism indistinguishability relation [11, 31]. This fundamental connection between comonads and homomorphism counting relations is exactly the reason why we can conclude the impossibility of the former from the impossibility of the latter: There is no finite-rank comonad for linear-algebraic logic.

Hence, linear-algebraic logic seems to be of a very different nature than the weaker counting logic as it does not connect with the theory revolving around homomorphism indistinguishability and game comonads. This raises the question as to what is the precise reason for this situation. What makes a logic "nice enough" to fit within the homomorphism indistinguishability and comonadic framework? We can at least say that the shortcomings of LA in this respect are not due to it being strictly stronger than counting logic. There does exist an extension of counting logic which admits a comonad construction and thereby a homomorphism indistinguishability relation: This is k-variable infinitary FO enriched with all possible n-ary *generalised quantifiers* over one-dimensional interpretations [30]. An n-ary generalised quantifier (also known as *Lindström quantifier*) is essentially a membership oracle for a class $\mathcal{K}$ (of at most n-ary structures) that allows to test whether some structure $\mathfrak{B}$ interpretable in the given structure $\mathfrak{A}$ is in $\mathcal{K}$. LA lies somewhere between counting logic and its extension by *all* binary Lindström quantifiers because LA is infinitary FO extended with a *proper subclass* of binary Lindström quantifiers. Counting logic itself is nothing but the extension of FO with *all unary* Lindström quantifiers [22]. Hence, we can describe the situation as follows: Whenever a Lindström-extension of infinitary FO contains *all* one-dimensional Lindström quantifiers up to a given arity n, then it admits a comonad. If it only contains a subset of these Lindström quantifiers, then this is not necessarily the case (our Theorem 1 is true even when we restrict LA to one-dimensional interpretations).

Finally, another direction that we explore in this paper is counting homomorphisms in *finite prime fields*. A large part of the theory of homomorphism indistinguishability that has been established so far works over the natural numbers. Given the fact that the linear-algebraic operators in LA are over finite fields, one might a priori suspect that the appropriate homomorphism indistinguishability relation must be based on homomorphism counts modulo a prime. However, this can also be ruled out, even when the homomorphisms are counted modulo several primes (Theorem 25).

As a positive result concerning homomorphism counting modulo primes, we find that Dvořák's proof [13] can be adapted to finite fields: Two graphs admit the same numbers of homomorphisms modulo p from all graphs of treewidth less than k if and only if they are equivalent with respect to k-variable modular counting logic (Theorem 26).

2 Preliminaries

All structures in this paper are relational and finite. General relational structures are usually denoted $\mathfrak{A}$ or $\mathfrak{B}$, with A or B, respectively, being used for the universe. When we speak of graphs, we mean $\{E\}$-structures, where E is binary, and we will write $V(G)$ and $E(G)$ for the vertices respectively edges of a graph G. When nothing else is specified, graphs are undirected and we may write $uv \in E(G)$ for edges $\{u, v\} \in E(G)$. The set $\{1, 2, ..., n\}$ is denoted by $[n]$, and $\mathbb{P} \subseteq \mathbb{N}$ denotes the set of primes.

Counting logic. The logic $\mathcal{C}^k$ is the k-variable fragment of first-order logic with counting quantifiers of the form $\exists^{\geq i}x$, for every $i \in \mathbb{N}$. The semantics is as expected, i.e., a structure $\mathfrak{A}$ satisfies a sentence $\exists^{\geq i}x\varphi(x)$ if there exist at least i distinct $a \in A$ such that $\mathfrak{A} \models \varphi(a)$. We write $\mathfrak{A} \equiv_{\mathcal{C}^k} \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{C}^k$-equivalent, i.e., they satisfy exactly the same $\mathcal{C}^k$-sentences.

Lindström quantifiers and interpretations. A more general way to extend FO is with *Lindström* quantifiers (also known as *generalised* quantifiers). A Lindström quantifier is essentially a membership oracle for a class of structures. Before introducing Lindström quantifiers, we need the concept of logical interpretations. Let σ and τ be relational vocabularies with $\tau = \{R_1, ..., R_m\}$ where each R_i is a relation symbol of arity r_i, and let $\mathcal{L}$ be a logic. An *ℓ-dimensional $\mathcal{L}[\sigma, \tau]$-interpretation* I is an $\mathcal{L}$-definable mapping from σ-structures to τ-structures. The elements of the τ-structure are sets of ℓ-tuples in the original σ-structure. Generally, interpretations can take a tuple of parameters $\overline{z}$: An ℓ-dimensional $\mathcal{L}$-interpretation (with parameters) is a tuple

$$I(\overline{z}) = \big(\varphi_\delta(\overline{x}, \overline{z}), \varphi_\approx(\overline{x}, \overline{y}, \overline{z}), \varphi_{R_1}(\overline{x}_1, ..., \overline{x}_{r_1}, \overline{z}), ..., \varphi_{R_m}(\overline{x}_1, ..., \overline{x}_{r_m}, \overline{z})\big),$$

where $\overline{x}$, $\overline{y}$, $\overline{x}_i$ are ℓ-tuples of variables, and $\varphi_\delta, \varphi_\approx, \varphi_{R_i}$ are σ-formulas of the logic $\mathcal{L}$. The interpretation $I(\overline{z})$ defines a partial mapping from σ-structures to τ-structures. For a given σ-structure $\mathfrak{A}$ and an assignment $\overline{z} \mapsto \overline{a}$, we define $\mathfrak{B}$ to be the τ-structure that has universe $B := \{\overline{b} \in A^k \mid \mathfrak{A} \models \varphi_\delta(\overline{b}, \overline{a})\}$ and, for all $i \in [m]$, has the relations $R_i^{\mathfrak{B}} := \{(\overline{b}_1, ..., \overline{b}_{r_i}) \in B^{r_i} \mid \mathfrak{A} \models \varphi_{R_i}(\overline{b}_1, ..., \overline{b}_{r_i}, \overline{a})\}$. From this structure, we obtain the "output" $I(\mathfrak{A}, \overline{z} \mapsto \overline{a})$ of I by factoring out the equivalence classes defined by $\varphi_\approx$. Formally, let $\mathcal{E} := \{(\overline{b}_1, \overline{b}_2) \in A^{2k} \mid \mathfrak{A} \models \varphi_\approx(\overline{b}_1, \overline{b}_2, \overline{a})\}$. If $\mathcal{E}$ is not a congruence relation on $\mathfrak{B}$, then $I(\mathfrak{A}, \overline{z} \mapsto \overline{a})$ is undefined. Otherwise, $I(\mathfrak{A}, \overline{z} \mapsto \overline{a})$ is defined as the quotient structure $\mathfrak{B}/\mathcal{E}$.

Let $\mathcal{K}$ be a class of τ-structures and $\mathcal{L}$ be a logic. The extension $\mathcal{L}(\mathcal{Q}_\mathcal{K})$ of $\mathcal{L}$ by the *Lindström quantifier* for $\mathcal{K}$ is obtained by closing $\mathcal{L}$ under the following formula formation rule: Whenever $I(\overline{z})$ is an $\mathcal{L}(\mathcal{Q}_\mathcal{K})[\sigma, \tau]$-interpretation, then $\mathcal{Q}_\mathcal{K} I(\overline{z})$ is a σ-formula of $\mathcal{L}(\mathcal{Q}_\mathcal{K})$ with free variables $\overline{z}$. For a σ-structure $\mathfrak{A}$ and an assignment $\overline{z} \mapsto \overline{a}$, it holds $(\mathfrak{A}, \overline{a}) \models \mathcal{Q}_\mathcal{K} I(\overline{z})$ if $I(\mathfrak{A}, \overline{x} \mapsto \overline{a}) \in \mathcal{K}$. If $\mathbf{Q}$ is a class of Lindström quantifiers, then $\mathcal{L}(\mathbf{Q})$ denotes the extension by all Lindström quantifiers in $\mathbf{Q}$. When we speak of the one-dimensional restriction of such a logic, we mean that in formulas $\mathcal{Q}_\mathcal{K} I(\overline{z})$, the interpretation I has to be one-dimensional.

Linear-algebraic logic and invertible-map equivalences. Linear-algebraic logic (LA) was introduced by Dawar, Grädel, and Pakusa [8] as an extension of infinitary first-order logic with all isomorphism-invariant linear-algebraic operators. As such, it extends *rank logic* [9, 16]. Rank logic in turn is an extension of FO with operators for determining the rank of a matrix that is definable in the input structure. In linear-algebraic logic, formulas have access to *any* isomorphism-invariant parameter of a definable matrix and not only to the rank. This logic was studied to show that no linear-algebraic operators whatsoever can enhance the power of FO such that its k-variable fragment distinguishes all non-isomorphic structures, for some fixed k. For the detailed definition of LA, we refer to [8]. In short, a *linear-algebraic* function over some field $\mathbb{F}$ with some arity $m \geq 1$ is a function f that maps tuples $(M_1, ..., M_m)$ of linear transformations/matrices over $\mathbb{F}$ to natural numbers such that f is invariant under vector space isomorphisms. Formally, this means that whenever two sequences of matrices $M_1, ..., M_m$ and $M'_1, ..., M'_m$ over $\mathbb{F}$ are *simultaneously similar*, then $f(M_1, ..., M_m) = f(M'_1, ..., M'_m)$. Simultaneous similarity means that there is an invertible matrix S over $\mathbb{F}$ such that $M_i \cdot S = S \cdot M'_i$ for all $i \in [m]$. That is to say, there exists an isomorphism between the underlying vector spaces that maps each linear transformation M_i to the corresponding M'_i that operates on the isomorphic space. For instance, the rank operator is such a function with arity $m = 1$ that maps a given matrix to its rank.

With every m-ary linear-algebraic function f and every natural number r, we associate the class $\mathcal{K}^t_f$ of structures (for some appropriate vocabulary) that encode tuples of matrices $(M_1, ..., M_m)$ satisfying $f(M_1, ..., M_m) \geq t$. Now, linear-algebraic logic LA is the closure of FO under infinite conjunctions and disjunctions and under Lindström quantifiers for all classes $\mathcal{K}^t_f$: A structure $\mathfrak{A}$ satisfies $\mathcal{Q}_{\mathcal{K}^t_f} I(\overline{x})$ if $I(\mathfrak{A})$ is a structure that encodes a tuple $(M_1, ..., M_m)$ of matrices and satisfies $f(M_1, ..., M_m) \geq t$.

Fragments of LA yield interesting equivalence relations between structures, which are approximations of isomorphism. The fragments that are studied in the literature (e.g. in [8, 23]) are parametrized by $k \in \mathbb{N}$ and $Q \subseteq \mathbb{P}$. The logic $\text{LA}^k(Q)$ is the k-variable fragment of LA that only uses linear-algebraic operators over finite fields of characteristic $p \in Q$. The equivalence relation induced by $\text{LA}^k(Q)$ is called *invertible-map equivalence*. We write $\mathfrak{A} \equiv^{\text{IM}}_{k,Q} \mathfrak{B}$ if the structures $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same $\text{LA}^k(Q)$-sentences. Invertible-map equivalence of two given structures can be tested in polynomial time [8].

The logic $\text{LA}^k(Q)$ is at least as expressive as $\mathcal{C}^k$ because the quantifier $\exists^{\geq i} x \varphi(x)$ can be simulated with the rank operator [8]: We have $\mathfrak{A} \models \exists^{\geq i} x \varphi(x)$ if and only if the diagonal matrix that has a 1-entry at exactly those positions $(a, a) \in A^2$ such that $\mathfrak{A} \models \varphi(a)$ has rank at least i. This works irrespective of which primes are in Q. Hence, for every non-empty Q, the relation $\equiv^{\text{IM}}_{k,Q}$ is at least as fine as $\equiv_{\mathcal{C}^k}$. In fact, it is strictly finer because there exist generalised CFI-structures that are $\equiv_{\mathcal{C}^k}$-equivalent but distinguishable in rank logic [9] using ranks over $\mathbb{F}_p$ for each $p \in Q$.

Invertible-map equivalence is also characterized by a Spoiler-Duplicator game called the *invertible-map game* [10]. We follow the exposition in [23]. Let $Q \subseteq \mathbb{P}$ and $k \in \mathbb{N}$. The IM-game $\mathcal{M}^{k,Q}$ is played on two structures $\mathfrak{A}$ and $\mathfrak{B}$. There are k pairs of pebbles labelled with $1, \dots, k$. A position in the game is a pair $\overline{a}, \overline{b}$ of tuples $\overline{a} \in A^m$ and $\overline{b} \in B^m$ for some $m \leq k$. In position $\overline{a}, \overline{b}$ corresponding pebbles, i.e., pebbles with the same label, are placed on a_i and b_i for every $i \in [k]$. Initially, the pebbles are not on the board. If $|A| \neq |B|$, then Spoiler wins immediately. Otherwise, a round of the game is played as follows:

1. Spoiler chooses a prime $p \in Q$ and a number ℓ satisfying $2\ell \leq k$. Next, Spoiler picks up 2ℓ pebbles from $\mathfrak{A}$ and the corresponding pebbles (with the same labels) from $\mathfrak{B}$.
2. Duplicator picks a partition $\mathcal{P}$ of $A^\ell \times A^\ell$ and another one $\mathcal{P}'$ of $B^\ell \times B^\ell$ such that $|\mathcal{P}| = |\mathcal{P}'|$. Furthermore, Duplicator picks a bijection $h \colon \mathcal{P} \to \mathcal{P}'$ and an invertible $(A^\ell \times B^\ell)$-matrix S over $\mathbb{F}_p$ such that $\chi^P = S \cdot \chi^{h(P)} \cdot S^{-1}$ for every $P \in \mathcal{P}$. Here, χ^P denotes the characteristic matrix of P, which has a 1-entry at position $(\overline{u}, \overline{v})$ if $\overline{uv} \in P$ and is 0 otherwise.
3. Spoiler chooses a block $P \in \mathcal{P}$, a tuple $\overline{u} \in P$, and a tuple $\overline{v} \in h(P)$. Then for each $i \in [2\ell]$, Spoiler places one of the pebbles picked up from $\mathfrak{A}$ on u_i and the corresponding one picked up from $\mathfrak{B}$ on v_i.

After a round, Spoiler wins the game if the pebbles do not define a partial isomorphism or if Duplicator was not able to respond with a matrix satisfying the condition above. Note that this condition states that the characteristic matrices of the blocks are simultaneously similar.

▶ **Lemma 3.** *Let $k \in \mathbb{N}$, $Q \subseteq \mathbb{P}$, $\mathfrak{A}$ and $\mathfrak{B}$ be structures, $\overline{a} \in A^k$, and $\overline{b} \in B^k$. Then $(\mathfrak{A}, \overline{a}) \equiv^{\mathrm{IM}}_{k,Q} (\mathfrak{A}, \overline{b})$ if and only if Duplicator has a winning strategy in the invertible-map game $\mathcal{M}^{k,Q}$ on $\mathfrak{A}$ and $\mathfrak{B}$ in position $\overline{a}, \overline{b}$.*

The lemma follows from a combination of [10, 8], in which the game is also parametrised by the dimension 2ℓ of the interpretations. In [10], only finite sets of primes are considered because the logics considered there are not infinitary. The arguments straight-forwardly apply to arbitrary sets of primes.

Homomorphism Indistinguishability. Let F and G be graphs. A *homomorphism* ψ from F to G is a map $\psi\colon V(F) \to V(G)$ such that $\psi(u)\psi(v) \in E(G)$ for every edge $uv \in E(F)$. We write $\mathrm{Hom}(F, G)$ for set of homomorphisms from F to G and $\mathrm{hom}(F, G) := |\mathrm{Hom}(F, G)|$. Homomorphism counts induce equivalence relations on graphs: Let $\mathcal{F}$ be a class of graphs. Two graphs G and H are *homomorphism indistinguishable* over $\mathcal{F}$, denoted by $G \equiv_{\mathcal{F}} H$, if for every $F \in \mathcal{F}$, it holds that $\mathrm{hom}(F, G) = \mathrm{hom}(F, H)$. An equivalence relation $\approx$ between graphs is a *homomorphism indistinguishability relation* if there exists a graph class $\mathcal{F}$ such that $\approx$ and $\equiv_{\mathcal{F}}$ coincide.

In this paper, we call two graphs *quantum isomorphic* if they are homomorphism indistinguishable over all planar graphs. The term was originally introduced as a quantum information theoretic notion [4]. The titular result of Mančinska's and Roberson's seminal work [26] asserts that it is the same as homomorphism indistinguishability over all planar graphs. Note that our results do not depend on [4, 26].

3 Homomorphisms to CFI-Like Graphs over Finite Abelian Groups

Roberson [32] studied homomorphisms to CFI-like graphs constructed over $\mathbb{Z}_2$. This variant of CFI graphs was introduced by Fürer [15]. Neuen and Schweitzer [29] generalised the more classical CFI construction from $\mathbb{Z}_2$ to arbitrary finite abelian groups. We combine both constructions and generalise the CFI construction from [15, 32] to arbitrary finite abelian groups. The goal of these efforts is to show that certain CFI graphs over planar base graphs and w.r.t. to arbitrary finite abelian groups are not quantum isomorphic.

We fix such a group Γ throughout this section and write its operation as addition. For a graph G and a vertex $v \in V(G)$, write $E(v) := \{e \in E(G) \mid v \in e\}$ for the set of edges incident to v. We consider vectors $U \in \Gamma^X$ for finite sets X. For an element $x \in X$, we write $U(x) \in \Gamma$ for the x-th entry of U. We write $\sum U$ for $\sum_{x \in X} U(x)$.

▶ **Definition 4.** *A* base graph *is a connected graph. Let G be a base graph and $U \in \Gamma^{V(G)}$. For every vertex u of G, we define*

$$V_u := \left\{ (u, S) \;\middle|\; S \in \Gamma^{E(u)}, \sum S = U(u) \right\}.$$

The CFI graph $\mathsf{CFI}[\Gamma, G, U]$ *over the finite abelian group Γ and the base graph G has vertex set $\bigcup_{u \in V(G)} V_u$ and edge set*

$$\left\{ \{(u, S), (v, T)\} \;\middle|\; (u, S) \in V_u, (v, T) \in V_v, uv \in E(G), S(uv) + T(uv) = 0 \right\}.$$

We say that the vertices in V_u have origin *u.*

The proof of the following Lemma 5 uses well-known arguments for CFI graphs [15, 29, 23].

▶ **Lemma 5.** *Let G be a base graph and $U, U' \in \Gamma^{V(G)}$. If $\sum U = \sum U'$, then* $\mathsf{CFI}[\Gamma, G, U] \cong \mathsf{CFI}[\Gamma, G, U']$.

Proof. Let $uv \in E(G)$. Denote the vertex set of $\mathsf{CFI}[\Gamma, G, U]$ respectively $\mathsf{CFI}[\Gamma, G, U']$ by V_U and $V_{U'}$. First consider $U' := U + u - v$ where u and v denote the vectors in $\Gamma^{V(G)}$ with one at the u-th and v-th component, respectively, and zero otherwise. Define the map $\varphi \colon V_U \to V_{U'}$ by

$$\varphi((w, S)) := \begin{cases} (u, S + uv), & \text{if } w = u, \\ (v, S - uv), & \text{if } w = v, \\ (w, S), & \text{otherwise.} \end{cases}$$

where uv denotes the vector in $\Gamma^{E(u)}$ in the first case or in $\Gamma^{E(v)}$ in the second case with one at the uv-th component and zero otherwise. Observe that $\sum_{e \in E(v)} (S - uv)(e) = U(v) - 1 = U'(v)$ and analogously for u. Hence, φ is indeed a well-defined map to $V_{U'}$. Clearly, φ is a bijection. Let $(x, S), (y, T) \in V_U$ be arbitrary vertices of $\mathsf{CFI}[\Gamma, G, U]$ and define $(x, S') := \varphi(x, S)$ and $(y, T') := \varphi(y, T)$. Then $S'(xy) + T'(xy) = S(xy) + T(xy)$. Hence, (x, S) and (y, T) are adjacent in $\mathsf{CFI}[\Gamma, G, U]$ if and only if they are adjacent in $\mathsf{CFI}[\Gamma, G, U']$.

Since G is connected, the maps constructed above can be composed to yield $\mathsf{CFI}[\Gamma, G, U] \cong \mathsf{CFI}[\Gamma, G, U + u - v]$ for every pair of vertices u, v. This yields $\mathsf{CFI}[\Gamma, G, U] \cong \mathsf{CFI}[\Gamma, G, U']$ as desired. ◀

We proceed by counting homomorphisms into the CFI graphs. For a graph G and $U \in \Gamma^{V(G)}$, consider the *projection map* $\rho \colon \mathsf{CFI}[\Gamma, G, U] \to G$ sending (v, S) to v. Clearly, ρ is a homomorphism. For a graph F and a homomorphism $\psi \colon F \to G$, define

$$\mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U]) := \left\{ \varphi \in \mathrm{Hom}(F, \mathsf{CFI}[\Gamma, G, U]) \;\middle|\; \rho \circ \varphi = \psi \right\}.$$

The sets $\mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$ for all $\psi \colon F \to G$ partition the set $\mathrm{Hom}(F, \mathsf{CFI}[\Gamma, G, U])$ of homomorphisms $F \to \mathsf{CFI}[\Gamma, G, U]$. Write $\mathrm{hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$ for the cardinality of $\mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$.

▶ **Lemma 6.** *Let F be a graph and G be a base graph. Let $U \in \Gamma^{V(G)}$ and fix $\psi \in \mathrm{Hom}(F, G)$. Consider the system of equations $\mathsf{Hom}(F, G, U, \psi)$ with variables x_e^a for all $a \in V(F)$ and $e \in E(\psi(a))$ and equations*

$$\sum_{e \in E(\psi(a))} x_e^a = U(\psi(a)) \qquad \textit{for all } a \in V(F), \tag{1}$$

$$x_e^a + x_e^b = 0 \qquad \textit{for all } ab \in E(F) \textit{ and } e = \psi(ab) \in E(G). \tag{2}$$

Then the number of solutions to $\mathsf{Hom}(F, G, U, \psi)$ over Γ is equal to $\mathrm{hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$.

Proof. The proof is by giving a bijection between the solution set and $\mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$. Let $\xi = (\xi_e^a)_{a \in V(F), e \in E(\psi(a))}$ be a solution to $\mathsf{Hom}(F, G, U, \psi)$ over Γ. Define a homomorphism $\varphi_\xi \in \mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$ via $\varphi_\xi(a) := \big(\psi(a), (\xi_e^a)_{e \in E(\psi(a))}\big)$. Equation (1) guarantees that this is indeed a map from the vertices of F to the ones of $\mathsf{CFI}[\Gamma, G, U]$. If a and b are adjacent in F, then so are $\psi(a)$ and $\psi(b)$ in G. Furthermore, $\xi^a_{\psi(ab)} + \xi^b_{\psi(ab)} = 0$ by Equation (2). Hence, $\varphi_\xi(a)$ and $\varphi_\xi(b)$ are adjacent in $\mathsf{CFI}[\Gamma, G, U]$.

It is easy to see that this construction is injective, i.e., if $\varphi_\xi = \varphi_\zeta$, then $\xi = \zeta$. For surjectivity, let $\varphi \in \mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, U])$. For every $a \in V(F)$ and $e \in E(\psi(a))$, define ξ_e^a as the second component of $\varphi(a)$, i.e. $\xi_e^a := S_a(e)$ where $\varphi(a) = (\psi(a), S_a)$. Clearly, $\xi := (\xi_e^a)$ is such that $\varphi_\xi = \varphi$. The fact that ξ satisfies Equations (1) and (2) is easily verified. ◀

▶ **Theorem 7.** *For a base graph G, $U \in \Gamma^{V(G)}$, and $\psi \in \mathrm{Hom}(F, G)$ for some graph F, the following hold:*

1. $\hom_\psi(F, \mathsf{CFI}[\Gamma, G, 0]) > 0$.
2. *If* $\mathsf{Hom}(F, G, U, \psi)$ *has a solution, then* $\hom_\psi(F, \mathsf{CFI}[\Gamma, G, 0]) = \hom_\psi(F, \mathsf{CFI}[\Gamma, G, U])$.
3. *If* $\mathsf{Hom}(F, G, U, \psi)$ *has no solution, then* $\hom_\psi(F, \mathsf{CFI}[\Gamma, G, U]) = 0$.

Proof. The system $\mathsf{Hom}(F, G, U, \psi)$ can be compressed into a matrix equation as follows: For $\psi \in \mathrm{Hom}(F, G)$ and $P := \{(a, e) \mid a \in V(F), e \in E(\psi(a))\}$, let $A^\psi \in \Gamma^{V(F)\times P}$ and $B^\psi \in \Gamma^{E(F)\times P}$ be the matrices defined by

$$A^\psi_{b,(a,e)} := \delta_{b=a} \quad \text{and} \quad B^\psi_{bc,(a,e)} := \delta_{a\in\{b,c\}\wedge e=\psi(bc)}, \tag{3}$$

where δ_C, similar to the Kronecker delta, evaluates to 1 if the condition C is satisfied and is 0 otherwise. Then Equations (1) and (2) are equivalent to

$$\begin{pmatrix} A^\psi \\ B^\psi \end{pmatrix} x = \begin{pmatrix} U \circ \psi \\ 0 \end{pmatrix}. \tag{4}$$

If $U = 0$, then this system always has a solution, namely $\xi = 0$. In particular, by Lemma 6, $\mathrm{Hom}_\psi(F, \mathsf{CFI}[\Gamma, G, 0]) \neq \emptyset$. By Lemma 6, it remains to give a bijection between the sets of solutions to $\left(\begin{smallmatrix} A^\psi \\ B^\psi \end{smallmatrix}\right) x = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and the set of solutions to $\left(\begin{smallmatrix} A^\psi \\ B^\psi \end{smallmatrix}\right) x = \left(\begin{smallmatrix} U\circ\psi \\ 0 \end{smallmatrix}\right)$: Provided with a solution ξ to the latter system, $x \mapsto x + \xi$ can be taken to be this bijection. ◀

Theorem 7 yields Corollary 8 which gives a criterion for a CFI graph $\mathsf{CFI}[\Gamma, G, U]$ to have $\sum U = 0$ in terms of homomorphism counts from G. The condition in Item 3 is what allows us to infer the ultimate Corollary 9.

▶ **Corollary 8.** *Let G be a base graph and $U \in \Gamma^{V(G)}$. Then the following are equivalent:*

1. $\sum U = 0$,
2. $\mathsf{CFI}[\Gamma, G, U] \cong \mathsf{CFI}[\Gamma, G, 0]$,
3. $\hom(G, \mathsf{CFI}[\Gamma, G, U]) = \hom(G, \mathsf{CFI}[\Gamma, G, 0])$, *and*
4. $\hom_{\mathrm{id}}(G, \mathsf{CFI}[\Gamma, G, U]) = \hom_{\mathrm{id}}(G, \mathsf{CFI}[\Gamma, G, 0])$, *where* id *is the identity map on G.*

Proof. The fact that Item 1 implies Item 2 follows from Lemma 5. It is immediate that Item 2 implies Item 3. The fact that Item 3 implies Item 4 follows from Theorem 7.

It thus remains to prove that Item 4 implies Item 1. By Theorem 7, let ξ be a solution to Equation (4) for $\psi = \mathrm{id}\colon G \to G$. Then,

$$\sum_{a\in V(G)} U(a) \overset{(1)}{=} \sum_{a\in V(G)} \sum_{e\in E(a)} \xi^a_e = \sum_{e=ab\in E(G)} \xi^a_e + \xi^b_e \overset{(2)}{=} 0.$$

Hence, Item 1 holds. ◀

Thus, if G is planar, then G witnesses quantum non-isomorphism of its CFI graphs.

▶ **Corollary 9.** *If G is a planar base graph and $\sum U \neq 0$, then* $\mathsf{CFI}[\Gamma, G, 0]$ *and* $\mathsf{CFI}[\Gamma, G, U]$ *are not quantum isomorphic.*

4 Invertible-Map Equivalence and Homomorphism Indistinguishability

In this section we prove that, for every $k \geq 6$, the invertible-map equivalence $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ over the set of all primes is not a homomorphism indistinguishability relation. The proof idea is the following: Using techniques from [23], we will construct, for every $k \in \mathbb{N}$, a planar base graph G such that we obtain non-isomorphic but $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$-equivalent generalised CFI graphs over G and $\mathbb{Z}_{2^i}$ for some $i \geq 1$. By Corollary 9, the two CFI graphs are not quantum isomorphic. Exploiting [34], we will see that this implies that $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ is not a homomorphism-indistinguishability relation.

▶ **Lemma 10.** *Let $k \geq 6$. If $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ (over graphs) is a homomorphism indistinguishability relation, then all $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$-equivalent graphs are quantum isomorphic.*

Proof. For every (self-complementary) logic $\mathcal{L}$, the following holds [34, Theorem 22]: If $\mathcal{L}$-equivalence is a homomorphism indistinguishability relation, and if, for every $\ell \in \mathbb{N}$, there are $\mathcal{C}^{\ell}$-equivalent but not $\mathcal{L}$-equivalent graphs H and H', then all $\mathcal{L}$-equivalent graphs are quantum isomorphic. Here, $\mathcal{L}$ is $\mathrm{LA}^k(\mathbb{P})$. We show that for every $\ell \in \mathbb{N}$, there are $\mathcal{C}^{\ell}$-equivalent but not $\mathrm{LA}^k(\mathbb{P})$-equivalent graphs H and H'. Let $\ell \in \mathbb{N}$. It is well-known [6] that there is a base graph G such that the two non-isomorphic CFI graphs H and H' over $\mathbb{Z}_2$ and G, using the classical CFI construction [6] (which we have not presented in this paper), are $\mathcal{C}^{\ell}$-equivalent. However, the CFI graphs H and H' are not equivalent in rank logic [9]. The interpretation defining the distinguishing matrices is actually one-dimensional and requires 6 variables [21]. Thus, H and H' are not $\mathrm{LA}^k(\mathbb{P})$-equivalent. ◀

The missing fundamental lemma to prove Theorem 1 is the following:

▶ **Lemma 11.** *For every $k \in \mathbb{N}$, there is a planar base graph G and an $i \in \mathbb{N}$ such that, for all $U, U' \in \mathbb{Z}_{2^i}^{V(G)}$ satisfying $\sum U = \sum U' + 2^{i-1}$, we have $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathrm{IM}}_{k,\mathbb{P}} \mathsf{CFI}[\mathbb{Z}_{2^i}, G, U']$.*

We first show how Theorem 1 can be proved using Lemma 11 and afterwards spend the rest of this section on the proof of Lemma 11.

Proof of Theorem 1. Let $k \geq 6$. By Lemma 11, there is a planar base graph G and an $i \in \mathbb{N}$ such that $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, 0] \equiv^{\mathrm{IM}}_{k,\mathbb{P}} \mathsf{CFI}[\mathbb{Z}_{2^i}, G, U]$ for some $U \in \mathbb{Z}_{2^i}$ with $\sum U = 2^{i-1}$. These two CFI graphs are not quantum isomorphic by Corollary 9. Hence, the invertible-map equivalence $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ is not a homomorphism indistinguishability relation by Lemma 10. ◀

Because the interpretation in the proof of Lemma 10 is one-dimensional, the result of Theorem 1 also holds for equivalence in the fragment of k-variable linear-algebraic logic that is restricted to one-dimensional interpretations.

It remains to prove Lemma 11. Without the planarity requirement, non-isomorphic but $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$-equivalent generalised CFI structures were constructed in [23]. By a careful analysis of the proof, the construction can be adapted to certain planar base graphs, which we will show now. However, we first have to extend our CFI graphs by additional relations. An *ordered graph* is a pair $(G, \leq)$ of a graph G and a total order $\leq$ on $V(G)$. If G is an ordered graph, we denote its vertex set, its edge set, and its order by $V(G)$, $E(G)$, and $\leq^G$, respectively.

▶ **Definition 12.** *Let i be a positive integer, G be an ordered base graph, and $U \in \mathbb{Z}_{2^i}^{V(G)}$. We define the* CFI structure $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ *on the same vertex set as* $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U]$, *that is, on $\bigcup_{u \in V(G)} V_u$ (recall Definition 4). We first define a total preorder $\preceq$ on the vertices: $(u, S) \preceq (v, T)$ if and only if $u \leq^G v$. For every $uv \in E(G)$, we define the following relations:*

$$N_{u,v} := \{ \big((u,S),(u,T)\big) \in V_u^2 \mid S(uv) = T(uv) \},$$
$$C_{u,v} := \{ \big((u,S),(u,T)\big) \in V_u^2 \mid S(uv) + 1 = T(uv) \}.$$

Finally, we add for every $j \in \mathbb{Z}_{2^i}$ the following relation:

$$I_j := \{ \{(u,S),(v,T)\} \mid (u,S) \in V_u, (v,T) \in V_v, uv \in E(G), S(uv) + T(uv) = j \}.$$

The structure $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ can be seen as a vertex-coloured and edge-coloured directed graph (with an order on the colours), where two vertices receive the same colour if and only if they are $\preceq$-equivalent. This means that precisely vertices with the same origin receive the same colour. The other relations colour edges by the set of relations in which they are contained. Note that I_0 coincides with the edge relation of the CFI graph $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U]$. The additional relations are, apart from the preorder, already implicit in $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U]$ and are made explicit to ensure definability of certain properties in logics.

Non-isomorphic but $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$-equivalent CFI graphs were constructed using a class of regular base graphs, in which the degree, the girth, and the vertex-connectivity are simultaneously unbounded [23]. We will show that it suffices that the graph only satisfies these properties "locally". The *r-ball around a vertex $v \in V$* is the set of vertices with distance at most r to v.

▶ **Definition 13.** *Let G be a base graph and $r, d, g, c \in \mathbb{N}$. We say that G is* (r, d, g, c)-nice *if there is some vertex $w \in V(G)$ such that the r-ball W around w satisfies the following:*
1. *Every vertex in W has degree at least d.*
2. *Every cycle in G containing a vertex of W as length at least g.*
3. *For every set $V' \subseteq V(G)$ of size at most c, all vertices in $W \setminus V'$ are contained in the same connected component of $G - V'$.*
4. *For every set $V' \subseteq V(G)$ of size $c' \leq c$, there is at most one connected component $X \subseteq V(G)$ of $G - V'$ such that $G[X]$ has treewidth[1] larger than c'.*

▶ **Lemma 14.** *For every $n \in \mathbb{N}$, there is a planar graph G that is $(n, 2n, 2n, n)$-nice.*

Proof. We start with a complete $2n$-ary tree (with fixed root w) of depth $4n$. For every $i \geq 1$, the i-th level of the tree consists of $(2n)(2n-1)^{i-1}$ vertices. In particular, the tree has $(2n)(2n-1)^{4n-1}$ leaves. Next, we attach a grid of height $2n$ and width $(2n)(2n-1)^{4n-1}$ to the tree as follows: The i-th leaf from the left (according to the usual drawing of a tree in the plane) is identified with the i-th vertex of the grid in the first row. Denote this graph by G. It is easy to see that G is planar. We prove that G is $(n, 2n, 2n, n)$-nice, which is witnessed by the root w. Let W be the n-ball around w, that is, the set of vertices whose level is at most $n+1$ in the tree. By construction, every vertex in W has degree $2n$ and every cycle, in which a vertex of W is contained, has length at least $2n$ because the tree has depth $4n$.

For every vertex $u \in W$, there are at least $2n$ paths from u into the grid that are disjoint apart from u. Let $V' \subset V(G)$ be a set of at most n vertices. We show that all vertices in $W \setminus V'$ are connected in $G - V'$. Let $u, v \in W \setminus V'$. If there is a path from u to v only using vertices of the tree, we are done. Otherwise, there are at most n paths disjoint apart from u respectively v into the grid (because there were $2n$ such paths for u respectively v before removing n vertices). Let V_u and V_v be the sets of endpoints of these paths, i.e., sets of size at least n of vertices in the first row of the grid. Because there is no path between u and v

[1] For a definition of treewidth, the reader is referred to [5].

in the tree, at most $n-1$ vertices of the grid are removed in $G - V'$ (we count the leaves of the tree as vertices of the grid). By removing $n-1$ vertices from a grid of height $2n$ (and larger width) it is not possible to separate the sets V_u and V_v because they are of size at least n each. Hence, some vertex of V_u is connected to some vertex of V_v in $G - V'$ and thus u and v are connected in $G - V'$.

We finally show that at most one connected component of $G - V'$ is not an induced subgraph of a grid of height at most $|V'|$. First, we claim that all vertices of the tree are in the same connected component of $G - V'$ (again, we count the leaves as vertices of the grid). One easily sees that the argument above actually works for all vertices of the tree because for all vertices of the tree there are $2n$ disjoint paths into the grid. So there is a component containing all vertices of the tree and some vertices of the grid. Second, because the grid has height and length greater than n, by removing $|V'| \leq n$ vertices from G we can only "cut out" holes or corners of the grid. This means that the component containing the tree vertices also contains all grid vertices apart from the holes and corners cut out. Each of them contains at most $|V'|$ vertices per column and thus all these holes and corners are induced subgraphs of a grid of height $|V'|$. It is well-known [5] that grids of height at most $|V'|$ have treewidth at most $|V'|$ and the same holds for induced subgraphs of them. ◀

We now analyse properties of CFI structures over nice base graphs. The following proofs assume that the reader is familiar with the CFI construction. For more details we refer for example to [6, 15, 16, 23]. For some number $c \in \mathbb{N}$, a *c-orbit* of a structure $\mathfrak{A}$ is a maximal set of c-tuples of $\mathfrak{A}$ that are all related by an automorphism of $\mathfrak{A}$. That is, $\overline{x}, \overline{y} \in A^c$ are in the same orbit if and only if there is an automorphism φ of $\mathfrak{A}$ such that $\varphi(\overline{x}) = \overline{y}$. The set of c-orbits is a partition of A^c.

We often need isomorphisms of a particular kind between generalised CFI structures. We have seen in Lemma 5 that two CFI graphs $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U]$ and $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U']$ over some base graph G are isomorphic if, and actually only if, $\sum U = \sum U'$. The same reasoning applies to the CFI structures $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ and $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U']$ (see also [23]). Let $p = u_1, \dots, u_m$ be a path in G and $j \in \mathbb{Z}_{2^i}$. Now we can construct an isomorphism φ between $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ and $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U - ju_1 + ju_m]$ (where jv denotes the vector in $V(G)^{\mathbb{Z}_{2^i}}$ that has entry j at position v and is zero otherwise) such that φ is the identity map on all vertices whose origin is not contained in p. This isomorphism can be composed out of the maps constructed in Lemma 5 by following the path p. We call such isomorphisms *path-isomorphisms*. If p is a closed cycle, then the associated path-isomorphism is an automorphism of the structure, which we call *cycle-automorphism* (again see [23]).

▶ **Lemma 15.** *Let $i \in \mathbb{N}$, G be an (r, d, g, c)-nice ordered base graph, and $U \in \mathbb{Z}_{2^i}^{V(G)}$. Then two tuples of length $c' \leq c$ of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ are $\mathcal{C}^{3c'}$-equivalent if and only if they are in the same c'-orbit.*

Proof. We start with the following special case:

▷ Claim 16. Let $\overline{a} = \overline{\gamma}x$ and $\overline{b} = \overline{\gamma}y$ be tuples of length $c' \leq c$ of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$. If $\overline{a}$ and $\overline{b}$ are $\mathcal{C}^{3c'}$-equivalent, then $\overline{a}$ and $\overline{b}$ are in the same c'-orbit.

Proof Sketch. The vertices x and y must have the same origin v because otherwise they are easily distinguished in $\mathcal{C}^3$. So let $x = (v, S)$ and $y = (v, T)$ for some $S, T \in \mathbb{Z}_{2^i}^{E(v)}$. To construct an automorphism π that pointwise fixes $\overline{\gamma}$ and maps x to y, we have to shift the edges $F := \{e \in E(v) \mid S(e) \neq T(e)\}$. Let B be the set of all origins of vertices in $\overline{\gamma}$. Let $\mathcal{P}$ be the partition of the edges F such that two edges are in the same part of $\mathcal{P}$ if and only if they lead into the same connected component of $G - B - \{v\}$. Such an automorphism π

exists if and only if every $P \in \mathcal{P}$ satisfies $\sum_{e \in P} S(e) - T(e) = 0$. Suppose this is not the case. At least two parts of $\mathcal{P}$ do not satisfy the condition, since $\sum S = \sum T$. Because G is nice, the corresponding connected component of at least one of the parts is an induced subgraph of a grid of height c'. Because non-isomorphic CFI graphs over base graphs of treewidth at most c' are not $\mathcal{C}^{c'+1}$-equivalent [5, 17, 19], the tuples $\overline{a}$ and $\overline{b}$ are not $\mathcal{C}^{3c'}$-equivalent, which is a contradiction. $\lhd$

To prove the lemma, first note that if two tuples are in the same orbit, then they are equivalent in every logic. So it remains to prove the other direction. We show by induction on the length c' of the tuples $\overline{a}$ and $\overline{b}$ that if $\overline{a}$ and $\overline{b}$ are $\mathcal{C}^{3c'}$-equivalent, then they are in the same c'-orbit, i.e., there is an automorphism of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ that maps $\overline{a}$ to $\overline{b}$.

For $c' = 1$, the result follows from Claim 16 using γ as the empty tuple. For the inductive step, assume $\overline{a} = \overline{a}'x$ and $\overline{b} = \overline{b}'y$ are $\mathcal{C}^{3(c'+1)}$-equivalent. Then $\overline{a}'$ and $\overline{b}'$ are $\mathcal{C}^{3c'}$-equivalent. By induction, there exists an automorphism π' such that $\pi'(\overline{a}') = \overline{b}'$. Then the tuples $\pi'(\overline{a})$ and $\overline{b}$ agree on all entries except potentially the last one. They are $\mathcal{C}^{3(c'+1)}$-equivalent because logical formulas do not distinguish between tuples in the same orbit. By Claim 16, there is an automorphism π such that $\pi(\pi'(\overline{a})) = \overline{b}$. So $\overline{a}$ and $\overline{b}$ are in the same orbit. ◀

For a graph G, we call two sets $V, W \subseteq V(G)$ *adjacent* if there are $v \in V$ and $w \in W$ such that v and w are adjacent in G.

▶ **Lemma 17.** *Let $i \in \mathbb{N}$, G be an (r, d, g, c)-nice ordered base graph witnessed by a vertex $w \in V(G)$, and let $U \in \mathbb{Z}_{2^i}^{V(G)}$. Furthermore, let φ be an automorphism of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$. If $\overline{x}$, $\overline{y}$, and $\overline{z}$ are tuples of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ such that*

1. *$|\overline{xyz}| \leq c$,*
2. *the sets of all origins of vertices in $\overline{x}$, $\overline{y}$, and $\overline{z}$, respectively, are pairwise not adjacent in G, and*
3. *all origins of vertices in $\overline{x}$ and $\overline{y}$ are contained in the $(r-1)$-ball around w,*

then $\overline{xyz}$, $\varphi(\overline{x})\overline{yz}$, and $\overline{x}\varphi(\overline{y})\overline{z}$ are in the same orbit of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$.

The proof of Lemma 17 makes use of standard arguments for CFI graphs and cycle-automorphisms [23]. Such cycles can always be found for $\overline{x}$ and $\overline{y}$ because removing all origins of vertices in $\overline{x}$ and $\overline{y}$ does not disconnect G because G is nice.

▶ **Lemma 18.** *For every $k \in \mathbb{N}$, there are $r, d, g, c, i \in \mathbb{N}$ such that, for every (r, d, g, c)-nice ordered base graph G and every $U, U' \in \mathbb{Z}_{2^i}^V$ such that $\sum U = \sum U' + 2^{i-1}$, we have $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathsf{IM}}_{k,\{2\}} \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U']$.*

Proof. The proof is based on a close inspection of the proof in [23]: For every $2m \leq k$, base graphs of degree at least $d(m, k-2m)$, girth at least $g(m, k-2m)$, and vertex-connectivity at least $c(m, k-2m)$ are considered (for the definitions of d, g, and c, see [23]). Of particular interest is the $r(m, k-2m)$-ball around some vertex, which we will see later. The CFI graphs are constructed over $\mathbb{Z}_{2^i}$, for some $i(m, k-2m) \in \mathbb{N}$. Define $d = d(k) := \max_{2m \leq k} d(m, k-2m)$ and define $g = g(k)$, $c = c(k)$, $r = r(k)$, and $i = i(k)$ analogously. We finally define $r' := 2r+4$ and $g' := \max\{r', g\}$.

Assume G is a(r', d, g', c)-nice and ordered base graph and let $u \in V(G)$ be a vertex witnessing this. We call the r'-ball around u the *nice region* of G. Let $U, U' \in \mathbb{Z}_{2^i}^V$ with $\sum U = \sum U' + 2^{i-1}$ and consider $\mathfrak{A} := \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ and $\mathfrak{B} := \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U']$. To prove $\mathfrak{A} \equiv^{\mathsf{IM}}_{k,\{2\}} \mathfrak{B}$, we show that Duplicator wins the characteristic 2 IM-game with k-pebbles $\mathcal{M}^{k,\{2\}}$ played on $\mathfrak{A}$ and $\mathfrak{B}$. Duplicator maintains as invariant that in position $\overline{v}, \overline{v}'$, there is an isomorphism $\varphi\colon \mathfrak{B} \to \mathfrak{B}'$ where $\mathfrak{B}' := \mathsf{CFI}[\mathbb{Z}_{2^i}, G, U'']$ for some $U'' \in \mathbb{Z}_{2^i}^V$ such that

1. $\varphi(\overline{v}') = \overline{v}$,
2. there is only a single vertex $w \in V$ such that $U(w) \neq U''(w)$ that we call *twisted*, and
3. the $(r+1)$-ball around w is contained in the nice region and does not contain the origin of a vertex in $\overline{v}$.

Clearly, the invariant holds initially. So assume that the invariant holds by the inductive hypothesis and that it is Spoiler's turn. Up to isomorphism, we can assume to play on $\mathfrak{A}$ and $\mathfrak{B}'$ in position $\overline{v}, \overline{v}$. Spoiler chooses an arity $2m \leq k$ and picks up $2m$ pebbles from $\mathfrak{A}$ and the corresponding ones (with the same labels) from $\mathfrak{B}'$. Duplicator picks the $2m$-orbit partition $\mathcal{P}$ of $(\mathfrak{A}, \overline{v})$, and the $2m$-orbit partition $\mathcal{P}'$ of $(\mathfrak{B}, \overline{v})$. We construct a suitable bijection $\mathcal{P} \to \mathcal{P}'$ using the techniques of [23]. If G was regular with degree at least $d(m, k-2m)$, of girth at least $g(m, k-2m)$, and of vertex-connectivity at least $c(m, k-2m)$, then there would indeed be a similarity matrix as required by the game [23]. One crucial property of base graphs with vertex-connectivity strictly larger than k is the following: Let $\overline{xy}$ be a tuple of $\mathfrak{A}$ of length at most k such that the set of all origins of vertices in $\overline{x}$ is not adjacent to the same set for $\overline{y}$. In this case, automorphisms can be applied independently, that is, if φ is an automorphism, then $\overline{xy}$ is in the same orbit as $\varphi(\overline{x})\overline{y}$, $\overline{x}\varphi(\overline{y})$, and $\varphi(\overline{xy})$. The construction of the similarity matrix in [23] heavily depends on this fact. However, non-trivial automorphisms are only applied to such parts of tuples, for which all entries are contained in the $r(m, k-2m)$-ball around the twisted vertex (called the "active region" in [23]). This still holds for the (r', d, g', c)-nice base graph G, if the $r(m, k-2m)$-ball around the twisted vertex w is contained in the nice region: Let $\overline{xyz}$ be a tuple of vertices of $\mathfrak{A}$ of length at most k such that the sets of all origins of vertices of $\overline{x}$, $\overline{y}$, and respectively $\overline{z}$ are pairwise not adjacent and the sets of all origins of vertices of $\overline{x}$ and $\overline{y}$ are contained within the r-ball around w. Then automorphisms can be applied independently in the sense above (Lemma 17). Hence, the same construction of the similarity matrix of [23] can also be applied here. All arguments requiring large girth and degree only consider vertices in the "active region", for which we also have long cycles and large degree in the nice region.

Spoiler places $2m$ pebbles on the vertices in a $2m$-tuple in some block $P \in \mathcal{P}$ and the corresponding ones on a $2m$-tuple in $f(P) \in \mathcal{P}'$ resulting in the position $\overline{v}''$ and $\overline{v}'''$. By the properties of the similarity matrix and the bijection from [23], the pebbles define a partial isomorphism, and there is an isomorphism $\psi\colon \mathfrak{B}' \to \mathfrak{B}''$ such that $\psi(\overline{v}''') = \overline{v}''$ and there is only a single twisted vertex between $\mathfrak{A}$ and $\mathfrak{B}''$. Hence, Conditions 1 and 2 of the invariant are satisfied.

To satisfy Condition 3, we move the twist to a vertex that has distance at least $r+2$ to the origins of all vertices in $\overline{v}$ using a path-isomorphism as follows. Let $O \subseteq V(G)$ be this set of at most k origins. Because G is nice, we can move the twist to all vertices in the nice region apart from whose in O (because removing the vertics in O does not separate the nice region). Since the nice region is an r'-ball around u and $g' \geq r'$, there are no cycles in the nice region. Hence, the nice region induces a tree T of height r' with root u where each non-leaf has degree at least d and every leaf has distance r' to u. We call a neighbour u' of u *blocked*, if the subtree $T_{u'}$ of T rooted at u' contains some vertex from O. Because $k < d$ [23], there is a neigbor u' of u that is not blocked. Hence, if a vertex w' in $T_{u'}$ has distance at least $r+1$ to u' in $T_{u'}$, then w' has distance at most $r+2$ in T to all vertices in O. Such a vertex w' of distance exactly $r+1$ to u' always exists in T beause every leaf has distance r' to u. Because $r' = 2r+4$, the $(r+1)$-ball around w' in G is contained in the nice region. Because $g' \geq r'$ and w' is in the nice region, w' has distance at least $r+2$ to each vertex of O in G. We move the twist to this vertex w'. Duplicator maintains the invariant and thus wins the invertible-map game. ◀

▶ **Lemma 19.** *For every $k \in \mathbb{N}$, there is a planar ordered base graph G and an $i \in \mathbb{N}$ such that, for all $U, U' \in \mathbb{Z}_{2^i}^{V(G)}$ with $\sum U = \sum U' + 2^{i-1}$, we have $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathrm{IM}}_{k,\mathbb{P}} \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U']$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary. Let r, d, g, c, and i be the constants given by Lemma 18 for $k' := 3k+1$ and let $\ell := \max\{r, d, g, c\}$. By Lemma 14, there is a planar graph G that is $(\ell, 2\ell, 2\ell, \ell)$-nice. One easily sees that G is also (r, d, g, c)-nice. Hence,

$$\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathsf{IM}}_{3k+1,\{2\}} \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U']$$

by Lemma 18 for all $U, U' \in \mathbb{Z}_{2^i}^{V(G)}$ with $\sum U = \sum U' + 2^{i-1}$. By Lemma 15, the k'-orbits of these CFI structures are $\mathcal{C}^{3k'}$-definable and hence the class of CFI structures over $(\ell, 2\ell, 2\ell, \ell)$-nice and ordered base graphs is homogeneous in the sense of [8]. From [8] it follows that

$$\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathsf{IM}}_{3k+1,\mathbb{P}\setminus\{2\}} \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U'].$$

To show that these two equivalences imply

$$\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathsf{IM}}_{k,\mathbb{P}} \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U'],$$

we use the arguments from [7, Lemma 10]. The authors prove for $k'' = k+2$ the following: If the k-orbits of two structures H and H' are definable in $\mathcal{C}^{k''}$ and for two sets of primes P and Q we have $H \equiv^{\mathsf{IM}}_{k''+1,P} H'$ and $H \equiv^{\mathsf{IM}}_{k''+1,Q} H'$, then $H \equiv^{\mathsf{IM}}_{k,P\cup Q} H'$. The same argument also applies for $k'' = 3k$ and the claim of the lemma is proven. ◀

Proof of Lemma 11. Because $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U]$ is up to renaming relation symbols a reduct of $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U]$ (only the relation I_0 is kept), $\mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathsf{IM}}_{k,\mathbb{P}} \mathsf{CFI}^*[\mathbb{Z}_{2^i}, G, U']$ (Lemma 19) implies $\mathsf{CFI}[\mathbb{Z}_{2^i}, G, U] \equiv^{\mathsf{IM}}_{k,\mathbb{P}} \mathsf{CFI}[\mathbb{Z}_{2^i}, G, U']$. ◀

5 Comonads

Certain comonads on the category of relational structures capture equivalences over certain fragments of first-order logic [1]. For example, the *pebbling comonad* $\mathbb{T}_k$ has the property that two structures $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences over k-variable first-order logic with counting quantifiers if and only if they are isomorphic in the co-Kleisli-category of $\mathbb{T}_k$. We refer the reader to [11, 31] and the previously mentioned references for formal definitions. The following Lovász-type theorem for comonads allows us to derive Theorem 2 from Theorem 1:

▶ **Theorem 20** ([31]). *Let $\mathbb{C}$ be a finite-rank comonad on the category of (not necessarily finite) graphs. Then there exists a graph class $\mathcal{F}$ such that two finite graphs are isomorphic in the co-Kleisli category of $\mathbb{C}$ if and only if they are homomorphism indistinguishable over $\mathcal{F}$.*

For a definition of finite rank, see [31, Definition B.3]. Less generally, one may think of a finite-rank comonad as a comonad which sends finite structures to finite structures, cf. [11]. Note that Theorem 2 does not rule out that invertible-map equivalence can be characterised comondically in a different way, i.e., not as co-Kleisli isomorphism but via a more involved construction.

Proof of Theorem 2. Towards a contradiction, suppose that $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ coincides with the isomorphism relation in the co-Kleisli category of some finite-rank comonad. Then, by Theorem 20, $\equiv^{\mathrm{IM}}_{k,\mathbb{P}}$ is a homomorphism indistinguishability relation contradicting Theorem 1. ◀

6 Modular Homomorphism Indistinguishability

In this section, we consider homomorphism indistinguishability modulo integers $n \in \mathbb{N}$. For a graph class $\mathcal{F}$, two graphs G and H are said to be *homomorphism indistinguishable over $\mathcal{F}$ modulo n*, in symbols $G \equiv_{\mathcal{F}}^{n} H$, if $\hom(F, G) \equiv \hom(F, H) \mod n$ for every $F \in \mathcal{F}$. We write $G \equiv_{\mathcal{F}}^{N} H$ for a set $N \subseteq \mathbb{N}$ if $G \equiv_{\mathcal{F}}^{n} H$ for every $n \in N$.

In contrary to the classical result of Lovász [24] asserting that two graphs are homomorphism indistinguishable over all graphs if and only if they are isomorphic, homomorphism counts modulo a prime p do not suffice to determine a graph up to isomorphism. In [14], homomorphism indistinguishability over all graphs modulo p was characterised as follows: For a graph G with automorphism σ, write G^σ for the subgraph of G induced by the fixed points of σ. Write $G \to_p G'$ for two graphs G and G' if there is an automorphism σ of G of order p such that $G^\sigma \cong G'$ and write $G \to_p^* H$ if there is a sequence of graphs $G_1, \dots, G_n$ such that $G \to_p G_1 \to_p G_2 \to_p \cdots \to_p G_n \to_p H$. By [14, Theorem 3.7], for every graph G and prime p, there is a graph G_p^*, unique up to isomorphism, such that G_p^* has no automorphisms of order p and $G \to_p^* G_p^*$. Furthermore, by [14, Theorem 3.4], G and G_p^* are homomorphism indistinguishable over all graphs modulo p. A characterisation of homomorphism indistinguishability over all graphs modulo p can now be stated as follows:

▶ **Theorem 21** ([14, Lemma 3.10])**.** *Let p be a prime. Two graphs G and H are homomorphism indistinguishable over all graphs modulo p if and only if G_p^* and H_p^* are isomorphic.*

In general, modular homomorphism indistinguishability relations are rather oblivious to striking differences between graphs:

▶ **Example 22.** For $n \in \mathbb{N}$, the one-vertex graph K_1 and the coclique $\overline{K_{n+1}}$ are homomorphism indistinguishable over all graphs modulo n, i.e. $\hom(F, K_1) \equiv \hom(F, \overline{K_{n+1}}) \mod n$ for all graphs F.

Proof. If F is an edgeless graph, then $\hom(F, K_1) = 1 \equiv (n+1)^{|V(F)|} = \hom(F, \overline{K_{n+1}}) \mod n$. If otherwise F contains an edge, then $\hom(F, K_1) = 0 = \hom(F, \overline{K_{n+1}})$. ◀

Before we move to modular homomorphism indistinguishability characterisations for certain logic fragments, we clarify the relationship between the various notions introduced so far:

▶ **Lemma 23.** *Let $\mathcal{F}$ and $\mathcal{K}$ be graph classes. Let $N \subseteq \mathbb{N}$ and $n \in \mathbb{N}$.*

1. *If N is infinite, then $\equiv_{\mathcal{F}}^{N}$ and $\equiv_{\mathcal{F}}$ coincide.*
2. *If N is finite and m is the least common multiple of the numbers in N, then $\equiv_{\mathcal{F}}^{N}$ and $\equiv_{\mathcal{F}}^{m}$ coincide.*
3. *If $\equiv_{\mathcal{F}}$ and $\equiv_{\mathcal{K}}^{n}$ coincide, then $\mathcal{F} = \emptyset$, i.e., all graphs are $\equiv_{\mathcal{F}}$-equivalent.*

Proof. For the first claim, let G and H be graphs and $F \in \mathcal{F}$. Since N is infinite, there exists $n \in N$ greater than $|V(G)|^{|V(F)|}$ and $|V(H)|^{|V(F)|}$. Then $\hom(F, G) \equiv \hom(F, H) \mod n$ implies that $\hom(F, G) = \hom(F, H)$.

For the second claim, first observe that $G \equiv_{\mathcal{F}}^{m} H$ entails $G \equiv_{\mathcal{F}}^{N} H$ since all $n \in N$ divide m. Conversely, for a prime p write $\nu(p)$ for the greatest integer $k \geq 0$ such that there is an $n \in N$ that is divisible by p^k. Then $m = \prod_{p \in \mathbb{P}} p^{\nu(p)}$, where the product ranges over all primes. Hence, if $\hom(F, G) \equiv \hom(F, H) \mod n$ for all $n \in N$, then $\hom(F, G) \equiv \hom(F, H) \mod p^{\nu(p)}$ for all primes p appearing as divisors of elements in N, i.e., $\nu(p) > 0$. Hence, by the Chinese Remainder Theorem [28, Theorem 2.10], also $\hom(F, G) \equiv \hom(F, H) \mod m$.

Towards the third claim, we first show the following Claim 24: Write ℓ for the maximum integer such that p^ℓ divides n for some prime p. Write $\varphi \colon \mathbb{N} \to \mathbb{N}$ for Euler's totient function [28, Section 2.3] and $G^{\times k}$ for the k-th categorical power of the graph G, cf. [25, p. 40].

▷ Claim 24. For every graph G, the graphs $G^{\times(\varphi(n)+\ell)}$ and $G^{\times\ell}$ are homomorphism indistinguishable over all graphs modulo n.

Proof. We show that $a^\ell(a^{\varphi(n)}-1) \equiv 0 \mod n$ for every $a \in \mathbb{N}$. By the Chinese Remainder Theorem [28, Theorem 2.10], writing $n = \prod p_i^{\ell_i}$ as product of powers of distinct primes, it suffices to show that this equality holds modulo $p_i^{\ell_i}$ for every i. By Euler's Theorem [28, Theorem 2.12], $a^{\varphi(p_i^{\ell_i})} \equiv 1 \mod p_i^{\ell_i}$ if a and p_i are coprime. Since $\varphi(n) = \prod \varphi(p_i^{\ell_i})$ [28, Theorem 2.7], also $a^{\varphi(n)} \equiv 1 \mod p_i^{\ell_i}$. If p_i divides a, then $a^\ell \equiv 0 \mod p_i^{\ell_i}$ as $\ell_i \leq \ell$. Finally, for every graph F, $\hom(F, G^{\times(\varphi(n)+\ell)}) = \hom(F,G)^{\varphi(n)+\ell} \equiv \hom(F,G)^\ell \mod n$ by [25, (5.30)]. ◁

Let $F \in \mathcal{F}$, $m := |V(F)|$, and write K_m for the clique on m vertices. Then $\hom(F, K_m) > 1$. Define $G := K_m^{\times(\varphi(n)+\ell)}$ and $H := K_m^{\times\ell}$. By [25, (5.30)] and $\varphi(n) \geq 1$, it holds that $\hom(F,G) = \hom(F,K_m)^{\varphi(n)+\ell} \neq \hom(F,K_m)^\ell = \hom(F,H)$. Hence, $G \not\equiv_{\mathcal{F}} H$. However, $G \equiv_{\mathcal{K}}^n H$ by Claim 24 contradicting that $\equiv_{\mathcal{F}}$ and $\equiv_{\mathcal{K}}^n$ coincide. ◀

Lemma 23 shows that non-trivial modular homomorphism indistinguishability relations cannot be expressed by (non-modular) homomorphism indistinguishability relations. Furthermore, considering sets of moduli does not yield more relations. We may restrict our attention to homomorphism indistinguishability relations modulo some not necessarily prime $n \in \mathbb{N}$. In the remainder of this section, we give an example and a non-example of a logic whose equivalence can be characterised as modular homomorphism indistinguishability relation.

We have seen already that the relation $\equiv_{k,\mathbb{P}}^{\mathsf{IM}}$ is not a homomorphism indistinguishability relation over any graph class. But since $\equiv_{k,\mathbb{P}}^{\mathsf{IM}}$ is a relation based on linear algebra over finite fields, it might a priori be that it can be characterised as a homomorphism indistinguishability relation *modulo a prime*. This can be ruled out, at least in the following sense:

▶ **Theorem 25.** *Let $k \geq 2$ and Q be a set of primes. Then there exists no graph class $\mathcal{F}$ and no $n \in \mathbb{N}$ such that $\equiv_{k,Q}^{\mathsf{IM}}$ and $\equiv_{\mathcal{F}}^n$ coincide.*

Proof. Towards a contradiction, suppose that $\equiv_{k,Q}^{\mathsf{IM}}$ and $\equiv_{\mathcal{F}}^n$ coincide for some graph class $\mathcal{F}$ and some $n \in \mathbb{N}$. Recall from Example 22 that the clique K_1 and the coclique $\overline{K_{n+1}}$ are homomorphism indistinguishable over all graphs modulo n. However K_1 and $\overline{K_{n+1}}$ are easily distinguished in 2-variable FO and thus $K_1 \not\equiv_{k,Q}^{\mathsf{IM}} \overline{K_{n+1}}$. ◀

By extending techniques of [13], we prove that homomorphism indistinguishability over graphs of bounded treewidth counted modulo a prime characterises equivalence in first-order logic with modular counting quantifiers. The strategy is to construct, for every graph F of treewidth $\leq k$ and every $m \in \mathbb{F}_p$, a modular counting logic formula with $\leq k+1$ variables such that a graph satisfies the formula if and only if it admits $m \mod p$ many homomorphisms from F. Conversely, counting logic formulas are translated into $\mathbb{F}_p$-linear combinations of graphs of bounded treewidth such that the linear combination of their homomorphism counts in a graph is $1 \mod p$ if and only if the formula is satisfied. In this direction, it is crucial that $\mathbb{F}_p$ is a field for an interpolation argument to carry through.

Modular counting logic is defined as follows: Let p be a prime. Formulas of $\mathcal{C}[p]$ are boolean combinations of atomic formulas, equality, and modular counting quantifiers $\exists^c x\varphi$ for every $c \in \mathbb{F}_p$. The semantics of modular counting quantifiers is as expected, i.e., a structure $\mathfrak{A}$ satisfies a sentence $\exists^c x\varphi(x)$ if there exist $c \mod p$ distinct $a \in A$ such that $\mathfrak{A} \models \varphi(a)$. Let $\mathcal{C}^{k+1}[p]$ denote the $(k+1)$-variable fragment of this logic.

▶ **Theorem 26.** *Let p be a prime and $k \geq 0$. Two arbitrary graphs G and H are homomorphism indistinguishable over all graphs of treewidth at most k modulo p if and only if G and H are $\mathcal{C}^{k+1}[p]$-equivalent.*

As a consequence, Theorem 26 in conjunction with Lemma 23 and Theorem 20 yields that $\mathcal{C}^{k+1}[p]$-equivalence cannot be characterised as a co-Kleisli isomorphism with respect to a finite-rank comonad.

7 Conclusion

We studied linear-algebraic logic, a logic stronger than first-order logic with counting, and proved that equivalence with respect to it can neither be characterised as a homomorphism indistinguishability relation, nor as co-Kleisli isomorphism for a finite-rank comonad. The latter answers an open question of Ó Conghaile and Dawar [30] and shows a limitation of the game comonad programme for capturing logical equivalences. It would be desirable to understand more generally which properties are responsible for making a logic suitable for a homomorphism indistinguishability or game comonad characterisation. We know that game comonads can be defined for FO with all Lindström quantifiers up to a fixed arity [30]. What we do not know is whether these are the only Lindström extensions of FO admitting such a characterisation. Other interesting classes of Lindström quantifiers to look at besides the linear-algebraic ones could be CSP quantifiers. The corresponding logic defined in [20] comes with a fairly natural game characterising equivalence. Thus, one may ask whether this CSP logic admits a game comonad or if this can be ruled out with similar methods as in this paper. The same question is also open for (bounded variable fragments of) counting monadic second order logic CMSO. In principle, our approach works for every extension of counting logic for which there exists a CFI-like lower bound construction that works over planar base graphs and with only one binary relation. It remains to devise such a construction for CSP logic and CMSO.

A different topic, that we have merely touched upon, is homomorphism counting in prime fields. We have shown that the corresponding homomorphism indistinguishability relations do not characterise IM-equivalence. On the other hand, we stated an example of a logic that is captured by a modular homomorphism indistinguishability relation, namely modular counting logic. A more comprehensive theory of modular homomorphism counting is yet to be developed. A particularly interesting question, which is not in the scope of this paper, is whether the known connections between homomorphism counting and solutions to semidefinite/linear programs for graph isomorphism [33] have a meaningful generalisation to prime fields.

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