# Local Operators in Topos Theory and Separation of Semi-Classical Axioms in Intuitionistic Arithmetic

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**Abstract**

There has been work on the strength of semi-classical axioms over Heyting arithmetic such as $\Sigma_n$-**DNE** (double negation elimination) and $\Pi_n$-**LEM** (law of excluded middle). Among other things, Akama et al. show that $\Sigma_n$-**DNE** does not imply $\Pi_n$-**LEM** for any $n \geq 1$ by using Kleene realizability relativized to Turing degrees. These realizability notions are expressed by subtoposes of the effective topos $\mathcal{E}ff$ and thus by corresponding local operators (a.k.a. Lawvere-Tierney topologies).

Our purpose is to provide a topos-theoretic explanation for separation of semi-classical axioms. It consists of determining the least dense local operator of a given axiom $\varphi$ in a topos $\mathcal{E}$, which completely characterizes the dense subtoposes of $\mathcal{E}$ satisfying $\varphi$. This idea is motivated by Caramello's study of intermediate propositional logics and van Oosten's study of Lifschitz realizability.

We first investigate sufficient conditions for an arithmetical formula to have a least dense operator. In particular, we show that each semi-classical axiom has a least dense operator in every elementary topos with natural number object. This is a generalization of van Oosten's result for $\Pi_1 \vee \Pi_1$-**DNE** in $\mathcal{E}ff$. We next determine least dense operators of semi-classical axioms in $\mathcal{E}ff$ in terms of (generalized) Turing degrees. Not only does it immediately imply some separation results of Akama et al. but also explain that realizability notions they used are optimal in the sense of minimality. We finally point out a negative consequence that $\Pi_n$-**LEM**, $\Sigma_n$-**LEM** and $\Sigma_{n+1}$-**DNE** are never separable by any subtopos of $\mathcal{E}ff$ for any $n \geq 0$.

**2012 ACM Subject Classification** Theory of computation → Constructive mathematics

**Keywords and phrases** local operator, elementary topos, effective topos, realizability, intuitionistic arithmetic

**Digital Object Identifier** 10.4230/LIPIcs.CSL.2024.42

**Funding** This work is supported by JST Grant Number JPMJFS2123.

**Acknowledgements** I would like to thank my supervisor, Kazushige Terui, for careful reading and many helpful suggestions. I am also grateful to Hisashi Aratake, Yutaka Maita, Takayuki Kihara and the anonymous referees for their invaluable comments.

## 1 Introduction

Toposes are useful as semantics for logical systems and programming languages. In this context, the *effective topos* of Hyland [10] and its generalization, *realizability toposes* [11], have multiple applications. In particular, it is well known that the interpretation of logic and arithmetic in realizability toposes corresponds to the traditional realizability interpretation in intuitionistic proof theory. Van Oosten and others deeply investigate this correspondence and analyze various realizability notions from a topos-theoretic perspective [25].

In this paper, we mainly focus on toposes as models of *first-order intuitionistic arithmetic*, which is rich enough to encode and reason about programs and computations.

### 1.1 Various realizability methods and semi-classical axioms

Since Kleene [15] defined the first realizability interpretation (*Kleene realizability*) for *Heyting arithmetic* **HA**, many variants have been proposed in the literature. For example,

**(1)** Relativization to Turing degree $d$ (*$d$-realizability*) [20].
**(2)** *Lifschitz realizability* [18, 23].
**(3)** Kreisel's *modified realizability* [16, 24].

These realizability methods are strongly related to the hierarchy of *semi-classical axioms* introduced by Akama, Berardi, Hayashi and Kohlenbach [1].

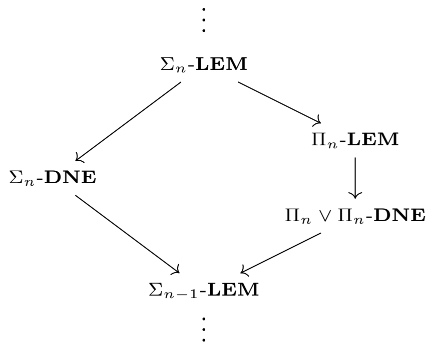


**Figure 1** The hierarchy of semi-classical axioms.

In Figure 1, **DNE** and **LEM** stand for the *double negation elimination* and the *law of excluded middle*, respectively. Of course, **DNE** is equivalent to **LEM** in intuitionistic propositional logic. However, a difference arises when restricted to a class of arithmetical formulas such as $\Sigma_n$ and $\Pi_n$. Indeed, $\Sigma_n$-**LEM** implies $\Sigma_n$-**DNE** in **HA** but the converse does not hold. More interestingly, an axiom scheme often corresponds to a semi-constructive principle such as the *lesser limited principle of omniscience*, the *constant domain axiom*, and even some variant of *Ramsey theorem* (*constructive reverse mathematics* over **HA**) [1, 2, 9].

Akama et al. separate the axioms in Figure 1 by using the realizability notions (1), (2) and a monotone variant of (3) above [1]. For instance, they show that $\Sigma_n$-**DNE** is realizable while $\Sigma_n$-**LEM**, $\Pi_n$-**LEM** and $\Pi_n \vee \Pi_n$-**DNE** are not under $\emptyset^{(n-1)}$-realizability, meaning that the former does not imply the latter. Similarly, Lifschitz realizability relativized to degree $\emptyset^{(n-1)}$ is used to separate $\Pi_n \vee \Pi_n$-**DNE** and $\Pi_n$-**LEM**.

It is known that the realizability notions (1), (2) and (3) correspond to subtoposes of (extensions of) the effective topos $\mathcal{E}\!f\!f$ [10, 20, 23, 24]. Above all, van Oosten studied the *Lifschitz topos* $\mathcal{L}if \subseteq \mathcal{E}\!f\!f$, where a first-order arithmetical formula $\varphi$ is true iff $\varphi$ is Lifschitz realizable. This representation leads to a topos-theoretic approach to the separation problem.

## 1.2 Least dense operators of logical and arithmetical formulas

Given an elementary topos $\mathcal{E}$ with subobject classifier $\Omega$, the subtoposes of $\mathcal{E}$ are in one-to-one correspondence with the *local operators* in $\mathcal{E}$, that is, the meet-preserving closure operators $j\colon \Omega \to \Omega$ (a.k.a. *Lawvere-Tierney topologies*). A typical example is the *double negation operator* $\neg\neg\colon \Omega \to \Omega$, which exists in every topos. No matter which logic $\mathcal{E}$ models, the corresponding subtopos $\mathcal{E}_{\neg\neg}$ is always a model of classical logic. This link between the local operators and the intermediate logics has been further explored by Caramello [5, 6] in the context of categorical logic. A key notion there is what we call the *least dense operator* of a formula $\varphi$, that completely determines the dense subtoposes of $\mathcal{E}$ which satisfy $\varphi$. The same notion appears in van Oosten's study of categorical realizability. He showed that the local operator $j_{\mathcal{L}if}$ in $\mathcal{E}\!f\!f$ corresponding to the Lifschitz topos $\mathcal{L}if$ is the least dense operator of an arithmetical formula, which is equivalent to $\Pi_1 \vee \Pi_1$-**DNE** over **HA** [23].

The specifics of least dense operators are explained in Subsection 3.1. Our emphasis here is the following observation: given two axioms, if their least dense operators are different, then it automatically follows that they are separable.

### 1.3 Contents of this paper

We investigate the least dense operators of arithmetical formulas in relation to the separation of semi-classical axioms. Throughout the investigation, our aim is to demonstrate the utility of such a topos-theoretic notion in the study of intuitionistic proof theory. For the readers interested in proof theory, we try to make this paper as self-contained as possible.

In Section 2, we give some background. In Section 3, we study sufficient conditions for an arithmetical formula $\varphi$ to have a least dense operator. For this purpose, we introduce two properties for formulas: *transparency* and *closedness*. Transparency ensures that a formula has a least dense operator under a mild assumption, while closedness is an intermediary means to show that a formula is transparent. Our argument here is a reconstruction and generalization of van Oosten's [23] for all toposes. The main result of Section 3 is that all semi-classical axioms in Figure 1 have least dense operators in every elementary topos with natural number object (Theorem 33, Corollary 34).

In Section 4, we apply the general theory in the previous section to the effective topos $\mathcal{E}\!f\!f$. As shown by Hyland [10], the poset of Turing degrees can be embedded into the poset of local operators in $\mathcal{E}\!f\!f$ (Figure 4). This allows us to relate the least dense operators of semi-classical axioms to (generalized) Turing degrees. For example, the least operator of $\Sigma_n$-**DNE** corresponds to Turing degree $\emptyset^{(n-1)}$, while that of $\Pi_n \vee \Pi_n$-**DNE** corresponds to another degree $\mathcal{L}\!i\!f^{(n-1)}$ (Theorems 45, 46, Figure 5). Noting that least dense operators characterize separability by dense subtoposes, these expressions not only immediately imply some separation results of [1] but also a negative consequence that $\Pi_n$-**LEM**, $\Sigma_n$-**LEM** and $\Sigma_{n+1}$-**DNE** are never separable by any subtopos of $\mathcal{E}\!f\!f$ for any $n \geq 0$ (Corollary 47).

## 2 Preliminary

In this section, we review some basic facts on first-order intuitionistic arithmetic and interpretation of first-order logic and arithmetic in a topos. Most of the facts mentioned here can be found in standard textbooks [22, 21, 13, 19].

### 2.1 First-order intuitionistic arithmetic

Let $\mathcal{L}_{\mathrm{A}}$ be the language of arithmetic that consists of constant 0, successor Suc, and function symbols for all primitive recursive functions. *Heyting arithmetic*, written as **HA**, is a first-order intuitionistic $\mathcal{L}_{\mathrm{A}}$-theory consisting of $\forall x\, \neg(\mathrm{Suc}(x) = 0)$, defining equations for all primitive recursive functions, and the induction axiom scheme for all $\mathcal{L}_{\mathrm{A}}$-formulas. *Peano arithmetic*, written as **PA**, is defined by **PA** = **HA** + **LEM**. We inductively define the classes $\Sigma_n$, $\Pi_n$ of $\mathcal{L}_{\mathrm{A}}$-formulas as follows: $\Sigma_0 = \Pi_0$ are the set of all quantifier-free formulas, while $\Sigma_{n+1}$, $\Pi_{n+1}$ are defined by $\Sigma_{n+1} :=\{\, \exists x_1 \cdots \exists x_k \varphi \mid \varphi \in \Pi_n,\ 0 \leq k \,\}$ and $\Pi_{n+1} := \{\, \forall x_1 \cdots \forall x_k \varphi \mid \varphi \in \Sigma_n,\ 0 \leq k \,\}$. $\Pi_n \vee \Pi_n$ denotes the set of formulas of the form $\varphi \vee \psi$ with $\varphi, \psi \in \Pi_n$.

Given a formula $\varphi$, the *universal closure* of $\varphi$ is denoted by ${}^{\forall}\varphi$. As in [1], we define some semi-classical axiom schemes as follows: for a subclass $\Gamma$ of $\mathcal{L}_{\mathrm{A}}$-formulas, let

$$\Gamma\text{-}\mathbf{DNE} := \{\, {}^{\forall}(\neg\neg\varphi \to \varphi) \mid \varphi \in \Gamma \,\}, \quad \Gamma\text{-}\mathbf{LEM} := \{\, {}^{\forall}(\varphi \vee \neg\varphi) \mid \varphi \in \Gamma \,\}.$$

It is well known that **HA** proves $\Sigma_0$-**DNE** and $\Sigma_0$-**LEM**, and that for every $n$, $\Pi_{n+1}$-**DNE** is equivalent to $\Sigma_n$-**DNE** over **HA**. In this paper, a *semi-classical axiom* refers to either $\Gamma$-**DNE** or $\Gamma$-**LEM**, where $\Gamma$ is $\Sigma_n$, $\Pi_n$, or $\Pi_n \vee \Pi_n$ for some $n \geq 0$.

Recall that **HA** formalizes a bijective primitive recursive pairing function $\langle -, - \rangle \colon \mathbb{N}^2 \to \mathbb{N}$. This allows us to code a finite sequence of natural numbers by a single one. Hence, without loss of generality, we can assume that each $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$ has at most one free variable.

By formalizing Post's theorem in **HA**, we obtain a *universal formula* $\varphi_{\Sigma_n}(e, x)$ for $\Sigma_n$ with $n \geq 1$. That is, for any $\Sigma_n$-formula $A(x)$, there exists a numeral $e_A$ such that $^{\forall}(A(x) \leftrightarrow \varphi_{\Sigma_n}(e_A, x))$ is provable in **HA** (folklore). The same holds for $\Pi_n$ and $\Pi_n \vee \Pi_n$. For example, universal formulas $\varphi_{\Sigma_1}$, $\varphi_{\Pi_1}$ for $\Sigma_1$, $\Pi_1$ are given by $\varphi_{\Sigma_1}(e, x) \coloneqq \exists w T(e, x, w)$, $\varphi_{\Pi_1}(e, x) \coloneqq \forall w \neg T(e, x, w)$, where $T(e, x, w)$ is Kleene's $T$-predicate. Thus, each axiom scheme in Figure 1 is finitely axiomatizable in **HA**.

## 2.2 Interpretation of intuitionistic logic in a topos

An *(elementary) topos* is a cartesian closed category with all finite limits and subobject classifier $\mathrm{true} \colon 1 \rightarrowtail \Omega$. According to the standard interpretation of many-sorted first-order logic in a topos, each formula is interpreted by a subobject in a suitable subobject poset. So let us first review the logical structure of a subobject poset.

For an object $X$ in a topos $\mathcal{E}$, we write $U \rightarrowtail X$ for a subobject $U$ of $X$ and write $(\mathrm{Sub}_{\mathcal{E}}(X), \leq)$ for the poset of subobjects of $X$. Given a morphism $f \colon X \to Y$, $f^*$ stands for the pullback functor along $f$. In a topos $\mathcal{E}$, $(\mathrm{Sub}_{\mathcal{E}}(X), \leq)$ forms a Heyting algebra.

▶ **Theorem 1.** *Let $\mathcal{E}$ be a topos.*

**(1)** *$\mathcal{E}$ is a* coherent category*. In particular, for any object $X$, $\mathrm{Sub}_{\mathcal{E}}(X)$ forms a distributive lattice with meet $\wedge$, join $\vee$, top $1$ and bottom $0$. In addition, for any morphism $f \colon X \to Y$, $f^* \colon \mathrm{Sub}_{\mathcal{E}}(Y) \to \mathrm{Sub}_{\mathcal{E}}(X)$ has a left adjoint $\exists_f \colon \mathrm{Sub}_{\mathcal{E}}(X) \to \mathrm{Sub}_{\mathcal{E}}(Y)$.*

**(2)** *$\mathcal{E}$ is further a* Heyting category*. In particular, for any $f \colon X \to Y$, $f^*$ has a right adjoint $\forall_f \colon \mathrm{Sub}_{\mathcal{E}}(X) \to \mathrm{Sub}_{\mathcal{E}}(Y)$.*

The last $\forall_f$ induces Heyting implication $\Rightarrow$ on $\mathrm{Sub}_{\mathcal{E}}(X)$. In fact, $U \Rightarrow V$ can be defined to be $\forall_{m_U}(U \wedge V)$, where $m_U$ is a representative of $U \rightarrowtail X$. If $f$ is a projection $\pi \colon X \times Z \to Z$, $\forall_\pi$ (resp. $\exists_\pi$) provides an interpretation of first-order quantification $\forall x$ (resp. $\exists x$).

Now assume that to each sort $A$ is assigned an object $[\![A]\!]$ and to each function symbol $f \colon A_1 \times \cdots \times A_n \to B$ a morphism $[\![f]\!] \colon [\![\vec{A}]\!] \to [\![B]\!]$, where $[\![\vec{A}]\!] \coloneqq [\![A_1]\!] \times \cdots \times [\![A_n]\!]$. Then each term $t = t(x_1^{A_1}, \cdots, x_n^{A_n}) \colon B$ is interpreted by a morphism $[\![t]\!] \colon [\![\vec{A}]\!] \to [\![B]\!]$ and equality $t =_B u$ by the equalizer $[\![t =_B u]\!] \rightarrowtail [\![\vec{A}]\!]$ of $[\![t]\!]$, $[\![u]\!] \colon [\![\vec{A}]\!] \rightrightarrows [\![B]\!]$. Interpretation of logical connectives is as above.

For a formula $\varphi = \varphi(x_1^{A_1}, \cdots, x_n^{A_n})$, if the interpretation $[\![\varphi]\!] \rightarrowtail [\![\vec{A}]\!]$ is identical to the greatest element of $\mathrm{Sub}_{\mathcal{E}}([\![\vec{A}]\!])$ (that is the equivalence class of the identity $\mathrm{id} : [\![\vec{A}]\!] \to [\![\vec{A}]\!]$), we say that $\varphi$ *is true in $\mathcal{E}$ (under $[\![-]\!]$)* and write $\mathcal{E} \models \varphi$. Under this interpretation, every topos satisfies all axioms of first-order intuitionistic logic.

## 2.3 Local operators and subtoposes

*Local operator* (a.k.a. *Lawvere-Tierney topology*) is one of the most important tools for creating a new topos from a given one.

▶ **Theorem 2.** *In a topos $\mathcal{E}$, there is a one-to-one correspondence among the following notions:*

**(1)** Local operator *$j \colon \Omega \to \Omega$, that is an endomorphism on the subobject classifier $\Omega$ of $\mathcal{E}$ which is an "internal"* nucleus *(recall that a nucleus on a lattice is a meet-preserving closure operator).*

**(2)** Universal closure operation $c \coloneqq \{\, c^X : \mathrm{Sub}_{\mathcal{E}}(X) \to \mathrm{Sub}_{\mathcal{E}}(X)\,\}_{X \in \mathcal{E}}$, *that is a family of nuclei on subobject posets which is natural in $X \in \mathcal{E}$. When it is clear from the context, we will omit superscript $X$ in $c^X$.*

**(3)** Subtopos $\mathcal{F} \hookrightarrow \mathcal{E}$, *that is a full subcategory of $\mathcal{E}$ which is itself a topos such that the inclusion functor $i \colon \mathcal{F} \hookrightarrow \mathcal{E}$ has a cartesian left adjoint $L \colon \mathcal{E} \to \mathcal{F}$. Such an $L$ is called a* sheafification functor (or associated sheaf functor) *on $\mathcal{F}$.*

Hereafter, $\mathcal{E}_j$, $c_j$ and $L_j$ denote the subtopos, the universal closure operation and the sheafification functor associated with a local operator $j$, respectively. (Note that the subtopos is usually denoted by $\mathbf{sh}_j(\mathcal{E})$.) We write $\mathbf{Lop}(\mathcal{E})$ for the class of local operators in $\mathcal{E}$.

▶ **Example 3.**

**(1)** The identity $\mathrm{id}_\Omega \colon \Omega \to \Omega$ and $\top \coloneqq \mathrm{true} \circ ! \colon \Omega \to \Omega$ are local operators in $\mathcal{E}$, where $!$ is the unique morphism from $\Omega$ to the terminal object $1$. The corresponding subtoposes are $\mathcal{E}$ itself and the degenerate topos, respectively. $\top$ is called the *degenerate local operator*.

**(2)** For every topos $\mathcal{E}$, the family $\{\, ((\cdot \Rightarrow 0) \Rightarrow 0) \colon \mathrm{Sub}_{\mathcal{E}}(X) \to \mathrm{Sub}_{\mathcal{E}}(X)\,\}_{X \in \mathcal{E}}$ always forms a universal closure operation. The associated local operator is called the *double negation operator* $\neg\neg \colon \Omega \to \Omega$. The corresponding subtopos $\mathcal{E}_{\neg\neg}$ is a model of classical logic.

The correspondence in Theorem 2 induces a natural order on $\mathbf{Lop}(\mathcal{E})$.

▶ **Lemma 4.** *For $j$, $k \in \mathbf{Lop}(\mathcal{E})$, the following are equivalent:*

**(1)** *$\mathcal{E}_k$ is a subtopos of $\mathcal{E}_j$.*

**(2)** *For any object $X \in \mathcal{E}$ and any subobject $U \rightarrowtail X$, $c_j(U) \leq c_k(U)$.*

We write $j \leq k$ if the above equivalent conditions hold. $(\mathbf{Lop}(\mathcal{E}), \leq)$ forms a poset with the bottom element $\mathrm{id}_\Omega$ and the top element $\top$.

Next, let us introduce two important notions.

▶ **Definition 5.** *Let $X$ be an object of $\mathcal{E}$, $U \rightarrowtail X$ a subobject of $X$ and $j \in \mathbf{Lop}(\mathcal{E})$. We say that $U$* is $j$-dense *if $c_j(U) = X$, and that $U$* is $j$-closed *if $c_j(U) = U$. Let $\mathrm{Cl}_j\mathrm{Sub}_{\mathcal{E}}(X)$ denote the class of $j$-closed subobjects of $X$.*

The $j$-dense elements in $\mathrm{Sub}_{\mathcal{E}}(X)$ are sent to the greatest element in $\mathrm{Sub}_{\mathcal{E}_j}(L_jX)$ by $L_j \colon \mathcal{E} \to \mathcal{E}_j$.

▶ **Lemma 6.** *For a subobject $U \rightarrowtail X$ and a local operator $j$, $U$ is $j$-dense if and only if $L_jU \rightarrowtail L_jX$ is an isomorphism.*

On the other hand, the $j$-closed objects form a subobject lattice in $\mathcal{E}_j$: whenever $F$ is an object of $\mathcal{E}_j \subseteq \mathcal{E}$, we have $\mathrm{Cl}_j\mathrm{Sub}_{\mathcal{E}}(F) = \mathrm{Sub}_{\mathcal{E}_j}(F)$. Moreover, the logical operations ($\wedge_j$, $\vee_j$, $\Rightarrow_j$, $\neg_j$, $\forall^j_f$, $\exists^j_f$) on $\mathrm{Sub}_{\mathcal{E}_j}(F)$ are derived from ($\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\forall_f$, $\exists_f$) on $\mathrm{Sub}_{\mathcal{E}}(F)$ by means of the closure operation $c_j$ as follows:

▶ **Lemma 7.** *Let $F$, $G$ be objects of $\mathcal{E}_j$, $f \colon F \to G$ a morphism of $\mathcal{E}_j$, and $A$, $B \in \mathrm{Sub}_{\mathcal{E}_j}(F) = \mathrm{Cl}_j\mathrm{Sub}_{\mathcal{E}}(F)$. Then*

$$\begin{aligned} &A \wedge_j B = A \wedge B, && A \vee_j B = c_j(A \vee B), && A \Rightarrow_j B = A \Rightarrow B,\\ &\forall^j_f A = \forall_f A, && \exists^j_f A = c_j(\exists_f A), && \neg_j(A) = A \Rightarrow c_j(0). \end{aligned}$$

## 2.4 Preservation of logical operations and degrees of openness

In this subsection, we have a look at when a sheafification functor $L_j$ preserves a logical operation. This leads to a distinction of various degrees of openness of local operators. Note that $L_j$ always preserves finite limits by definition, hence it preserves monomorphisms. Thus $L_j$ induces a map $L_j\colon (\mathrm{Sub}_{\mathcal{E}}(X), \leq) \to (\mathrm{Sub}_{\mathcal{E}_j}(L_j X), \leq)$ for each object $X \in \mathcal{E}$. As is well known in categorical logic, $L_j$ always preserves $\wedge$, $\vee$, 0 and $\exists_f$.

▶ **Proposition 8** ([19, Chapter IX]). *For any $j \in \mathbf{Lop}(\mathcal{E})$, $L_j$ is a* coherent functor*. In particular, for any objects $X$, $Y$, morphism $f\colon X \to Y$ and subobjects $U$, $V \in \mathrm{Sub}_{\mathcal{E}}(X)$,*

$$L_j(U \circ V) = L_j U \circ_j L_j V, \quad L_j 0 = 0_j, \quad L_j(\exists_f U) = \exists^j_{L_j f} L_j U,$$

*where $\circ \in \{\wedge, \vee\}$ and $\circ_j$, $0_j$, $\exists^j_{L_j f}$ are logical operations on $\mathrm{Sub}_{\mathcal{E}_j}(L_j X)$.*

In addition, the following proposition shows that $\forall_f$ and $\Rightarrow$ are preserved by $L_j$ under an assumption of closedness.

▶ **Proposition 9** ([10, Theorem 5.1]). *Let $X$ be an object of $\mathcal{E}$, $U$, $V \rightarrowtail X$ subobjects of $X$ and $j \in \mathbf{Lop}(\mathcal{E})$. If $V$ is $j$-closed, then $L_j(\forall_f V) = \forall^j_{L_j f}(L_j V)$ and $L_j(U \Rightarrow V) = L_j U \Rightarrow_j L_j V$.*

However, $L_j$ does not in general preserve universal quantification $\forall_f$, Heyting implication $\Rightarrow$ and negation $\neg$. Preservation of these operations is related to *opneness* of geometric morphisms [13, Proposition A4.5.1]. Motivated by this observation, Caramello gave the following definitions.

▶ **Definition 10** ([6, Definition 3.2]). *Let $j$ be a local operator in $\mathcal{E}$.*
**(1)** *$j$ is* open *if $L_j$ preserves universal quantification on every subobject lattice.*
**(2)** *$j$ is* implicationally open *if $L_j$ preserves Heyting implication on every subobject lattice.*
**(3)** *$j$ is* weakly open *if $L_j$ preserves negation on every subobject lattice.*

We finally introduce another openness notion, *denseness*. This should not be confused with the notion of $j$-dense subobject in Definition 5.

▶ **Definition 11** ([6, Proposition 3.1]). *For $j \in \mathbf{Lop}(\mathcal{E})$, we say that $j$ (or the corresponding subtopos $\mathcal{E}_j$) is* dense *if it satisfies one of the following equivalent conditions:*
**(1)** *The inclusion functor $i : \mathcal{E}_j \hookrightarrow \mathcal{E}$ preserves the initial object 0.*
**(2)** *$j \leq \neg\neg$, where $\neg\neg$ is the double negation operator.*
**(3)** *$c_j$ preserves negation on every subobject lattice.*
**(4)** *The least subobject 0 of the terminal object 1 is $j$-closed.*

Considering the least element 0 as a subobject $V$ in Proposition 9, the fourth condition of Definition 11 implies $L_j(\neg U) = L_j(U \Rightarrow 0) = (L_j(U) \Rightarrow_j 0_j) = \neg_j L_j(U)$ (weak openness). As a consequence, we have the following implications among the openness notions.

▶ **Proposition 12** ([6, Section 3]). *The following implications hold for local operators:*

$$\textit{open} \implies \textit{implicationally open} \implies \textit{weakly open} \impliedby \textit{dense}.$$

In fact, the implications are strict as there is a topos in which all the openness notions are different. The effective topos $\mathcal{E}\!f\!f$ provides such an example.

## 2.5 Preservation of arithmetical equality

If a topos $\mathcal{E}$ has a *natural number object (NNO)* $N$, it is possible to interpret $\mathcal{L}_{\mathrm{A}}$-terms and $\mathcal{L}_{\mathrm{A}}$-formulas in it. That is, we can assign to each function symbol $f \in \mathcal{L}_{\mathrm{A}}$ a morphism $[\![f]\!]\colon N^k \to N$ so that the defining equation for $f$ is true in $\mathcal{E}$ by the universal property of NNO. Other axioms of Heyting arithmetic can also be verified ([17, Theorem 4.1]). Thus we can regard every topos with NNO as a model of **HA**.

Every sheafification functor $L_j\colon \mathcal{E} \to \mathcal{E}_j$ preserves NNO ([13, Lemma A2.5.6]). That is, if $N$ is an NNO in $\mathcal{E}$, then so is $N_j := L_j N$ in $\mathcal{E}_j$. It automatically follows that $L_j$ preserves the interpretation of an atomic formula $f(\vec{x}) = g(\vec{x})$.

▶ **Lemma 13.** *Let $f(\vec{x})$ and $g(\vec{x})$ be $k$-ary function symbols. We have*

$$[\![f(\vec{x}) = g(\vec{x})]\!]_{\mathcal{E}_j} = \theta^*(L_j[\![f(\vec{x}) = g(\vec{x})]\!]_{\mathcal{E}}) \quad \text{in} \quad \mathrm{Sub}_{\mathcal{E}_j}(N_j^k),$$

*where $\theta\colon N_j^k \to L_j(N^k)$ is the canonical isomorphism.*

**Proof.** One can show that $[\![f]\!]_{\mathcal{E}_j} : N_j^k \to N_j$ coincides with $(L_j[\![f]\!]_{\mathcal{E}}) \circ \theta$ by induction on the construction of primitive recursive functions. The result then follows since the interpretation of $f(\vec{x}) = g(\vec{x})$ in $\mathcal{E}$ (resp. $\mathcal{E}_j$) is given by an equalizer of $[\![f]\!]_{\mathcal{E}}$, $[\![g]\!]_{\mathcal{E}} : N^k \to N$ (resp. $[\![f]\!]_{\mathcal{E}_j}$, $[\![g]\!]_{\mathcal{E}_j}$), which is preserved by $L_j$. ◀

# 3 Least dense operators of arithmetical formulas

Given a topos $\mathcal{E}$ and a formula $\varphi$, it is often possible to associate a local operator $j^{\mathcal{E}}_{\varphi}$ that completely determines the subtoposes of $\mathcal{E}$ which validate $\varphi$, in the sense that $j^{\mathcal{E}}_{\varphi} \leq k$ if and only if $\mathcal{E}_k \models \varphi$ for any local operator $k$ in $\mathcal{E}$. Our purpose in this section is to develop a general theory of such local operators. Although the main focus of this paper lies on the effective topos $\mathcal{E}\!f\!f$, we anticipate that our general theory will find a wide range of applications in future, as will be discussed in Section 5. To achieve this, we need to restrict our treatment of local operators to *dense* ones. This restriction is not essential since all nondegenerate operators are dense in $\mathcal{E}\!f\!f$. See Example 16 and Remark 35 for further justifications.

In Subsection 3.1, we introduce the notion of *least dense operator* and see how it is relevant to the separation of subclassical axioms. In Subsections 3.2 and 3.3, we look at two properties of arithmetical formulas: *transparency* and *closedness*. Transparent formulas have least dense operators under a mild assumption, while the class of transparent and closed formulas enjoys good closure properties. In particular, all $\Sigma_2$-formulas are transparent and thus have least dense operators (under a mild assumption). However, there is a non-transparent formula in $\Pi_3$ which does not have a least dense operator, so the above result is optimal. These considerations lead us to a new technique to iterate the transparency argument. We show in Subsection 3.4 that all axioms in Figure 1 have least dense operators in an arbitrary topos with natural number object.

Throughout this section, we fix a topos $\mathcal{E}$ with natural number object $N$. By the assumption that each $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$ has at most one free variable (Subsection 2.1), the interpretation of $\varphi$ in $\mathcal{E}$ can be simply regarded as subobject $[\![\varphi]\!]_{\mathcal{E}} \rightarrowtail N$.

## 3.1 Least dense operators

▶ **Notation 14.** *Let* $\mathbf{DLop}(\mathcal{E}) := \{\, k \in \mathbf{Lop}(\mathcal{E}) \mid k \leq \neg\neg \,\}$ *denote the class of dense local operators in $\mathcal{E}$. For a local operator $j$, define* $\mathbf{DLop}(\mathcal{E})^{\geq j} := \{\, k \in \mathbf{DLop}(\mathcal{E}) \mid j \leq k \,\}$. $\mathbf{Lop}(\mathcal{E})^{\geq j}$ *is similarly defined. For an $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$, let* $\langle\varphi\rangle^{\mathcal{E}} := \{\, k \in \mathbf{DLop}(\mathcal{E}) \mid \mathcal{E}_k \models \varphi \,\}$.

▶ **Definition 15.** *Let* $\varphi$ *be an* $\mathcal{L}_{\mathrm{A}}$*-formula. A dense local operator* $j$ *in* $\mathcal{E}$ *is called the* least dense operator of $\varphi$ in $\mathcal{E}$, *written* $j^{\mathcal{E}}_{\varphi}$, *if it satisfies*

$$\langle\varphi\rangle^{\mathcal{E}} = \mathbf{DLop}(\mathcal{E})^{\geq j}.$$

*Similarly, for an* $\mathcal{L}_{\mathrm{A}}$*-theory* $T$*, we use notations* $\langle T\rangle^{\mathcal{E}} := \{ k \in \mathbf{DLop}(\mathcal{E}) \mid \mathcal{E}_k \models T \}$ *and* $j^{\mathcal{E}}_{T}$*.*

The concept of least dense operator is illustrated in Figure 2. The filled region corresponds to the class of dense operators whose associated subtoposes satisfy $\varphi$.

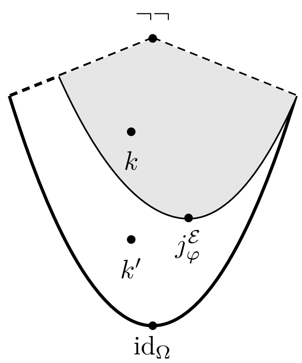


**Figure 2** Least dense operator in $\mathbf{DLop}(\mathcal{E})$.

▶ **Example 16.** Similar concepts have been studied in various contexts.

**(1)** Blass and Ščedrov [4], in their investigation of categorical logic, proved that in any topos, propositional formula $p \vee \neg p$ has a unique local operator $\neg\neg$, which is nothing but the least dense operator in our terminology. It is significant for the study of the *classifying toposes* of geometric theories ([12, 4, 5], [7, Section 4.2.3]). Caramello [6] further investigated least dense operators for more general propositional formulas. She revealed that propositional formulas have least dense operators in any topos as far as they are *implication-free*. The reason for this restriction is that dense local operators are not implicationally open in general.

**(2)** Least local operators of arithmetical formulas in $\mathcal{E}\!\mathit{ff}$ are studied by van Oosten [23]. It is well known that in this topos, all nondegenerate operators are dense [10]. He identified a certain restricted class of $\mathcal{L}_{\mathrm{A}}$-formulas which have least local operators in $\mathcal{E}\!\mathit{ff}$. He then showed that the local operator $j_{\mathcal{L}if}$ corresponding to the *Lifschitz topos* $\mathcal{L}if \subseteq \mathcal{E}\!\mathit{ff}$ is the least operator of an $\mathcal{L}_{\mathrm{A}}$-formula $(O)$ in that class. It is known that $(O)$ is equivalent to $\Pi_1 \vee \Pi_1$-**DNE** over **HA**, so $j_{\mathcal{L}if}$ is the least operator of $\Pi_1 \vee \Pi_1$-**DNE** in $\mathcal{E}\!\mathit{ff}$ too.

We remark that both lines of work rely on the denseness of local operators, either explicitly or implicitly.

We also remark that least dense operators provide a good notion of "invariant", which is useful for the separation of subclassical axioms. In fact, the subset $\langle\varphi\rangle^{\mathcal{E}} \subseteq \mathbf{DLop}(\mathcal{E})$ is invariant under **HA**-provable equivalence:

$$\mathbf{HA} \vdash \varphi \leftrightarrow \psi \implies \langle\varphi\rangle^{\mathcal{E}} = \langle\psi\rangle^{\mathcal{E}}.$$

If $\varphi$ and $\psi$ further have least dense operators, we have

$$j^{\mathcal{E}}_{\varphi} \neq j^{\mathcal{E}}_{\psi} \implies \mathbf{HA} \nvdash \varphi \leftrightarrow \psi.$$

That is, we can separate two axioms just by showing that their least dense operators are different. Moreover, even if $j^{\mathcal{E}}_{\varphi} = j^{\mathcal{E}}_{\psi}$, we obtain a negative consequence that $\varphi$ and $\psi$ are never separable by a dense subtopos of $\mathcal{E}$ (Corollary 47). Thus, least dense operators provide us with sufficient information on separability by dense subtoposes.

All least operators in Example 16 are obtained based on the theorem below due to Joyal.

▶ **Theorem 17** ([13, Corollary A4.5.13]). *Let $X$ be an object in $\mathcal{E}$ and $U \rightarrowtail X$. There is a unique local operator $\ell$ in $\mathcal{E}$ such that $\mathbf{Lop}(\mathcal{E})^{\geq \ell} = \{ j \in \mathbf{Lop}(\mathcal{E}) \mid U \rightarrowtail X : j\text{-dense} \}$ holds.*

▶ **Definition 18.** *The above $\ell$ is called the* least operator of $U$ in $\mathcal{E}$ *and written as $\ell^{\mathcal{E}}_{U}$.*

In particular, suppose that $X$ is the natural number object $N$, $U$ is the interpretation $[\![\varphi]\!]_{\mathcal{E}} \rightarrowtail N$ of an $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$, and $j \in \mathbf{Lop}(\mathcal{E})$ satisfies the following condition:

$$L_j[\![\varphi]\!]_{\mathcal{E}} = [\![\varphi]\!]_{\mathcal{E}_j}. \qquad (j\text{-transparency})$$

Then Lemma 6 implies that $\mathcal{E}_j \models \varphi$ if and only if $[\![\varphi]\!]_{\mathcal{E}}$ is $j$-dense. Hence by Theorem 17 we have $\mathbf{Lop}(\mathcal{E})^{\geq \ell^{\mathcal{E}}_{U}} = \{ j \in \mathbf{Lop}(\mathcal{E}) \mid \mathcal{E}_j \models \varphi \}$. What is critical here is the assumption of $j$-transparency, which will be the subject of the following subsections.

## 3.2 Transparency and closedness

▶ **Definition 19.** *For a local operator $j$ and an $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$,*

**(1)** $\varphi$ is $j$-transparent *if $L_j[\![\varphi]\!]_{\mathcal{E}} = [\![\varphi]\!]_{\mathcal{E}_j}$ holds. Let $\mathrm{Trp}^{\mathcal{E}}_j := \{ \varphi \mid \varphi \text{ is } j\text{-transparent} \}$.*

**(2)** $\varphi$ is $j$-closed *if $c_j[\![\varphi]\!]_{\mathcal{E}} = [\![\varphi]\!]_{\mathcal{E}}$ holds. Let $\mathrm{Cl}^{\mathcal{E}}_j := \{ \varphi \mid \varphi \text{ is } j\text{-closed} \}$.*

*Let $\mathrm{Trp}^{\mathcal{E}} := \bigcap_{j \leq \neg\neg} \mathrm{Trp}^{\mathcal{E}}_j$ and $\mathrm{Cl}^{\mathcal{E}} := \bigcap_{j \leq \neg\neg} \mathrm{Cl}^{\mathcal{E}}_j$. We say $\varphi$* is transparent in $\mathcal{E}$ *if $\varphi \in \mathrm{Trp}^{\mathcal{E}}$.*

As an example, every quantifier-free formula is transparent and closed in $\mathcal{E}$.

▶ **Lemma 20.** $\mathrm{Cl}^{\mathcal{E}} = \mathrm{Cl}^{\mathcal{E}}_{\neg\neg}$ *holds. Hence, for every $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$,*

**(1)** *$\varphi \in \mathrm{Cl}^{\mathcal{E}}$ if and only if $\mathcal{E} \models \neg\neg\varphi \to \varphi$.*

**(2)** *$\Sigma_0, \Pi_0 \subseteq \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$.*

**Proof.** Notice that $j \leq \neg\neg$ implies $c_j[\![\varphi]\!]_{\mathcal{E}} \leq c_{\neg\neg}[\![\varphi]\!]_{\mathcal{E}}$ (Lemma 4). This leads to $\mathrm{Cl}^{\mathcal{E}} = \mathrm{Cl}^{\mathcal{E}}_{\neg\neg}$. (1) immediately follows from the fact that $\mathcal{E} \models \neg\neg\varphi \to \varphi$ if and only if $\varphi$ is $\neg\neg$-closed.

To show (2), suppose that $\varphi \in \Sigma_0 (= \Pi_0)$. We then obtain $\varphi \in \mathrm{Cl}^{\mathcal{E}}$ by (1) since $\mathbf{HA} \vdash \varphi \leftrightarrow \neg\neg\varphi$. By noting that every quantifier-free formula is equivalent to an atomic formula in $\mathbf{HA}$, $\varphi \in \mathrm{Trp}^{\mathcal{E}}$ follows from Lemma 13. ◀

Transparency and closedness are strongly related to the concept of least dense operator. More precisely, we will show the following correspondence: under a natural assumption,

- $\mathrm{Trp}^{\mathcal{E}}$ is a class of formulas which have least dense operators.
- $\mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ is a class of formulas whose least dense operators are trivial.

Let us consider the latter first.

▶ **Lemma 21.** *Let $j \in \mathbf{DLop}(\mathcal{E})$ and $\varphi \in \mathrm{Trp}^{\mathcal{E}}_j \cap \mathrm{Cl}^{\mathcal{E}}_j$. Then $\mathcal{E}_j \models \varphi$ if and only if $\mathcal{E} \models \varphi$.*

**Proof.** The backward direction is clear since $L_j$ preserves isomorphisms and $L_j[\![\varphi]\!]_{\mathcal{E}} = [\![\varphi]\!]_{\mathcal{E}_j}$.

For the forward direction, first note that $[\![\varphi]\!]_{\mathcal{E}_j} = L_j[\![\varphi]\!]_{\mathcal{E}}$ is the greatest element in $\mathrm{Sub}_{\mathcal{E}_j}(N_j)$. It then follows from Lemma 6 that $[\![\varphi]\!]_{\mathcal{E}}$ is $j$-dense. Since $\varphi$ is $j$-closed, $[\![\varphi]\!]_{\mathcal{E}}$ is also the greatest element in $\mathrm{Sub}_{\mathcal{E}}(N)$. ◀

Therefore, we obtain the following.

▶ **Theorem 22.** *Let $\varphi$ be an $\mathcal{L}_{\mathrm{A}}$-formula. The following are equivalent:*
**(1)** $\varphi \in \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ *and* $\langle\varphi\rangle^{\mathcal{E}}$ *is nonempty.*
**(2)** $\langle\varphi\rangle^{\mathcal{E}} = \mathbf{DLop}(\mathcal{E})$*, that is,* $j^{\mathcal{E}}_{\varphi} = \mathrm{id}_{\Omega}$.

**Proof.** If (1) holds, then we have $\mathcal{E}_j \models \varphi$ for every dense local operator $j$ by Lemma 21. Hence $\langle\varphi\rangle^{\mathcal{E}} = \mathbf{DLop}(\mathcal{E})$. Conversely, assume that (2) holds. By assumption, $[\![\varphi]\!]_{\mathcal{E}_j}$ is the greatest element in $\mathrm{Sub}_{\mathcal{E}_j}(N_j)$ for any $j \in \mathbf{DLop}(\mathcal{E})$, so in particular $[\![\varphi]\!]_{\mathcal{E}}$ is also the greatest in $\mathrm{Sub}_{\mathcal{E}}(N)$. This implies $[\![\varphi]\!]_{\mathcal{E}_j} = L_j[\![\varphi]\!]_{\mathcal{E}}$ since $L_j$ preserves isomorphisms. It is obvious that $\varphi$ is in $\mathrm{Cl}^{\mathcal{E}}$ by $\mathcal{E} \models \varphi$. ◀

▶ **Corollary 23.** *Let $\varphi$ be an $\mathcal{L}_{\mathrm{A}}$-formula.*
**(1)** *If $\varphi$ is provable in* **HA**, $\varphi \in \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ *and* $j^{\mathcal{E}}_{\varphi} = \mathrm{id}_{\Omega}$.
**(2)** *If $\varphi$ is provable in* **PA**, $\varphi \in \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ *iff* $j^{\mathcal{E}}_{\varphi} = \mathrm{id}_{\Omega}$.

**Proof.** If $\varphi$ is provable in **HA**, then we have $\langle\varphi\rangle^{\mathcal{E}} = \mathbf{DLop}(\mathcal{E})$ since $\varphi$ is true in any topos. Hence we also have $\varphi \in \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ by Theorem 22.

On the other hand, if $\varphi$ is provable in **PA**, then the double negation operator $\neg\neg$ is in $\langle\varphi\rangle^{\mathcal{E}}$ since the corresponding subtopos $\mathcal{E}_{\neg\neg}$ satisfies any classically true formula, including $\varphi$. This implies that $\langle\varphi\rangle^{\mathcal{E}}$ is nonempty. ◀

## 3.3 Transparency yields least dense operators

We now turn our attention to transparent (but not necessarily closed) formulas. The argument below is a reconstruction of van Oosten's [23] in terms of transparency and closedness. The following lemma (cf. [23, Proposition 2.1]) plays a crucial role.

▶ **Lemma 24** (MAIN LEMMA). *Let $\mathcal{E}$ be an arbitrary topos with natural number object.*
**(1)** *Suppose that an $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$ is transparent in $\mathcal{E}$. Then either $\langle\varphi\rangle^{\mathcal{E}} = \emptyset$ or $\varphi$ has least dense operator $j^{\mathcal{E}}_{\varphi} = \ell^{\mathcal{E}}_{[\![\varphi]\!]_{\mathcal{E}}}$, where $\ell^{\mathcal{E}}_{[\![\varphi]\!]_{\mathcal{E}}}$ is the least operator of $[\![\varphi]\!]_{\mathcal{E}}$.*
**(2)** *Suppose that $\mathcal{L}_{\mathrm{A}}$-formulas $\varphi$, $\psi$ are transparent in $\mathcal{E}$ and* $\mathbf{HA} \vdash \varphi \to \psi$*. Then for $\rho := \psi \to \varphi$, either $\langle\rho\rangle^{\mathcal{E}} = \emptyset$ or $\rho$ has least dense operator $j^{\mathcal{E}}_{\rho}$.*

**Proof.** By taking an **HA**-provable formula as $\psi$, (1) can be regarded as a special case of (2) (notice Corollary 23 (1)).

Let $j$ be a dense local operator in $\mathcal{E}$. The assumption $\mathbf{HA} \vdash \varphi \to \psi$ implies that $\varphi \to \psi$ is true in any topos, including $\mathcal{E}$ and $\mathcal{E}_j$. So $[\![\varphi]\!]_{\mathcal{E}} \leq [\![\psi]\!]_{\mathcal{E}}$ and $[\![\varphi]\!]_{\mathcal{E}_j} \leq [\![\psi]\!]_{\mathcal{E}_j}$ hold. Now consider a subobject $U := [\![\varphi]\!]_{\mathcal{E}} \rightarrowtail [\![\psi]\!]_{\mathcal{E}}$. Since $\varphi$ and $\psi$ are transparent, we get the equation $L_jU = (L_j[\![\varphi]\!]_{\mathcal{E}} \rightarrowtail L_j[\![\psi]\!]_{\mathcal{E}}) = ([\![\varphi]\!]_{\mathcal{E}_j} \rightarrowtail [\![\psi]\!]_{\mathcal{E}_j})$. Thus the following equivalence holds: $\mathcal{E}_j \models \rho$ iff $L_jU = [\![\varphi]\!]_{\mathcal{E}_j} \rightarrowtail [\![\psi]\!]_{\mathcal{E}_j}$ is an isomorphism iff $U$ is $j$-dense (the last equivalence follows from Lemma 6).

By Theorem 17, we have the least operator $\ell^{\mathcal{E}}_U$ such that $U$ is dense. If $\ell^{\mathcal{E}}_U \notin \mathbf{DLop}(\mathcal{E})$, then $\langle\rho\rangle^{\mathcal{E}} = \emptyset$. If $\ell^{\mathcal{E}}_U \in \mathbf{DLop}(\mathcal{E})$, $\ell^{\mathcal{E}}_U$ is the least dense operator of $\rho$ in $\mathcal{E}$. ◀

The next step of our reconstruction is to examine the closure properties satisfied by $\mathrm{Trp}^{\mathcal{E}}$ and $\mathrm{Cl}^{\mathcal{E}}$. The proof of the case $\neg$ relies on the restriction to dense local operators.

▶ **Lemma 25.**
**(1)** $\mathrm{Trp}^{\mathcal{E}}$ *is closed under* $\wedge$, $\vee$, $\exists$, $\neg$.
**(2)** $\mathrm{Cl}^{\mathcal{E}}$ *is closed under* $\wedge$, $\to$, $\forall$, $\neg$.
**(3)** *Suppose that* $\varphi \in \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ *and* $\psi \in \mathrm{Trp}^{\mathcal{E}}$. *Then* $\psi \to \varphi$ *and* $\forall x\, \varphi$ *belong to* $\mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$. *In particular,* $\mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ *is closed under* $\wedge$, $\to$, $\forall$, $\neg$.

**Proof.** (1) Closure under $\wedge$, $\vee$, $\exists$ is due to Proposition 8. For $\neg$, just recall that dense local operators are weakly open (Proposition 12). (2) $\mathrm{Cl}^{\mathcal{E}}$ is closed under $\wedge$, $\to$, $\forall$ by Lemma 7, and under $\neg$ by denseness (Definition 11 (3)). (3) holds by Proposition 9. ◀

Together with Lemma 20, we obtain the following:

▶ **Corollary 26.** $\Pi_1 \subseteq \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$ *and* $\Sigma_2 \subseteq \mathrm{Trp}^{\mathcal{E}}$.

▶ **Corollary 27.** *Suppose that an $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$ is transparent in $\mathcal{E}$. Then $\varphi \vee \neg\varphi$ and $\neg\neg\varphi \to \varphi$ have least dense operators in $\mathcal{E}$.*

**Proof.** Note that $\langle \varphi \vee \neg\varphi \rangle^{\mathcal{E}}$ and $\langle \neg\neg\varphi \to \varphi \rangle^{\mathcal{E}}$ are always nonempty because both formulas are provable in **PA**, so true in $\mathcal{E}_{\neg\neg}$. $\varphi \vee \neg\varphi$ belongs to $\mathrm{Trp}^{\mathcal{E}}$ by Lemma 25 (1), hence $\varphi \vee \neg\varphi$ has least dense operator in $\mathcal{E}$ by Lemma 24 (1).

For $\neg\neg\varphi \to \varphi$, we apply the case (2) of Lemma 24. ◀

This corollary can be extended to semi-classical axioms. Recall that for $n \geq 1$, there exist universal formulas $\varphi_{\Sigma_n}$ and $\varphi_{\Pi_n}$ for the classes $\Sigma_n$ and $\Pi_n$. Hence axiom scheme $\Sigma_n$-**LEM**, for example, is equivalent to formula ${}^{\forall}(\varphi_{\Sigma_n} \vee \neg\varphi_{\Sigma_n})$ over **HA**. Moreover, we have $\mathcal{E} \models^{\forall} (\varphi_{\Sigma_n} \vee \neg\varphi_{\Sigma_n})$ iff $\mathcal{E} \models \varphi_{\Sigma_n} \vee \neg\varphi_{\Sigma_n}$. Thus we obtain:

▶ **Lemma 28.** *Let $\Gamma$ be one of $\Sigma_n$, $\Pi_n$ and $\Pi_n \vee \Pi_n$ and assume that $\Gamma \subseteq \mathrm{Trp}^{\mathcal{E}}$. Then $\Gamma$-**LEM** and $\Gamma$-**DNE** have least dense operators in $\mathcal{E}$.*

As we saw in Corollary 26, $\Sigma_2 \subseteq \mathrm{Trp}^{\mathcal{E}}$ always holds. This fact ensures that $\Sigma_2$-**DNE** and $\Sigma_2$-**LEM** have least dense operators in $\mathcal{E}$. On the other hand, we can show that $\Pi_3 \subseteq \mathrm{Trp}^{\mathcal{E}}$ does not hold in general (see Theorem 49 in Appendix A). Therefore, to show the existence of least dense operators of semi-classical axioms for $n \geq 3$, we need another technique, that is to be discussed in the next subsection.

## 3.4 Iteration argument and least dense operators of semi-classical axioms

In this subsection, we prove that all semi-classical axioms have least dense operators in every topos. The key idea is to iterate the construction of Lemma 28.

▶ **Lemma 29.** *Let $n \geq 0$. If $\mathcal{E} \models \Sigma_n$-**DNE**, then $\Pi_{n+1} \subseteq \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$, so $\Sigma_{n+2} \subseteq \mathrm{Trp}^{\mathcal{E}}$.*

**Proof.** By induction on $n$, recalling that $\mathcal{E} \models \Sigma_n$-**DNE** iff $\Sigma_n \subseteq \mathrm{Cl}^{\mathcal{E}}$ (Lemma 20). The base case is true by Corollary 26. Next assume that $\Sigma_{n+1} \subseteq \mathrm{Cl}^{\mathcal{E}}$. By the induction hypothesis $\Pi_n \subseteq \Pi_{n+1} \subseteq \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$, so $\Sigma_{n+1} \subseteq \mathrm{Trp}^{\mathcal{E}}$ by Lemma 25 (1), that is, $\Sigma_{n+1} \subseteq \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$. By Lemma 25 (3), we conclude $\Pi_{n+2} \subseteq \mathrm{Trp}^{\mathcal{E}} \cap \mathrm{Cl}^{\mathcal{E}}$. ◀

For the sake of the argument below, let us note a natural correspondence between $\mathbf{Lop}(\mathcal{E})^{\geq j}$ and $\mathbf{Lop}(\mathcal{E}_j)$.

▶ **Notation 30.** *Let $j \in \mathbf{Lop}(\mathcal{E})$. There is a one-to-one correspondence between $\mathbf{Lop}(\mathcal{E})^{\geq j}$ and $\mathbf{Lop}(\mathcal{E}_j)$. We write $k_j$ for the local operator in $\mathbf{Lop}(\mathcal{E}_j)$ corresponding to $k \in \mathbf{Lop}(\mathcal{E})^{\geq j}$. Under this notation we have $(\mathcal{E}_j)_{k_j} = \mathcal{E}_k$.*

▶ **Lemma 31** ([7, Corollary 4.2.9])**.** *If $j$ is a dense local operator in $\mathcal{E}$, $\neg\neg_j$ is identical to the double negation operator in $\mathbf{Lop}(\mathcal{E}_j)$. Thus, the correspondence in Notation 30 holds even if restricted to the dense local operators (Figure 3).*

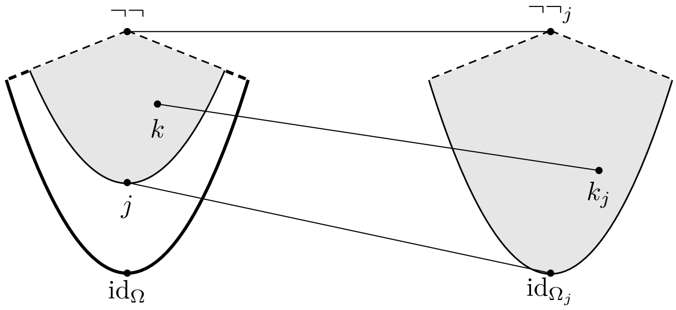


**Figure 3** Correspondence between $\mathbf{DLop}(\mathcal{E})^{\geq j}$ and $\mathbf{DLop}(\mathcal{E}_j)$.

The following lemma allows us to iterate Lemma 28.

▶ **Lemma 32.** *Let $T$, $S$ be $\mathcal{L}_{\mathrm{A}}$-theories such that $\mathbf{HA} + T \vdash S$. Suppose further that*
**(1)** *$S$ has least dense operator $j_S := j_S^{\mathcal{E}} \in \mathbf{DLop}(\mathcal{E})$ in $\mathcal{E}$.*
**(2)** *$T$ has least dense operator $j'_T := j_T^{\mathcal{E}_{j_S}} \in \mathbf{DLop}(\mathcal{E}_{j_S})$ in $\mathcal{E}_{j_S}$.*
*Then $j_T \in \mathbf{DLop}(\mathcal{E})^{\geq j_S}$ corresponding to $j'_T$ is the least dense operator of $T$ in $\mathcal{E}$.*

**Proof.** Let $k \in \mathbf{DLop}(\mathcal{E})$. We show that $\mathcal{E}_k \models T$ if and only if $j_T \leq k$. Assume that $\mathcal{E}_k \models T$. $j_S \leq k$ clearly follows from the assumption (1) and $\mathbf{HA} + T \vdash S$. So $k \in \mathbf{DLop}(\mathcal{E})^{\geq j_S}$ and the corresponding operator $k_{j_S} \in \mathbf{DLop}(\mathcal{E}_{j_S})$ yields a dense subtopos of $\mathcal{E}_{j_S}$ that satisfies $T$. Hence, by the assumption (2), we get $j'_T \leq k_{j_S}$. This implies $j_T \leq k$.

Conversely, suppose $j_T \leq k$. Since $j_T \in \mathbf{DLop}(\mathcal{E})^{\geq j_S}$, the corresponding operator $j'_T \in \mathbf{DLop}(\mathcal{E}_{j_S})$ satisfies $j'_T \leq k_{j_S}$. By the assumption (2) again, $(\mathcal{E}_{j_S})_{k_{j_S}} = \mathcal{E}_k \models T$ holds. ◀

▶ **Theorem 33.** *For every topos $\mathcal{E}$ with natural number object and $n \geq 0$, $\Sigma_n$-**DNE** has least dense operator $j_n := j^{\mathcal{E}}_{\Sigma_n\text{-}\mathbf{DNE}}$ in $\mathcal{E}$.*

**Proof.** By induction on $n$. For $n = 0$, let $j_0 = \mathrm{id}_\Omega$ (See Corollary 23).

Next, assume that $\Sigma_n$-**DNE** has least dense operator $j_n$ in $\mathcal{E}$. Then $\Sigma_n$-**DNE** is true in $\mathcal{E}_{j_n}$, hence we have $\Sigma_{n+1} \subseteq \Sigma_{n+2} \subseteq \mathrm{Trp}^{\mathcal{E}_{j_n}}$ by Lemma 29. Thus, it follows from Lemma 28 that $\Sigma_{n+1}$-**DNE** has least dense operator in $\mathcal{E}_{j_n}$. Since $\mathbf{HA} + \Sigma_{n+1}$-**DNE** implies $\Sigma_n$-**DNE**, all assumptions of Lemma 32 are satisfied. We therefore conclude that $\Sigma_{n+1}$-**DNE** has least dense operator in $\mathcal{E}$. ◀

The local operators $\{ j_n \}$ can be used as the "footholds" to obtain least dense operators of other semi-classical axioms.

▶ **Corollary 34.** *For every topos $\mathcal{E}$ with natural number object and $n \geq 0$, $\Sigma_n$-**LEM**, $\Pi_n$-**LEM** and $\Pi_n \vee \Pi_n$-**DNE** have least dense operators in $\mathcal{E}$.*

**Proof.** We here focus on $\Pi_{n+1}$-**LEM**. Considering the least dense operator $j_n$ of $\Sigma_n$-**DNE** in Theorem 33, we have $\Pi_{n+1} \subseteq \Sigma_{n+2} \subseteq \mathrm{Trp}^{\mathcal{E}_{j_n}}$. Thus, it follows from Lemma 28 that $\Pi_{n+1}$-**LEM** has least dense operator in $\mathcal{E}_{j_n}$. Since $\mathbf{HA} + \Pi_{n+1}$-**LEM** proves $\Sigma_n$-**DNE** (Figure 1), it also has least dense operator in $\mathcal{E}$ by Lemma 32. ◀

▶ Remark 35. Let us finally discuss (dis)advantages of the restriction to dense operators. One clear disadvantage is that it forces us to introduce an additional assumption $\langle\varphi\rangle^{\mathcal{E}} \neq \emptyset$ in Lemma 24 (MAIN LEMMA). Although this may appear inconvenient, it does not cause any problem as long as semi-classical axioms are concerned (See the proof of Corollary 27).

On the other hand, the restriction is really essential for Lemma 20. In fact, this lemma does not hold without the assumption of denseness, as indicated by the following:

▶ **Theorem 36.** *Let $j$ be a local operator. All $\Sigma_0$-formulas are $j$-closed iff $j$ is dense.*

**Proof.** The backward direction is shown in Lemma 20. To see the forward direction, consider $\Sigma_0$-formula $\neg(x = x)$, whose interpretation is the least element 0 in the subobject poset of NNO. If it is $j$-closed, then the least subobject 0 of the terminal object 1 is also $j$-closed. The latter condition is equivalent to being a dense local operator (Definition 11 (4)). ◀

Failure of Lemma 20 would affect most of the subsequent theorems in Section 3. For example, there is no guarantee that all $\Sigma_2$-formulas are transparent. We would say that denseness is a price to pay to obtain these theorems in the general setting.

## 4 Least operators in the effective topos

In this section, we apply the general theory developed in the previous section to the effective topos $\mathcal{E}\!f\!f$. As explained in Example 16, any non-degenerate operator is dense in this topos. Henceforth, we speak of *least operators* instead of least dense ones.

In Subsection 4.1, we briefly review the structure of subobjects and that of local operators in $\mathcal{E}\!f\!f$. We also mention that there are local operators corresponding to Turing degrees. In Subsection 4.2, we express all least dense operators of semi-classical axioms in terms of (generalized) Turing degrees. This immediately leads to some separation results conforming to [1]. For details on $\mathcal{E}\!f\!f$, the reader is referred to [25].

### 4.1 Subobjects of NNO and local operators in $\mathcal{E}\!f\!f$

Let us first recall the effective topos and associated concepts.

▶ **Notation 37.** *We fix a primitive recursive pairing function $\langle -, - \rangle : \mathbb{N}^2 \to \mathbb{N}$ with the associated projections $(-)_0$, $(-)_1 : \mathbb{N} \to \mathbb{N}$. For natural numbers $e$ and $n$, we write $e \cdot n$ for the result of applying the $e$-th partial computable function to $n$, and write $e \cdot n \downarrow$ if the computation terminates. If $\psi$ is a closed $\mathcal{L}_{\mathrm{A}}$-formula, $n \mathrel{\mathbf{r}_{\mathrm{K}}} \psi$ means that $n$ realizes $\psi$ under Kleene realizability. Also we use $\lambda$-notation: for a partial computable function $t :\subseteq \mathbb{N} \to \mathbb{N}$ in variable $x$, $\lambda x.t$ denotes an index of $t$. Similarly, for $u :\subseteq \mathbb{N}^2 \to \mathbb{N}$ in variable $x$ and $y$, $\lambda xy.u$ is an abbreviation for $\lambda x.(\lambda y.u)$.*

Given a set $X$, we consider the following operations on $\mathcal{P}(\mathbb{N})^X$. For any $\varphi, \psi : X \to \mathcal{P}(\mathbb{N})$,

$$\begin{aligned}
\varphi \wedge \psi(x) &:= \{ \langle n, m \rangle \mid n \in \varphi(x) \ \wedge\ m \in \psi(x) \}, & \top(x) &:= \mathbb{N},\\
\varphi \vee \psi(x) &:= \{ \langle 0, n \rangle \mid n \in \varphi(x) \} \cup \{ \langle 1, m \rangle \mid m \in \psi(x) \}, & \bot(x) &:= \emptyset,\\
\varphi \to \psi(x) &:= \{ e \mid \forall n \in \varphi(x)\ (e \cdot n \downarrow \wedge e \cdot n \in \psi(x)) \}, & \neg\varphi(x) &:= \varphi \to \bot(x).
\end{aligned}$$

In addition, we define a preorder $\sqsubseteq$ on $\mathcal{P}(\mathbb{N})^X$: $\varphi \sqsubseteq \psi$ if $\bigcap_{x \in X}(\varphi \to \psi(x))$ is nonempty. Then $(\mathcal{P}(\mathbb{N})^X, \sqsubseteq)$ forms a Heyting prealgebra and induces the *effective tripos* $\mathrm{P}_{\mathcal{E}\!f\!f} : X \mapsto (\mathcal{P}(\mathbb{N})^X, \sqsubseteq)$. The *effective topos* $\mathcal{E}\!f\!f$ is given by the *tripos-to-topos construction* on $\mathrm{P}_{\mathcal{E}\!f\!f}$. For example, an object $X$ of $\mathcal{E}\!f\!f$ is a pair $X = (X, =_X)$ where $X$ is a set and $=_X : X \times X \to \mathcal{P}(\mathbb{N})$ is a "$\mathcal{P}(\mathbb{N})$-valued equality" with respect to $\mathrm{P}_{\mathcal{E}\!f\!f}$, and a morphism $f : X \to Y$ of $\mathcal{E}\!f\!f$ is an (equivalence class of) "$\mathcal{P}(\mathbb{N})$-valued functional relation" $F : X \times Y \to \mathcal{P}(\mathbb{N})$ that respects $=_X$ and $=_Y$ ([25, Chapter 2]).

$\mathcal{E}\!f\!f$ has a natural number object $N = (\mathbb{N}, =_N)$, where $[n =_N m] \coloneqq \{ n \}$ if $n = m$ and $\coloneqq \emptyset$ otherwise. A subobject classifier $\Omega = (\mathcal{P}(\mathbb{N}), =_\Omega)$ is given by defining $[p =_\Omega q] \coloneqq (p \to q) \wedge (q \to p)$, where $\wedge$, $\to$ are operations on $\mathcal{P}(\mathbb{N}) \cong \mathcal{P}(\mathbb{N})^{\{ * \}}$.

Similarly, the structure of subobjects in $\mathcal{E}\!f\!f$ is determined by $\mathrm{P}_{\mathcal{E}\!f\!f}$. Indeed, a subobject $U$ of $X \in \mathcal{E}\!f\!f$ can be described by a "strict relational" function $U : X \to \mathcal{P}(\mathbb{N})$ with respect to $=_X$. As far as the subobjects of $N$ are concerned, relationality is trivial so that they admit much simpler descriptions below.

▶ **Definition 38.** *A function $U : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is called a* (partial) multifunction *and written as $U :\subseteq \mathbb{N} \rightrightarrows \mathbb{N}$. Let* **Mfunc** *denote the set of all multifunctions. A preorder on* **Mfunc** *can be defined by invoking Notation 37 for the case of $X = \mathbb{N}$: $U \sqsubseteq_s V$ iff $U' \sqsubseteq V$, where $U'(e) \coloneqq \{ \langle e, n\rangle \mid n \in U(e) \}$. The latter means that*

$$\exists f \in \mathbb{N}\, \forall e, n \in \mathbb{N} \quad (n \in U(e) \implies f \cdot \langle e, n\rangle \in V(e)).$$

*We write $U \equiv_s V$ if $U \sqsubseteq_s V$ and $V \sqsubseteq_s U$.*

This preorder induces a correspondence between the multifunctions and the subobjects of $N$. The reason for using $U'$ is that $U$ is not strict with respect to $=_N$ in general.

▶ **Proposition 39.** $(\mathbf{Mfunc}, \sqsubseteq_s) \simeq (\mathrm{Sub}_{\mathcal{E}\!f\!f}(N), \leq)$.

We thus think of a multifunction $U :\subseteq \mathbb{N} \rightrightarrows \mathbb{N}$ as a subobject of $N$. In the sequel, we are mainly interested in the subobjects of $N$ that are interpretations of $\mathcal{L}_{\mathrm{A}}$-formulas. Given an $\mathcal{L}_{\mathrm{A}}$-formula $\varphi$, the interpretation $[\![\varphi]\!]_{\mathcal{E}\!f\!f}$ corresponds to a multifunction $[\![\varphi]\!]_{\mathbf{Mfunc}}$ as follows:

$$[\![\varphi]\!]_{\mathbf{Mfunc}}(e) \coloneqq \{ n \in \mathbb{N} \mid n \ \mathbf{r}_{\mathrm{K}}\ \varphi(e) \}.$$

Hence, $\varphi$ is true in $\mathcal{E}\!f\!f$ iff $[\![\varphi]\!]_{\mathcal{E}\!f\!f}$ is the greatest element in $\mathrm{Sub}_{\mathcal{E}\!f\!f}(N)$ iff $\top \sqsubseteq_s [\![\varphi]\!]_{\mathbf{Mfunc}}$ iff $\exists f \in \mathbb{N}\ \forall e \in \mathbb{N}\ (f \cdot e \downarrow$ and $f \cdot e\ \mathbf{r}_{\mathrm{K}}\ \varphi(e))$ iff ${}^{\forall}\varphi$ is Kleene realizable [25]. Notice that $[\![\varphi]\!]_{\mathbf{Mfunc}}$ is coherent with the operations introduced in Notation 37. That is, for each $\circ \in \{ \wedge, \vee, \to \}$,

$$[\![\varphi \circ \psi]\!]_{\mathbf{Mfunc}} = [\![\varphi]\!]_{\mathbf{Mfunc}} \circ [\![\psi]\!]_{\mathbf{Mfunc}}, \quad [\![\neg\varphi]\!]_{\mathbf{Mfunc}} = \neg[\![\varphi]\!]_{\mathbf{Mfunc}}.$$

Each local operator in $\mathcal{E}\!f\!f$ has a simple expression as an endofunction on $\mathcal{P}(\mathbb{N})$, but we omit its details here. It is important that there are local operators corresponding to *Turing degrees*. Recall Joyal's theorem (Theorem 17) in the previous section, which shows that every subobject $U \rightarrowtail N$ has least local operator $\ell_U^{\mathcal{E}}$ that makes $U$ dense. The following observation is due to Hyland [10].

▶ **Theorem 40** ([10, Theorem 17.2])**.** *Let $A$, $B \subseteq \mathbb{N}$. $A$ is* Turing reducible *to $B$ if and only if $j_A \leq j_B$ in $\mathbf{Lop}(\mathcal{E}\!f\!f)$, where $j_A$ is the least operator of the characteristic function $\chi_A$ of $A$ in* **Mfunc***. Hence the poset of Turing degrees can be embedded into $(\mathbf{Lop}(\mathcal{E}\!f\!f), \leq)$.*

In other words, the local operators in $\mathcal{E}\!f\!f$ can be regarded as *generalized Turing degrees*. For example, the Lifschitz operator $j_{\mathcal{L}\!i\!f}$ in Example 16 is in between $j_\emptyset$ and $j_{\emptyset^{(1)}}$, but it is never equal to $j_d$ for any Turing degree $d$ (see the notation below and Figure 4). This connection with degree theory is further extended and refined by Faber and van Oosten [8] and Kihara [14].

▶ **Notation 41.** *Given a set $D \subseteq \mathbb{N}$ with Turing degree $d$ (i.e., $D \in d$), we write $j_d$ to denote the least operator $\ell_{\chi_D}^{\mathcal{E}\!f\!f}$ of $\chi_D : \mathbb{N} \to \mathbb{N}$. For example, when $D$ is a decidable set, we have $j_d = \ell_{\chi_D}^{\mathcal{E}\!f\!f} = j_\emptyset = \mathrm{id}_\Omega$. It is known that a closed formula is true in the subtopos $\mathcal{E}\!f\!f_{j_d}$ if and* *only if it is realizable under the Kleene realizability relativized to* $d$ *(*$d$-realizability*) [20]. For* $n \in \mathbb{N}$, $\emptyset^{(n)}$ *denotes the* $n$*-th Turing jump of* $\emptyset$ *and* $\mathcal{E}\!f\!f^{(n)}$ *the subtopos* $\mathcal{E}\!f\!f_{j_{\emptyset^{(n)}}}$. *By letting* $\mathbf{r}_{\mathrm{K}}^{(n)}$ *be the* $\emptyset^{(n)}$*-realizability relation, the structure of subobjects in* $\mathcal{E}\!f\!f^{(n)}$ *can be described by* $\mathbf{r}_{\mathrm{K}}^{(n)}$. *For instance,* $[\![\varphi]\!]_{\mathcal{E}\!f\!f^{(n)}}$ *corresponds to a multifunction* $[\![\varphi]\!]^{(n)}_{\mathbf{Mfunc}}(e) := \{\, m \mid m\ \mathbf{r}_{\mathrm{K}}^{(n)}\ \varphi(e) \,\}$.

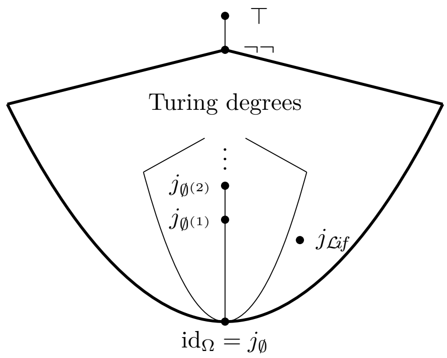


**Figure 4** Turing degrees embedded in $\mathbf{Lop}(\mathcal{E}\!f\!f)$.

## 4.2 Turing degrees and least operators of semi-classical axioms

In this subsection, we give concrete representations to the least operators of semi-classical axioms in terms of generalized Turing degrees. First of all, the following is straightforward by Corollary 23 and Figure 1.

▶ **Lemma 42.** *For every topos* $\mathcal{E}$ *with natural number object and* $n \geq 0$,

**(1)** $j^{\mathcal{E}}_{\Sigma_0\text{-}\mathbf{DNE}} = j^{\mathcal{E}}_{\Pi_0\text{-}\mathbf{LEM}} = j^{\mathcal{E}}_{\Sigma_0\text{-}\mathbf{LEM}} = \mathrm{id}_\Omega$.

**(2)** $j^{\mathcal{E}}_{\Pi_n\text{-}\mathbf{LEM}} \leq j^{\mathcal{E}}_{\Sigma_n\text{-}\mathbf{LEM}} \leq j^{\mathcal{E}}_{\Sigma_{n+1}\text{-}\mathbf{DNE}}$.

It is well known that $\Sigma_1$-**DNE** is Kleene realizable, so the least dense operator of $\Sigma_1$-**DNE** in $\mathcal{E}\!f\!f$ is just $\mathrm{id}_\Omega$. Using Notation 41, this can be expressed by Turing degree $\emptyset$:

$$j^{\mathcal{E}\!f\!f}_{\Pi_0\text{-}\mathbf{LEM}} = j^{\mathcal{E}\!f\!f}_{\Sigma_0\text{-}\mathbf{LEM}} = j^{\mathcal{E}\!f\!f}_{\Sigma_1\text{-}\mathbf{DNE}} = j_\emptyset. \tag{$\clubsuit$}$$

This equation (♣) can be extended to any $n \geq 1$ (Theorem 45). Here we give only a proof of the case $n = 1$ because there is no great difficulty in generalizing the following arguments. The complete proof can be found in Appendix B.

Let us recall Subsection 2.1 and a primitive recursive function $S(e, x)$ obtained by formalizing the *parameter theorem* in **HA**. $S(e, x)$ has the ability to "shift" a variable, that is, "$S(e, x) \cdot y\downarrow$ iff $e \cdot \langle x, y\rangle \downarrow$" is provable in **HA**. Then we can describe the universal formula for $\Sigma_2$ by $\varphi_{\Sigma_2}(e, x) := \exists y\, \varphi_{\Pi_1}(S(e, x), y) = \exists y \forall w\, \neg T(S(e, x), y, w)$, which is equivalent to $\exists y \forall w\, \neg T(e, \langle x, y\rangle, w)$ in **HA**.

Now let $p_1(e, x) := \lambda w.0$ and define $s_2(e, x) := \langle y_0, \lambda w.0\rangle$, where $y_0$ is the least number such that $\mathbb{N} \models \varphi_{\Pi_1}(S(e, x), y_0)$ holds if it exists. Note that $p_1$ is a total computable function and $s_2$ a partial $\emptyset^{(1)}$-computable one. Then, by the standard definition of realizability interpretation, we can easily verify that

**(1)** For any $e$, $x \in \mathbb{N}$, $\mathbb{N} \models \varphi_{\Pi_1}(e, x)$ implies $p_1(e, x)\ \mathbf{r}_{\mathrm{K}}\ \varphi_{\Pi_1}(e, x)$.

**(2)** For any $e$, $x \in \mathbb{N}$, $\mathbb{N} \models \varphi_{\Sigma_2}(e, x)$ implies $s_2(e, x)\ \mathbf{r}^{(1)}_{\mathrm{K}}\ \varphi_{\Sigma_2}(e, x)$.

The second property immediately implies that $\lambda ex.\lambda m.s_2(e,x)$ realizes ${}^{\forall}(\neg\neg\varphi_{\Sigma_2} \to \varphi_{\Sigma_2})$ under $\emptyset^{(1)}$-realizability, where $\lambda m.$ is a dummy abstraction. Hence $\Sigma_2$-**DNE** is true in $\mathcal{E}\mathit{ff}^{(1)}$. This allows us to estimate an upper bound for $j^{\mathcal{E}\mathit{ff}}_{\Sigma_2\text{-}\mathbf{DNE}}$. This argument can be straightforwardly extended to $j^{\mathcal{E}\mathit{ff}}_{\Sigma_{n+1}\text{-}\mathbf{DNE}}$ (Lemma 50 in Appendix B). So we have

▶ **Lemma 43.** *For any $n \geq 0$, $\mathcal{E}\mathit{ff}^{(n)} \models \Sigma_{n+1}$-**DNE**. Thus $j^{\mathcal{E}\mathit{ff}}_{\Sigma_{n+1}\text{-}\mathbf{DNE}} \leq j_{\emptyset^{(n)}}$.*

Next, notice that if a formula $\varphi$ is transparent in $\mathcal{E}\mathit{ff}$, then we have $j^{\mathcal{E}\mathit{ff}}_{\varphi\vee\neg\varphi} = \ell^{\mathcal{E}\mathit{ff}}_{[\![\varphi\vee\neg\varphi]\!]_{\mathcal{E}\mathit{ff}}}$, where the latter is Joyal's least operator of subobject $[\![\varphi \vee \neg\varphi]\!]_{\mathcal{E}\mathit{ff}}$ (Lemma 24, Corollary 27). The following lemma gives us a simple description of $[\![\varphi_{\Pi_1} \vee \neg\varphi_{\Pi_1}]\!]_{\mathcal{E}\mathit{ff}}$.

▶ **Lemma 44.** *Let $\varphi(x)$ be an $\mathcal{L}_{\mathrm{A}}$-formula and let $\chi_\varphi$ be the characteristic function of $\{\, m \in \mathbb{N} \mid \mathbb{N} \models \varphi(m)\,\}$. Suppose further that there is a* total *computable function $p(x)$ such that the following two conditions hold:*

**(a)** *For any $m \in \mathbb{N}$, $\mathcal{E}\mathit{ff} \models \varphi(m)$ implies $\mathbb{N} \models \varphi(m)$.*

**(b)** *For any $m \in \mathbb{N}$, $\mathbb{N} \models \varphi(m)$ implies $p(m)\ \mathbf{r}_{\mathrm{K}}\ \varphi(m)$.*

*Then $[\![\neg\varphi \vee \varphi]\!]_{\mathbf{Mfunc}} \equiv_s \chi_\varphi$ in $\mathbf{Mfunc} \simeq \mathrm{Sub}_{\mathcal{E}\mathit{ff}}(N)$.*

**Proof.** Assume that $\varphi(x)$ and $p(x)$ satisfy (a) and (b), and let $m \in \mathbb{N}$. The assumptions imply that $p(m)\ \mathbf{r}_{\mathrm{K}}\ \varphi(m)$ iff $\mathbb{N} \models \varphi(m)$. Because $\neg\psi$ is Kleene realizable by any natural number iff $\psi$ is not Kleene realizable for a closed formula $\psi$, we have $p(m)\ \mathbf{r}_{\mathrm{K}}\ \neg\varphi(m)$ iff $\mathbb{N} \models \neg\varphi(m)$. Hence we obtain that $\langle i, p(m)\rangle \in [\![\neg\varphi \vee \varphi]\!]_{\mathbf{Mfunc}}(m)$ iff $\chi_\varphi(m) = i$ for any $i \in \{\,0,1\,\}$. Therefore, $\lambda x.(x_1)_0$ and $\lambda x.\langle x_1, p(x_0)\rangle$ witness $[\![\neg\varphi \vee \varphi]\!]_{\mathbf{Mfunc}} \sqsubseteq_s \chi_\varphi$ and $\chi_\varphi \sqsubseteq_s [\![\neg\varphi \vee \varphi]\!]_{\mathbf{Mfunc}}$, respectively. ◀

Let us apply Lemma 44 to $\varphi \coloneqq \varphi_{\Pi_1}$ and $p \coloneqq p_1$. We have already mentioned that (b) holds. Moreover, (a) follows from the categorical equivalence $\mathcal{E}\mathit{ff}_{\neg\neg} \simeq \mathbf{Set}$ ([10, Proposition 4.4]) and Lemma 21 (this is another transparency argument). Thus we obtain $\ell^{\mathcal{E}\mathit{ff}}_{[\![\varphi\vee\neg\varphi]\!]_{\mathcal{E}\mathit{ff}}} = \ell^{\mathcal{E}\mathit{ff}}_{\chi_\varphi}$.

In addition, the subset of natural numbers defined by $\varphi_{\Pi_1}$ has Turing degree $\emptyset^{(1)}$ by *Post's theorem*, so $\ell^{\mathcal{E}\mathit{ff}}_{\chi_{\varphi_{\Pi_1}}} = j_{\emptyset^{(1)}}$ holds in the sense of Notation 41. Hence we have the equation:

$$j^{\mathcal{E}\mathit{ff}}_{\Pi_1\text{-}\mathbf{LEM}} = j^{\mathcal{E}\mathit{ff}}_{\varphi_{\Pi_1}\vee\neg\varphi_{\Pi_1}} = \ell^{\mathcal{E}\mathit{ff}}_{[\![\varphi_{\Pi_1}\vee\neg\varphi_{\Pi_1}]\!]_{\mathcal{E}\mathit{ff}}} = \ell^{\mathcal{E}\mathit{ff}}_{\chi_{\varphi_{\Pi_1}}} = j_{\emptyset^{(1)}}.$$

This argument can also be extended to $j^{\mathcal{E}\mathit{ff}}_{\Pi_n\text{-}\mathbf{LEM}}$ (Lemma 51, 52 in Appendix B). Therefore, combining this with Lemma 42 (2) and Lemma 43, we obtain

▶ **Theorem 45.** *For any $n \geq 0$, $j^{\mathcal{E}\mathit{ff}}_{\Pi_n\text{-}\mathbf{LEM}} = j^{\mathcal{E}\mathit{ff}}_{\Sigma_n\text{-}\mathbf{LEM}} = j^{\mathcal{E}\mathit{ff}}_{\Sigma_{n+1}\text{-}\mathbf{DNE}} = j_{\emptyset^{(n)}}$.*

On the other hand, as in van Oosten's work [23], the least operator of $\Pi_1 \vee \Pi_1$-**DNE** is equal to Lifschitz operator $j_{\mathcal{L}\mathit{if}}$ in $\mathcal{E}\mathit{ff}$. His argument can be lifted to $\mathcal{E}\mathit{ff}^{(n)}$. We define a multifunction $U_{\mathcal{L}\mathit{if}^{(n)}} :\subseteq \mathbb{N} \rightrightarrows \mathbb{N}$ by

$$U_{\mathcal{L}\mathit{if}^{(n)}}(e) \coloneqq \{\, \langle 0, e\rangle \mid \mathbb{N} \models \varphi_{\Pi_{n+1}}(e_0)\,\} \cup \{\, \langle 1, e\rangle \mid \mathbb{N} \models \varphi_{\Pi_{n+1}}(e_1)\,\}.$$

Following his observation in [23], we see that the least operator of $U_{\mathcal{L}\mathit{if}^{(n)}}$ in $\mathcal{E}\mathit{ff}^{(n)}$ represents Lifschitz realizability relativized to degree $\emptyset^{(n)}$. Hence we write $j_{\mathcal{L}\mathit{if}^{(n)}}$ for that least operator. Recall that $j_{\mathcal{L}\mathit{if}}$ is strictly in between $j_\emptyset$ and $j_{\emptyset^{(1)}}$. This can be generalized to $j_{\emptyset^{(n)}} < j_{\mathcal{L}\mathit{if}^{(n)}} < j_{\emptyset^{(n+1)}}$ for any $n \geq 0$ since the proof can be relativized to $\emptyset^{(n)}$.

By using a "realizer" of $\varphi_{\Pi_{n+1}}$ ($p_{n+1}$ in Lemma 50), we can prove the equivalence of $[\![\psi]\!]_{\mathcal{E}\mathit{ff}^{(n)}} \rightarrowtail [\![\neg\neg\psi]\!]_{\mathcal{E}\mathit{ff}^{(n)}}$ and $U_{\mathcal{L}\mathit{if}^{(n)}}$ as subobject in $\mathcal{E}\mathit{ff}^{(n)}$, where $\psi$ is a universal formula for $\Pi_{n+1} \vee \Pi_{n+1}$. By reasoning similarly to the proof of Theorem 45, we conclude

▶ **Theorem 46.** *For any $n \geq 0$, $j^{\mathcal{E}\mathit{ff}}_{\Pi_{n+1}\vee\Pi_{n+1}\text{-}\mathbf{DNE}} = j_{\mathcal{L}\mathit{if}^{(n)}}$.*

We have thus determined all least operators of semi-classical axioms in $\mathcal{E}\mathit{ff}$ (Figure 5).

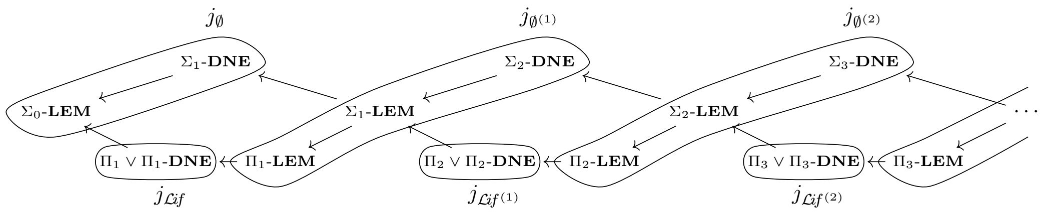


**Figure 5** Summary of least operators in $\mathbf{Lop}(\mathcal{Eff})$.

## 5 Conclusion

As we explained in Subsection 3.1, least dense operators behave as "invariants" under **HA**-provability. Figure 5 gives us the complete information about separation of semi-classical axioms in Figure 1 by subtoposes of $\mathcal{Eff}$.

▶ **Corollary 47.**

**(1)** *Any two axioms belonging to different circles in Figure 5 are separable.*

**(2)** *Those in the same circle are never separable by any subtopos of $\mathcal{Eff}$.*

While the first part of Corollary 47 is already established in [1], the second part is genuinely our original contribution. In addition:

- We have a complete characterization of separability of semi-classical axioms by a subtopos. Take $\Sigma_n$-**DNE** and $\Pi_n \vee \Pi_n$-**DNE** as an example. It follows from the nature of least operator that for any $k \in \mathbf{Lop}(\mathcal{Eff})$, $\mathcal{Eff}_k$ separates them if and only if $j_{\emptyset^{(n-1)}} \leq k$ and $j_{\mathcal{Lif}^{(n-1)}} \not\leq k$. This is a refinement of [1] and indicates that realizability notions they used are "optimal" in the sense of minimality.
- In addition to the separation results explained above, [1] also separates $\Pi_n$-**LEM** and $\Sigma_n$-**LEM** by using *monotone modified realizability*. As a by-product of Corollary 47 (2), we find that this realizability notion cannot be captured by a subtopos of $\mathcal{Eff}$. We are thus led to look for another suitable topos. A candidate is the topos $\mathcal{Eff}_{\cdot\to\cdot}$ proposed by van Oosten [24], that contains the *modified realizability topos* in addition to $\mathcal{Eff}$. In future work, we plan to explore such richer toposes and to determine the least dense operators of various axioms in them.
- Our detailed analysis of transparency gives a systematic account on previous work on least dense operators by Caramello and van Oosten. Since we have worked on an arbitrary topos, our results in Section 3 may also be applied to another semantics instead of realizability, *sheaf semantics* including Kripke frame semantics and Beth semantics [3].
- The major advantage of our framework is that we acquire a new methodology to prove impossibility of separation (Corollary 47 (2)). In further study of intuitionistic arithmetic, other variants of semi-classical axioms have been proposed, but many of them have not yet been separated [9]. The least dense operators may allow us to analyze the "difficulty" of separation from a topos-theoretic point of view.

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# A $\mathrm{Trp}^{\mathcal{E}}$ and $\Pi_3$

We here give an example of a $\Pi_3$-formula that is non-transparent. In fact, non-transparent formulas are easily found as in the following lemma.

▶ **Lemma 48.** *Let $\mathcal{E}$ be a topos and $\varphi$ a formula such that $\mathcal{E} \models \varphi$ but $\mathcal{E}_{\neg\neg} \not\models \varphi$. Then $\langle\varphi\rangle^{\mathcal{E}} \neq \emptyset$ and $\varphi$ has no least dense operator in $\mathcal{E}$. So $\varphi$ is not transparent in $\mathcal{E}$.*

**Proof.** By assumption, we have $\mathrm{id}_\Omega \in \langle\varphi\rangle^{\mathcal{E}}$, so $\langle\varphi\rangle^{\mathcal{E}} \neq \emptyset$. Now assume that $\varphi$ has least dense operator $j^{\mathcal{E}}_\varphi$ in $\mathcal{E}$. Then, $j^{\mathcal{E}}_\varphi = \mathrm{id}_\Omega$ holds by minimality. This leads to $\langle\varphi\rangle^{\mathcal{E}} = \mathbf{DLop}(\mathcal{E})$, and in particular $\neg\neg \in \langle\varphi\rangle^{\mathcal{E}}$, which contradicts the assumption that $\mathcal{E}_{\neg\neg} \not\models \varphi$. ◀

This suggests that "constructively true" but "classically false" formulas are likely to be non-transparent. Let us consider the effective topos $\mathcal{E}\mathit{ff}$ and recall that for a closed $\mathcal{L}_{\mathrm{A}}$-formula $\psi$, $\mathcal{E}\mathit{ff} \models \psi$ iff $\psi$ is Kleene realizable. With this in mind, define $\Sigma_2$-formula $H(x, y)$ to be the graph of the characteristic function of *Halting problem*. That is,

$$H(x, y) := \exists v \forall u((T(x, x, u) \to y = 1) \wedge (\neg T(x, x, v) \to y = 0)).$$

Furthermore, define *Church's thesis* with respect to $H$ by

$$\mathrm{CT}^H_0 := \forall x \exists y H(x, y) \to \exists e \forall x \exists w(H(x, U(w)) \wedge T(e, x, w)),$$

where $U$ is Kleene's $U$-function. Then $\mathcal{E}\mathit{ff} \models \mathrm{CT}^H_0$ is obtained by the fact that Church's thesis is Kleene realizable [22]. However, $\mathcal{E}\mathit{ff}_{\neg\neg} \not\models \mathrm{CT}^H_0$ because $\mathcal{E}\mathit{ff}_{\neg\neg}$ is equivalent to the category **Set** of sets.

▶ **Theorem 49.** $\Pi_3 \not\subseteq \mathrm{Trp}^{\mathcal{E}\mathit{ff}}$ *in the effective topos $\mathcal{E}\mathit{ff}$.*

**Proof.** It is obvious that the antecedent of $\mathrm{CT}^H_0$ is $\Pi_3$ and the consequent is equivalent to a $\Sigma_4$-formula in **HA**. In addition, the converse of $\mathrm{CT}^H_0$ is provable in **HA**.

Now assume that $\Pi_3 \subseteq \mathrm{Trp}^{\mathcal{E}\mathit{ff}}$. Then $\Sigma_4 \subseteq \mathrm{Trp}^{\mathcal{E}\mathit{ff}}$ by Lemma 25 (1), hence $\mathrm{CT}^H_0$ has least dense operator in $\mathcal{E}\mathit{ff}$ by Lemma 24 (2). This contradicts Lemma 48. ◀

# B Proof of Theorem 45 in the general case

To show Theorem 45, we provide generalized versions of $p_1$, $s_2$ and Lemma 44 in Subsection 4.2. Recall that $S(e, x)$ denotes the function from the parameter theorem. Then we can inductively define universal formulas for $\Sigma_n$ and $\Pi_n$ by $\varphi_{\Sigma_{n+1}} := \exists y\, \varphi_{\Pi_n}(S(e, x), y)$ and $\varphi_{\Pi_{n+1}} := \forall y\, \varphi_{\Sigma_n}(S(e, x), y)$.

▶ **Lemma 50.** *For every $n \geq 0$, there are a* total *$\emptyset^{(n)}$-computable function $p_{n+1}(e, x)$ and a* partial *$\emptyset^{(n)}$-computable function $s_{n+1}(e, x)$ such that the following two conditions hold:*

**(1)** *For any $e$, $x \in \mathbb{N}$, $\mathbb{N} \models \varphi_{\Pi_{n+1}}(e, x)$ implies $p_{n+1}(e, x)\ \mathbf{r}^{(n)}_{\mathrm{K}}\ \varphi_{\Pi_{n+1}}(e, x)$.*

**(2)** *For any $e$, $x \in \mathbb{N}$, $\mathbb{N} \models \varphi_{\Sigma_{n+1}}(e, x)$ implies $s_{n+1}(e, x)\ \mathbf{r}^{(n)}_{\mathrm{K}}\ \varphi_{\Sigma_{n+1}}(e, x)$.*

**Proof.** By induction on $n$. For $n = 0$, we have already given $p_1(e, x) := \lambda w.0$. Define a partial computable function $s_1$ by $s_1(e, x) := \langle w_0, 0\rangle$, where $w_0$ is the code of computation history of $e \cdot x$ when $e \cdot x \downarrow$. It is clear from the description of $\varphi_{\Sigma_1}(e, x)$ that (2) holds.

Next assume that the statement holds for $n$. Let us define $p_{n+2}(e, x) := \lambda y.s_{n+1}(S(e, x), y)$ and $s_{n+2}(e, x) := \langle y_0, p_{n+1}(S(e, x), y_0)\rangle$, where $y_0$ is the least number such that $\varphi_{\Pi_{n+1}}(S(e, x), y_0)$ is true in $\mathbb{N}$ if it exists. Note that $p_{n+2}$ can be obtained as a total $\emptyset^{(n)}$-computable function and $s_{n+2}$ as a partial $\emptyset^{(n+1)}$-computable one. Then, for any $e, x \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{N} \models \varphi_{\Pi_{n+2}}(e,x) &\implies \forall y \in \mathbb{N} \quad \mathbb{N} \models \varphi_{\Sigma_{n+1}}(S(e,x),y) \\
&\implies \forall y \in \mathbb{N} \quad s_{n+1}(S(e,x),y)\ \mathbf{r}_{\mathrm{K}}^{(n)}\ \varphi_{\Sigma_{n+1}}(S(e,x),y) \\
&\implies p_{n+2}(e,x)\ \mathbf{r}_{\mathrm{K}}^{(n+1)}\ \forall y\, \varphi_{\Sigma_{n+1}}(S(e,x),y) \\
\mathbb{N} \models \varphi_{\Sigma_{n+2}}(e,x) &\implies \exists y \in \mathbb{N} \quad \mathbb{N} \models \varphi_{\Pi_{n+1}}(S(e,x),y) \\
&\implies \exists y \in \mathbb{N} \quad (\mathbb{N} \models \varphi_{\Pi_{n+1}}(S(e,x),y) \wedge p_{n+1}(S(e,x),y)\ \mathbf{r}_{\mathrm{K}}^{(n)}\ \varphi_{\Pi_{n+1}}(S(e,x),y)) \\
&\implies s_{n+2}(e,x)\ \mathbf{r}_{\mathrm{K}}^{(n+1)}\ \exists y\, \varphi_{\Pi_{n+1}}(S(e,x),y). \qquad \blacktriangleleft
\end{aligned}
$$

The condition (2) of Lemma 50 implies that $\lambda ex.\lambda m.s_{n+1}(e,x)$ realizes $^\forall(\neg\neg\varphi_{\Sigma_{n+1}} \to \varphi_{\Sigma_{n+1}})$ under $\emptyset^{(n)}$-realizability. This means that we have confirmed Lemma 43.

Next, let us generalize Lemma 44. It is simply given by relativizing to $\emptyset^{(n)}$, so the proof is obtained exactly in the same way.

▶ **Lemma 51.** *Let $\varphi(x)$ be an $\mathcal{L}_{\mathrm{A}}$-formula and let $\chi_\varphi$ be the characteristic function of $\{\, m \in \mathbb{N} \mid \mathbb{N} \models \varphi(m) \,\}$. Suppose further that there is a* total *$\emptyset^{(n)}$-computable function $p(x)$ such that the following two conditions hold:*
**(a)** *For any $m \in \mathbb{N}$, $\mathcal{E}\!f\!f^{(n)} \models \varphi(m)$ implies $\mathbb{N} \models \varphi(m)$.*
**(b)** *For any $m \in \mathbb{N}$, $\mathbb{N} \models \varphi(m)$ implies $p(m)\ \mathbf{r}_{\mathrm{K}}^{(n)}\ \varphi(m)$.*
*Then $[\![\neg\varphi \vee \varphi]\!]^{(n)}_{\mathbf{Mfunc}} \equiv_s \chi_\varphi$ in $\mathrm{Sub}_{\mathcal{E}\!f\!f^{(n)}}(N)$.*

Let $\varphi := \varphi_{\Pi_{n+1}}$ and $p := p_{n+1}$. Note that $\varphi \in \Pi_{n+1}$ is transparent in $\mathcal{E}\!f\!f^{(n)}$ by Lemma 29 and Lemma 43, so $\varphi \vee \neg\varphi$ has least operator $j^{\mathcal{E}\!f\!f^{(n)}}_{\varphi\vee\neg\varphi}$ in $\mathcal{E}\!f\!f^{(n)}$. By Lemma 51 and the same reasoning as in the proof for $n = 0$, we have the equation below in $\mathbf{Lop}(\mathcal{E}\!f\!f^{(n)})$:

$$j^{\mathcal{E}\!f\!f^{(n)}}_{\varphi\vee\neg\varphi} = \ell^{\mathcal{E}\!f\!f^{(n)}}_{[\![\varphi\vee\neg\varphi]\!]_{\mathcal{E}\!f\!f^{(n)}}} = \ell^{\mathcal{E}\!f\!f^{(n)}}_{\chi_\varphi}. \qquad (\blacklozenge)$$

The following lemma is the final piece to establish Theorem 45.

▶ **Lemma 52.** *Let $j$ be a local operator in a topos $\mathcal{E}$ and $U \rightarrowtail X$ a subobject in $\mathcal{E}$. Further suppose that $j \le \ell^{\mathcal{E}}_U$, where $\ell^{\mathcal{E}}_U$ is the least operator of $U$ in $\mathcal{E}$. Then $\ell^{\mathcal{E}}_U \in \mathbf{Lop}(\mathcal{E})^{\ge j}$ corresponds to the least operator $\ell^{\mathcal{E}_j}_{L_jU} \in \mathbf{Lop}(\mathcal{E}_j)$ of the subobject $L_jU \rightarrowtail L_jX$ in $\mathcal{E}_j$ in the sense of Notation 30.*

**Proof.** Fix $k \in \mathbf{Lop}(\mathcal{E})^{\ge j}$ and the corresponding local operator $k_j$ in $\mathbf{Lop}(\mathcal{E}_j)$. Note that $L_k \simeq L_{k_j} \circ L_j$ holds since $\mathcal{E}_k$ is equal to the subtopos $(\mathcal{E}_j)_{k_j}$ of $\mathcal{E}_j$ corresponding $k_j$. Hence $L_kU \rightarrowtail L_kX$ is an isomorphism if and only if so is $L_{k_j}(L_jU) \rightarrowtail L_{k_j}(L_jX)$. Recalling Lemma 6, we obtain that $\ell^{\mathcal{E}}_U \le k$ in $\mathbf{Lop}(\mathcal{E})^{\ge j}$ iff $U$ is $k$-dense in $\mathcal{E}$ iff $L_{k_j}(L_jU)$ is $k_j$-dense in $\mathcal{E}_j$ iff $\ell^{\mathcal{E}_j}_{L_jU} \le k_j$ in $\mathbf{Lop}(\mathcal{E}_j)$. This means that $\ell^{\mathcal{E}}_U$ corresponds to $\ell^{\mathcal{E}_j}_{L_jU}$. ◀

In particular, considering $X := N$, $U := \chi_\varphi$ and $j := j_{\emptyset^{(n)}}$ in $\mathcal{E}\!f\!f$, we have the correspondence between $\ell^{\mathcal{E}\!f\!f}_U = j_{\emptyset^{(n+1)}} \in \mathbf{Lop}(\mathcal{E}\!f\!f)$ and $\ell^{\mathcal{E}\!f\!f^{(n)}}_{L_jU} \in \mathbf{Lop}(\mathcal{E}\!f\!f^{(n)})$. In addition, $\ell^{\mathcal{E}\!f\!f^{(n)}}_{L_jU}$ can be regarded as the least operator $\ell^{\mathcal{E}\!f\!f^{(n)}}_{\chi_\varphi}$ because the subobject $L_jU \rightarrowtail L_jN(= N_j)$ can be described by a multifunction $\chi_\varphi$ in $\mathcal{E}\!f\!f^{(n)}$. So we have the following correspondence:

$$\ell^{\mathcal{E}\!f\!f^{(n)}}_{\chi_\varphi} \in \mathbf{Lop}(\mathcal{E}\!f\!f^{(n)}) \quad \longleftrightarrow \quad j_{\emptyset^{(n+1)}} \in \mathbf{Lop}(\mathcal{E}\!f\!f). \qquad (\spadesuit)$$

Thus, combining this correspondence ($\spadesuit$) with the equation ($\blacklozenge$), we obtain

▶ **Theorem 53.** *For any $n \ge 0$, $j^{\mathcal{E}\!f\!f^{(n)}}_{\Pi_{n+1}\text{-}\mathbf{LEM}} \in \mathbf{Lop}(\mathcal{E}\!f\!f^{(n)})$ corresponds to $j_{\emptyset^{(n+1)}} \in \mathbf{Lop}(\mathcal{E}\!f\!f)$.*

Finally, we prove Theorem 45 by using the iteration argument developed in Subsection 3.4.

**Proof of Theorem 45.** It is sufficient to show that $j^{\mathcal{E}f\!f}_{\Pi_n\text{-}\mathbf{LEM}} = j_{\emptyset^{(n)}}$ by induction on $n$. The base case is already discussed in Subsection 4.2.

Assume that it holds for $n$; in particular, $k \coloneqq j^{\mathcal{E}f\!f}_{\Sigma_n\text{-}\mathbf{LEM}} = j_{\emptyset^{(n)}}$ holds. Recall that $\Pi_{n+1}$-**LEM** implies $\Sigma_n$-**LEM** over **HA** (Figure 1). By Lemma 32, the least operator $j^{\mathcal{E}f\!f}_{\Pi_{n+1}\text{-}\mathbf{LEM}}$ of $\Pi_{n+1}$-**LEM** in $\mathcal{E}f\!f$ corresponds to the least operator $j^{\mathcal{E}f\!f_k}_{\Pi_{n+1}\text{-}\mathbf{LEM}}$ of that in $\mathcal{E}f\!f_k$, which is described as $j^{\mathcal{E}f\!f^{(n)}}_{\Pi_{n+1}\text{-}\mathbf{LEM}}$. Therefore, we conclude $j^{\mathcal{E}f\!f}_{\Pi_{n+1}\text{-}\mathbf{LEM}} = j_{\emptyset^{(n+1)}}$ by Theorem 53. ◀